Chapter 7
Fractional Derivatives for Erdélyi-Kober Operators and Statistical Densities in the Real and Complex Matrix-Variate Cases

7.1 Introduction

Fractional integrals and fractional derivatives in the real scalar variable case and their applications in stochastic processes and random walk problems may be seen from many papers, see for example Gorenflo and Mainardi [1]. Solutions of fractional differential equations in the real variable case may be seen, for example, from Haubold, Mathai and Saxena [2]. There are not many papers on fractional integrals in the matrix-variate case. Some discussions on functions of matrix argument may be seen from the earlier chapters and from Mathai [8, 10, 12], Mathai and Haubold [14]. Some aspects of fractional derivatives in the matrix-variate case are discussed in Mathai [8, 12, 13]. Some generalized fractional calculus is discussed in Kiryakova [4], Kiryakova et al. [5]. Some geometrical aspects are discussed in Herrmann [3], Mathai [11]. Some aspects of Erdélyi-Kober fractional calculus are discussed in Luchko [6], Luchko et al. [7], Pagnini [16], Plociniczak [17], Sneddon [18]. In the present chapter we introduce fractional differential operators in the real and complex matrix-variate cases and applicable when the arbitrary function of matrix argument has certain structures. As an illustration of the matrix-variate differential operators, Erdélyi-Kober fractional differential operators of the first and second kinds are discussed in the Riemann-Liouville and Caputo senses. The standard notations from earlier chapters and various results from earlier chapters will be made use of in the present chapter. We consider the real matrix-variate case first and then we will look into matrices on the complex domain.
7.2 Some Fractional Differential Operators in the Real Matrix-Variate Case

Let \( U = (u_{ij}) \) be \( p \times p \) matrix of distinct real variables. Let \( \frac{\partial^*}{\partial U} = (\eta_{ij} \frac{\partial}{\partial u_{ij}}) \) where
\[
\eta_{ij} = \begin{cases} 
1, & i = j \\
\frac{1}{2}, & i \neq j.
\end{cases}
\]
Let \( U = U' \) and \( X = X' \) be \( p \times p \) real symmetric matrices. Then
\[
\frac{\partial^*}{\partial U} \left[ e^{-\text{tr}(UX)} \right] = -\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\
x_{21} & x_{22} & \cdots & x_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
x_{p1} & x_{p2} & \cdots & x_{pp} \end{bmatrix} \ e^{-\text{tr}(UX)} = -X e^{-\text{tr}(UX)},
\]
for \( x_{ij} = x_{ji}, u_{ij} = u_{ji} \) for all \( i \) and \( j \). Let us consider the case \( p = 2 \). Then
\[
\frac{\partial^*}{\partial u_{12}} \left[ e^{-\text{tr}(UX)} \right] = -x_{12} e^{-\text{tr}(UX)} \quad \text{and} \quad \frac{1}{2} \frac{\partial}{\partial u_{12}} e^{-\text{tr}(UX)} = -\frac{1}{2} x_{12} e^{-\text{tr}(UX)}, \quad i \neq j.
\]
Then
\[
\left[ \frac{\partial^*}{\partial u_{22}} \frac{\partial^*}{\partial u_{11}} - \left( \frac{\partial^*}{\partial u_{12}} \right)^2 \right] e^{-\text{tr}(UX)} = (-1)^2 [x_{22} x_{11} - x_{12}^2] e^{-\text{tr}(UX)} = (-1)^2 [\det(X)] e^{-\text{tr}(UX)}.
\]

For the general \( p \), consider the determinant of the operator \( \frac{\partial^*}{\partial U} \), that is \( [\det(\frac{\partial^*}{\partial U})] \) operating on \( e^{-\text{tr}(UX)} \) then the result is \((-1)^p [\det(X)] e^{-\text{tr}(UX)} \). Then the operator \( [(-1)^p \det(\frac{\partial^*}{\partial U})]^n = [(-1)^p \det(\frac{\partial^*}{\partial U})] \ldots [(-1)^p \det(\frac{\partial^*}{\partial U})] \) operating on \( e^{-\text{tr}(UX)} \) gives \([\det(X)]^p e^{-\text{tr}(UX)} \). This determinant operator will be denoted by \( D_{U-} \). Then
\[
D_{U-}^n e^{-\text{tr}(UX)} = [\det(X)]^n e^{-\text{tr}(UX)}.
\]

Similarly, \( \frac{\partial^*}{\partial U} \) operating on \( e^{\text{tr}(UX)} \) gives \( X e^{\text{tr}(UX)} \). Consider the operator \( D_{U+}^n = [\det(\frac{\partial^*}{\partial U})]^n \). Then
\[
D_{U+}^n e^{\text{tr}(UX)} = [\det(X)]^n e^{\text{tr}(UX)}.
\]

With the help of these two operators we will establish a few basic results which will be stated as lemmas.

**Lemma 7.1** Let \( X = (x_{ij}) = X' \) be \( p \times p \) real positive definite with \( p(p + 1)/2 \) distinct real variables \( x_{ij} \)'s as elements. Then
\[
D_{U-}^n [\det(U)]^{-\gamma} = [\det(U)]^{-(\gamma+n)} \frac{\Gamma_p(\gamma+n)}{\Gamma_p(\gamma)}
\]
for \( \Re(\gamma) > \frac{p-1}{2} \), \( n = 0, 1, 2, \ldots \).
Proof Consider the following integral, for $X = X' > O$ and $U = U' > O$:

$$
\int_{X > O} [\det(X)]^{\gamma - \frac{p+1}{2}} e^{-\text{tr}(UX)} dX = \int_{X > O} [\det(X)]^{\gamma - \frac{p+1}{2}} e^{-\text{tr}(U^{\frac{1}{2}}XU^{\frac{1}{2}})} dX
$$

where $U^{\frac{1}{2}}$ is the unique positive definite square root of the positive definite matrix $U$. Let $V = U^{\frac{1}{2}}XU^{\frac{1}{2}} \Rightarrow dV = [\det(U)]^{\frac{1}{2}} dX$. Then

$$
\int_{X > O} [\det(X)]^{\gamma - \frac{p+1}{2}} e^{-\text{tr}(U^{\frac{1}{2}}XU^{\frac{1}{2}})} dX = [\det(U)]^{-\gamma} \int_{V > O} [\det(V)]^{\gamma - \frac{p+1}{2}} e^{-\text{tr}(V)} dV
$$

Hence we have the following identity:

$$
[\det(U)]^{-\gamma} = \frac{1}{\Gamma_p(\gamma)} \int_{V > O} [\det(V)]^{\gamma - \frac{p+1}{2}} e^{-\text{tr}(UV)} dV, \Re(\gamma) > \frac{p-1}{2}.
$$

Now, operate on both sides with $D_{U-}^n$ that is,

$$
D_{U-}^n[\det(U)]^{-\gamma} = \frac{1}{\Gamma_p(\gamma)} \int_{V > O} [\det(V)]^{\gamma - \frac{p+1}{2}} [D_{U-}^n e^{-\text{tr}(UV)}] dV
$$

$$
= \frac{1}{\Gamma_p(\gamma)} \int_{V > O} [\det(V)]^{\gamma + n - \frac{p+1}{2}} e^{-\text{tr}(UV)} dV
$$

$$
= \frac{\Gamma_p(\gamma + n)}{\Gamma_p(\gamma)} [\det(U)]^{-(\gamma + n)} , \Re(\gamma) > \frac{p-1}{2}, n = 0, 1, 2, ..
$$

This establishes the result.

Now, let us look at a basic result of $D_{U+}^n$ operating on $e^{\text{tr}(UX)}$, which will be stated as a lemma.

**Lemma 7.2** Let $X$ and $U$ be $p \times p$ real positive definite matrices. Let $D_{U+}^n$ be the operator defined in (7.2). Then

$$
D_{U+}^n [\det(V)]^{\gamma - \frac{p+1}{2}} = \frac{[\det(U)]^{\gamma - n - \frac{p+1}{2}}}{\Gamma_p(\gamma - n)} , \Re(\gamma) > n + \frac{p-1}{2}.
$$

Proof Observe that (7.3) can be taken as the Laplace transform of the function $\frac{[\det(V)]^{\gamma - \frac{p+1}{2}}}{\Gamma_p(\gamma)}$ with Laplace parameter matrix $U$. If $U = U' = (\eta_{ij}u_{ij})$ with
\( \eta_{ij} = \begin{cases} 1, & i = j \\ \frac{1}{2}, & i \neq j \end{cases} \)

then it is the multivariate Laplace transform of all elements in \( V \), taking each element once. If \( U = (u_{ij}) \), \( U = U' \) then it is the Laplace transform of all elements in \( V \), taking the diagonal elements once and the off-diagonal elements twice. Hence as an inverse Laplace transform we can write, for \( \Re(\gamma) > \frac{p-1}{2} \),

\[
\frac{[\det(X)]^{\gamma - \frac{p+1}{2}}}{\Gamma_p(\gamma)} = \begin{cases} \frac{1}{(2\pi i)^{\frac{p(p+1)}{2}}} \int_{\Re(U) > U_o} [\det(U)]^{-\gamma} e^{\text{tr}(UX)} \, dU, & U = (\eta_{ij} u_{ij}) \\ \frac{2}{(2\pi i)^{\frac{p(p+1)}{2}}} \int_{\Re(U) > U_o} [\det(U)]^{-\gamma} e^{\text{tr}(UX)} \, dU, & U = (u_{ij}). \end{cases}
\]

(7.6)

Then, operating on both sides with the operator \( D^n X_+ \) we have

\[
D^n_{X,+} \frac{[\det(X)]^{\gamma - \frac{p+1}{2}}}{\Gamma_p(\gamma)} = \frac{2}{(2\pi i)^{\frac{p(p-1)}{2}}} \int_{\Re(U) > U_o} [\det(U)]^{-\gamma} (D^n_{X,+} e^{\text{tr}(UX)}) \, dU
\]

Interpreting the right side as an inverse Laplace transform the right side corresponds to \( \frac{[\det(X)]^{\gamma - n - \frac{p+1}{2}}}{\Gamma_p(\gamma - n)} \) for \( \Re(\gamma - n) > \frac{p-1}{2} \) or \( \Re(\gamma) > n + \frac{p-1}{2} \). Hence the result.

With the help of Lemmas 7.1 and 7.2 we can look at some fractional derivatives when the arbitrary function \( f(X) \) is of the form \( \det(V)^{-\gamma} \) or of the form \( \det(V)^{\gamma - \frac{p+1}{2}} \) or of the form \( e^{\pm \text{tr}(V)} \). Let \( D^{-\alpha}_{2,U} f \) and \( D^\alpha_{2,U} f \) denote the fractional integral and fractional derivative of order \( \alpha \) of the second kind or right-sided situation respectively. Similarly, let \( D^{-\alpha}_{1,U} f \) and \( D^\alpha_{1,U} f \) be the fractional integral and fractional derivative of the first kind (left-sided) and of order \( \alpha \) respectively. The following symbolic representations will be used to write fractional derivatives from fractional integrals:

\[
D^\alpha_{2,U} f = D^n_{U,-} [D^{-\alpha}_{2,U} f] =
\]

fractional derivative of order \( \alpha \) of the second kind, in the Riemann-Liouville sense for \( n > \Re(\alpha) + \frac{p-1}{2} \);

\[
D^\alpha_{2,U} f = D^{-\alpha}_{2,U} (D^n_{U,-} f) =
\]

fractional derivative of order \( \alpha \), of the second kind, in the Caputo sense for \( n > \Re(\alpha) + \frac{p-1}{2} \);

\[
D^\alpha_{1,U} f = D^n_{U,-} [D^{-\alpha}_{1,U} f] =
\]
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fractional derivative of order \( \alpha \), of the first kind, in the Riemann-Liouville sense for \( n > \Re(\alpha) + \frac{p-1}{2} \):

\[
D_{1,U}^\alpha f = D_{1,U}^{-(n-\alpha)} [D_{U-}^n f] =
\]

fractional derivative of order \( \alpha \), of the first kind, in the Caputo sense. The operator of the second kind is also called right-sided operator and the operator of the first kind is also called the left-sided operator.

7.3 Fractional Derivatives in Some Special Cases

We will examine a few cases of the arbitrary function with reference to first and second kinds of fractional derivatives of order \( \alpha \).

**Case 7.3a:** \( f(V) = e^{-\text{tr}(V)} \), right-sided fractional derivative in the Riemann-Liouville sense

For \( n > \Re(\alpha) + \frac{p-1}{2} \),

\[
D_{2,U}^\alpha f = D_{U-}^n [D_{2,U}^{-(n-\alpha)} f] = D_{U-}^n \frac{1}{\Gamma_p(n-\alpha)} \int_{V > U > O} [\text{det}(V - U)]^{n-\alpha-\frac{p+1}{2}} e^{-\text{tr}(V)} dV
\]

\[
= D_{U-}^n \frac{e^{-\text{tr}(U)}}{\Gamma_p(n-\alpha)} \int_{W > O} [\text{det}(W)]^{n-\alpha-\frac{p+1}{2}} e^{-\text{tr}(W)} dW, W = V - U
\]

\[
= D_{U-}^n e^{-\text{tr}(U)} = e^{-\text{tr}(U)}
\] (7.7)

where \( D_{U-}^n e^{-\text{tr}(U)} = [(-1)^p \text{det}((-I)^n)] e^{-\text{tr}(U)} = 1 e^{-\text{tr}(U)} \). In this case, the right-sided fractional derivative in Caputo sense is the following for \( n > \Re(\alpha) + \frac{p-1}{2} \):

\[
D_{2,U}^\alpha f = D_{2,U}^{-(n-\alpha)} [D_{U-}^n f] = \frac{1}{\Gamma_p(n-\alpha)} \int_{V > U > O} [\text{det}(V - U)]^{n-\alpha-\frac{p+1}{2}} e^{-\text{tr}(V)} dV.
\]

But \( D_{U-}^n e^{-\text{tr}(V)} = e^{-\text{tr}(V)} \) and

\[
\frac{1}{\Gamma_p(n-\alpha)} \int_{V > U > O} [\text{det}(V - U)]^{n-\alpha-\frac{p+1}{2}} e^{-\text{tr}(V)} dV = e^{-\text{tr}(V)} \frac{\Gamma_p(n-\alpha)}{\Gamma_p(n-\alpha)} = e^{-\text{tr}(U)}.
\]

In this case, both Riemann-Liouville and Caputo derivatives are the same.

**Note 7.1** In this case it is easy to note that the semigroup property holds for both Riemann-Liouville and Caputo derivatives. That is,

\[
D_{2,U}^\alpha D_{2,U}^\beta e^{-\text{tr}(U)} = D_{2,U}^\beta D_{2,U}^\alpha e^{-\text{tr}(U)} = D_{2,U}^{\alpha+\beta} e^{-\text{tr}(U)}.
\]
Case 7.3b: $f(V) = [\det(V)]^{-\gamma}, \Re(\gamma) > \frac{p-1}{2}$, right-sided fractional derivative of order $\alpha$ in Riemann-Liouville sense.

This is the following:

$$D^\alpha_{2,U}f = D^n_{U-}[D^\alpha_{2,U} f]$$

$$= D^n_{U-} \left[ \frac{1}{\Gamma_p(n-\alpha)} \int_{V>U>0} [\det(V-U)]^{n-\alpha-p+\frac{1}{2}} [\det(V)]^{-\gamma} dV \right]$$

$$= D^n_{U-} \left[ \frac{1}{\Gamma_p(n-\alpha)} \int_{W>0} [\det(W)]^{n-\alpha-p+\frac{1}{2}} [\det(U+W)]^{-\gamma} dW \right]$$

$$= D^n_{U-}[\det(U)]^{-\gamma+n-\alpha} \Gamma_p(n-\alpha) \Gamma_p(\gamma-n+\alpha) \Gamma_p(n-\alpha) \Gamma_p(\gamma) \quad T = U^{-\frac{1}{2}} V U^{-\frac{1}{2}}$$

$$= D^n_{U-}[\det(U)]^{-(\gamma+n+\alpha)} \Gamma_p(\gamma-n+\alpha) \Gamma_p(\gamma).$$

(7.9)

But from Lemma 7.1

$$D^n_{U-}[\det(U)]^{-(\gamma+n+\alpha)} = \frac{\Gamma_p(\gamma+\alpha)}{\Gamma_p(\gamma-n+\alpha)} [\det(U)]^{-(\gamma+\alpha)}$$

(7.10)

for $\Re(\gamma) > \frac{p-1}{2}, \Re(\gamma+\alpha) > \frac{p-1}{2}$. Substituting (7.10) in (7.9) we have for $\Re(\gamma) > \frac{p-1}{2},$

$$D^\alpha_{2,U}[\det(U)]^{-\gamma} = [\det(U)]^{-(\gamma+\alpha)} \frac{\Gamma_p(\gamma+\alpha)}{\Gamma_p(\gamma)}. $$

Note 7.2 It is not difficult to see that the semigroup property holds here. Note that, for $n > \Re(\alpha) + \frac{p-1}{2}$ and $n > \Re(\beta) + \frac{p-1}{2},$

$$D^\beta_{2,U} \{ D^\alpha_{2,U} [\det(U)]^{-\gamma} \} = D^\beta_{2,U} \left\{ \frac{\Gamma_p(\gamma+\alpha)}{\Gamma_p(\gamma)} [\det(U)]^{-(\gamma+\alpha)} \right\}$$

$$= \frac{\Gamma_p(\gamma+\alpha)}{\Gamma_p(\gamma)} D^\beta_{2,U-} \left[ \frac{1}{\Gamma_p(m-\beta)} \int_{V>U>0} [\det(V-U)]^{m-\beta-p+\frac{1}{2}} [\det(V)]^{-(\gamma+\alpha)} dV \right]$$

$$= \frac{\Gamma_p(\gamma+\alpha)}{\Gamma_p(\gamma)} D^\beta_{U-} \left[ [\det(U)]^{-(\gamma+\alpha+m-\beta)} \frac{\Gamma_p(\gamma+\alpha+\beta-m)}{\Gamma_p(\gamma+\alpha)} \right]$$

$$= \frac{\Gamma_p(\gamma+\alpha+\beta-m)}{\Gamma_p(\gamma)} D^\beta_{U-} \left[ [\det(U)]^{-(\gamma+\alpha+\beta-m)} \right]$$

$$= \frac{\Gamma_p(\gamma+\alpha+\beta)}{\Gamma_p(\gamma)} [\det(U)]^{-(\gamma+\alpha+\beta)} = D^\alpha_{2,U} \left[ [\det(U)]^{-\gamma} \right].$$

(7.11)

Thus, semigroup property is proved for Riemann-Liouville type derivative of order $\alpha$ and of the second kind.
Case 7.3c: $f(V) = [\det(V)]^{-\gamma}, \Re(\gamma) > \frac{p-1}{2}$, right-sided fractional derivative of order $\alpha$ in Caputo sense

Here

$$D_{2,U}^\alpha f = D_{2,U}^{-(n-\alpha)}[D_{U+}^n f] = \frac{1}{\Gamma_p(n - \alpha)} \times \int_{V > U > O} [\det(V - U)]^{n - \alpha - \frac{p+1}{2}} D_{U+}^n [\det(V)]^{-\gamma} dV$$

for $n > \Re(\alpha) + \frac{p-1}{2}$. But

$$D_{U+}^n [\det(V)]^{-\gamma} = \frac{\Gamma_p(\gamma + n)}{\Gamma_p(\gamma)} [\det(V)]^{-(\gamma + n)}.$$

Therefore,

$$D_{2,U}^\alpha f = \frac{1}{\Gamma_p(n - \alpha)} \int_{V > U > O} [\det(V - U)]^{n - \alpha - \frac{p+1}{2}} \frac{\Gamma_p(\gamma + n)}{\Gamma_p(\gamma)} [\det(V)]^{-(\gamma + n)} dV$$

$$= \frac{\Gamma_p(\gamma + \alpha)}{\Gamma_p(\gamma)} [\det(U)]^{-(\gamma + \alpha)}$$

(7.12)

for $\Re(\gamma) > \frac{p-1}{2}, \Re(\gamma + \alpha) > \frac{p-1}{2}$. This is the same result in the Riemann-Liouville case also. Hence for both the cases here we have the same expression for the fractional derivative of order $\alpha$ and also the semigroup property holds good in both the cases.

Note 7.3 If $f(V) = [\det(V)]^\gamma, \Re(\gamma) > \frac{p-1}{2}$ then it is easy to see that the conditions in the above procedure are violated. In fact the right-sided or second kind fractional integral diverges for this situation whereas the left-sided integrals will be available in this case.

7.4 First Kind Fractional Derivative for Some Special Cases

Here we consider two special cases of the arbitrary function.

Case 7.4a: $f(V) = e^{tr(V)}$, fractional integral of order $\alpha$ in the Riemann-Liouville sense

$$D_{1,U}^\alpha f = D_{U+}^n [D_{1,U}^{-(n-\alpha)} f]$$

$$= D_{U+}^n \frac{1}{\Gamma_p(n - \alpha)} \int_{O < V < U} [\det(U - V)]^{n - \alpha - \frac{p+1}{2}} e^{tr(V)} dV$$
\[
D^n_{U+} \left\{ \frac{e^{\tr(U)}}{\Gamma_p(n - \alpha)} \int_{W > O} [\det(W)]^{n - \alpha - \frac{p+1}{2}} e^{-\tr(W)} dW, \ U - V = W \right\} = D^n_{U+} e^{\tr(U)} \frac{\Gamma_p(n - \alpha)}{\Gamma_p(n - \alpha)} = e^{\tr(U)}. \]

It is trivial to see that the semigroup property holds. Since \(D^n_{U+} e^{\tr(U)} = e^{\tr(U)}\) the derivative in the Caputo sense also gives the same result.

**Note 7.4** If \(f(V) = e^{-\tr(V)}\) then the above procedure does not hold. But we can take out \(U\) from \(\det(U - V)\), make a transformation \(W = U^{-\frac{1}{2}} V U^{-\frac{1}{2}}\). Then expand \(e^{-\tr(UW)}\) for \(O < W < I\) and integrate out to obtain a confluent hypergeometric series of matrix argument, see Mathai [8] for details.

**Case 7.4b:** \(f(V) = \frac{[\det(V)]^{\gamma - \frac{p+1}{2}}}{\Gamma_p(\gamma)}\), left-sided fractional derivative in the Riemann-Liouville sense. In this case, for \(n > \Re(\alpha) + \frac{p-1}{2}\),

\[
D^\alpha_{1,U} f = D^n_{U+}[D^{-n-\alpha}_{-1,U} f]
\]

\[
= D^n_{U+} \frac{1}{\Gamma_p(n - \alpha)} \int_{O < V < U} [\det(U - V)]^{n - \alpha - \frac{p+1}{2}} \frac{[\det(V)]^{\gamma - \frac{p+1}{2}}}{\Gamma_p(\gamma)} dV
\]

\[
= D^n_{U+} \frac{[\det(U)]^{\gamma + n - \alpha - \frac{p+1}{2}}}{\Gamma_p(\gamma + n - \alpha)}, T = U^{-\frac{1}{2}} V U^{-\frac{1}{2}}
\]

\[
= \frac{[\det(U)]^{\gamma - \alpha - \frac{p+1}{2}}}{\Gamma_p(\gamma - \alpha)}
\]

by Lemma 7.2, for \(\Re(\gamma - \alpha) > \frac{p-1}{2}\), \(n > \Re(\alpha) + \frac{p-1}{2}\). Let us see whether it satisfies the semigroup property.

\[
D^\beta_{1,U}[D^\alpha_{1,U} f] = D^\beta_{1,U}\left\{ \frac{[\det(U)]^{\gamma - \alpha - \frac{p+1}{2}}}{\Gamma_p(\gamma - \alpha)} \right\}, \Re(\gamma - \alpha) > \frac{p-1}{2}
\]

\[
= D^m_{U+} \frac{[\det(U)]^{\gamma - \alpha - \beta + m - \frac{p+1}{2}}}{\Gamma_p(\gamma - \alpha - \beta + m)} \frac{\Gamma_p(m - \beta)\Gamma_p(\gamma - \alpha)}{\Gamma_p(\gamma - \alpha - \beta + m)}
\]

\[
= D^m_{U+} \frac{[\det(U)]^{\gamma - \alpha - \beta + m - \frac{p+1}{2}}}{\Gamma_p(\gamma - \alpha - \beta + m)} = \frac{[\det(U)]^{\gamma - \alpha - \beta - \frac{p+1}{2}}}{\Gamma_p(\gamma - \alpha - \beta)}
\]

\[
= D^{\alpha+\beta}_{1,U} f = D^\alpha_{1,U}[D^\beta_{1,U} f]
\]

for \(\Re(\gamma - \alpha - \beta) > \frac{p-1}{2}\). Hence the semigroup property is satisfied.
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Note 7.5 If $f(V) = [\det(V)]^{-\gamma}$, $\Re(\gamma) > \frac{p-1}{2}$ then it is easy to see that the above conditions on the parameters are violated. Hence the left-sided fractional derivatives are not available for this situation.

7.5 Fractional Derivatives of Erdélyi-Kober Operators

Erdélyi-Kober fractional integral of the second kind and of order $\alpha$, in the real matrix-variate case is given by the following:

$$K_{2,U,\rho}^{-\alpha} f = \frac{[\det(U)]^\rho}{\Gamma_p(\alpha)} \int_{V > U > O} [\det(V)]^{-\rho-\alpha} [\det(V - U)]^{\alpha - \frac{p+1}{2}} f(V) dV$$

(7.13)

for $\Re(\alpha) > \frac{p-1}{2}$. Then, Erdélyi-Kober fractional derivative of order $\alpha$ and of the second kind in the Riemann-Liouville sense is given by the following, see Mathai [12]:

$$K_{2,U,\rho}^\alpha f = D_{U-}^n K_{2,U,\rho}^{-(n-\alpha)} f$$

$$= D_{U-}^n \frac{[\det(U)]^\rho}{\Gamma_p(n-\alpha)} \int_{V > U > O} [\det(V)]^{-\rho+\alpha-n} [\det(V - U)]^{n-\alpha - \frac{p+1}{2}} f(V) dV$$

(7.14)

for $n > \Re(\alpha) + \frac{p-1}{2}$. Erdélyi-Kober fractional derivative of order $\alpha$ and of the second kind, in the Caputo sense is given by the following:

$$K_{2,U,\rho}^\alpha f = K_{2,U,\rho}^{-(n-\alpha)} \{ D_{U-}^n f \}$$

$$= \frac{[\det(U)]^\rho}{\Gamma_p(n-\alpha)} \int_{V > U > O} [\det(V)]^{-\rho-(n-\alpha)} [\det(V - U)]^{n-\alpha - \frac{p+1}{2}} [D_{V-}^n f(V)] dV$$

(7.15)

for $n > \Re(\alpha) + \frac{p-1}{2}$. Now, we will evaluate (7.14) and (7.15) for various cases for $f(V)$.

Case 7.5a: $f(V) = [\det(V)]^{-\gamma}$, $\Re(\gamma) > \frac{p-1}{2}$, evaluation of (7.14)

$$K_{2,U,\rho}^\alpha f = D_{U-}^n K_{2,U,\rho}^{-(n-\alpha)} f$$

$$= D_{U-}^n \{ \frac{[\det(U)]^\rho}{\Gamma_p(n-\alpha)} \int_{V > U > O} [\det(V)]^{-\rho+\alpha-n-\gamma} [\det(V - U)]^{n-\alpha - \frac{p+1}{2}} dV \}$$

$$= D_{U-}^n \{ \frac{[\det(U)]^{-\gamma}}{\Gamma_p(n-\alpha)} \int_{W > O} [\det(W)]^{n-\alpha - \frac{p+1}{2}} [\det(I + W)]^{-\rho+\alpha-n-\gamma} dW \}$$
\[ = D^n_{U_\gamma} [\det(U)]^{-\gamma} \frac{\Gamma_p(\rho + \gamma)}{\Gamma_p(\rho + \gamma + n - \alpha)} Y = V - U \]

\[ = \frac{\Gamma_p(\rho + \gamma)}{\Gamma_p(\rho + \gamma + n - \alpha)} \frac{1}{\Gamma_p(\gamma)} \int_{S > O} [\det(S)]^{-\gamma - \frac{p+1}{2}} D^n_{U_\gamma} e^{-\tr(U S)} dS \]

\[ = \frac{\Gamma_p(\rho + \gamma)}{\Gamma_p(\rho + \gamma + n - \alpha)} \frac{\Gamma_p(\gamma + n)}{\Gamma_p(\gamma)} [\det(U)]^{-(\gamma + n)}, \quad (7.16) \]

for \( \Re(\gamma) > \frac{p-1}{2} \), \( \Re(\rho + \gamma) > \frac{p-1}{2} \). This is the \( \alpha \)th order Erdélyi-Kober fractional derivative of the second kind in the Riemann-Liouville sense.

**Case 7.5b:** Caputo derivative in Case 7.5a

Consider

\[ K^{\alpha}_{\gamma, U_\gamma, \rho} f = K^{-(n - \alpha)}_{\gamma, U_\gamma, \rho} [D^n_{V_\gamma} f] \]

\[ = \frac{[\det(U)]^\rho}{\Gamma_p(n - \alpha)} \int_{U > V > O} [\det(V - U)]^{n - \alpha - \frac{p+1}{2}} [\det(V)]^{-\rho - n + \alpha} [D^n_{V_\gamma} [\det(V)]^{-\gamma}] dV \]

\[ = \frac{[\det(U)]^\rho}{\Gamma_p(n - \alpha)} \frac{\Gamma_p(\gamma + n)}{\Gamma_p(\gamma)} \int_{U > V > O} [\det(V - U)]^{n - \alpha - \frac{p+1}{2}} [\det(V)]^{-\rho - 2n - \gamma + \alpha} dV \]

\[ = \frac{\Gamma_p(\gamma + n)}{\Gamma_p(\gamma)} \frac{\Gamma_p(\rho + n + \gamma)}{\Gamma_p(\rho + 2n - \alpha)} [\det(U)]^{-(\gamma + n)} \]

for \( \Re(\gamma) > \frac{p-1}{2} \), \( n > \Re(\alpha) + \frac{p-1}{2} \). Note that Caputo derivative is different from Riemann-Liouville derivative in this case.

**Case 7.5c:** \( f(V) = e^{-\tr(V)} \)

This will go to Whittaker function of matrix argument, both in Riemann-Liouville and Caputo senses. For the final integrals, see Mathai [8]. Final integrals will be of the form

\[ \int_{S > O} [\det(S)]^{\alpha_1 - \frac{p+1}{2}} [\det(I + S)]^{-(\alpha_1 + \beta_1)} e^{-\tr(S)} dS \]

for some \( \Re(\alpha_1) > \frac{p-1}{2} \), \( \Re(\beta_1) > \frac{p-1}{2} \). Hence this case will not be discussed here.

### 7.6 Erdélyi-Kober Fractional Derivatives of the First Kind of Order \( \alpha \)

We will consider fractional derivative of order \( \alpha \) in the Riemann-Liouville and Caputo senses for some special cases of \( f(V) \). Erdélyi-Kober fractional integral of the first kind is given by the following, see Mathai [12]:

\[ K^{-\alpha}_{1, U_\gamma, \rho} f = \frac{[\det(U)]^{-\rho - \alpha}}{\Gamma_p(\alpha)} \int_{O < V < U} [\det(V)]^\rho [\det(U - V)]^{\alpha - \frac{p+1}{2}} f(V) dV. \]

\[ (7.17) \]
The $\alpha$th order fractional derivative of the first kind in the Riemann-Liouville sense is then given by the following:

$$K_{1,U,\rho}^{\alpha} f = D_{U-}^{\alpha} \{ K_{1,U,\rho}^{-(n-\alpha)} f \}$$

$$= D_{U-}^{\alpha} \left\{ \frac{[\det(U)]^{-\rho-n+\alpha}}{\Gamma_p(n-\alpha)} \int_{O<V<U} [\det(V)]^{\rho} [\det(U-V)]^{\rho-n-\frac{\rho+1}{2}} f(V) dV \right\}$$

(7.18)

for $n > \Re(\alpha) + \frac{\rho+1}{2}$, and that in the Caputo sense is given by the following:

$$K_{1,U,\rho}^{\alpha} f = \left[ \frac{[\det(U)]^{-\rho-n+\alpha}}{\Gamma_p(n-\alpha)} \int_{O<V<U} [\det(V)]^{\rho} [\det(U-V)]^{\rho-n-\frac{\rho+1}{2}} \{ D_{V-}^{\alpha} f(V) \} dV \right]$$

(7.19)

Let us examine these two types of derivatives for some special cases of $f(V)$.

**Case 7.6a**: $f(V) = e^{\pm \text{tr}(V)}$

In this case the integrals to be evaluated, corresponding to (7.18) and (7.19) will be of the form

$$\int_{O<W<I} [\det(W)]^{\gamma_1-\frac{\rho+1}{2}} [\det(I-W)]^{\gamma_2-\frac{\rho+1}{2}} e^{\pm \text{tr}(UW)} dW$$

for $\Re(\gamma_i) > \frac{\rho-1}{2}$, $i = 1, 2$ and the integral will go to confluent hypergeometric function of matrix argument, see Mathai [8], and hence it will not be discussed here.

**Case 7.6b**: $f(V) = \left[ \frac{[\det(V)]^{\rho}}{\Gamma_p(\gamma)} \right]^{\frac{\rho+1}{2}}, \Re(\gamma) > \frac{\rho-1}{2}$

In this case (7.18) will reduce to the following for $n > \Re(\alpha) + \frac{\rho+1}{2}, \Re(\gamma) > \frac{\rho-1}{2}$:

$$K_{1,U,\rho}^{\alpha} f = D_{U+}^{n} \left\{ K_{1,U,\rho}^{-(n-\alpha)} f \right\}$$

$$= D_{U+}^{n} \left\{ \frac{[\det(U)]^{-\rho-n+\alpha}}{\Gamma_p(n-\alpha)} \times \int_{O<V<U} [\det(V)]^{\rho} [\det(U-V)]^{\rho-n-\frac{\rho+1}{2}} \left[ \frac{[\det(V)]^{\gamma-\frac{\rho+1}{2}}}{\Gamma_p(\gamma)} \right]^{\frac{\rho+1}{2}} dV \right\}$$

$$= D_{U+}^{n} \left\{ \frac{[\det(U)]^{\gamma-\frac{\rho+1}{2}}}{\Gamma_p(\gamma)} \frac{1}{\Gamma_p(n-\alpha)} \times \int_{O<W<I} [\det(W)]^{\rho+\gamma-\frac{\rho+1}{2}} [\det(I-W)]^{\rho-n-\frac{\rho+1}{2}} dW \right\}$$

$$= \left[ \frac{[\det(U)]^{\gamma-n-\frac{\rho+1}{2}}}{\Gamma_p(\gamma-n)} \right]^{\frac{\rho+1}{2}} \frac{\Gamma_p(\rho+\gamma)}{\Gamma_p(n-\alpha+\rho+\gamma)}$$

(7.20)
for \( \Re(\gamma) > n + \frac{p-1}{2}, \Re(\rho + \gamma) > \frac{p-1}{2}, n > \Re(\alpha) - \Re(\gamma + \rho) \). This is the Erdélyi-Kober fractional derivative of order \( \alpha \) of the first kind in the Riemann-Liouville sense. The result in the Caputo sense is given as an exercise.

7.7 Fractional Derivatives of Matrix-Variate Statistical Densities

In Chap. 3 it is shown that matrix-variate statistical densities are directly connected to Erdélyi-Kober fractional integrals. Let \( X_1 \) and \( X_2 \) be \( p \times p \) statistically independently distributed real matrix-variate random variables. Let \( U_2 = X_2^\frac{1}{2} X_1 X_2^\frac{1}{2} \) and \( U_1 = X_2^\frac{1}{2} X_1^{-1} X_2^\frac{1}{2} \). Then \( U_2 \) and \( U_1 \) are called product and ratio of matrices \( X_1 \) and \( X_2 \) where \( X_2^\frac{1}{2} \) denotes the real positive definite square root of the real positive definite matrix \( X_2 \). If the densities of \( U_2 \) and \( U_1 \) are denoted by \( g_2(U_2) \) and \( g_1(U_1) \) respectively then it is shown in Chap. 3 that

\[
g_2(U_2) = \frac{\Gamma_p(\alpha + \gamma + \frac{p+1}{2})}{\Gamma_p(\gamma + \frac{p+1}{2})} K_{\alpha,2U,\gamma} f \quad \text{and} \quad g_1(U_1) = \frac{\Gamma_p(\alpha + \gamma)}{\Gamma_p(\gamma)} K_{\alpha,1U,\gamma} f
\]

(7.21)

where \( K_{\alpha,2U,\gamma} \) is the Erdélyi-Kober fractional integral operator of order \( \alpha \) and of the second kind, given in (7.13), and \( K_{\alpha,1U,\gamma} \) is the Eedélyi-Kober fractional integral operator of order \( \alpha \) and of the first kind, given in (7.17). Hence, fractional derivatives of order \( \alpha \) of the second kind, in Riemann-Liouville and Caputo senses, of the density \( g_2(U_2) \), are available from (7.14) and (7.15) by multiplying with the constant \( \frac{\Gamma_p(\alpha + \gamma + \frac{p+1}{2})}{\Gamma_p(\gamma + \frac{p+1}{2})} \). For the density \( g_1(U_1) \) the fractional derivative of order \( \alpha \) and of the first kind, in Riemann-Liouville and Caputo senses, are available from (7.18) and (7.19) by multiplying with the constant \( \frac{\Gamma_p(\alpha + \gamma)}{\Gamma_p(\gamma)} \).

For \( f(V) \) we have considered two special cases in Sect. 7.6. One was \( [\det(V)]^{-\rho} \). Note that the results will go through for the function \( [\det(I + V)]^{-\rho} \) also. This can be made into a statistical density by multiplying with a constant. Note that

\[
f(V) = \frac{\Gamma_p(\rho)}{\Gamma_p(\frac{p+1}{2}) \Gamma_p(\rho - \frac{p+1}{2})} [\det(I + V)]^{-\rho}
\]

(7.22)
for $\Re(\rho) > p$, which is a real matrix-variate type-2 beta density with the parameters $(\frac{p+1}{2}, \rho - \frac{p+1}{2})$. Hence the fractional derivative of $g_2(U_2)$ is available from those of the Erdélyi-Kober fractional integral of order $\alpha$, namely $K_{2,U,\rho}^{-\alpha}$. Thus, we can define the Erdélyi-Kober fractional derivative of the density, $K_{2,U,\rho}^{-\alpha} g_2$, by using the density in (7.21).

The second function that we have considered for $f(V)$ was of the form

$$f(V) = \frac{[\det(V)]^{\rho - \frac{p+1}{2}}}{\Gamma_p(\rho)} = \frac{1}{\Gamma_p(\rho)}[\det(V)]^{\rho - \frac{p+1}{2}}[\det(I - U)]^{\frac{p+1}{2} - \frac{p+1}{2}}. \quad (7.23)$$

Hence $\frac{\Gamma_p(\rho + \frac{p+1}{2})}{\Gamma_p(\frac{p+1}{2})} f(V)$ is a statistical density, which is a real matrix-variate type-1 beta density with the parameters $(\rho, \frac{p+1}{2})$. Hence the fractional derivative of order $\alpha$ for the density $g_1(U_1)$ in (7.21) is available from the corresponding derivative of $K_{1,U,\rho}^{-\alpha}$ with $f(V)$ in (7.23) multiplied by the constant $\frac{\Gamma_p(\rho + \frac{p+1}{2})}{\Gamma_p(\frac{p+1}{2})}$. Thus, we can define the fractional derivative of the density $g_1(U_1)$, that is, $K_{1,U,\rho}^{-\alpha} g_1$, with the help of $K_{1,U,\rho}^{-\alpha}$ of (7.21) and the density in (7.23).

Fractional derivatives in the complex domain will be considered next. For some matrix-variate statistical densities in the complex domain see Mathai and Provost [15].

### 7.8 Fractional Differential Operators in the Complex Matrix-Variate Case

The following standard notations will be used. All matrices appearing here are $p \times p$ Hermitian positive definite when in the complex domain and positive definite in the real case, unless otherwise stated. $\tr(\cdot)$ and $\det(\cdot)$ denote the trace and determinant of the square matrix $\cdot$ respectively. $|\det(\cdot)|$ denotes the absolute value of the determinant of $\cdot$ as in Chap. 6. Wedge product of differential, matrix-variate gamma in the real and complex domain will be denoted as in Chap. 6. All the results and notations from Chap. 5 will be made use of here also. Some additional basic results will be listed here as lemmas.

**Lemma 7.3** Let $X$ and $Y$ be $p \times p$ matrices with real elements where $X = X'$ (symmetric) and $Y = -Y'$ (skew symmetric). Then

$$\tr(XY) = 0. \quad (7.24)$$

**Proof** For any square matrix $C$, $\tr(C) = \tr(C')$. For any two $p \times p$ matrices $A$ and $B$, $\tr(AB) = \tr(BA)$ even if $AB \neq BA$. Consider
\[ \text{tr}(XY) = \text{tr}(XY)' = \text{tr}(Y'X') = -\text{tr}(XY) \]
since \( Y' = -Y \) and \( X' = X \). But
\[ \text{tr}(XY) = \text{tr}(YX) \Rightarrow \text{tr}(YX) = -\text{tr}(YX) \Rightarrow \text{tr}(YX) = 0. \]

Lemma 7.4 Let \( \tilde{X} \) and \( \tilde{A} \) be \( p \times p \) Hermitian matrices. Then when \( \tilde{X} = X + iY, \tilde{A} = A + iB \) where \( i = \sqrt{(-1)} \), \( X, Y, A, B \) are real then \( X = X', A = A', Y = -Y', B = -B' \) where a prime denotes the transpose, with \( y_{ji} = -y_{ij}, b_{ji} = -b_{ij} \) for all \( i < j \). Then
\[ \text{tr}(\tilde{X} \tilde{A}) = \sum_{i=1}^{p} x_{ii} a_{ii} + 2 \sum_{i<j} x_{ij} a_{ij} + 2 \sum_{i<j} y_{ij} b_{ij} \]  
(7.25)
where \( X = (x_{ij}), Y = (y_{ij}), A = (a_{ij}), B = (b_{ij}) \).

Proof
\[ \text{tr}(\tilde{X} \tilde{A}) = \text{tr}[(X + iY)(A + iB)] = \text{tr}[(XA - YB) + i(XB + YA)] = \text{tr}[(XA) - (YB)] \]  
from (7.24)
since \( \text{tr}(XB) = 0, \text{tr}(YA) = 0 \) from Lemma 7.3 being symmetric and skew symmetric matrices. Also, by taking dot products
\[ \text{tr}(XA) = \sum_{i=1}^{p} x_{ii} a_{ii} + 2 \sum_{i<j} x_{ij} a_{ij} \]

\[ -\text{tr}(YB) = 2 \sum_{i<j} y_{ij} b_{ij}. \]

7.9 Differential Operators in the Complex Domain for Hermitian Matrices

Let \( \tilde{X} = X + iY, \tilde{A} = A + iB, i = \sqrt{(-1)} \) where \( X, Y, A, B \) are real matrices. Let \( X \) and \( A \) be symmetric and \( Y \) and \( B \) be skew symmetric, with \( y_{ji} = -y_{ij}, b_{ji} = -b_{ij} \) for all \( i < j \), so that \( \tilde{X} \) and \( \tilde{A} \) are Hermitian. Let us define the following operators.
\[ \frac{\partial}{\partial X} = (\eta_{ij} \frac{\partial}{\partial x_{ij}}), \quad X = (x_{ij}) = X', \quad \eta_{ij} = \begin{cases} 1, & i = j \\ \frac{1}{2}, & i \neq j \end{cases} \]
7.9 Differential Operators in the Complex Domain for Hermitian Matrices

\[
\frac{\partial}{\partial Y} = (\hat{\eta}_{ij} \frac{\partial}{\partial y_{ij}}), \quad Y = (y_{ij}) = -Y', \quad \hat{\eta}_{ij} = \begin{cases} 0, & i = j \\ \frac{1}{2}, & i \neq j \end{cases}
\]

Let

\[
\frac{\partial}{\partial \tilde{X}} = \frac{\partial}{\partial X} + i \frac{\partial}{\partial Y}, \quad i = \sqrt(-1).
\]

Then \(\frac{\partial}{\partial \tilde{X}}\) operating on \(e^{-\text{tr}(\tilde{X}\tilde{A})}\) gives \(-\tilde{A}e^{-\text{tr}(\tilde{X}\tilde{A})}\). Let \(\frac{\partial^*}{\partial \tilde{X}} = \frac{\partial}{\partial X} - i \frac{\partial}{\partial Y}\) denote the conjugate transpose operator. Then \(\frac{\partial^*}{\partial \tilde{X}}\) operating on \(e^{-\text{tr}(\tilde{X}\tilde{A})}\) gives \(-\tilde{A}^*e^{-\text{tr}(\tilde{X}\tilde{A})}\)
where \(\tilde{A}^*\) is the conjugate transpose of \(A\). Here \(X' = X\) and \(Y' = -Y\). Then

\[
\frac{\partial}{\partial \tilde{X}} \frac{\partial^*}{\partial \tilde{X}} e^{-\text{tr}(\tilde{X}\tilde{A})} = \tilde{A}\tilde{A}^* e^{-\text{tr}(\tilde{X}\tilde{A})}.
\]

Consider the determinants of the operators operating on this exponential function. We will have the following:

\[
[\det(\frac{\partial}{\partial \tilde{X}} \frac{\partial^*}{\partial \tilde{X}})] e^{-\text{tr}(\tilde{X}\tilde{A})} = [\det(\tilde{A}\tilde{A}^*)] e^{-\text{tr}(\tilde{X}\tilde{A})}.
\]

Therefore we will define the operator

\[
D_{\tilde{X}} = [\det(\frac{\partial}{\partial \tilde{X}} \frac{\partial^*}{\partial \tilde{X}})]^{\frac{1}{2}} = [\det(\frac{\partial}{\partial X})] = (7.26)
\]

absolute value of the determinant of the operator \(\frac{\partial}{\partial \tilde{X}}\). Then

\[
D_{\tilde{X}}[e^{-\text{tr}(\tilde{X}\tilde{A})}] = [\det(\tilde{A})] e^{-\text{tr}(\tilde{X}\tilde{A})}. \quad (7.27)
\]

Consider this operator operating \(n\) times, denoted by \(D^n_{\tilde{X}}\). Then

\[
D^n_{\tilde{X}}[e^{-\text{tr}(\tilde{X}\tilde{A})}] = [\det(\tilde{A})]^n e^{-\text{tr}(\tilde{X}\tilde{A})};
\]

\[
D^n_{\tilde{X}} e^{\text{tr}(\tilde{X}\tilde{A})} = [\det(\tilde{A})]^n e^{\text{tr}(\tilde{X}\tilde{A})}. \quad (7.28)
\]

Then we have the following lemma:

**Lemma 7.5** Let \(D^n_{\tilde{X}}\) be as defined in (7.26). Then

\[
D^n_{\tilde{X}}[e^{\pm\text{tr}(\tilde{X}\tilde{A})}] = [\det(\tilde{A})]^n e^{\pm\text{tr}(\tilde{X}\tilde{A})}. \quad (7.29)
\]
From the integral representation of complex matrix-variate gamma function
\[
\tilde{\Gamma}_p(\alpha) = \int_{\tilde{X} > O} |\det(\tilde{X})|^{\alpha-p} e^{-\text{tr}(\tilde{X})} d\tilde{X}, \Re(\alpha) > p - 1
\]
we have
\[
|\det(\tilde{X})|^{-\alpha} = \frac{1}{\tilde{\Gamma}_p(\alpha)} \int_{\tilde{S} > O} |\det(\tilde{S})|^{\alpha-p} e^{-\text{tr}(\tilde{X}\tilde{S})} d\tilde{S}, \Re(\alpha) > p - 1 \tag{7.30}
\]
where \(\tilde{X}\) and \(\tilde{S}\) are Hermitian positive definite \(p \times p\) matrices and the notation \(\tilde{S} > O\) means that \(\tilde{S}\) is Hermitian positive definite.

**Lemma 7.6** Let \(\tilde{X}\) and \(\tilde{S}\) be Hermitian positive definite \(p \times p\) matrices. Let \(D^n_{\tilde{X}}\) be the operator defined in (7.26) for \(n = 0, 1, 2, \ldots\) Then
\[
D^n_{\tilde{X}}|\det(\tilde{X})|^{-\alpha} = \frac{\tilde{\Gamma}_p(\alpha + n)}{\tilde{\Gamma}_p(\alpha)} |\det(\tilde{X})|^{-(\alpha+n)} , n = 0, 1, 2, \ldots, \Re(\alpha) > p - 1. \tag{7.31}
\]

**Proof** Consider the integral representation of \(|\det(\tilde{X})|^{-\alpha}\) in (7.30). Let us operate on both sides by \(D^n_{\tilde{X}}\). That is,
\[
D^n_{\tilde{X}} |\det(\tilde{X})|^{-\alpha} = \frac{1}{\tilde{\Gamma}_p(\alpha)} \int_{\tilde{S} > O} |\det(\tilde{S})|^{\alpha-p} [D^n_{\tilde{X}} e^{-\text{tr}(\tilde{X}\tilde{S})}] d\tilde{S}
\]
\[
= \frac{1}{\tilde{\Gamma}_p(\alpha)} \int_{\tilde{S} > O} |\det(\tilde{S})|^{\alpha-p} |\det(\tilde{S})|^n e^{-\text{tr}(\tilde{X}\tilde{S})} d\tilde{S}
\]
\[
= \frac{1}{\tilde{\Gamma}_p(\alpha)} \int_{\tilde{S} > O} |\det(\tilde{S})|^{\alpha+n-p} e^{-\text{tr}(\tilde{X}\tilde{S})} d\tilde{S}
\]
\[
= \frac{\tilde{\Gamma}_p(\alpha + n)}{\tilde{\Gamma}_p(\alpha)} |\det(\tilde{X})|^{-(\alpha+n)}, \Re(\alpha) > p - 1, n = 0, 1, 2, \ldots
\]

Note that the left-side of (7.30) can also be looked upon as the Laplace transform of the function \(\frac{|\det(\tilde{S})|^{\alpha-p}}{\tilde{\Gamma}_p(\alpha)}\) with the Laplace parameter matrix \(\tilde{X}\). In this case either \(\tilde{X}\) shall be written as \(\tilde{X} = X + iY, i = \sqrt{-1}\) where \(X\) and \(Y\) are real with
\[
X = (\eta_{ij}x_{ij}), Y = (\hat{\eta}_{ij}y_{ij}), \eta_{ij} = \begin{cases} 1, & i = j \\ \frac{1}{2}, & i \neq j \end{cases}, \hat{\eta}_{ij} = \begin{cases} 0, & i = j \\ \frac{1}{2}, & i \neq j \end{cases}
\]
so that in the Laplace transform, all elements of \( \tilde{S} \) will be taken once each. Otherwise if \( X = (x_{ij}) = X' \) and \( Y = (y_{ij}) = -Y' \) then the diagonal elements will be taken once each and the off-diagonal elements will be taken twice each. In this case the inverse Laplace transform must be multiplied by \( 2^{\frac{p(p-1)}{2}} \). Therefore from (7.30), as an inverse Laplace transform, we have

\[
\left| \det(\tilde{S}) \right|^{\alpha-p} \Gamma_p(\alpha) = \frac{2^{\frac{p(p-1)}{2}}}{(2\pi i)^\frac{p(p+1)}{2}} \int_{\Re(\tilde{X}) > X_0} |\det(\tilde{X})|^{-\alpha} e^{\text{tr}(\tilde{X}\tilde{S})} d\tilde{X}, \quad i = \sqrt{-1}.
\]

**Lemma 7.7** Let \( \tilde{S} \) and \( \tilde{X} \) be \( p \times p \) Hermitian positive definite matrices. Let \( |\det(\tilde{S})| \) be the absolute value of the determinant of \( \tilde{S} \). Let \( \tilde{\Gamma}_p(\alpha) \) be the complex matrix-variate gamma function. Let \( D^n_S \) be the operator defined in (7.28). Then

\[
D^n_S \left[ \frac{|\det(\tilde{S})|^{\alpha-p}}{\tilde{\Gamma}_p(\alpha)} \right] = \frac{|\det(\tilde{S})|^{\alpha-n-p}}{\tilde{\Gamma}_p(\alpha - n)}, \quad \Re(\alpha) > n + p - 1, \quad n = 0, 1, 2, \ldots
\]

**Proof** Let us operate on both sides of (7.32) with \( D^n_S \). Then

\[
D^n_S \left[ \frac{|\det(\tilde{S})|^{\alpha-p}}{\tilde{\Gamma}_p(\alpha)} \right] = \frac{2^{\frac{p(p-1)}{2}}}{(2\pi i)^\frac{p(p+1)}{2}} \int_{\Re(\tilde{X}) > X_0} |\det(\tilde{X})|^{-\alpha} [D^n_S e^{\text{tr}(\tilde{S})}] d\tilde{X}
\]

Interpreting the right-side integral as an inverse Laplace transform the function is the left-side function with \( \alpha \) replaced by \( \alpha - n \). That is,

\[
D^n_S \left[ \frac{|\det(\tilde{S})|^{\alpha-p}}{\tilde{\Gamma}_p(\alpha)} \right] = \frac{|\det(\tilde{S})|^{\alpha-n-p}}{\tilde{\Gamma}_p(\alpha - n)}, \quad \Re(\alpha) > n + p - 1, \quad n = 0, 1, \ldots
\]

Hence the result.

**Lemma 7.8**

\[
D^n_U e^{\pm \text{tr}(\tilde{U})} = e^{\pm \text{tr}(\tilde{U})}.
\]

The proof is straightforward.
7.10 Fractional Derivative in Riemann-Liouville and Caputo Senses for Weyl Integrals

Here all the matrices appearing are \( p \times p \) Hermitian positive definite unless stated otherwise. The Riemann-Liouville fractional integral of order \( \alpha \) and of the second kind in the complex matrix-variate case is denoted by \( \tilde{D}_2^{-\alpha} f \) and it is defined in Mathai [10] as

\[
\tilde{D}_2^{-\alpha} f = \frac{1}{\Gamma_p(\alpha)} \int_{\tilde{V} > \tilde{U} > 0} |\det(\tilde{V} - \tilde{U})|^{\alpha - p} f(\tilde{V})d\tilde{V}, \Re(\alpha) > p - 1.
\] (7.35)

Hence the fractional derivative of order \( \alpha \) of the second kind may be symbolically written as \( \tilde{D}_2^{\alpha} f \). With the help of the operator \( D^n_\tilde{U} \) of Sect. 7.9 we can symbolically write

\[
\tilde{D}_2^{\alpha} f = D^n_\tilde{U} [ \tilde{D}_2^{-(n-\alpha)} f ] =
\] (7.36)

fractional derivative of order \( \alpha \), of the second kind in the Riemann-Liouville sense. We can also symbolically write \( \tilde{D}_2^{\alpha} f \) as

\[
\tilde{D}_2^{\alpha} f = \tilde{D}_2^{-(n-\alpha)} [ D^n_\tilde{V} f ] =
\] (7.37)

fractional derivative of order \( \alpha \), of the second kind in the Caputo sense. In (7.36) we take the fractional integral first and then operate on the left with \( D^n_\tilde{U} \) whereas in (7.37) we operate on the arbitrary function \( f(\tilde{V}) \) with the operator \( D^n_\tilde{V} \) first and then take the fractional integral. The former is the fractional derivative in the Riemann-Liouville sense whereas the latter is the fractional derivative in the Caputo sense. For the symbolic operation to be consistent with the definition, we must have \( \Re(n - \alpha) > p - 1 \) or \( n > \Re(\alpha) + p - 1 \). The smallest possible such \( n \) is \( m = [\Re(\alpha)] + 1 + p - 1 = [\Re(\alpha)] + p \) where \( [\Re(\alpha)] \) denotes the integer part of \( \Re(\alpha) \).

Fractional integral of order \( \alpha \), of the first kind in the complex matrix-variate case is denoted by \( \tilde{D}_1^{-\alpha} f \) and the Riemann-Liouville fractional integral of the first kind is given as

\[
\tilde{D}_1^{-\alpha} f = \frac{1}{\Gamma_p(\alpha)} \int_{\Omega < \tilde{V} < \tilde{U}} |\det(\tilde{U} - \tilde{V})|^{\alpha - p} f(\tilde{V})d\tilde{V}, \Re(\alpha) > p - 1.
\] (7.38)

Here also we can define fractional derivative of order \( \alpha \) of the first kind in Riemann-Liouville and Caputo senses.

\[
\tilde{D}_1^{\alpha} f = D^n_\tilde{U} [ \tilde{D}_1^{-(n-\alpha)} ] =
\] (7.39)
fractional derivative of order $\alpha$, of the first kind in the Riemann-Liouville sense, whereas

$$\tilde{D}^\alpha_{1,\tilde{U}} f = \tilde{D}^{-(n-\alpha)}_{1,\tilde{U}} [D^n_{\tilde{V}} f(\tilde{\mathbf{V}})] = $$

fractional derivative of order $\alpha$ and of the first kind in the Caputo sense.

We will examine the fractional derivatives for a few situations of the arbitrary function $f(\tilde{\mathbf{V}})$ in the cases of Riemann-Liouville right-sided (second kind) and left-sided (first kind) fractional derivatives. Note that general definitions of right and left-aided fractional integrals are given by this author in terms of M-convolutions of a product and ratio or in terms of statistical densities of product and ratio of matrices in the real as well as in the complex cases. For tackling the Caputo situation also the steps are parallel and hence we will illustrate the procedure for Riemann-Liouville fractional derivatives.

**Case 7.10a**: $f(\tilde{\mathbf{V}}) = |\det(\tilde{\mathbf{V}})|^{-\gamma}$, $\Re(\gamma) > p - 1$, second kind fractional derivative of order $\alpha$ in the Riemann-Liouville sense

Fractional derivative of order $\alpha$ in this case is given by the following:

$$\tilde{D}^\alpha_{2,\tilde{U}} f = D^n_{\tilde{U}} \left[ \frac{1}{\tilde{\Gamma}_p(n-\alpha)} \right]$$

$$\times \int_{\tilde{\mathbf{V}} > \tilde{\mathbf{U}} > 0} |\det(\tilde{\mathbf{V}} - \tilde{\mathbf{U}})|^{n-\alpha-p} |\det(\tilde{\mathbf{V}})|^{-\gamma} d\tilde{\mathbf{V}}, \ (n > \Re(\alpha) + p - 1)$$

$$= D^n_{\tilde{U}} \left[ \frac{1}{\tilde{\Gamma}_p(n-\alpha)} \right] \int_{\tilde{\mathbf{W}} > 0} |\det(\tilde{\mathbf{W}})|^{n-\alpha-p} |\det(\tilde{\mathbf{U}} + \tilde{\mathbf{W}})|^{-\gamma} d\tilde{\mathbf{W}}$$

$$= D^n_{\tilde{U}} \left[ \frac{|\det(\tilde{\mathbf{U}})|^{-\alpha-\gamma+n}}{\tilde{\Gamma}_p(n-\alpha)} \right] \int_{\tilde{\mathbf{T}} > 0} |\det(\tilde{\mathbf{T}})|^{n-\alpha-p} |\det(I + \tilde{\mathbf{T}})|^{-\gamma} d\tilde{\mathbf{T}}$$

$$\tilde{D}^\alpha_{2,\tilde{U}} f = D^n_{\tilde{U}} \left[ \frac{\tilde{\Gamma}_p(n-\alpha)}{\tilde{\Gamma}_p(n-\alpha)} \right] |\det(\tilde{\mathbf{U}})|^{-\alpha-\gamma+n} \text{ from type-2 beta integral}$$

$$= \frac{\tilde{\Gamma}_p(\gamma + \alpha - n)}{\tilde{\Gamma}_p(\gamma)} D^n_{\tilde{U}} |\det(\tilde{\mathbf{U}})|^{-(\gamma+\alpha-n)}, \Re(\gamma) > p - 1$$

$$= \frac{\tilde{\Gamma}_p(\gamma + \alpha)}{\tilde{\Gamma}_p(\gamma)} |\det(\tilde{\mathbf{U}})|^{-(\gamma+\alpha)}$$

by using Lemma 7.6, for $\Re(\gamma) > p - 1, \Re(\gamma + \alpha) > p - 1, n > \Re(\alpha) + p - 1$. This is the $\alpha$th order fractional derivative of the second kind in the Riemann-Liouville sense. Now let us look at the Caputo derivative for the same case of $f(\tilde{\mathbf{V}})$.
But from Lemma 7.6

\[ D^n_{\tilde{V}} |\det(\tilde{V})|^{-\gamma} = \frac{\tilde{\Gamma}_p(\gamma + n)}{\tilde{\Gamma}_p(\gamma)} |\det(\tilde{V})|^{-(\gamma + \alpha)} , \Re(\gamma) > p - 1. \]

Hence

\[
\tilde{D}^\alpha_{2,\tilde{U}} f = \frac{\tilde{\Gamma}_p(\gamma + n)}{\tilde{\Gamma}_p(\gamma) \tilde{\Gamma}_p(n - \alpha)} \int_{\tilde{V} > \tilde{U} > O} |\det(\tilde{V} - \tilde{U})|^{n - \alpha - p} |\det(\tilde{V})|^{-(\gamma + n)} d\tilde{V} \\
= \frac{\tilde{\Gamma}_p(\gamma + n)}{\tilde{\Gamma}_p(\gamma) \tilde{\Gamma}_p(n - \alpha)} \int_{\tilde{U} > O} |\det(\tilde{T})|^{n - \alpha - p} |\det(I + \tilde{T})|^{-(\gamma + n)} d\tilde{T} \\
= \frac{\tilde{\Gamma}_p(\gamma + n)}{\tilde{\Gamma}_p(\gamma) \tilde{\Gamma}_p(n - \alpha) \tilde{\Gamma}_p(\gamma + \alpha)} |\det(\tilde{U})|^{-(\gamma + \alpha)} \\
= \frac{\tilde{\Gamma}_p(\gamma + \alpha)}{\tilde{\Gamma}_p(\gamma)} |\det(\tilde{U})|^{-(\gamma + \alpha)}
\]

for \( \Re(\gamma) > p - 1, \Re(\gamma + \alpha) > p - 1, n > \Re(\alpha) + p - 1, \Re(\alpha) > p - 1 \). This is the same result as in the Riemann-Liouville sense also. Thus we have the following result:

**Theorem 7.1** Consider the second kind of Riemann-Liouville fractional derivative of order \( \alpha \) denoted by \( \tilde{D}^\alpha_{2,\tilde{U}} f \). If the arbitrary function \( f(\tilde{V}) = |\det(\tilde{V})|^{-\gamma} , \Re(\gamma) > p - 1 \) then

\[
\tilde{D}^\alpha_{2,\tilde{U}} f = \frac{\tilde{\Gamma}_p(\gamma + \alpha)}{\tilde{\Gamma}_p(\gamma)} |\det(\tilde{U})|^{-(\gamma + \alpha)} , \Re(\gamma) > p - 1, \Re(\gamma + \alpha) > p - 1 \quad (7.41)
\]

for both the Riemann-Liouville sense, that is \( \tilde{D}^\alpha_{2,\tilde{U}} f = D^n_{\tilde{V}} \{ \tilde{D}^{-(n - \alpha)}_{2,\tilde{U}} f \} \) and in the Caputo sense, that is, \( \tilde{D}^\alpha_{2,\tilde{U}} f = \tilde{D}^{-(n - \alpha)}_{2,\tilde{U}} \{ D^n_{\tilde{V}} f(\tilde{V}) \} \).

Since (7.41) is free of \( n \), it is not difficult to show that the semigroup property holds here. This will be stated as a theorem.

**Remark 7.1** We have called (7.35) and (7.37) as Riemann-Liouville integral and derivative. But strictly speaking, they are Weyl integral type. For Riemann-Liouville integrals there is a finite upper bound for the second kind integrals and a finite lower bound for first kind integrals. In the matrix-variate case, there will be constant Hermitian positive definite matrices as upper bound for integrals of the second kind and lower bound for integrals of the first kind. These bounds do not affect our procedures because we have only made use of the property that \( \tilde{V} - \tilde{U} \) is positive definite in the case of the second kind integrals and \( \tilde{U} - \tilde{V} \) is positive definite in the case of first kind integrals.
Case 7.10b: \( f(\tilde{V}) = e^{-\text{tr}(\tilde{V})} \), right-sided Weyl fractional derivatives of order \( \alpha \) in the Riemann-Liouville and Caputo Senses

\[
\tilde{W}_{2,\tilde{U}}^\alpha f = D^n_{\tilde{U}}[\tilde{W}_{2,\tilde{U}}^{-(n-\alpha)} f] \text{ in the Riemann-Liouville sense}
\]

\[
= D^n_{\tilde{U}} \left[ \frac{1}{\Gamma_p(n-\alpha)} \int_{\tilde{V} > \tilde{U} > O} |\text{det}(\tilde{V} - \tilde{U})|^{n-\alpha-p} e^{-\text{tr}(\tilde{V})} d\tilde{V}, n > \Re(\alpha) + p - 1 \right]
\]

\[
= D^n_{\tilde{U}} \left[ \frac{e^{-\text{tr}(\tilde{U})}}{\Gamma_p(n-\alpha)} \int_{\tilde{T} > O} |\text{det}(\tilde{T})|^{n-\alpha-p} e^{-\text{tr}(\tilde{T})} d\tilde{T} \right] = D^n_{\tilde{U}}[e^{-\text{tr}(\tilde{U})}] = e^{-\text{tr}(\tilde{U})}.
\]

Since \( D^n_{\tilde{U}} e^{-\text{tr}(\tilde{U})} = e^{-\text{tr}(\tilde{U})} \) both in the Riemann-Liouville and Caputo senses the second kind Weyl fractional derivative of order \( \alpha \) for \( f(\tilde{V}) = e^{-\text{tr}(\tilde{V})} \) remains as \( e^{-\text{tr}(\tilde{V})} \). Also it is evident that the semigroup property holds here. Note that if \( f(\tilde{V}) = e^{\text{tr}(\tilde{V})} \) then the integral is not convergent.

7.11 Fractional Derivatives of the First Kind for Weyl Operators in the Riemann-Liouville and Caputo Senses

The Weyl fractional integral of order \( \alpha \) of the first kind, denoted by \( \tilde{W}_{1,\tilde{U}}^{-\alpha} \), is given by the integral

\[ \tilde{W}_{1,\tilde{U}}^{-\alpha} f = \frac{1}{\Gamma_p(\alpha)} \int_{O < \tilde{V} < \tilde{U}} |\text{det}(\tilde{U} - \tilde{V})|^{\alpha-p} f(\tilde{V}) d\tilde{V}, \Re(\alpha) > p - 1. \quad (7.42) \]

We will consider some special cases of \( f(\tilde{V}) \) and look into Riemann-Liouville and Caputo type derivatives of order \( \alpha \) and of the first kind.

Case 7.11a: \( f(\tilde{V}) = \frac{|\text{det}(\tilde{V})|^{\gamma-p}}{\tilde{T}_p(\gamma)} \), \( \Re(\gamma) > p - 1 \). Weyl fractional derivative of order \( \alpha \) in the Riemann-Liouville sense

\[ \tilde{W}_{1,\tilde{U}}^{\alpha} f = D^n_{\tilde{U}}[\tilde{W}_{1,\tilde{U}}^{-(n-\alpha)} f], n > \Re(\alpha) + p - 1 \]

\[
= D^n_{\tilde{U}} \left[ \frac{1}{\tilde{T}_p(n-\alpha)} \int_{O < \tilde{V} < \tilde{U}} |\text{det}(\tilde{U} - \tilde{V})|^{n-\alpha-p} |\text{det}(\tilde{V})|^{\gamma-p} \frac{1}{\tilde{T}_p(\gamma)} d\tilde{V} \right]
\]

\[
= D^n_{\tilde{U}} \left[ \frac{|\text{det}(\tilde{U})|^{\gamma+n-\alpha-p}}{\tilde{T}_p(n-\alpha) \tilde{T}_p(\gamma)} \int_{\tilde{T} > O} |\text{det}(\tilde{T})|^{\gamma-p} |\text{det}(1 - \tilde{T})|^{n-\alpha-p} d\tilde{T} \right]
\]

\[
= D^n_{\tilde{U}} \left[ \frac{|\text{det}(\tilde{U})|^{\gamma+n-\alpha-p}}{\tilde{T}_p(\gamma + n - \alpha)} \right] = \frac{|\text{det}(\tilde{U})|^{\gamma-\alpha-p}}{\tilde{T}_p(\gamma - \alpha)}, \text{ from Lemma 7.7}
\]

\( \Re(\gamma - \alpha) > p - 1, \Re(\gamma) > p - 1, n > \Re(\alpha) + p - 1 \) \quad (7.43)
by using Lemma 7.7. Let us look into the fractional derivative of order $\alpha$ here in the Caputo sense.

\[
\tilde{W}_\alpha f = \tilde{W}_\alpha^{-(n-\alpha)} [D^n_U f]
\]

\[
= \frac{1}{\Gamma_p(n-\alpha)} \int_{\tilde{V} < \tilde{U}} |\det(\tilde{U} - \tilde{V})|^{n-\alpha-p} |\det(\tilde{V})|^{-p} d\tilde{V}, n > \Re(\alpha) + p - 1
\]

\[
= \frac{1}{\Gamma_p(n-\alpha)} \int_{\tilde{V} < \tilde{U}} |\det(\tilde{U} - \tilde{V})|^{n-\alpha-p} |\det(\tilde{V})|^{-p} d\tilde{V} (\text{Lemma 7.7}),
\]

\[
= \frac{|\det(\tilde{U})|^{\gamma - \alpha - p}}{\Gamma_p(\gamma - n)} \frac{\tilde{\Gamma}_p(n - \alpha)}{\tilde{\Gamma}_p(n - \alpha)} \int_{\tilde{T} > 0} |\det(\tilde{T})|^{\gamma - n - p} |\det(I - \tilde{T})|^{n-\alpha-p} d\tilde{T}
\]

\[
\tilde{W}_\alpha f = |\det(\tilde{U})|^{\gamma - \alpha - p} \frac{\tilde{\Gamma}_p(\gamma - n)}{\tilde{\Gamma}_p(n - \alpha)} \frac{\tilde{\Gamma}_p(n - \alpha)}{\tilde{\Gamma}_p(n - \alpha)} , \Re(\gamma) > n + p - 1
\]

\[
= \frac{|\det(\tilde{U})|^{\gamma - \alpha - p}}{\tilde{\Gamma}_p(\gamma - \alpha)} , \Re(\gamma - \alpha) > p - 1.
\]

(7.44)

It is the same result as in the Riemann-Liouville sense also. Since (7.43) and (7.44) are free of $n$ it is evident that the semigroup property holds for both Riemann-Liouville and Caputo senses. This can be stated as a theorem.

**Theorem 7.2** When $f(\tilde{V}) = |\det(\tilde{V})|^{\gamma - p}$ for $\Re(\gamma) > p - 1$ then Weyl fractional derivative of the first kind and of order $\alpha$ in the complex matrix-variate case, namely $\tilde{W}_\alpha f$ is such that

\[
\tilde{W}_\alpha [\tilde{W}_\beta f] = \tilde{W}_{\alpha + \beta} f = \tilde{W}_\alpha [\tilde{W}_\beta f]
\]

both in the Riemann-Liouville and Caputo senses, that is, operating on the left as well as on the right with the differential operator $D^n_U$.

**Case 7.11b:** $f(\tilde{V}) = e^{tr(\tilde{V})}$, left-sided Weyl fractional derivative of order $\alpha$ in the Riemann-Liouville and Caputo senses in the complex matrix-variate case

Fractional derivative in the Riemann-Liouville sense is given by the following:

\[
\tilde{D}_\alpha f = D^n_U \left( \frac{1}{\Gamma_p(n-\alpha)} \int_{\tilde{V} < \tilde{U}} |\det(\tilde{U} - \tilde{V})|^{n-\alpha-p} e^{tr(\tilde{V})} d\tilde{V}, \tilde{T} = \tilde{U} - \tilde{V}
\]

\[
= D^n_U \frac{e^{tr(\tilde{U})}}{\Gamma_p(n-\alpha)} \int_{\tilde{T} > 0} |\det(\tilde{T})|^{n-\alpha-p} e^{-tr(\tilde{T})} d\tilde{T}, n > \Re(\alpha) + p - 1
\]

\[
= D^n_U e^{tr(\tilde{U})} = e^{tr(\tilde{U})}.
\]
Since $D^n_U e^{\nu(U)} = e^{\nu(U)}$ we have the same result in Riemann-Liouville and Caputo senses. Also it is evident that the semigroup property holds here.

7.12 General Definitions

General definitions of fractional integral of the second kind is given by Mathai [12] as the following for complex domain:

\[ \tilde{D}_2^{-\alpha} f = \frac{1}{\Gamma_p(\alpha)} \int_{\tilde{V}} |\det(\tilde{V})|^{-p} \phi_1(\tilde{V}^{-\frac{1}{2}} \tilde{U} \tilde{V}^{-\frac{1}{2}}) |\det(I - \tilde{V}^{-\frac{1}{2}} \tilde{U} \tilde{V}^{-\frac{1}{2}})|^{\alpha-p} \phi_2(\tilde{V}) f(\tilde{V}) d\tilde{V}, \]  
(7.45)

where $\phi_1$ and $\phi_2$ are given functions and $f$ is arbitrary. Hence the fractional derivative of order $\alpha$, of the second kind in the Riemann-Liouville sense is given by the following:

\[ \tilde{D}_2^{\alpha} f = D^n_{\tilde{U}} [\tilde{D}_2^{-(n-\alpha)} f], \ n > \Re(\alpha) + p - 1 \]

\[ = D^n_{\tilde{U}} \left( \frac{1}{\Gamma_p(n-\alpha)} \int_{\tilde{V} > \tilde{U}} |\det(\tilde{V})|^{-p} \phi_1(\tilde{V}^{-\frac{1}{2}} \tilde{U} \tilde{V}^{-\frac{1}{2}}) \right. \]

\[ \times |\det(I - \tilde{V}^{-\frac{1}{2}} \tilde{U} \tilde{V}^{-\frac{1}{2}})|^{n-\alpha-p} \phi_2(\tilde{V}) f(\tilde{V}) d\tilde{V} \} \]  
(7.46)

and that in the Caputo sense is given by

\[ \tilde{D}_2^{\alpha} f = \tilde{D}_2^{-(n-\alpha)} [D^n_V f(\tilde{V})], \ n > \Re(\alpha) + p - 1 \]

\[ = \frac{1}{\Gamma_p(n-\alpha)} \int_{\tilde{V} > \tilde{U}} |\det(\tilde{V})|^{-p} \phi_1(\tilde{V}^{-\frac{1}{2}} \tilde{U} \tilde{V}^{-\frac{1}{2}}) \]

\[ \times |\det(I - \tilde{V}^{-\frac{1}{2}} \tilde{U} \tilde{V}^{-\frac{1}{2}})|^{n-\alpha-p} \phi_2(\tilde{V}) [D^n_{\tilde{V}} f(\tilde{V})] d\tilde{V}. \]  
(7.47)

Erdélyi-Kober fractional differential operator is available by taking $\phi_1(\tilde{X}_1) = |\det(\tilde{X}_1)|^{\rho}$ for $\Re(\rho) > -1$ and $\phi_2 = 1$. Erdélyi-Kober fractional integral becomes a statistical density when

\[ \phi_1(\tilde{X}_1) = \frac{\tilde{\Gamma}_p(\rho + p + \alpha)}{\tilde{\Gamma}_p(\rho + p)} |\det(\tilde{X}_1)|^{\rho}, \ \phi_2(\tilde{X}_2) = 1 \text{ and } f(\tilde{X}_2) \]
is an arbitrary density of $\tilde{X}_2$, and $\tilde{X}_1$ and $\tilde{X}_2$ are statistically independently distributed. Erdélyi-Kober fractional integral of the second kind and of order $\alpha$ in the complex matrix-variate case is defined as follows:

$$
\tilde{K}_{2,\tilde{U},\rho}^{-\alpha} f = \frac{|\det(\tilde{U})|^\rho}{\Gamma_p(\alpha)} \int_{\tilde{V} > \tilde{U} > 0} |\det(\tilde{V})|^{-\alpha-\rho} |\det(\tilde{V} - \tilde{U})|^{\alpha-p} f(\tilde{V}) d\tilde{V}, \Im(\alpha) > p-1.
$$

This Erdélyi-Kober fractional integral is also a constant multiple of a statistical density of a product of the form $\tilde{U} = \tilde{X}_2^{1/2} \tilde{X}_1 \tilde{X}_2^{-1/2}$, $\tilde{X}_1 = \tilde{V}$ where $\tilde{X}_1$ and $\tilde{X}_2$ are statistically independently distributed $p \times p$ Hermitian positive definite complex matrix-variate random variables, with $\tilde{X}_1$ having a matrix-variate type-1 beta density with the parameters $(\rho + p, \alpha)$ and $\tilde{X}_2$ having an arbitrary density $f(\tilde{X}_2)$.

When the arbitrary functions are as given for the real matrix-variate case in this chapter, the results for Erdélyi-Kober fractional derivatives of the first and second kind are parallel. Hence these materials are not included here.

References


Note added in proof: The authors were made aware that there is a series of publications regarding the topic of an interpretation of the Erdélyi–Kober operators as the so-called H-transforms, their properties, multiple Erdélyi–Kober fractional integrals and derivatives (compositions of the suitable Erdélyi-Kober fractional integrals and derivatives), operational calculus for these operators and fractional differential equations with the multiple Erdélyi–Kober fractional derivatives were treated: