Chapter 3: Two-dimensional Steady State Conduction

3.1 The Heat Conduction Equation
- Examples of two-dimensional steady state, constant properties heat equations.

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - \frac{\rho c_p U}{k} \frac{\partial T}{\partial x} + \frac{q}{k} = 0
\]

(3.1)

- This equation includes the effects of motion and heat generation. If these factors are neglected, we obtain

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0
\]

(3.2)

- Equations (3.1) and (3.2) are special cases of a more general partial differential equation of the form

\[
f_2(x) \frac{\partial^2 T}{\partial x^2} + f_1(x) \frac{\partial T}{\partial x} + f_0(x)T + g_2(y) \frac{\partial^2 T}{\partial y^2} + g_1(y) \frac{\partial T}{\partial y} + g_0(y)T = 0
\]

(3.4)

- This equation is: (1) second order PDE, (2) homogeneous, (3) linear, and (4) has variable coefficients.

- Solution to this type of PDE can be obtained by the method of separation of variables.

3.2 Method of Solution and Limitations
- The method of separation of variables is limited to: (1) linear PDE, and (2) the geometry is described by orthogonal coordinates.

- Basic idea: Replace PDE with two sets of ODE.

3.3 Homogeneous Differential Equations and Boundary Conditions
- The key to the application of the method of separation of variables is understanding the definition of homogeneous DE and BC.

3.4 Sturm-Liouville Boundary-Value Problem: Orthogonality
- One of the two sets of the ODE replacing the PDE is of a type known as Sturm-Liouville problem.

- The general form of this problem is

\[
\frac{d^2 \phi_n}{dx^2} + a_1(x) \frac{d \phi_n}{dx} + \left[ a_2(x) + \lambda_n a_3(x) \right] \phi_n = 0
\]

(3.5a)

- \( \lambda_n \) takes on many values depending on \( n \).
Note that this equation represents a set of \( n \) equations corresponding to \( n \) values of \( \lambda_n \). Thus there are \( \phi_n \) solutions. These solutions are known as characteristic functions.

An important property of eq. (3.5a) is called orthogonality. This property is
\[
\int_{a}^{b} \phi_n(x) \phi_m(x) w(x) \, dx = 0 \quad n \neq m
\]  
(3.7)

The function \( w(x) \) is known as the weighting function. It is obtained from eqs. (3.5) and (3.6).

Eq. (3.7) is valid if: the boundary conditions at \( x = a \) and \( x = b \) are homogenous of the form
\[
\phi_n = 0 \quad \text{(3.8a)}
\]
\[
\frac{d\phi_n}{dx} = 0 \quad \text{(3.8b)}
\]
\[
\phi_n + \beta \frac{d\phi_n}{dx} = 0 \quad \text{(3.8c)}
\]

Note the physical significance of these three conditions.

### 3.5 Procedure for the Application of Separation of Variables Method

**Example 3.1 Conduction in a Rectangular Plate**

In studying this example pay particular attention to:

1. The selection of an origin and coordinate axes.
2. Identifying the applicable PDE (eq. 3.2).
3. Identifying the variable with two homogeneous BC.
4. Writing down four BC in the recommended order.
5. Assuming a product solution (eq. a).
6. Substituting the product solution into PDE and constructing two sets of ODE (eqs. e and f).
7. **Deciding which equation (e or f) should take the positive sign.** Review instruction on how to make this decision.
8. Constructing two sets of ODE (eqs. g and h).
9. Including the special case of \( \lambda_n = 0 \) (eqs. i and j).
10. Solving the four ODE.
11. Applying BC **in the order listed**.

### 3.6 Cartesian Coordinates: Examples

Study Examples 2-4 noting the systematic steps followed in obtaining solutions.

### 3.7 Cylindrical Coordinates: Examples
Two-dimensional steady states conduction in cylindrical systems is treated in the same manner as conduction in Cartesian systems: The PDE is replaced by a set of two ODE.

One of the two sets of ODE often turns out to be a Bessel equation. If the Bessel equation has two homogeneous boundary conditions, orthogonality is applied to Bessel functions. Note how the weighting function is determined.

Make use of Table 3.1 and Appendix B in evaluating Bessel function integrals resulting from the application of orthogonality.

Study Examples 3.5 and 3.6 noting the systematic steps followed in obtaining solutions.

The heat equation in Example 3.6 is non-homogeneous

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \theta}{\partial r} \right) + \frac{\partial^2 \theta}{\partial z^2} + \frac{q^m}{k} = 0
\]  

(3.22)

To deal with the non-homogeneous term in eq. (3.22) we assume a solution of the form

\[
\theta(r, z) = \psi(r, z) + \phi(z)
\]  

(a)

Substituting (a) into eq. (3.22) and separating the resulting equation into two equations, we obtain

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial z^2} = 0
\]  

(c)

\[
\frac{d^2 \phi}{dz^2} + \frac{q^m}{k} = 0
\]  

(d)

The four boundary conditions must generate six conditions for eqs. (c) and (d) by substituting solution (a) into the four boundary conditions.