Parametric decay of non-linear circularly polarised Alfvén waves

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Abstract. We study the evolution of non-linear monochromatic circularly polarised Alfvén waves by solving numerically the time-dependent equations of magnetohydrodynamics in one dimension. We find that in a low $\beta$ plasma the waves may undergo a parametric decay. This is because the wave excites a density enhancement that travels slower than the wave itself and thus interacts with the wave. When $\beta \sim 1$ the density enhancement does not interact with the wave and no decay takes place.

INTRODUCTION

Large amplitude, low frequency, Alfvén waves have been observed in the solar corona for over 30 years (e.g. Belcher & Davis [1]) and are thought to play a role in heating the corona and accelerating the solar wind. During the last decade Ulysses has provided plasma and magnetic field measurements that have allowed extensive investigations on the behaviour of Alfvénic turbulence in the high-latitude solar wind. The data shows a strong correlation between the fluctuations in velocity and magnetic fields [3], which reveals the presence of both inward and outward propagating Alfvén waves (e.g Bavassano et al. [2]).

An infinitely long circularly polarised Alfvén wave is an exact solution to the equations of magnetohydrodynamics (MHD). Therefore circularly polarised Alfvén waves are more likely to reach high altitude, whereas the linearly polarised Alfvén wave can form current sheets at the nodes of the fluctuating magnetic field and dissipate at a lower altitude (Boynton & Torkelsson [6]).

Different mechanisms, in which the wave may lose its linearity have been studied, like phase mixing due to a transverse gradient in the phase velocity (e.g. Heyvaerts & Priest [4]) or the nonlinear coupling of the Alfvén wave with other modes (e.g. Wentzel [5]). In the high latitude solar wind with a smooth density profile the parametric decay of circularly polarised Alfvén waves has been proposed to play a role in generating turbulence and inward propagating Alfvén waves beyond the critical point where the Alfvén speed becomes lower than the solar wind speed.

A forward propagating Alfvén wave can generate a forward propagating acoustic wave and a backward propagating Alfvén wave through a parametric instability in the presence of a density fluctuation. The parametric decay of circularly polarised Alfvén waves has been studied both analytically (e.g. [7] and [8]) and numerically by several groups (e.g. [9], [10], [11] and [12]). Most of these studies were restricted to periodic boundary conditions (except [9] and [11]), and external noise in the form of density fluctuations were added to excite the decay.

In this paper we study the evolution of non-linear circularly polarised Alfvén waves by solving numerically the time-dependent MHD-equations in one dimension assuming plane parallel geometry. We examine the behaviour of the waves for different values of beta without adding any external perturbations, which makes our model an almost ideal case. We therefore expect our model to underestimate the importance of the parametric decay compared to a real medium with density inhomogeneities. In all our models we extend the box so that the right boundary does not affect the solution. The plan of the paper is the following: in Sec. II we describe the basic MHD equations that we are solving numerically and some of the properties of the parametric decay. In Sec. III we describe the parameters of our models and discuss the results, and the concluding remarks are in Sec. IV.

MATHEMATICAL FORMULATION

Fundamental properties of Alfvén waves

The equations of ideal isothermal MHD can be written as
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,
\]
\[
\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = -\nabla \rho + \mathbf{J} \times \mathbf{B},
\]
\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}),
\]
\[
\nabla \cdot \mathbf{B} = 0,
\]
where \( \rho \) is the density, \( \mathbf{v} \) the velocity, \( \rho \) the pressure, \( \mathbf{B} \) the magnetic field, and \( \mathbf{J} = \nabla \times \mathbf{B} / \mu_0 \) the current density. The constraint \( \nabla \cdot \mathbf{B} = 0 \) is fulfilled by Eq. (3) if it is imposed as an initial condition.

In a homogeneous medium with a density \( \rho_0 \) and a background magnetic field \( \mathbf{B}_0 = B_0 \hat{z} \), a circularly polarised forward propagating Alfvén wave is described by the transverse magnetic field
\[
\mathbf{B}_\perp = B_\perp \left[ \cos(kz - \omega t) \hat{x} + \sin(kz - \omega t) \hat{y} \right],
\]
and the velocity
\[
\mathbf{v}_\perp = -\frac{\mathbf{B}_\perp}{\sqrt{\rho_0 \rho_0}}.
\]
The wave obeys the dispersion relation
\[
\omega = v_A k
\]
with the Alfvén velocity
\[
v_A = \frac{B_0}{\sqrt{\rho_0 \rho_0}}.
\]
For this wave the magnetic pressure \( \frac{B_\perp^2}{2\mu_0} \) is the same everywhere inside the wave. This is the physical reason why an infinitely extended circularly polarised Alfvén wave is incompressible and an exact solution of the nonlinear MHD equations. In the one-dimensional problem that we study, there is only one additional wave mode, an acoustic wave obeying the dispersion relation
\[
\omega = c_s k,
\]
where the isothermal speed of sound is
\[
c_s = \sqrt{\frac{\rho_0}{\rho_0}}.
\]
The amplitude of the density oscillation \( \Delta \rho \) is related to that of the longitudinal velocity, \( v_z \), through
\[
v_z = \frac{\Delta \rho}{\rho} c_s.
\]

### The modulational instability

Galeev & Oraevskii [13] showed that an Alfvén wave with a frequency \( \omega_0 = v_A k_0 \) and a wave number \( k_0 \) can decay into a backward propagating Alfvén wave with a frequency \( \omega_- \) and a wave number \( k_- \) and a forward propagating acoustic wave with a frequency \( \omega \) and a wave number \( k \) that fulfill the resonance conditions
\[
\omega_0 = \omega + |\omega_-|,
\]
and
\[
k = k_- + k_0.
\]
In the limit of low \( \beta \) and high wave amplitude, the growth rate of the reflected wave is (Galeev & Oraevskii 1983).

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**TABLE 1.** Simulations of Alfvén waves in a homogeneous medium. The wave is characterised by the two quantities \( \eta \), the amplitude of the imposed Alfvén wave in terms of the vertical magnetic field, and \( \beta = 2 \mu_0 \rho_0 \) the plasma beta. For the different Runs we further specify the length of the time step \( \Delta t \) in terms of the period of the wave, \( P \), the length of the computational domain, \( L \) in terms of the wave length, \( \lambda \), and the number of grid points, \( N \).

<table>
<thead>
<tr>
<th>Run</th>
<th>( \Delta t/P )</th>
<th>( N )</th>
<th>( L/\lambda )</th>
<th>( \eta )</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a</td>
<td>0.0033</td>
<td>3600</td>
<td>78</td>
<td>0.007</td>
<td>0.042</td>
</tr>
<tr>
<td>1b</td>
<td>0.0033</td>
<td>3600</td>
<td>78</td>
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<tr>
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<td>3600</td>
<td>78</td>
<td>0.7</td>
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<tr>
<td>2a</td>
<td>0.0033</td>
<td>13500</td>
<td>193</td>
<td>0.007</td>
<td>0.96</td>
</tr>
<tr>
<td>2b</td>
<td>0.0033</td>
<td>13500</td>
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<td>2c</td>
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<td>0.96</td>
</tr>
</tbody>
</table>

**FIGURE 1.** MHD waves propagating through a homogeneous medium of low \( \beta \) (Run 1a). a \( \Delta \rho/\rho \) versus \( z \) at \( t = 4.2P \). b \( B_x/B_0 \) (solid line) and \( v_x/v_A \) (dashed line) versus \( z/\lambda \) at the same time.
FIGURE 2. MHD waves propagating through a homogeneous medium of low $\beta$ (Run 1b). a, c, e, g $\Delta \rho / \rho$ as a function of distance $z$ at $t / P = 16.7, 23.3, 33.3, 46.7$. Note the changes in the $\Delta \rho / \rho$-scale between the frames. b, d, f, h $z_\perp / v_A$ (solid line) and $z_\parallel / v_A$ (dashed line) as a function of distance $z$ at the same times.

$\gamma \simeq \frac{\omega_p}{2^{3/4}} \frac{\eta^{1/2}}{\beta^{1/4}}$,  

where $\beta = 2\mu_0 \rho_0 / B_0^2$, and $\eta = B_\perp / B_0$.

Since we want to be able to separate the forward and backward propagating Alfvén waves in our simulations, it is useful to introduce the Elsässer variables

$$z_\pm = \left| v_\perp \mp \frac{B_\perp}{\sqrt{\mu_0 \rho}} \right|,$$

which describe the forward and backward propagating Alfvén waves, respectively.

RESULTS

We modify the numerical code of Boynton & Torkelsson [6] to simulate circularly polarised Alfvén waves. The waves are driven on the left boundary. In all the runs the grid is sufficiently extended that the wave does not hit the right boundary. The different models are described in Tab. 1.

The propagation of a low amplitude wave in a low $\beta$ medium (model 1a) is shown in Fig. 1b. The Alfvén wave excites a density enhancement (Fig. 1a), whose right edge coincides with the front of the Alfvén wave, while the left edge propagates with the lower speed $c_s$. The density enhancement is a second order effect, and its amplitude is consequently proportional to the square of the amplitude of the Alfvén wave. The Alfvén wave is propagating through the medium at a speed equal to $v_A$ without decaying.

The density discontinuity at the front of the Alfvén wave excites a secondary compressional wave, which can be seen as a weak modulation of the density in Fig. 1a, and also of the magnetic pressure $|B_\perp|^2 / (2\mu_0)$. This density fluctuation serves as the necessary seed for the parametric instability, but due to the low amplitude of the

\[\gamma \simeq \frac{\omega_p}{2^{3/4}} \frac{\eta^{1/2}}{\beta^{1/4}},\]

\[z_\pm = \left| v_\perp \mp \frac{B_\perp}{\sqrt{\mu_0 \rho}} \right|,\]
fluctuation the instability grows slowly. The evolution of the backward-propagating Alfvén wave that is generated by the instability can be followed in Figures 2b, d, f and h. The backward-propagating wave grows in amplitude away from the wave front, which enhances the growth rate of the parametric instability upstream. Eventually the instability is so strong that it becomes an efficient source of a forward-propagating sound wave (Figs 2a, c, e and g). In agreement with the theoretical prediction the sound wave has a wave number $k_{0}$. Figure 2h shows that there is a phase shift of $\frac{\pi}{2}$ between the backward- and the forward-propagating Alfvén waves as we expect if the backward-propagating wave is generated through the parametric instability.

The sound waves stimulate the parametric decay of the forward-propagating Alfvén wave, but since the Alfvén speed is larger than the sound speed, the first section of the Alfvén wave will remain unaffected. On the other hand we see a region with $z < 13\lambda$ in Fig. 2g and h, in which the waves are strongly interacting. In this region there is a strong damping of the forward-propagating Alfvén wave, and the acoustic wave is amplified until it becomes so nonlinear that it steepens into shocks.

When $\beta \sim 1$ the density enhancement remains ahead of the Alfvén wave, and the parametric decay does not take place regardless of the amplitude of the wave. Instead the Alfvén wave is reflected off the density gradient at the back of the density enhancement (Fig. 3). This process will be studied in more details in a future paper.

CONCLUSION

Circularly polarised Alfvén waves are subject to a parametric instability that generates a backward propagating Alfvén wave and a forward propagating sound wave. We find that the compression of the background medium that takes place at the wave front of an Alfvén wave in a low $\beta$ plasma is sufficient to trigger this instability. However when $\beta \sim 1$ the Alfvén wave generates an acoustic precursor that remains ahead of the Alfvén wave, which therefore is not subject to the parametric decay.

The aim of our model is to demonstrate some of the basic properties of a propagating circularly polarised Alfvén waves. These results can then guide us in future investigations of the dynamics of Alfvén waves in more realistic configurations.

REFERENCES