IDENTIFYING PRODUCT BEHAVIOUR USING CONSTITUTIVE EQUATION GAP METHOD

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Abstract: The Constitutive Equation Gap Method (CEGM) is a well-known concept which, until now, has been used mainly for the verification of finite element simulations. This has led to many developments, especially concerning the techniques for constructing statically admissible stress fields. At the same time, the identification of material model parameters is based more and more on full-field measurements. The originality of the present study resides in the application of construction of a CEGM functional to identify heterogeneous isotropic elastic parameters. The results obtained are compared with the actual material parameters. This is made possible with the use of synthetic data: the experimental data is generated numerically from a set of exact material parameters. The data is without any noise, this is a first step. Next step will consist in studying the influence of the noise on data.

Key words: Identification · Finite Element.

1- Introduction

Experimental techniques for the identification of the parameters of material models have evolved considerably with the recent development of new experimental devices that provide full-field measurements of various physical quantities of interest. Now, affordable numerical devices enabling discrete measurements of 2D and/or 3D displacement fields, thermal fields...are available [SW1, SC1]. Due to the complexity of such corpuses of information, new post-processing techniques must be developed in order to extract the quantities of interest from the experimental data.

Many works focus on the identification of mechanical material properties based on displacement fields [KP1, PL1, PG1]. Some interesting techniques have been proposed for dealing with nonlinear material behavior [LC1, GP1]. Other interesting techniques based on an appropriate filtering of the experimental data coupled with global sensitivity analysis are available [L1], but these remain marginal. All these techniques rely on fundamental and well-known works on linear material behavior. Up to now, five techniques have been proposed: the virtual field method [G1, GT1, AH1], the finite element model updating method [KC1, LV1], the reciprocity gap method [I1], the equilibrium gap method [CH1, CH2], and the constitutive equation gap method [GH1, CD1, FL1]. A very good review of these techniques can be found in [AB1]. Here, we will focus on the CEGM.

Initially, the CEGM concept was not developed in the framework of identification, but for the verification and validation (V&V) of finite element simulations. Its use has led to interesting results concerning the estimation of global or local [FG2] errors in finite element simulations. An important feature of this technique is its ability to build admissible stress fields when the boundary conditions of the reference problem are known.

The objective of this paper is to present the improvements which can be expected from the application of the specific techniques developed in the field of V&V to the identification framework.

Following this introduction, Section 2 presents a brief review of the identification problem. The principle of CEGM technique is described in Section 3. In the last section, we present an illustration. We identify the material parameters of a heterogeneous elastic material simultaneously (Young’s modulus and Poisson’s ratio). The results show that enhanced CEGM leads to a very good identification of both Young’s modulus and Poisson’s ratio.

We use synthetic data, in order to qualify the quality of the proposed method. This is clearly a simplistic point of view, that avoid the classical problems linked to actual incomplete data, but that makes it possible to know the actual values of the material parameter. This is a first step that shows the basic qualities of the developed method. Next step is to use experimental data, with noise as in [AF1], and another step is to extend to non-linear cases. The CEGM method is easily extendable to non-linear mechanics using the dissipation error concept, which is a natural prolongation of the constitutive relation error.
2- Identification problem

2.1 – Governing equations

Let us consider a bounded domain \( \Omega \) and its boundary \( \partial \Omega \). More precisely, we assume that the boundary \( \partial \Omega \) is the union of two subsets \( S_f \) and \( S_u \), such that \( \partial \Omega = S_f \cup S_u \) and \( S_f \cap S_u = \emptyset \). Over \( S_f \), the traction field \( \mathbf{T} \) is prescribed, while over \( S_u \), the displacement field \( \mathbf{u} \) is prescribed. Domain \( \Omega \) is assumed to be filled with a heterogeneous isotropic elastic material.

![Figure 1: Domain \( \Omega \)](image)

The equations governing this linear elastic problem are based on three sets of differential equations for the displacement field (denoted \( u \)) and the stress field (denoted \( \sigma \)).

The first set contains the classical equilibrium equations with the traction field \( \mathbf{T} \), assuming zero body forces:

\[
\begin{align*}
\text{div} \mathbf{\sigma} &= \mathbf{0} & \text{in} \ \Omega \\
\mathbf{\sigma} \cdot \mathbf{n} &= \mathbf{T} & \text{on} \ \partial \Omega
\end{align*}
\]  

where \( \text{div} \) is the divergence operator and \( n \) the external unit vector normal to \( S_f \).

The second set contains the kinematic compatibility equations, consisting of the strain-displacement equation and the compatibility with the prescribed value \( \mathbf{u} \):

\[
\begin{align*}
\varepsilon[u] &= \frac{1}{2} (\nabla \mathbf{u} + \nabla^T \mathbf{u}) & \text{in} \ \Omega \\
\mathbf{u} &= \mathbf{u}_p & \text{on} \ S_u
\end{align*}
\]  

where \( \nabla \) and \( \nabla^T \) denote the gradient and its transpose respectively.

The last set of equations contains the constitutive relations, using Hooke’s law:

\[
\mathbf{\sigma} = \mathbf{A} : \varepsilon \ \text{in} \ \Omega
\]  

where \( \mathbf{A} \) is Hooke’s operator.

Classically, one introduces the associated admissible spaces \( S \) and \( \mathcal{C} \).

- \( S \) is the set of the statically admissible stress fields corresponding to Equation (1):

\[
S = \{ \mathbf{\tau} \in \mathcal{V}_\sigma, \ \text{div} \mathbf{\tau} = \mathbf{0} \ \text{in} \ \Omega \ \text{and} \ \mathbf{\tau} \cdot \mathbf{n} = \mathbf{T} \ \text{on} \ S_f \} \tag{4}
\]

where \( \mathcal{V}_\sigma = [L^2(\Omega)]^6 \) is the space in which the stress is sought.

- \( \mathcal{C} \) is the set of the kinematically admissible displacement fields corresponding to Equation (2):

\[
\mathcal{C} = \{ \mathbf{v} \in \mathcal{V}_u, \ \mathbf{v} = \mathbf{u} \ \text{on} \ S_u \} \tag{5}
\]

where \( \mathcal{V}_u = [H^1(\Omega)]^3 \) is the space in which the displacement field is sought.

Another space is introduced: \( \mathcal{A} \) is the set of the admissible elastic tensor fields, which corresponds to fourth-order tensor functions verifying the symmetry, positive definiteness and coercivity properties expected from any linear elastic material.

2.2 – The identification problem

Let us consider the problem of the identification of material properties using full-field measurements. The objective is to determine the best parameters of \( S \) given these measurements. Let \( \mathbf{u} \) denote the experimental measurements, which have been obtained using a numerical video device and the associated digital image correlation process. The sampled displacements are assigned to the vertices of a coarse grid (Figure 2). Then, it is interpolated classically over a classical quadrangle Finite. Other interpolation options are possible, but we will study only this classical choice.

![Figure 2: Measurement grid (10x10 example)](image)

3- The Constitutive Equation Gap Method

3.1 – General presentation

The CEGM measures the distance between an admissible stress \( \mathbf{\tau} \) (i.e., a stress belonging to \( S \)) and an admissible displacement \( \mathbf{v} \) (i.e., a displacement belonging to \( S_u \)). This distance is quantified by an energy norm, which defines what is called the Constitutive Equation Gap (CEG):

\[
\mathcal{E}(\mathbf{v}, \mathbf{\tau}, \mathbf{A}) = \frac{1}{2} \int_{\Omega} \left( (\mathbf{\tau} - \mathbf{A} : \varepsilon[\mathbf{v}]) : \mathbf{A}^{-1} : (\mathbf{\tau} - \mathbf{A} : \varepsilon[\mathbf{v}]) \right) \, d\Omega
\]  

(6)

Remark: this functional is zero if the stress field corresponds to the actual stress field and the material field corresponds to the actual field.

The identification procedure consists in minimizing the CEG:

\[
\mathbf{A}^* = \arg \min_{\mathbf{A}^* \in \mathcal{A}} J(\mathbf{A}^*)
\]
The key point of the method is now the technique used for describing the space. The method that we use here come from the improved construction used classically in verification. It had term to function.

The identification procedure consists in minimizing the CEG:

\[ \varepsilon_l(A^*) \]

with \( J(A^*) = \min_{(\nu, \tau) \in C \times S} \varepsilon(\nu, \tau, A^*) \) (7)

The material is linear elastic, and we assume plane stresses in the x–y plane. This continuous medium was divided into a grid of 10x10 sub-domains (100 finite elements).

The material was considered to be homogenous within each sub-domain and, thus, completely defined by the local Young’s modulus and Poisson’s ratio. Figure 4 shows the material configuration retained.

### 3.2 – Construction of \( \tau \)

The method we use is derived from the improved construction used classically in verification [FG2]. This method has been used to control and adapt meshes [FG3] and, more recently, to verify problems in the stochastic framework [LF1]. The main points for building an admissible stress is:

a) Construct equilibrated tractions from the data on each edge of each element.

b) Solve an elastic problem on each element using a very fine mesh.

The technical construction is not detailed in this paper, the reader can refer to [LC2, FG1] for more details about this technical construction. Two variant are presented in these works. The first [LC2] has a very low cost and leads to a good global estimate of the error. The second [FG1] is more costly, especially in 3D, but has a very strong mechanical content at the local level. We chose the second approach in this work because, we are in 2D and the problem of cost vanishes here, and the local quality of this functional is important here (due to heterogeneity of the material).

The main ideas of the construction used here is to solve the dual problem to construct the admissible stress field. In the first step (a), we chose a discretization for the equilibrated traction and then minimize the complementary energy. In the second step (b), we chose a discretization for the displacement (finite element type) and then minimize the potential energy.

### 4- Numerical illustration

#### 4.1 – Test case considered

In this section, we consider an idealized reference problem in order to validate the method discussed above. The structure being studied is a square (10mm×10mm) in the x–y plane (Figure 3). The loading we consider is traction:

![Figure 3: Problem configuration.](image3)

The material was used to control and adapt meshes [FG3] and, more recently, to verify problems in the stochastic framework [FG1]. For example if \( \tau^r \) we’ll also denote \( \tau \).

The choice is here done to use \( \tau^r \) a kinematically admissible field:

\[ \tau^r u \in C \]

\[ \tau^r \subset \tau \]

\[ \tau^r \cap \tau \]

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4.2 – Numerical results

The results obtained by the CEGM are presented below. The identified values are plotted for the Young’s modulus on Figure 6. We can observe that the identification seems very representative of the actual material (Figure 4).

In order to be more precise, we introduce kind of effectivity index. The quantity identified in the sub-domains is normalized by the exact local values (Figure 7) and (Figure 8). This means that for any sub-domain the following quantities (effectivity index) are plotted:

\[ i_E = \frac{E}{E_{ex}} \quad \text{and} \quad i_\nu = \frac{\nu}{\nu_{ex}} \]  

\[ (9) \]

A clear observation the CEGM tends to return good results.

We also indicate, for quantifying purposes, the relative error between the identified and exact values, defined for each subdomain as:

\[ e_E = \left| \frac{E}{E_{ex}} - 1 \right| \quad \text{and} \quad e_\nu = \left| \frac{\nu}{\nu_{ex}} - 1 \right| \]  

\[ (10) \]

For the set of identified zone we can consider the mean and the standard distribution and the maximum of the error \( e_E \) and \( e_\nu \).

For the Young’s modulus, the mean error is 5.02%, the standard deviation of error is 6.04% and the maximum error is 22.05%.

For the Poisson’s ratio, the mean error is 11.35%, the standard deviation of error is 9.60% and the maximum error is 37%.

These results show that the Young’s modulus is identified with a sharper quality than the Poisson’s ratio. We can also observe that the global identification is good. Locally the error can be more important: the maximum error is obtained on the transition between the two materials. This problem should disappear using a finer grid for observation. The numerical problem designed here is an academic one for purpose of illustration.

7- Conclusion

This paper illustrates the use of the CEGM to identify the parameters of a structure’s linear elastic behavior from full-field measurements. We presented a new application of the CEGM. This method had already been proven effective in the V&V domain; the objective of the paper is to show that it can also be effective in the context of identification. The numerical example illustrates the quality of the identification. Future works will focus on the influence of noise on the identification, the use of actual experimental data, extension of the method to nonlinear problems.
7- References


