Some considerations on the accuracy of derivative computation from PIV vector fields.

J.M. Foucaut

Abstract  Particle Image Velocimetry is a measurement technique which allows to obtained velocity fields with a limited spatial resolution. This contribution concerns the derivative computation by differential method with the goal to obtain vorticity fields. A large variety of derivative schemes was studied by determining their transfer function taking into account the truncation and the noise amplification. The noise was characterized as a white one in the Fourier domain. The most significant scheme was applied to a velocity field containing a vortex. A comparison of the maximum of vorticity obtained by each schemes and by a least square fit with an Oseen vortex allows to quantify the effect of the band pass filter on this value and to select the best scheme.

1 Introduction
Particle Image Velocimetry is now a reliable measurement technique which allows to store large numbers of instantaneous velocity maps. Therefore, it has become a useful tool to study turbulence. In a previous paper, the problem of vector validation and interpolation was addressed (Foucaut et al (2000)). The present contribution focuses on the problem of derivative computation. This is an important subject in PIV post-processing as spatial derivatives give access to the vorticity, which is a quantity of importance in the study of turbulence. For derivative computation, a compromise has to be made between the order of the filter, the number of points used for the derivation, the frequency response and the noise amplification. Generally, for classical finite difference schemes, the order increases with the number of points used for the computation. The frequency response increases when the truncation error decreases which is directly linked to the order of the filter. The noise amplification increases also with the order of the scheme. For PIV vector map, the noise due to the measurement random error is not negligible. For the lowest frequencies, it can be come the main source of error. Lourenco and Krothapalli (1995) recommend to use the Richardson extrapolation, which takes the values far from the measurement point into account, to improve the accuracy by minimizing the noise amplification.

In the present contribution, a comparison of the transfer functions of different schemes is presented in order to assess the frequency range of each filter. The use of compact difference schemes (see Lele (1992) or Kim & Lee(1996)) is also proposed and tested.

Raffel et al (1998) have studied the effect of overlapping higher than 50% on the computation of vorticity. This effect was also discussed in Foucaut et al (2001) as far as the velocity is concerned: by use of a power spectra analysis, the effect of overlapping and of PIV measurement noise were characterized. Raffel et al (1998) describe sources of error in differencial estimation. These sources are the measurement uncertainty on velocity vectors, the over-sampled data when an overlapping higher than 50% is used, the interrogation window size which limits the spatial resolution and the truncation of the curvature of the particle motion due to the use of only two pulses in PIV. The main measurement error which can affect the derivative is the measurement noise. In the spectral domain its effect is coupled with the over-sampling and windowing effects. These effects are already discussed in the previous study on the velocity spectral analysis (Foucaut et al (2001)). The curvature truncation error can be minimized by decreasing the PIV laser pulse delay (Raffel et al (1998)).

J. M. Foucaut, Laboratoire de Mécanique de lille, France

Correspondence to:
Dr J. M. Foucaut, Ecole Centrale de Lille, LML URA 1441, BP 48, Cité Scientifique, F59651 Villeneuve d'Ascq Cedex, France,
E-mail : jean-marc.foucaut@ec-lille.fr
Transfer function of derivatives

The spatial data of PIV allow to compute directly the derivative by use of differential estimations. In the case of centered difference schemes on a regular grid, the $n$th order derivative at a point $j$ can be estimated by the following equation:

$$
\frac{\partial^nu}{\partial x^n} = \frac{1}{a\Delta x} \sum_{i=1,n/2} a_i (u_{j+i} - u_{j-i}) + \sum_{i=n+1,\infty} a_i \frac{\Delta x^{-1}}{(n+1)!} \frac{\partial^nu}{\partial x^n} + \varepsilon \frac{\sigma u}{\Delta x}
$$

(1)

Where $\Delta x$ is the spatial resolution. The second term on the right hand side is the truncation error and the last term is the noise error obtained by a probabilistic quadratic approach. $\sigma u$ is an estimation of the noise obtained in a specific experiment (generally of the order of 0.1 pixel) and $\varepsilon$ is the noise amplification obtained by:

$$
\varepsilon^2 = \frac{4}{a^2} \sum_{i=1,n/2} a_i^2
$$

(2)

In this case the PIV noise measurement is considered as a white noise. This hypothesis will be justified further downstream. Using this approach, the 4th order centered difference scheme can be shown as an example:

$$
\frac{\partial^nu}{\partial x^n} = \frac{u_{i-2} - 8u_{i-4} + 8u_{i+4} - u_{i+2}}{12\Delta x} + 4 \frac{\Delta x^4}{(5)!} \frac{\partial^nu}{\partial x^n} + 0.95 \frac{\sigma u}{\Delta x}
$$

(3)

The frequency response of this filter can be obtained by several methods. First, the Dirac response can be computed. The input signal is a step function, which gives a Dirac pulse as derivative. Due to the digitization of the signal, the exact derivative is a quasi Dirac function which is located at $-\Delta x/2$ from the step and is $\Delta x$ long. Figure 1 shows the Dirac response of the 4th order derivative filter (3) as compared to a sinc function giving the best transfer function and to the exact solution. The Fourier transform of the step response of the derivative operator multiplied by the Fourier transform of the exact derivative, which is a cardinal sine function : $\text{sinc}(\Delta x k)$, gives the transfer function of the derivative filter, $k$ is the spatial wave number.

Figure 1 : Dirac response of the 4th order centered difference derivative filter compared to a sinc function and to the exact derivation.

An other method to obtain the transfer function is to use directly the Fourier function : $u = u'\exp(ikx)$ in the derivative scheme (Lele(1992)), where $u'$ can be the turbulence intensity. The transfer function is then directly obtained by:

$$
\text{Tr} = \frac{8\sin(k\Delta x) - \sin(2k\Delta x)}{6k\Delta x}
$$

(4)

The comparison between these two methods is presented in figure 2, which shows a perfect agreement. This figure shows also an estimation of the transfer function given by the truncation error :
\[
\frac{\partial u}{\partial x}_{\text{est}} = 1 - \sum_{i=0}^{n+1} \alpha_i \frac{\Delta x^{i-1}}{(n+1)!} \frac{\partial u}{\partial x}^{i}
\]

(5)

where the index "est" means estimated. Figure 2 shows that if only one term of the truncation error \((n+1)^{\text{th}}\) order of the expansion) is considered, the transfer function is the same only for the lower frequency. If 3 terms of the truncation error are kept, the transfer function obtained is then close to that calculated by the direct method.

![Figure 2: Comparison of transfer functions of the 4th order centered difference derivative filter.](image)

This result shows that, if the noise is not taken into account, a derivative scheme is a low pass filter and that it is possible to deduce its transfer function from the truncation error. The same approach can be proposed to take into account the measurement noise. By means of a PIV experiment in zero velocity flow, Foucaut et al (2001) showed that the measurement noise is white. In that case, the transfer function of the noise error can be expressed as:

\[
T_n = 1 - \frac{\sigma_u}{\sigma_u} = 1 - \varepsilon \frac{\sigma_u}{\|u'\| k \Delta x}
\]

(6)

Which shows that the noise effect is comparable to a high pass filter. That expression evidence the fact that, due to the noise, when the frequency goes to zero, the error on the derivative can become very important. This error varies also with the inverse of the turbulence rate. Figure 3 presents an example, for the 4th order centered difference scheme (3), for the transfer functions for the low pass filter due to the truncation, the high pass filter due to the noise and the band pass filter deduced from both.

The method presented above has been generalized for different schemes. Four filters based on centered difference schemes have been studied (2nd, 4th, 6th and 8th orders). Table 1 gives their characteristics. In spite of the decrease of the truncation error when the order increases, \(\varepsilon\) is higher than 1 for the 6th and 8th order filters.

<table>
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<th>Order</th>
<th>(a)</th>
<th>(a_1)</th>
<th>(a_2)</th>
<th>(a_3)</th>
<th>(a_4)</th>
<th>(\alpha_{n+1})</th>
<th>(\varepsilon)</th>
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Table 1: Detailed characteristics of centered difference derivative filters.
Figure 3: transfer functions of the 4th order centered difference derivative filter.

Seven filters from Richardson extrapolation (Lourenco and Krothapalli (1995)) have also been tested. The Richardson extrapolation principle is to take a linear combination of 2nd order centered difference schemes computed with a variable data spacing $\Delta x$, $2\Delta x$, $4\Delta x$ and $8\Delta x$. The Richardson extrapolation schemes are equivalent to the following equation:

$$\frac{\partial u}{\partial x} = \frac{1}{a} \sum_{i=1,2,4,8} a_i u_i - u_{i-1} \frac{-i \Delta x \sigma}{(n+1)!} e^{\frac{\Delta x}{\Delta x}}$$

(7)

The coefficients given in table 2 are optimized in order to minimize the truncation error or the noise error. Four schemes has been computed with an optimization of the truncation error (2nd, 4th, 6th and 8th orders) and three with a minimization of the noise coefficient $\varepsilon$ (2nd, 4th and 6th orders). The 2nd and 4th order filters obtained with the minimized truncation error are exactly the same as the centered difference schemes. The 6th and 8th order with the truncation error minimized present also an $\varepsilon$ value greater than 1. The 2nd, 4th and 6th order filters with $\varepsilon$ minimized do not present a small truncation error. Generally $\alpha$ (see eq. (1)) is 10 times higher than in the previous cases. However, $\varepsilon$ is 10 times as small as than the filter with a truncation minimization for the 2nd order derivative scheme.

<table>
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<tr>
<th>Order</th>
<th>$a$</th>
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Table 2: detailed characteristics of Richardson extrapolation derivative filters. * indicates a noise minimization.

Three compact difference schemes have finally been studied. The compact difference scheme is presented by Lele (1992) and Kim and Lee(1996). It is implicit and needs thus a matrix inversion. It is given by:

$$\beta u'_{i-2} + \chi u'_{i-1} + u'_i + \chi u'_{i+1} + \beta u'_{i+2} = c \frac{u_{i+3} - u_{i-3}}{6\Delta x} + b \frac{u_{i+2} - u_{i-2}}{4\Delta x} + a \frac{u_{i+1} - u_{i-1}}{2\Delta x}$$

(8)

where $\beta$, $\chi$, $a$, $b$ and $c$ are optimized to obtain the selected order. A summary of the compact scheme characteristics is given in table 3. The derivative estimation presents the same types of errors as previously (see eq. (1)):

$$\frac{\partial u}{\partial x} = u'_i + \sum_{i=n+1, \infty} \alpha_i \frac{\Delta x^{i-1}}{(n+1)!} \frac{\partial^i u}{\partial i + \gamma \frac{\sigma u}{\Delta x}}$$

(9)
The equation for the low pass transfer function writes:

\[
\frac{\frac{c}{3}\sin(3k\Delta x) + \frac{b}{2}\sin(2k\Delta x) + a\sin(k\Delta x)}{(1 + 2\beta \cos(2k\Delta x) + 2\chi \cos(k\Delta x))k\Delta x}
\] (10)

As can be seen in equation (10), if \(\beta\) and \(\chi\) are equal to zero the explicit form of the centered difference transfer function is recovered. The compact difference scheme is interesting because for the 10\(^{th}\) order filter \(c\) is smaller than for the 4\(^{th}\) order centered scheme with a very small truncation error.

The most significant results of these comparison are presented in figure 4. This figure also shows the least squares filter given by

\[
\frac{\partial u}{\partial x} = -2u_{i-2} - u_{i-1} + u_{i+1} - 2u_{i+2} + \frac{3.4\Delta x^2}{(3)!} \frac{\partial^3 u}{\partial x^3} + 0.316 \frac{\sigma u}{\Delta x}
\] (11)

This filter, proposed in Raffel et al (1998), is second order and shows a decrease of the noise amplification coefficient as compared to classical centered schemes.

Figure 4: transfer functions of several derivative filters.

The scheme that shows the largest bandwidth is the 10\(^{th}\) order compact difference. It presents a real interest for turbulent flows which contain a wide range of scales. The Richardson extrapolation scheme is also interesting for flows with low turbulence rate (see eq. (6)) or with small frequency range. As can be seen in figure 4, the effect of a minimization of \(c\) leads to move the bandwidth toward the lower frequency. The consequence is not really a minimization of the noise on the frequency response (Lourencio and Krothapalli (1995)) but a strong filtering of the high frequency. Figure 4 also shows the cutoff wave number of PIV evidenced in Foucault et al (2001). PIV allows to resolve the frequency up to \(k\Delta x = 1.4\) (with an overlapping value of 50\%). The cutoff frequency of the schemes have thus to be greater than this value. It is the case for the compact and the centered schemes. The second order centered difference cutoff wave number is the same as the PIV one. The Richardson extrapolation filters have cutoff frequency significantly smaller.

For turbulence applications, the lowest frequency is the inverse of the length of the vector field. Typically, it is about 10 cm which gives \(k \sim 60\). With a step of 1 mm, that gives \(k\Delta x = 0.06\). This value has to be close to the low cutoff frequency of the derivative filters. The higher frequency can be obtained by the cutoff frequency of the PIV \(k\)
The window size is $\Delta x / (1-a)$ where $a$ is the overlapping. $k \Delta x$ is thus of 2.8 (1-a) corresponding to 1.4 with an overlapping of 50%. The bandwidth has thus to extend at least from $k \Delta x = 0.06$ to 1.4.

Looking at figure 4, it can be concluded that in order not to loose any signal in the PIV bandwidth, the compact difference scheme should be used, with a well known low pass filter with a cutoff frequency close to $1.4 / \Delta x$, before the derivative computation.

4 Result and discussion

The best derivative scheme necessary to compute the vorticity with a good accuracy is difficult to deduce. The approach detailed in this contribution is to determine the transfer function of each filter and to know the spectrum of the velocity field to optimize the cutoff frequency of the derivative filter to the PIV one. This study is under way. A first result can be shown in figure 5. It concerns a selected region of a velocity map in a plane parallel to a wall in a turbulent boundary layer located at 100 wall units. The interrogation windows are optimized following the method given in Foucaut et al (2001) at a size of 44 x 44 pixels. In this case the best filter is the second order centered difference scheme with presents the same cutoff wave number as turbulent flow. Figure 5 shows the velocity fluctuation vector field and presented in contour the vorticity field. The optimized filter used is the best as far as the frequency response is concerned. It seem logical if the frequency domain is optimized that the level of vorticity is computed with a good accuracy.

If no spectrum is computable it is difficult to choose a filter. For example, different schemes were tested on a real field contained only a vortex (figure 6 a). This map corresponds to the case A001 of the first PIV Challenge. The velocity field is computed by means of a classical single pass algorithm with a 3 points Gaussian peak fitting. A fit with an Oseen vortex using the least square method allows to obtain the value of the maximum of vorticity. This value, which is 0.168 with a coefficient $R^2 = 0.986$, gives an idea of the expected result. Figures 6 b, c, d, e and f present the vorticity fields computed respectively from the 2nd order Richardson extrapolation, the least square, the 2nd order centered difference, the 6th order compact difference schemes and the same last scheme applied on filtered data. The filter was a cut-off at $k \Delta x = 1.4$. Without any spectral information, the scheme can not be chosen a priori.

In figure 6 b the vorticity peak is smeared out. The maximum value of the vorticity goes from 0.069 for the Richardson extrapolation scheme which presents strong filtering effect, to 0.29 for 6th order compact difference
scheme which has a large bandwidth. The closest value to the Oseen fit is given this time by the least square filter and is of 0.19. As can be seen in figure 6 c the vorticity given by this filter seems littly affected by the noise. Looking at figure 4 allows to conclude that, in the considered velocity field, the signal is probably contained in a range of wave numbers smaller than $k \Delta x = 0.7$. The signal above this wave number contains probably only the noise, which is strongly amplified when the centered or compact filters are used.

As a conclusion, the transfer functions of different derivative filters have been determined, based on differential estimation. Using these transfer function applied to a PIV spectrum allow easily to select the best derivative scheme. Without spectrum, a test on a real case showed that the choice of derivative schemes is not evident and depends on the noise level and on the PIV cutoff frequency which is linked to this level. It seems that in the present state of PIV, a second order filter is sufficient, the choice between Richardson extrapolation, least square or centered differences depends only on the cutoff frequencies. The compact difference, which presents a large bandwidth, can be interesting if the PIV noise is strongly reduced.

In the case of turbulence, a systematic study of the PIV spectra should allow to deduce the filter cutoff frequency of an analysis taking the noise into account (like in Foucaut et al (2001)). The knowledge of this frequency will allow to choose the best suited derivative filter. In that case the frequency response is optimized but an uncertainty concerning the effective vorticity level is still to study. The best way is probably to make the complete analysis of vorticity computation on velocity fields obtain by synthetic images computed from DNS velocity field (see Lecordier et al). By this way the vorticity will be well known and allow to validate completely the derivative scheme choice.

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References


Figure 6: a) vortex velocity field with 32 x 32 interrogation windows, b), c), d) and e) vorticity field computed respectively with 2nd and 4th order Richardson extrapolation, 2nd order centered diff., 6th order compact diff. schemes, f) 6th order compact diff. from filtered data.