Chapter 2

Comparability of Projections

§ 11. Orthogonal Additivity of Equivalence

Let A be a Baer $*$-ring, let $(e_\iota)_{\iota\in I}$ and $(f_\iota)_{\iota\in I}$ be orthogonal families of projections indexed by the same set I, let $e=\sup e_\iota$, $f=\sup f_\iota$, and suppose that $e_\iota \sim f_\iota$ for all $\iota\in I$. Does it follow that $e\sim f$?

I don't know (see Exercise 3). If the index set I is finite, the question is answered affirmatively by trivial algebra [§ 1, Prop. 8]. The present section settles the question affirmatively under the added restriction that $ef=0$; this restriction is removed in Section 20, but only under an extra hypothesis on A. Some terminology helps to simplify the statements of these results:

Definition 1. Let A be a Baer $*$-ring (or, more generally, a $*$-ring in which the suprema in question are assumed to exist). If the answer to the question in the first paragraph is always affirmative, we say that equivalence in A is *additive* (or 'completely additive'); if it is affirmative whenever $\operatorname{card} I \le \aleph$, we say that equivalence in A is $\aleph$-*additive;* if it is affirmative whenever $ef=0$, we say that equivalence in A is *orthogonally additive* (see Theorem 1). The term *orthogonally* $\aleph$-*additive* is self-explanatory.

Suppose, more precisely, that the equivalences $e_\iota \sim f_\iota$ in question are implemented by partial isometries $w_\iota\,(\iota\in I)$. We say that partial isometries in A are *addable* if $e\sim f$ via a partial isometry w such that $we_\iota = w_\iota = f_\iota w$ for all $\iota\in I$. The terms $\aleph$-*addable*, *orthogonally addable*, and *orthogonally* $\aleph$-*addable* are self-explanatory. The main result of the section:

Theorem 1. *In any Baer $*$-ring, partial isometries are orthogonally addable; in particular, equivalence is orthogonally additive.*

Four lemmas prepare the way for the proof of Theorem 1.

Lemma 1. *In a weakly Rickart $*$-ring, suppose $(h_\iota)_{\iota\in I}$ is an orthogonal family of projections, and $(e_\iota)_{\iota\in I}$ is a (necessarily orthogonal) family*

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of projections such that $e_\iota \le h_\iota$ *for all* ι *and such that* $e = \sup e_\iota$ *exists. Then* $e_\iota = h_\iota e$ *for all* ι.

Proof. Fix ι and set $x = h_\iota e - e_\iota$; obviously $xe = x$. If $\varkappa \ne \iota$ then $xe_\varkappa = h_\iota e_\varkappa - e_\iota e_\varkappa = 0$ by the assumed orthogonality; also $xe_\iota = h_\iota e_\iota - e_\iota = 0$; therefore $xe = 0$ [§ 5, Exer. 4], that is, $x = 0$. ∎

Lemma 2. *If* A *is a Rickart* $*$*-ring containing a projection* e *such that* $e \sim 1 - e$, *then* $2 = 1 + 1$ *is invertible in* A.

Proof. Let w be a partial isometry such that $w^*w = e$, $ww^* = 1 - e$, and write $R(\{e - w^*\}) = fA$, f a projection; we show that $x = fe + wf - w$ satisfies $2x = 1$.

From $(e - w^*)f = 0$, we have $fw = fe$. Since $w \in (1 - e)Ae$, it follows that $(e - w^*)(e + w) = 0$, therefore $f(e + w) = e + w$; noting that $fw = fe$, this yields

$$(*) \qquad e = 2fe - w.$$

Right-multiplying $(*)$ by w^*, we have $w^* = 2fw^* - (1 - e)$; taking adjoints, we obtain

$$(**) \qquad 1 - e = 2wf - w.$$

Addition of $(*)$ and $(**)$ yields $1 = 2x$. ∎

Lemma 3. *Let* A *be a weakly Rickart* $*$*-ring in which every orthogonal family of projections of cardinality* $\le \aleph$ *has a supremum.*

Let $(h_\iota)_{\iota \in I}$ *be an orthogonal family of central projections, with* $\operatorname{card} I \le \aleph$, *and suppose that, for each* ι, e_ι *and* f_ι *are orthogonal projections with* $e_\iota + f_\iota = h_\iota$, $e_\iota \sim f_\iota$. *Let* $e = \sup e_\iota$, $f = \sup f_\iota$.

Then $e \sim f$. *More precisely, if the equivalences* $e_\iota \sim f_\iota$ *are implemented by partial isometries* w_ι, *then there exists a partial isometry* w *implementing* $e \sim f$, *such that* $we_\iota = w_\iota = f_\iota w$ *for all* ι.

Proof. Since $h_\iota A$ is a Rickart $*$-ring [§ 5, Prop. 6] and $w_\iota \in h_\iota A$, we have $e_\iota \sim f_\iota$ in $h_\iota A$. By Lemma 2, $2h_\iota$ is invertible in $h_\iota A$; say $a_\iota \in h_\iota A$ with $h_\iota = (2h_\iota)a_\iota = 2a_\iota$. Since $2h_\iota$ is self-adjoint and central in $h_\iota A$, so is a_ι.

Write $u_\iota = w_\iota + w_\iota^*$; clearly u_ι is a symmetry (= self-adjoint unitary) in $h_\iota A$, that is, $u_\iota^* = u_\iota$, $u_\iota^2 = h_\iota$. Defining

$$g_\iota = a_\iota(h_\iota + u_\iota)$$

(informally, $g_\iota = (1/2)(h_\iota + w_\iota + w_\iota^*)$), it is easy to check that g_ι is a projection in $h_\iota A$.

Define $g = \sup g_\iota$. Citing Lemma 1, we have $h_\iota g = g_\iota$, $h_\iota e = e_\iota$, $h_\iota f = f_\iota$. Finally, define

$$w = 2fge;$$

the proof will be concluded by showing that w is a partial isometry having the desired properties.

Note that $f_\iota u_\iota = w_\iota = u_\iota e_\iota$; it follows easily that

$$f_\iota g_\iota e_\iota = a_\iota w_\iota,$$

therefore $2 f_\iota g_\iota e_\iota = 2 a_\iota w_\iota = h_\iota w_\iota = w_\iota$. Then

$$h_\iota w = w_\iota; \tag{1}$$

for, $h_\iota w = h_\iota(2 fge) = 2(h_\iota f)(h_\iota g)(h_\iota e) = 2 f_\iota g_\iota e_\iota = w_\iota$. It follows that

$$w e_\iota = w_\iota = f_\iota w; \tag{2}$$

for example, $w e_\iota = w(h_\iota e_\iota) = (h_\iota w) e_\iota = w_\iota e_\iota = w_\iota$.

It remains to show that $w^* w = e$ and $w w^* = f$. Let $h = \sup h_\iota$. Since $e_\iota \le h_\iota \le h$, we have $e \le h$, and similarly $f \le h$, $g \le h$. It follows that $wh = w$, $w^* h = w^*$. For all ι, we have

$$(w^* w - e) h_\iota = (h_\iota w)^* (h_\iota w) - h_\iota e = w_\iota^* w_\iota - e_\iota = 0,$$

therefore $(w^* w - e) h = 0$ [§ 5, Exer. 4], thus $w^* w - e = 0$. Similarly $w w^* - f = 0$. ∎

Lemma 4. *If A is a weakly Rickart $*$-ring in which every orthogonal family of projections of cardinality $\le \aleph$ has a supremum, then partial isometries in A are orthogonally $\aleph$-addable.*

Proof. Let $(w_\iota)_{\iota \in I}$ be a family of partial isometries, $\operatorname{card} I \le \aleph$, with orthogonal initial projections e_ι, orthogonal final projections f_ι, and such that, setting $e = \sup e_\iota$, $f = \sup f_\iota$, we have $ef = 0$. We seek a partial isometry w such that $w^* w = e$, $w w^* = f$ and $w e_\iota = w_\iota = f_\iota w$ for all ι.

Write $h_\iota = e_\iota + f_\iota$, $S = \{h_\iota : \iota \in I\}$, and let $B = S'$. According to [§ 5, Prop. 5], B is also a weakly Rickart $*$-ring, with unambiguous RP's. Since the h_ι are orthogonal, S is a commutative set; it follows that $B \supset B'$, and that B has center $B \cap B' = B' = S''$. In particular, the h_ι are central projections in B.

Each w_ι belongs to B; for, $h_\iota w_\iota = w_\iota h_\iota$ $(= w_\iota)$ and $h_\varkappa w_\iota = w_\iota h_\varkappa$ $(=0)$ whenever $\varkappa \ne \iota$, thus $w_\iota \in S' = B$. In particular, $e_\iota \sim f_\iota$ in B. Finally, it is clear that $e, f \in B$ (cf. the proof of [§ 3, Prop. 11]), thus the desired w exists by Lemma 3. ∎

Proof of Theorem 1. When A is a Baer $*$-ring, the hypothesis of Lemma 4 is verified for every $\aleph$. ∎

We now take up several other useful consequences of Lemma 4.

Proposition 1. *Let A be a weakly Rickart $*$-ring in which every orthogonal family of projections of cardinality $\le \aleph$ has a supremum.*

Let $(e_\iota)_{\iota \in I}$ *be an infinite family of mutually equivalent, orthogonal projections, with* $\operatorname{card} I \leq \aleph$, *and let* $J \subset I$ *with* $\operatorname{card} J = \operatorname{card} I$. *Define*

$$e = \sup\{e_\iota : \iota \in I\}, \qquad f = \sup\{e_\iota : \iota \in J\}.$$

Then $e \sim f$.

Proof. Dropping down to eAe, we can suppose that A is a Rickart $*$-ring [§ 5, Prop. 6]. Write $J = J' \cup J''$, where $J' \cap J'' = \emptyset$ and $\operatorname{card} J' = \operatorname{card} J'' = \operatorname{card} J$ $(= \operatorname{card} I)$. Define

$$\begin{aligned} f' &= \sup\{e_\iota : \iota \in J'\}, \\ f'' &= \sup\{e_\iota : \iota \in J''\}, \\ g &= \sup\{e_\iota : \iota \in J'' \cup (I - J)\}. \end{aligned}$$

Since the e_ι are orthogonal, clearly $f'g = 0$, thus $\sup\{f', g\} = f' + g$; but $J' \cup [J'' \cup (I - J)] = I$, therefore

(1) $$e = f' + g$$

by the associativity of suprema. Similarly,

(2) $$f = f' + f''.$$

Since J' and $J'' \cup (I - J)$ have the same cardinality, and since $f'g = 0$, we have

(3) $$f' \sim g$$

by Lemma 4. Similarly,

(4) $$f'' \sim f'.$$

Adding (3) and (4), we have $f' + f'' \sim g + f'$, that is, citing (1) and (2), $f \sim e$. ∎

Proposition 2. *Let A be a Baer $*$-ring, e and f projections in A with* $ef = 0$, (h_α) *an orthogonal family of central projections, and* $h = \sup h_\alpha$. *If* $h_\alpha e \precsim h_\alpha f$ *for all* α, *then* $he \precsim hf$. *More precisely, if, for each* α, w_α *is a partial isometry such that* $w_\alpha^* w_\alpha = h_\alpha e$ *and* $w_\alpha w_\alpha^* = f_\alpha \leq h_\alpha f$, *then there exists a partial isometry* w *such that* $w^* w = he$ *and* $w(h_\alpha e) = w_\alpha = f_\alpha w$ *for all* α.

Proof. Since $(\sup h_\alpha e)(\sup f_\alpha) = he \sup f_\alpha \leq hef = 0$, and since the f_α are also mutually orthogonal (because the h_α are), Lemma 4 is applicable to the partial isometries w_α. ∎

In a weakly Rickart C^*-algebra, countable suprema are available [§ 8, Lemma 3], thus Lemma 4 holds with $\aleph = \aleph_0$, implying the obvious sequential forms of Propositions 1 and 2.

Exercises

1A. Let e, f be orthogonal projections in a Rickart $*$-ring. In order that $e \sim f$, it is necessary and sufficient that there exist a projection g such that $2ege = e$, $2fgf = f$, $2geg = 2gfg = g$.

2A. Let C be a Baer $*$-ring possessing a projection e such that $e \sim 1 - e$ (for example, the $*$-ring of all 2×2 complex matrices). Let B be the complete direct product of $\aleph_0$ copies of C, that is, $B = \prod_1^\infty A_n$ with $A_n = C$ for all n [cf. § 1, Exer. 13]. Let B_0 be the weak direct product of the A_n, that is, the ideal of all $x = (a_n)$ in B such that $a_n = 0$ for all but finitely many n. Write $f = 1 - e$, $\bar{e} = (e, e, e, \ldots)$, $\bar{f} = (f, f, f, \ldots) = 1 - \bar{e}$, and let S be the $*$-subring of B generated by $\bar{e}$ and 1; thus, $S = \{m\bar{e} + n\bar{f} : m, n \text{ integers}\}$. Define $A = B_0 + S$; thus, A is the $*$-subring of B generated by B_0, $\bar{e}$ and 1. If the additive group of C is torsion-free, then A is a Rickart $*$-ring.

For $m = 1, 2, 3, \ldots$ write $e_m = (\delta_{mn} e)$, $f_m = (\delta_{mn} f)$. Then, relative to the $*$-ring A, we have $e_m \sim f_m$ for all m, $\sup e_m = \bar{e}$, $\sup f_m = \bar{f}$, $\bar{e}\bar{f} = 0$, but $\bar{e}$ is not equivalent to $\bar{f}$. Thus Theorem 1 does not generalize to Rickart $*$-rings.

3D. Let C be a Baer $*$-ring, let C° be the reduced ring of C [§ 3, Exer. 18], and suppose there exists an equivalence $e \sim f$ in C which cannot be implemented by any partial isometry in C° (that is, $e \sim f$ but not $e \overset{\circ}{\sim} f$).

Let A be the P^*-sum of $\aleph_0$ copies of C [cf. § 4, Exer. 8]. For $m = 1, 2, 3, \ldots$ let e_m and f_m be the projections in A defined by the sequences $e_m = (\delta_{mn} e)$, $f_m = (\delta_{mn} f)$. Then $e_m \sim f_m$ for all m, but $\sup e_m$ is not equivalent to $\sup f_m$.

Problem: Does there exist such a Baer $*$-ring C? (Cf. [§ 17, Exer. 20].)

4D. *Problem:* Is equivalence $\aleph_0$-additive (i.e., 'countably additive') in a Rickart C^*-algebra?

5A. If A is a Baer $*$-ring such that the ring A_2 of all 2×2 matrices over A is also a Baer $*$-ring (with $*$-transpose as involution), then partial isometries in A are addable.

6A. In the notation of Definition 1, if there exists an element x such that $x e_\iota = w_\iota = f_\iota x$ for all ι, then the partial isometries w_ι are addable (the desired partial isometry is $w = xe$).

§ 12. A General Schröder-Bernstein Theorem

We say that *the Schröder-Bernstein theorem holds* in a $*$-ring if the relations $e \lesssim f$ and $f \lesssim e$ imply $e \sim f$. In Section 1, it was shown that the Schröder-Bernstein theorem holds in any $*$-ring whose set of projections is conditionally complete [§ 1, Th. 1]—in particular, it holds in any Baer $*$-ring [cf. § 4, Prop. 1]. The fixed-point theorem employed there requires lattice completeness; by reverting to the format of the classical set-theoretic proof, one can get along with countable lattice operations:

Proposition 1. *If A is a weakly Rickart $*$-ring in which every countable family of orthogonal projections has a supremum, then the Schröder-Bernstein theorem holds in A.*

It is convenient to separate out an elementary lemma:

Lemma. *If e_n is a decreasing sequence of projections in such a $*$-ring, that is, if $e_1 \geq e_2 \geq e_3 \geq \cdots$, then $\inf e_n$ exists. Explicitly,*

$$\inf e_n = e_1 - g,$$

where $g = \sup\{e_n - e_{n+1} : n = 1, 2, 3, \ldots\}$.

Proof of Proposition 1. Assuming $e \sim f' \leq f$ and $f \sim e' \leq e$, it is to be shown that $e \sim f$.

Let w be a partial isometry such that $w^* w = f$, $w w^* = e'$. Setting $v = wf'$, we have $v^* v = f'$; thus v is a partial isometry, and, writing $e'' = v v^*$, we have

$$f' \sim e'' \leq e'.$$

Combining this with $e \sim f'$, we have the following situation:

(1) $$e'' \leq e' \leq e \quad \text{and } e'' \sim e.$$

On the basis of (1), it will be shown that $e' \sim e$ (the observation $f \sim e' \sim e$ then ends the proof); no further reference to f is necessary.

By (1), there exists a partial isometry u such that

$$u^* u = e'', \qquad u u^* = e.$$

Since $g \mapsto \varphi(g) = u^* g u$ is an order-preserving bijection of $[0, e]$ onto $[0, e'']$ (see [§ 1, Prop. 9]), we may define a sequence $e_0, e_2, e_4, \ldots, e_{2n}, \ldots$ of subprojections of e as follows:

$$\begin{aligned}
&e_0 = e \\
&e_2 = u^* e_0 u = u^* e u = u^* u = e'' \leq e \\
&e_4 = u^* e_2 u \\
&\ldots \\
&e_{2n} = u^* e_{2(n-1)} u \qquad (n = 1, 2, 3, \ldots).
\end{aligned}$$

Define another sequence $e_1, e_3, e_5, \ldots, e_{2n-1}, \ldots$ of subprojections of e by the same technique, starting with e':

$$\begin{aligned}
&e_1 = e' \leq e \\
&e_3 = u^* e_1 u = u^* e' u \\
&e_5 = u^* e_3 u \\
&\ldots \\
&e_{2n+1} = u^* e_{2n-1} u \qquad (n = 1, 2, 3, \ldots).
\end{aligned}$$

Observe that

(2) $$e_0 \geq e_1 \geq e_2 \geq e_3 \geq \cdots.$$

(Indeed, (1) may be written $e_0 \geq e_1 \geq e_2$; application of φ yields $e_2 \geq e_3 \geq e_4$; etc.)

We now look at the 'gaps' in the decreasing sequence (2). Since, by definition, $u^* e_n u = e_{n+2}$ $(n=0,1,2,3,\ldots)$, we have $u^*(e_n - e_{n+1})u = e_{n+2} - e_{n+3}$, thus

(3) $$e_n - e_{n+1} \sim e_{n+2} - e_{n+3} \qquad (n=0,1,2,3,\ldots)$$

(the equivalence (3) is implemented by the partial isometry $u^*(e_n - e_{n+1})$).

By the lemma, we may define

(4) $$e_\infty = \inf\{e_n : n=0,1,2,3,\ldots\}.$$

Obviously any truncation of the sequence e_n has the same infimum, in particular,

(5) $$e_\infty = \inf\{e_n : n=1,2,3,\ldots\}.$$

Consider the following two sequences of orthogonal projections:

(*) $$e_\infty,\ e_0 - e_1,\ e_1 - e_2,\ e_2 - e_3, \ldots$$

(**) $$e_\infty,\ e_1 - e_2,\ e_2 - e_3,\ e_3 - e_4, \ldots$$

(the second sequence merely omits the second term of the first sequence). In view of (4), it follows from the lemma that

$$e_0 = e_\infty + \sup\{e_0 - e_1,\ e_1 - e_2,\ e_2 - e_3, \ldots\};$$

thus, by the associativy of suprema, the sequence (*) has supremum $e_\infty + (e_0 - e_\infty) = e_0 = e$. It follows similarly from (5) that the sequence (**) has supremum $e_\infty + (e_1 - e_\infty) = e_1 = e'$.

The desired equivalence $e \sim e'$ is obtained by putting together the pieces in (*) and (**) in another way. We define

$$g = \sup\{e_0 - e_1, e_2 - e_3, e_4 - e_5, \ldots\},$$
$$g' = \sup\{e_2 - e_3, e_4 - e_5, e_6 - e_7, \ldots\},$$
$$h = e_\infty + \sup\{e_1 - e_2, e_3 - e_4, e_5 - e_6, \ldots\}.$$

By the associativity of suprema, $g + h$ coincides with the supremum of the sequence (*), thus

(6) $$e = g + h;$$

similarly, $g' + h$ is the supremum of the sequence (**), thus

(7) $$e' = g' + h.$$

It follows from (3), and the definitions of g and g', that $g \sim g'$ [§11, Prop. 1]; in view of (6) and (7), this implies $e \sim e'$. ∎

The principal applications:

Corollary. *The Schröder-Bernstein theorem holds* (i) *in any Baer $*$-ring, and* (ii) *in any weakly Rickart C^*-algebra.*

Proof. Of course (i) is also covered by [§1, Th. 1]; (ii) follows from the fact that every sequence of projections in a weakly Rickart C^*-algebra has a supremum [§8, Lemma 3]. ∎

Exercises

1A. Let A be a weakly Rickart $*$-ring in which every countable family of orthogonal projections has a supremum. If e is any projection, write $[e]$ for the equivalence class of e with respect to $\sim$, that is, $[e]=\{f: f\sim e\}$. Define $[e]\le[f]$ iff $e\precsim f$. This is a partial ordering of the set of equivalence classes.

2A. The Schröder-Bernstein theorem holds trivially in any finite $*$-ring. (A $*$-ring with unity is said to be *finite* [§15, Def. 3] if $e\sim 1$ implies $e=1$.)

3A. Let $\mathscr{H}$ be a separable, infinite-dimensional Hilbert space, with orthonormal basis $e_1, e_2, e_3, \ldots$. Let T be the operator such that $Te_n = e_{n+2}$ for all n; thus $T^*T=I$, $TT^*=E$, where E is the projection with range $[e_3, e_4, e_5, \ldots]$. Let F be the projection with range $[e_2, e_3, e_4, \ldots]$. Finally, let $\mathscr{A}$ be the $*$-ring generated by T and F.

The relations $T^*T=I$, $TT^*=E\le F$ and $F\le I$ show that $I\precsim F$ and $F\precsim I$ relative to the $*$-ring $\mathscr{A}$. Is $F\sim I$ relative to $\mathscr{A}$?

§ 13. The Parallelogram Law (P) and Related Matters

The law in question is reminiscent of the 'second isomorphism theorem' of abstract algebra:

Definition 1. A $*$-ring whose projections form a lattice is said to satisfy the *parallelogram law* if

$$\text{(P)} \qquad e - e\cap f \sim e\cup f - f$$

for every pair of projections e, f.

The projections of every weakly Rickart $*$-ring form a lattice [§5, Prop. 7], but even a Baer $*$-ring may fail to satisfy the parallelogram law (Exercise 1). Occasionally, the following variant of (P) is more convenient:

Proposition 1. *Let A be a $*$-ring with unity, whose projections form a lattice. The following conditions are equivalent:*

(a) *A satisfies the parallelogram law* (P);

(b) $e - e\cap(1-f) \sim f - (1-e)\cap f$ *for every pair of projections e, f.*

Proof. Replacement of f by $1-f$ in the relation (P) yields

$$e-e\cap(1-f)\sim[e\cup(1-f)]-(1-f)=f-[1-e\cup(1-f)]$$
$$=f-(1-e)\cap f. \quad ∎$$

Proposition 1 may be interpreted as saying that, in the presence of (P), certain subprojections of e,f (indicated in (b)) are guaranteed to be equivalent; this conclusion reduces to the triviality $0\sim 0$ precisely when $e=e\cap(1-f)$, that is, when $ef=0$.

The projections that occur in (P) are familiar from [§5, Prop. 7]:

Proposition 2. *If A is a weakly Rickart $*$-ring such that* $\mathrm{LP}(x)\sim\mathrm{RP}(x)$ *for all* $x\in A$, *then A satisfies the parallelogram law* (P).

Proof. Apply the hypothesis to the element $x=e-ef$ [§5, Prop. 7]. ∎

An important application:

Corollary. *Every von Neumann algebra satisfies the parallelogram law* (P).

Proof. Let $\mathscr{A}$ be a von Neumann algebra of operators on a Hilbert space $\mathscr{H}$ [§4, Def. 5]. If T is any operator on $\mathscr{H}$, the 'canonical factorization' $T=WR$ is uniquely characterized by the following three properties: (i) $R\geq 0$, (ii) W is a partial isometry, and (iii) W^*W is the projection on the closure of the range of R, that is, $W^*W=\mathrm{LP}(R)$ as calculated in $\mathscr{L}(\mathscr{H})$. It follows that $W^*W=\mathrm{RP}(T)$, $WW^*=\mathrm{LP}(T)$, thus $\mathrm{LP}(T)\sim\mathrm{RP}(T)$ in $\mathscr{L}(\mathscr{H})$. The proof is concluded by observing that if $T\in\mathscr{A}$ then $W\in\mathscr{A}$ (therefore $\mathrm{LP}(T)\sim\mathrm{RP}(T)$ in $\mathscr{A}$).

Suppose $T\in\mathscr{A}$. If $U\in\mathscr{A}'$ is unitary, then $T=UTU^*=(UWU^*)(URU^*)$; since the properties (i), (ii), (iii) are satisfied by the positive operator URU^* and the partial isometry UWU^*, it follows from uniqueness that $UWU^*=W$, thus W commutes with U. Since $\mathscr{A}'$ is the linear span of its unitaries (as is any C^*-algebra with unity [cf. **23**, Ch. I, §1, No. 3, Prop. 3]), it follows that $W\in(\mathscr{A}')'=\mathscr{A}$. ∎

Later in the section it will be shown, more generally, that every AW^*-algebra—indeed, any weakly Rickart C^*-algebra—satisfies the parallelogram law (P). The proof will avoid the use of $\mathrm{LP}\sim\mathrm{RP}$ (known to hold in any AW^*-algebra [§20, Cor. of Th. 3], but of unknown status in Rickart C^*-algebras). The general strategy is to reduce the consideration of arbitrary pairs of projections e,f to pairs of projections in 'special position'; the following concept is central to such considerations:

Definition 2. Let A be a $*$-ring with unity, whose projections form a lattice (for example, A any Rickart $*$-ring). Projections e, f in A are said to be in *position* p' in case

$$e \cap (1-f) = (1-e) \cap f = 0.$$

{Equivalenty, $e \cap (1-f) = 0$ and $e \cup (1-f) = 1$; that is, the projections e, $1-f$ are complementary.} The condition is obviously symmetric in e and f.

In Rickart $*$-rings, the concept has a useful reformulation:

Proposition 3. *In a Rickart $*$-ring, the following conditions on a pair of projections e, f imply one another:*

(a) *e, f are in position p';*

(b) $\mathrm{LP}(ef) = e$ *and* $\mathrm{RP}(ef) = f$.

Proof. Let $x = ef = e[1-(1-f)]$. Citing [§3, Prop. 7], we have

$$\mathrm{LP}(x) = e - e \cap (1-f), \qquad \mathrm{RP}(x) = e \cup (1-f) - (1-f);$$

thus, the conditions (b) are equivalent to $e \cap (1-f) = 0$ and $e \cup (1-f) = 1$. ∎

In a Rickart $*$-ring, the parallelogram law can be reformulated in terms of position p':

Proposition 4. *The following conditions on a Rickart $*$-ring A are equivalent:*

(a) *A satisfies the parallelogram law* (P);

(b) *if e, f are projections in position p', then $e \sim f$.*

Proof. (a) implies (b): If $e \cap (1-f) = (1-e) \cap f = 0$, then, in the presence of (P), $e \sim f$ by Proposition 1.

(b) implies (a): Let e, f be any pair of projections, and set $e' = \mathrm{LP}(ef)$, $f' = \mathrm{RP}(ef)$. Since $ef = e'(ef)f' = e'f'$, it follows from Proposition 3 that e', f' are in position p'; therefore, by hypothesis, $e' \sim f'$, that is,

$$e - e \cap (1-f) \sim e \cup (1-f) - (1-f) = f - (1-e) \cap f.$$

Since e, f are arbitrary, it follows from Proposition 1 that A satisfies (P). ∎

The proof of Proposition 4 yields a highly useful decomposition theorem:

Proposition 5. *Let A be a Rickart $*$-ring satisfying the parallelogram law* (P). *If e, f is any pair of projections in A, there exist orthogonal decompositions*

$$e = e' + e'', \qquad f = f' + f''$$

with e', f' in position p' (hence $e' \sim f'$ by Proposition 4) and $e''f = ef'' = 0$.

Proof. Let $e' = \mathrm{LP}(ef)$, $f' = \mathrm{RP}(ef)$; as noted in the proof of Proposition 4, e', f' are in position p'. Set $e'' = e - e'$, $f'' = f - f'$; obviously $e''(ef) = (ef)f'' = 0$, thus $e''f = ef'' = 0$. ∎

The rest of the section is concerned with developing sufficient conditions for (P) to hold. With an eye on Proposition 4, we seek conditions ensuring that projections in position p' are equivalent. For the most part, victory hinges on being able to analyze position p' considerations in terms of the following more stringent relation:

Definition 3. Let A be a $*$-ring with unity, whose projections form a lattice. Projections e, f in A are said to be in *position p* in case

$$e \cap f = e \cap (1-f) = (1-e) \cap f = (1-e) \cap (1-f) = 0 .$$

{Equivalently, each of the pairs e, f and e, $1-f$ is in position p'.} The condition is obviously symmetric in e and f.

If e, f are in position p, then so is any pair g, h, where $g = e$ or $1-e$, and $h = f$ or $1-f$.

In Proposition 3, position p' is characterized in terms of the element ef; the characterization of position p involves both ef and its adjoint:

Proposition 6. *In a Rickart $*$-ring, the following conditions on a pair of projections e, f imply one another:*

(a) *e, f are in position p*;

(b) $\mathrm{RP}(ef - fe) = 1$.

Proof. (b) implies (a): Set $x = ef - fe$. Since $\mathrm{RP}(x) = 1$, the relations $e \cap f = 0$ and $e \cup f = 1$ are implied by the obvious computations

$$x(e \cap f) = 0, \qquad x(e \cup f) = x.$$

But $e(1-f) - (1-f)e = -x$ also has right projection 1, therefore $e \cap (1-f) = 0$ and $e \cup (1-f) = 1$. Thus $e \cap f = (1-e) \cap (1-f) = e \cap (1-f) = (1-e) \cap f = 0$.

(a) implies (b): Let $x = ef - fe$, $g = \mathrm{RP}(x)$; assuming e, f are in position p, it is to be shown that $g = 1$. {For an insight on the success of the following strategem, compute $(ab - ba)^2$ for a pair of 2×2 matrices a, b over a commutative ring.} Set $z = x^* x = -x^2$; by direct computation,

$$\begin{aligned} z = -(ef - fe)^2 &= -efef + efe + fef - fefe \\ &= ef(1-e)fe + (1-e)fef(1-e) \\ &= fe(1-f)ef + (1-f)efe(1-f) . \end{aligned}$$

From the last two formulas, it is clear that e and f commute with z. On the other hand,

$$g = \mathrm{RP}(x) = \mathrm{RP}(x^* x) = \mathrm{RP}(z) \in \{z\}''$$

[§3, Cor. 2 of Prop. 10], therefore g commutes with e and with f. Set $h = 1 - g$. Since $g = \mathrm{RP}(x)$ and since h commutes with e and f, we have

$$0 = xh = (eh)(fh) - (fh)(eh),$$

thus eh, fh are commuting projections; citing [§1, Prop. 3], we have

$$(eh)(fh) = (eh) \cap (fh) \leq e \cap f = 0,$$

thus

$$(1) \qquad (ef)h = 0.$$

Since $e(1-f) - (1-f)e = -x$ also has right projection g, and since $e \cap (1-f) = 0$ by hypothesis, the same reasoning yields

$$(2) \qquad [e(1-f)]h = 0.$$

Adding (1) and (2), we have $eh = 0$. Similarly $fh = 0$. Thus $e \leq 1 - h = g$ and $f \leq g$; since $e \cup f = 1$, we conclude that $g = 1$. ∎

An obvious way to fulfill condition (b) of Proposition 6 is to assume outright that $ef - fe$ is invertible; in the next proposition, it is shown that the invertibility of $ef - fe$ implies $e \sim f$, provided one also assumes a condition on the existence of 'square roots'. Historically, the first condition of this type, considered by I. Kaplansky ([**52**], [**54**]), was the following:

Definition 4. A $*$-ring is said to satisfy the *square-root axiom* (briefly, the (SR)-*axiom*) in case, for each element x, there exists $r \in \{x^* x\}''$ such that $r^* = r$ and $x^* x = r^2$.

Occasionally, the following weaker axiom suffices (later in the section, stronger axioms will be employed):

Definition 5. A $*$-ring is said to satisfy the *weak square-root axiom* (briefly, the (WSR)-*axiom*) in case, for each element x, there exists $r \in \{x^* x\}''$ (necessarily normal, but not necessarily self-adjoint) such that $x^* x = r^* r$ $(= r r^*)$.

A sample of the wholesome effect of square roots:

Lemma. *If A is a $*$-ring satisfying the* (WSR)-*axiom, and if the projections e, f are algebraically equivalent in the sense that $yx = e$ and $xy = f$ for suitable elements $x, y \in A$, then $e \sim f$.*

Proof. Replacing x and y by fxe and eyf, we can suppose $x \in fAe$, $y \in eAf$. We seek a partial isometry w such that $w^*w = e$, $ww^* = f$.

Choose $r \in \{y^*y\}''$ with $y^*y = r^*r = rr^*$, and set $w = rx$. Then

$$w^*w = x^*r^*rx = x^*y^*yx = (yx)^*(yx) = e^*e = e.$$

On the other hand, $ww^* = rxx^*r^*$; to proceed further, we show that r commutes with xx^*. Since $r \in \{y^*y\}''$, it suffices to note that $xx^* \in \{y^*y\}'$; indeed, xx^* and y^*y are self-adjoint elements whose product

$$(xx^*)(y^*y) = x(yx)^*y = xey = xy = f$$

is also self-adjoint. Thus $r \in \{x^*x\}'$, and

$$ww^* = rxx^*r^* = xx^*rr^* = (xx^*)(y^*y) = f. \quad \blacksquare$$

Armed with square roots, a considerable dent can be made on the parallelogram law problem:

Proposition 7. *If A is a $*$-ring with unity satisfying the* (WSR)*-axiom, and if e, f are projections in A such that $ef - fe$ is invertible, then $e \sim f \sim 1-e \sim 1-f$.*

Proof. Since the invertibility hypothesis for the pair e, f clearly holds also for the pairs e, $1-f$ and $1-e$, f, it is sufficient to show that $e \sim f$.

Let $z = (ef-fe)^*(ef-fe) = -(ef-fe)^2$ and write $B = \{z\}'$. As noted in the proof of Proposition 6, $e, f \in B$. Since $\{z\} \subset \{z\}'$, we have $B = \{z\}' \supset \{z\}'' = B'$, thus B has center $B \cap B' = B' = \{z\}''$. In particular, z is central in B.

We assert that efe is invertible in eBe. The proof begins by noting that $s = z^{-1}$ is also central in B; then $zs = sz = 1$ implies $(eze)(ese) = (ese)(eze) = e$, thus $eze = ez$ is invertible in eBe. From one of the formulas for z in the proof of Proposition 6, we have

$$ez = ef(1-e)fe = efe - (efe)^2 = efe(e-efe) = (e-efe)efe,$$

thus the invertibility of ez in eBe implies that of efe. Let $t \in eBe$ with $t(efe) = (efe)t = e$, that is,

$$(*) \qquad tfe = eft = e$$

(explicitly, $t = s(e-efe)$).

By the lemma, it will suffice to show that e and f are algebraically equivalent. To this end, define

$$x = ft, \quad y = ef.$$

Obviously $x \in fAe$, $y \in eAf$, and $yx=(ef)(ft)=eft=e$ by (*). On the other hand, $xy=(ft)(ef)=ftf$; citing (*) at the appropriate step, we have

$$\begin{aligned}(ef-fe)xy &= (ef-fe)ftf = eftf - feftf \\ &= (eft)f - f(eft)f = ef - fef = (ef-fe)f,\end{aligned}$$

therefore $xy=f$ by the invertibility of $ef-fe$. ∎

The technique of Proposition 7 suffices to establish the parallelogram law in the C^*-algebra case:

Theorem 1. *Every weakly Rickart C^*-algebra satisfies the parallelogram law* (P).

Proof. If A is a weakly Rickart C^*-algebra, then the projections of A form a lattice [§ 5, Prop. 7]. Let e, f be any pair of projections in A. To verify that e, f satisfy the relation (P), it suffices to work in the Rickart C^*-algebra $(e \cup f)A(e \cup f)$ [§ 5, Prop. 6]; dropping down, we can suppose without loss of generality that A has a unity element.

Set $z=(ef-fe)^*(ef-fe)=-(ef-fe)^2$ and consider the Rickart C^*-algebra $\{z\}'$ [§ 3, Prop. 10]. As noted in the proof of Proposition 7, $e, f \in \{z\}'$ and $\{z\}'$ has center $\{z\}''$. Dropping down to $\{z\}'$, we can suppose that z is in the center Z of A. (This will yield the sharper conclusion that the equivalence $e - e \cap f \sim e \cup f - f$ can be implemented by a partial isometry in $\{(ef-fe)^2\}'$.)

Write $Z=C(T)$, T a compact space with the properties noted in [§ 8, Prop. 1]. By C^*-algebra theory, we have $z \geq 0$ in Z (see the proof of [§ 7, Prop. 3]); setting

$$U = \{t \in T : z(t) > 0\},$$

it follows that $\bar{U}$ is a clopen set whose characteristic function h is $\mathrm{RP}(z)$ [§ 8, Prop. 1].

If U is empty, that is, if $z=0$, then $ef-fe=0$ and the desired relation (P) reduces to the triviality $e-ef=(e+f-ef)-f$ [§ 1, Prop. 3].

Assuming U is nonempty, write $U=\bigcup P_n$, where P_n is a sequence (possibly finite) of disjoint, nonempty clopen sets (cf. the proof of [§ 8, Prop. 3]). Let h_n be the characteristic function of P_n; thus the h_n are orthogonal central projections with $\sup h_n = h$ (cf. [§ 7, Lemma to Prop. 1]).

Since z is bounded below on the compact-open set P_n, it follows that zh_n is invertible in $h_n A$; but

$$zh_n = -(ef-fe)^2 h_n = -[(eh_n)(fh_n)-(fh_n)(eh_n)]^2,$$

therefore

$$eh_n \sim h_n - fh_n \tag{1}$$

by Proposition 7 (note that every C^*-algebra satisfies the (SR)-axiom by easy spectral theory [cf. § 2, Example 5]). By Proposition 6, eh_n and fh_n are in position p in $h_n A$; in particular,

$$(eh_n) \cap (fh_n) = 0, \qquad (eh_n) \cup (fh_n) = h_n,$$

therefore (1) may be rewritten as

$$eh_n - (eh_n) \cap (fh_n) \sim (eh_n) \cup (fh_n) - fh_n.$$

Since h_n is central, the foregoing relation can, by lattice-theoretic trivia, be rewritten as

$$(1') \qquad (e - e \cap f) h_n \sim (e \cup f - f) h_n.$$

Since hA is the C^*-sum of the $h_n A$ [§ 10, Prop. 3], and since every partial isometry has norm ≤ 1, it follows from the relation (1′) that

$$(2) \qquad (e - e \cap f) h \sim (e \cup f - f) h.$$

What happens on $1-h$? Since $h = \mathrm{RP}(z)$, we have

$$0 = z(1-h) = (ef - fe)^* (ef - fe)(1-h),$$

therefore $(ef - fe)(1-h) = 0$, that is, $e(1-h)$ and $f(1-h)$ commute. Write $e' = e(1-h)$, $f' = f(1-h)$; as noted earlier, the relation

$$e' - e' \cap f' \sim e' \cup f' - f'$$

holds trivially, thus

$$(3) \qquad (e - e \cap f)(1-h) \sim (e \cup f - f)(1-h).$$

Adding (2) and (3), we arrive at (P). ∎

To proceed further, it is necessary to sharpen the conclusion of Proposition 7 (the price, of course, is a sharper hypothesis). As it stands, the relations $e \sim f$ and $1 - e \sim 1 - f$ obviously imply that e and f are unitarily equivalent, that is, $ueu^* = f$ for a suitable unitary element u; the sharper conclusion needed is that u can be taken to be a symmetry in the sense of the following definition:

Definition 6. In a $*$-ring with unity, a *symmetry* is a self-adjoint unitary ($u^* = u$, $u^2 = 1$).

In a $*$-ring with unity, the mapping $e \mapsto u = 2e - 1$ transforms projections e into symmetries u; if, in addition, 2 is invertible, then this mapping is *onto* the set of all symmetries, with inverse mapping $u \mapsto (\frac{1}{2})(1+u)$.

Definition 7. If e, f are projections such that $ueu = f$ for some symmetry u (hence also $ufu = e$), we say that e and f are *exchanged* by the symmetry u.

It can be shown that if, in Proposition 7, one assumes the (SR)-axiom, then the projections e, f can be exchanged by a symmetry (see Exercise 5). We content ourselves with a much simpler result (it is complicated enough) based on a stronger axiom. The stronger axiom depends on a general notion of positivity available in any $*$-ring (and therefore generally useless), consistent with the usual notion of positivity in C^*-algebras:

Definition 8. In any $*$-ring, an element x is called *positive*, written $x \geq 0$, in case $x = y_1^* y_1 + \cdots + y_n^* y_n$ for suitable elements $y_1, \ldots, y_n$.

The following properties are elementary: (1) if $x \geq 0$ then $x^* = x$; (2) if $x \geq 0$ then $y^* x y \geq 0$ for all y; (3) if $x \geq 0$ and $y \geq 0$, then $x + y \geq 0$. {Warning: $x \geq 0$ and $-x \geq 0$ is possible for nonzero x; equivalently, the relations $x \geq 0$, $y \geq 0$ and $x + y = 0$ need not imply $x = y = 0$.}

In particular, elements of the form $x^* x$ are positive; thus the following is an obvious strengthening of the (SR)-axiom:

Definition 9. A $*$-ring is said to satisfy the *positive square-root axiom* (briefly, the (PSR)-*axiom*) in case, for every $x \geq 0$, there exists $y \in \{x\}''$ with $y \geq 0$ and $x = y^2$.

The axiom we want is still stronger:

Definition 10. A $*$-ring is said to satisfy the *unique positive square-root axiom* (briefly, the (UPSR)-*axiom*) in case, for every $x \geq 0$, there exists a unique element y such that (1) $y \geq 0$, and (2) $x = y^2$; we assume, in addition, that (3) $y \in \{x\}''$ (but conditions (1) and (2) are already assumed to determine y uniquely).

Every C^*-algebra A satisfies the (UPSR)-axiom. {Proof: If $x \in A$, $x \geq 0$, there exists a unique $y \in A$ such that $y \geq 0$ and $x = y^2$; since $x \geq 0$ as an element of the C^*-algebra $\{x\}''$, it follows from uniqueness that $y \in \{x\}''$.}

The key to the rest of the section is the following result:

Proposition 8. *Let A be a $*$-ring with unity and proper involution, satisfying the* (UPSR)-*axiom. If e, f are projections such that $ef - fe$ is invertible, then e and f can be exchanged by a symmetry.*

Of course the pair $e, 1-f$ also satisfies the hypothesis of Proposition 8, as do the pairs $1-e, f$ and $1-e, 1-f$; the statement of the conclusion is confined to the pair e, f for simplicity. {Proposition 8 holds more generally with (UPSR) weakened to (SR), but with a considerably more complicated proof (Exercise 5).} To break up the rather long proof of Proposition 8, we separate out some of the earlier steps, which are valid under a weaker hypothesis, in the form of an admittedly ugly lemma:

Lemma. *Let A be a $*$-ring with unity and proper involution, satisfying the* (WSR)-*axiom, and suppose e, f are projections such that $ef-fe$ is invertible in A. Define $x=fe$. Then*

(1) $$x^*x=efe \quad \textit{is invertible in } eAe.$$

Let a be the inverse of efe in eAe; thus,

(2) $$a\in eAe, \quad a^*=a, \quad a(efe)=(efe)a=e \quad (\textit{that is, } afe=efa=e).$$

Choose $r\in\{x^*x\}''$ with $x^*x=r^*r$. Then

(3) $$r\in eAe,$$

(4) $$r \textit{ is invertible in } eAe, \textit{ with inverse } ar^*=r^*a,$$

(5) $$ar=ra.$$

Define $v=xar^*$. Then

(6) $$v^*v=e,$$

(7) $$x=vr,$$

(8) $$vv^*=f.$$

Proof. (1) See the proof of Proposition 7.

(2) The self-adjointness of a follows from that of efe.

(3) By the (WSR)-axiom, we may choose $r\in\{x^*x\}''=\{efe\}''$ such that $efe=r^*r=rr^*$. Since $e\in\{efe\}'$, it follows that $re=er$; a straightforward calculation then yields $(re-r)^*(re-r)=0$, therefore $re-r=0$ (the involution is assumed proper). Thus $r=re=er$, $r\in eAe$.

(4), (5) Since $r^*r=rr^*=efe$ is invertible in eAe, so is r; explicitly, the calculations

$$e=(efe)a=(rr^*)a=r(r^*a),$$
$$e=a(efe)=a(r^*r)=(ar^*)r$$

show that the inverse of r in eAe is $r^*a=ar^*$. Taking adjoints in the last equation, we have $ar=ra$.

(6) Setting $v=xar^*$, we have $v^*v=rax^*xar^*=ra[(efe)a]r^*=raer^*=(ra)r^*=(ar)r^*=a(efe)=e$ by (5) and (2).

(7) $vr=(xar^*)r=x[a(r^*r)]=x[a(efe)]=xe=x$.

(8) Writing $g=vv^*$, it remains to show that $g=f$. At any rate, g is a projection [§ 2, Prop. 2] and $fg=g$ (because $v=xar^*=(fe)ar^*\in fA$), thus $g\le f$. To show that $f-g=0$, it will suffice, by the invertibility of $ef-fe$, to show that

$$(ef-fe)(f-g)=0;$$

in fact, it will be shown that $ef(f-g)=fe(f-g)=0$. A straightforward computation yields $g=faf$, therefore

$$eg = e(faf) = (efa)f = ef$$

by (2); thus $e(f-g)=0$. On the one hand, this implies $fe(f-g)=0$; on the other hand, since $f-g\leq f$ we have also $ef(f-g)=e(f-g)=0$. ∎

Proof of Proposition 8. With notation as in the lemma, we assume, in addition, that $r\geq 0$.

Similarly, let $y=-(1-f)(1-e)$ (the minus sign is intentional) and consider $y^*y=(1-e)(1-f)(1-e)$. Since

$$(1-e)(1-f)-(1-f)(1-e) = ef - fe$$

is invertible, the lemma is again applicable, as follows.

(1′) $\quad y^*y=(1-e)(1-f)(1-e)$ is invertible in $(1-e)A(1-e)$.

If b is the inverse of $(1-e)(1-f)(1-e)$ in $(1-e)A(1-e)$, then

(2′) $$b^*=b, \quad b(1-f)(1-e)=(1-e)(1-f)b=1-e.$$

Choosing $s\in\{y^*y\}''$ with $s\geq 0$ and $y^*y=s^2$, we have

(3′) $$s\in(1-e)A(1-e),$$

(4′) $\quad s$ is invertible in $(1-e)A(1-e)$, with inverse $bs=sb$

(recall that $s^*=s$; thus (5′) is redundant). Defining $w=ybs$, we have (the minus sign in the definition of y gives no trouble)

(6′) $$w^*w=1-e,$$

(7′) $$y=ws,$$

(8′) $$ww^*=1-f.$$

Define $u=v+w$. Obviously u is unitary and $ueu^*=f$; the proof will be concluded by showing that u is self-adjoint.

Set $t=r+s$. From (4) and (4′), it is clear that t is invertible in A (with $t^{-1}=ar+bs$). Since

$$ut = vr+vs+wr+ws = x+0+0+y,$$

and since

$$x+y = fe-(1-f)(1-e) = e+f-1,$$

we have $ut=e+f-1$. Thus, setting $z=e+f-1$, we have

(∗) $$z=ut,$$

where u and t are invertible and $z^*=z$. Since $t=r+s$, where $r\geq 0$ and $s\geq 0$, we have $t\geq 0$. Since, in addition, (∗) yields

$$z^2 = z^*z = tu^*ut = t^2,$$

it follows from the (UPSR)-axiom that t is the unique positive square root of z^2, and in particular $t \in \{z^2\}''$; but $z \in \{z^2\}'$, therefore $tz = zt$, that is, $zt^{-1} = t^{-1}z$. Citing (*), we see that $u = zt^{-1} = t^{-1}z$ is the product of commuting self-adjoints, therefore $u^* = u$. ∎

In a Rickart *-ring, a condition weaker than the invertibility of $ef - fe$ is $\mathrm{RP}(ef - fe) = 1$, that is, position p (Proposition 6); still weaker is position p'. To arrive at the parallelogram law (P), we must show that projections in position p' are equivalent (Proposition 4); it would suffice to show that they can be exchanged by a symmetry. Thus, to establish the parallelogram law, it would suffice to prove the conclusion of Proposition 8 under the weaker hypothesis that e, f are in position p'; this is done in the next group of results (but the proofs require added axioms on A). It is convenient to separate out the intermediate case of position p as a lemma:

Lemma. *Let A be a Baer *-ring satisfying the* (EP)*-axiom and the* (UPSR)*-axiom. If e, f are projections in position p, then e and f can be exchanged by a symmetry (in particular, $e \sim f \sim 1 - e \sim 1 - f$).*

Proof. We show that e and f can be exchanged by a symmetry; it is then automatic that $1 - e \sim 1 - f$, and the parenthetical assertion of the lemma follows from the observation that $e, 1 - f$ are also in position p.

Let $x = ef - fe$, $z = x^*x = -(ef - fe)^2$, and write $B = \{z\}'$. As noted in the proof of Proposition 7, B has center $B' = \{z\}''$, and B contains e and f (hence also x).

By hypothesis, $\mathrm{RP}(z) = \mathrm{RP}(x) = 1$ (Proposition 6); we shall reduce matters to the situation of Proposition 8 by constructing a central partition of 1 in B such that z is invertible in each direct summand. Let (h_ι) be a maximal orthogonal family of nonzero projections in $\{z\}''$ such that, for each ι, zh_ι is invertible in $h_\iota B$ (the Zorn's lemma argument is launched by an application of the (EP)-axiom). We assert that $\sup h_\iota = 1$ (recall that suprema in B are unambiguous [§ 4, Prop. 7]). Writing $h = \sup h_\iota$, it is to be shown that $1 - h = 0$; since $\mathrm{RP}(z) = 1$, it will suffice to show that $z(1-h) = 0$, equivalently, $x(1-h) = 0$. Assume to the contrary. Then, by the (EP)-axiom, there exists an element

$$y \in \{(1-h)x^*x(1-h)\}'' = \{z(1-h)\}'' \subset \{z\}''$$

such that $z(1-h)y = k$, k a nonzero projection. Obviously $k \in \{z\}''$, $k \leq 1 - h$, and zk is invertible in kB, contradicting maximality of the family (h_ι).

We propose to apply Proposition 8 in each $h_\iota B$; to this end, we note that the (UPSR)-axiom is satisfied by B (Exercise 2) and therefore by $h_\iota B = h_\iota B h_\iota$ (Exercise 3). Since

$$(eh_\iota)(fh_\iota) - (fh_\iota)(eh_\iota) = xh_\iota$$

is invertible in $h_\iota B$ (because $(xh_\iota)(xh_\iota)^* = xx^* h_\iota = (xh_\iota)^*(xh_\iota) = zh_\iota$ is invertible in $h_\iota B$), it follows from Proposition 8 that there exists a symmetry u_ι in $h_\iota B$ such that

$$(*) \qquad u_\iota(eh_\iota)u_\iota = fh_\iota.$$

It remains to join the u_ι into a symmetry u exchanging e and f. {If A were an AW^*-algebra, the C^*-sum technique would do the trick; in a Baer $*$-ring, we must be more deft.} The strategy is to express the symmetry u_ι in terms of a projection g_ι of $h_\iota A$ (see the remarks following Definition 6), take the supremum g of the g_ι, and define $u = 2g - 1$. Part of the conclusion of Proposition 8 is $eh_\iota \sim h_\iota - eh_\iota$; therefore $2h_\iota$ has an inverse a_ι in $h_\iota B$ [§ 11, Lemma 2], thus $g_\iota = a_\iota(h_\iota + u_\iota)$ is a projection in $h_\iota B$, such that $u_\iota = 2g_\iota - h_\iota$. Define $g = \sup g_\iota$, $u = 2g - 1$. Since $gh_\iota = g_\iota$ for all ι [§ 11, Lemma 1], it follows that

$$uh_\iota = 2gh_\iota - h_\iota = 2g_\iota - h_\iota = u_\iota;$$

thus $(*)$ yields $(ueu - f)h_\iota = 0$ for all ι, and $ueu - f = 0$ results from $\sup h_\iota = 1$. ∎

The above proof actually yields information for an arbitrary pair of projections:

Theorem 2. *Let A be a Baer $*$-ring satisfying the* (EP)*-axiom and the* (UPSR)*-axiom. If e, f is any pair of projections in A, there exists a projection h, central in the subring $B = \{-(ef - fe)^2\}'$, such that* (1) *eh and fh are in position p in Bh (hence may be exchanged by a symmetry in Bh), and* (2) *$e(1-h)$ and $f(1-h)$ commute. Explicitly, $h = \mathrm{RP}(ef - fe)$.*

Proof. With notation as in the proof of the lemma (but with the hypothesis $\mathrm{RP}(x) = 1$ suppressed), set $h = \sup h_\iota$; the argument given there shows that $h = \mathrm{RP}(x)$. On the one hand, $x(1-h) = 0$ shows that $e(1-h)$ and $f(1-h)$ commute. On the other hand, $(eh)(fh) - (fh)(eh) = xh = x$ has right projection h, therefore eh and fh are in position p in Bh (Proposition 6). ∎

We now advance to position p':

Lemma. *Notation as in Theorem 2. If, in addition, e, f are in position p', then $e(1-h) = f(1-h)$.*

Proof. Write $k = 1 - h$ and set $e'' = ek$, $f'' = fk$; we know from Theorem 2 that e'' and f'' commute. By hypothesis,

$$e \cap (1-f) = (1-e) \cap f = 0;$$

since k is central in B, it follows that

$$e'' \cap (k - f'') = (k - e'') \cap f'' = 0,$$

that is, in view of the commutativity of e'' and f'',

$$e''(k-f'')=(k-e'')f''=0.$$

Thus $e''=e''f''=f''$. ∎

Theorem 3. *Let A be a Baer $*$-ring satisfying the* (EP)*-axiom and the* (UPSR)*-axiom. If e,f are projections in position p', then e and f can be exchanged by a symmetry $2g-1$, g a projection.*

Proof. With notation as in the proof of Theorem 2, set $e'=eh$, $e''=e(1-h)$, $f'=fh$, $f''=f(1-h)$; thus

$$e=e'+e'', \quad f=f'+f''.$$

By Theorem 2, e' and f' are in position p in Bh, and there exists a symmetry u' in Bh such that $u'e'u'=f'$; by the lemma, $e''=f''$. Then $u=u'+(1-h)$ is a symmetry in B (hence in A) and it is straightforward to check that $ueu=f$. A second look at the proof of Theorem 2 (rather, its lemma) shows that $u'=2g'-h$ for a suitable projection g', thus $u=2g-1$, where $g=g'+(1-h)$. ∎

Combining Theorem 3 with Proposition 4, we arrive at the climax of the section (see also Exercise 7):

Theorem 4. *The parallelogram law* (P) *holds in any Baer $*$-ring satisfying the* (EP)*-axiom and the* (UPSR)*-axiom.*

Theorems 3 and 4, combined with Proposition 5, yield an important decomposition theorem (see also Exercise 8):

Theorem 5. *Let A be a Baer $*$-ring satisfying the* (EP)*-axiom and the* (UPSR)*-axiom. If e,f is any pair of projections in A, there exist orthogonal decompositions*

$$e=e'+e'', \quad f=f'+f''$$

such that $e'\sim f'$ and $e''f=ef''=0$. Explicitly, $e'=\mathrm{LP}(ef)$, $f'=\mathrm{RP}(ef)$, $e''=e-e'$, $f''=f-f'$; e' and f' are in position p', and can be exchanged by a symmetry.

Exercises

1A. In the Baer $*$-ring of all 2×2 matrices over the field of three elements [§ 1, Exer. 17], the parallelogram law (P) fails; so does the (SR)-axiom; so does the (EP)-axiom.

2A. Let A be a $*$-ring, B a $*$-subring such that $B=B''$. If A satisfies the (WSR)-axiom [(SR)-axiom, (PSR)-axiom, (UPSR)-axiom] then so does B.

3A. Let A be a $*$-ring with proper involution, and let e be a projection in A. If A satisfies the (WSR)-axiom [(SR)-axiom, (PSR)-axiom, (UPSR)-axiom] then so does eAe.

4A. If A is a weakly Rickart $*$-ring satisfying the (WSR)-axiom, and if e,f are projections such that $ef-fe$ is invertible in $(e\cup f)A(e\cup f)$, then $e\sim f\sim e\cup f-e \sim e\cup f-f$.

5C. Let A be a $*$-ring with unity and proper involution, satisfying the (SR)-axiom. If e,f are projections such that $ef-fe$ is invertible, then e and f can be exchanged by a symmetry. (This generalizes Proposition 8.)

6C. Let A be a Baer $*$-ring satisfying the (EP)-axiom and the (SR)-axiom. If e,f are projections in position p', then e and f can be exchanged by a symmetry. (This generalizes Theorem 3.)

7C. The parallelogram law (P) holds in every Baer $*$-ring satisfying the (EP)-axiom and the (SR)-axiom. (This generalizes Theorem 4.)

8C. Let A be a Baer $*$-ring satisfying the (EP)-axiom and the (SR)-axiom. If e,f is any pair of projections in A, there exist orthogonal decompositions $e=e'+e''$, $f=f'+f''$ with e',f' in position p' and $e''f=ef''=0$; in particular, $e'\sim f'$, indeed, e' and f' can be exchanged by a symmetry. (This generalizes Theorem 5.)

9A. The following conditions on a $*$-ring are equivalent: (a) the involution is proper, and the relations $x\geq 0$, $y\geq 0$, $x+y=0$ imply $x=y=0$; (b) $\sum_1^n x_i{}^*x_i=0$ implies $x_1=\cdots=x_n=0$ (n arbitrary).

10A. In a $*$-ring satisfying the conditions of Exercise 9, the (PSR)-axiom and the (UPSR)-axiom are equivalent.

11A. Let A be a $*$-ring with proper involution, satisfying the following *strong square-root axiom* (SSR): If $x\in A$, $x\geq 0$, then there exists $y\in\{x\}''$ with $y^*=y$ and $x=y^2$. (The (SR)-axiom provides such a y only for positives x of the special form $x=t^*t$.) Assume, in addition, that (1) A has a central element i such that $i^2=-1$ and $i^*=-i$, and (2) $2x=0$ implies $x=0$. Then $\sum_1^n x_k^*x_k=0$ implies $x_1=\cdots=x_n=0$ (cf. Exercise 9).

12A. Let A be a $*$-ring with proper involution, satisfying the conditions (1), (2) of Exercise 11. In such a $*$-ring, the (PSR)-axiom and the (UPSR)-axiom are equivalent.

13A. If A is a $*$-ring satisfying the (WEP)-axiom and the (SR)-axiom, then A satisfies the (EP)-axiom.

14A. Let A be a Baer $*$-ring, let $(e_\iota)_{\iota\in I}$ and $(f_\iota)_{\iota\in I}$ be equipotent families of orthogonal projections such that $e_\iota\sim f_\iota$ for all $\iota\in I$, and let $e=\sup e_\iota$, $f=\sup f_\iota$. We know that if $ef=0$ then $e\sim f$ [§11, Th. 1]. If A satisfies the parallelogram law (P), then the weaker condition $e\cap f=0$ also implies $e\sim f$.

15A. If e,f are projections in a $*$-ring A, such that $e\sim f$ and $ef=0$, then e and f can be exchanged by a symmetry in $(e+f)A(e+f)$.

16A. Theorems 2–5 hold in any Rickart C^*-algebra; in particular, any pair of projections in position p' can be exchanged by a symmetry.

17A. Suppose A is a Rickart $*$-ring in which every pair of projections in position p' can be exchanged by a symmetry. If e,f is any pair of projections in A, there exists a symmetry u such that $u(ef)u=fe$.

18A. In an arbitrary Baer $*$-ring, projections in position p need not be equivalent.

19C. In a von Neumann algebra A, projections e,f are in position p' (relative to A) if and only if $e \sim f$ relative to the von Neumann algebra generated by e and f.

§ 14. Generalized Comparability

Projections e,f in a $*$-ring A are said to be *comparable* if either $e \precsim f$ or $f \precsim e$. Rings in which any two projections are comparable are of interest in the same way that simply ordered sets are interesting examples of partially ordered sets [cf. § 12, Exer. 1]. In general, the concept of comparability is of limited use. (For example, if A contains a central projection h different from 0 and 1, and if e,f are nonzero projections such that $e \leq h$ and $f \leq 1-h$, then e and f cannot be comparable.) The pertinent concept in general $*$-rings is as follows:

Definition 1. Projections e,f in a $*$-ring A are said to be *generalized comparable* if there exists a central projection h such that

$$he \precsim hf, \qquad (1-h)f \precsim (1-h)e.$$

(When A has no unity element, the use of 1 is formal and the condition need not be symmetric in e and f.) We say that A has *generalized comparability* (briefly, A has GC) if every pair of projections is generalized comparable.

Generalized comparability may be reformulated in terms of the following concept, which generalizes, and is consistent with, an earlier definition [§ 6, Def. 2]:

Definition 2. Projections e,f in a $*$-ring A are said to be *very orthogonal* if there exists a central projection h such that $he=e$ and $hf=0$. (That is, $e \leq h$ and $f \leq 1-h$, where 1 is used formally when A has no unity element—in which case, the relation need not be symmetric in e and f.)

If e,f are projections in a Baer $*$-ring A, then the following conditions are equivalent: (a) e,f are very orthogonal; (b) $C(e)C(f)=0$; (c) $eAf=0$ [§ 6, Cor. 1 of Prop. 3].

The relevance of very orthogonality to generalized comparability is as follows:

Proposition 1. *If e,f are projections in a $*$-ring, the following conditions are equivalent:*

(a) *e,f are generalized comparable;*

(b) *there exist orthogonal decompositions $e=e_1+e_2$, $f=f_1+f_2$ with $e_1 \sim f_1$ and f_2, e_2 very orthogonal.*

Proof. (a) implies (b): Choose h as in Definition 1, say

$$he \sim f_1' \leq hf, \qquad (1-h)f \sim e_1'' \leq (1-h)e.$$

Writing $e_1' = he$ and $f_1'' = (1-h)f$, we have

$$(*) \qquad e_1' \sim f_1', \quad e_1'' \sim f_1''.$$

Obviously $e_1' e_1'' = 0$ and $f_1' f_1'' = 0$; setting

$$e_1 = e_1' + e_1'', \quad f_1 = f_1' + f_1'',$$

it follows from $(*)$ that $e_1 \sim f_1$ [§1, Prop. 8]. Since $e_1 \leq e$ and $f_1 \leq f$, we may define $e_2 = e - e_1, f_2 = f - f_1$; it is routine to check that $he_2 = 0$ and $hf_2 = f_2$.

(b) implies (a): Assuming there exists such a decomposition, let h be a central projection such that $hf_2 = f_2$ and $he_2 = 0$. Then $he = he_1 \sim hf_1 \leq hf$ [§1, Prop. 7], thus $he \precsim hf$, and similarly $(1-h)f \precsim (1-h)e$. ∎

If e, f are generalized comparable, but are not very orthogonal, then Proposition 1 shows that e, f have nonzero subprojections e_1, f_1 such that $e_1 \sim f_1$; this is a phenomenon worth formalizing:

Definition 3. Projections e, f in a $*$-ring A are said to be *partially comparable* if there exist nonzero subprojections $e_0 \leq e$, $f_0 \leq f$ such that $e_0 \sim f_0$. We say that A has *partial comparability* (briefly, A has PC) if $eAf \neq 0$ implies e, f are partially comparable.

GC is stronger than PC:

Proposition 2. *If A is a $*$-ring with* GC, *then A has* PC.

Proof. Assuming e, f are projections that are not partially comparable, it is to be shown that $eAf = 0$. Write $e = e_1 + e_2$, $f = f_1 + f_2$ as in Proposition 1. By the hypothesis on e, f, necessarily $e_1 = f_1 = 0$, thus f, e are very orthogonal; if h is a central projection with $hf = f$ and $he = 0$, then $eAf = eAhf = ehAf = 0$. ∎

PC is implied by axioms of 'existence of projections' type; for instance:

Proposition 3. *If A is a $*$-ring satisfying the* (VWEP)-*axiom, then A has* PC.

Proof. Suppose e, f are projections such that $eAf \neq 0$, equivalently, $fAe \neq 0$. Let $x \in fAe$, $x \neq 0$. By hypothesis, there exists an element $y \in \{x^*x\}'$ with $(y^*y)(x^*x) = e_0$, e_0 a nonzero projection [§7, Def. 3], thus $e_0 = y^*(x^*x)y = (xy)^*(xy)$. Writing $w = xy$, we have $w^*w = e_0$; since the involution of A is proper [§2, Exer. 6], w is a partial isometry [§2, Prop. 2]. Set $f_0 = ww^*$. Since $x \in fAe$, the formula $e_0 = (y^*y)(x^*x)$ shows that $e_0 \leq e$, and $f_0 = ww^* = (xy)w^*$ shows that $f_0 \leq f$. ∎

In Baer $*$-rings, generalized comparability is intimately related to additivity of equivalence [§ 11, Def. 1]; in fact, a Baer $*$-ring has GC if and only if it has PC and equivalence is additive [§ 20, Th. 2]. The "only if" part appears to be fairly difficult—the proof we give in Section 20 involves most of the structure theory discussed in Part 2. The "if" part is easy:

Proposition 4. *If A is a Baer $*$-ring with* PC *and if equivalence in A is additive, then A has* GC.

Proof. Let e,f be any pair of projections in A. If $eAf=0$ then e,f are very orthogonal and the generalized comparability of e and f is trivial. Assuming $eAf\neq 0$, let $(e_\iota)_{\iota\in I}$, $(f_\iota)_{\iota\in I}$ be a maximal pair of orthogonal families of nonzero projections such that $e_\iota\leq e$, $f_\iota\leq f$ and $e_\iota\sim f_\iota$ for all $\iota\in I$ (an application of PC starts the Zorn's lemma argument). Set $e'=\sup e_\iota$, $f'=\sup f_\iota$, $e''=e-e'$, $f''=f-f'$. On the one hand, $e'\sim f'$ by the assumed additivity of equivalence. On the other hand, $e''Af''=0$ (if not, an application of PC would contradict maximality), therefore e'',f'' are very orthogonal. In view of Proposition 1, the decompositions $e=e'+e''$, $f=f'+f''$ show that e,f are generalized comparable. ∎

It is a corollary that every von Neumann algebra A has GC; for, it is easy to see that partial isometries in A are addable (e. g., they can be summed in the strong operator topology), and the validity of the (EP)-axiom [§ 7, Cor. of Prop. 3] ensures, via Proposition 3, that A has PC. For AW^*-algebras, essentially the same argument may be employed (except that the proof of addability is harder—see Section 20), but an alternative proof will shortly be given.

Proposition 4, and the fact that equivalence is orthogonally additive in any Baer $*$-ring [§ 11, Th. 1], naturally suggest the following definition:

Definition 4. We say that a $*$-ring has *orthogonal* GC if every pair of orthogonal projections is generalized comparable.

This condition is automatically fulfilled in a Baer $*$-ring with PC:

Proposition 5. *If A is a Baer $*$-ring with* PC, *then A has orthogonal* GC.

Proof. Let e,f be projections with $ef=0$. The proofs proceeds as for Proposition 4, except that $e'\sim f'$ results from a theorem [§ 11, Th. 1] rather than an assumption. ∎

In the presence of the parallelogram law, GC and orthogonal GC are equivalent hypotheses:

Proposition 6. *If A is a Rickart $*$-ring with orthogonal* GC, *and if A satisfies the parallelogram law* (P), *then A has* GC.

Proof. Let e, f be any pair of projections in A. By the parallelogram law, write

$$e = e' + e'', \quad f = f' + f''$$

with $e' \sim f'$ and $e'' f = e f'' = 0$ [§ 13, Prop. 5]. Since, by hypothesis, the orthogonal projections e'', f'' are generalized comparable, Proposition 1 yields decompositions

$$e'' = e_1 + e_2, \quad f'' = f_1 + f_2$$

with $e_1 \sim f_1$ and e_2, f_2 very orthogonal. Then

$$e = (e' + e_1) + e_2, \quad f = (f' + f_1) + f_2,$$

where $e' + e_1 \sim f' + f_1$ and e_2, f_2 are very orthogonal, therefore e, f are generalized comparable by Proposition 1. ∎

In a Baer $*$-ring satisfying the parallelogram law, the concepts PC, GC and orthogonal GC merge:

Proposition 7. *If A is a Baer $*$-ring satisfying the parallelogram law* (P), *then the following conditions on A are equivalent:* (a) *A has* PC; (b) *A has orthogonal* GC; (c) *A has* GC.

Proof. (a) implies (b) by Proposition 5; in the presence of (P), (b) implies (c) by Proposition 6; and (c) implies (a) by Proposition 2. ∎

Corollary 1. *Every AW^*-algebra has* GC.

Proof. An AW^*-algebra A satisfies the parallelogram law (P) [§ 13, Th. 1]; since A satisfies the (EP)-axiom [§ 7, Cor. of Prop. 3], and therefore has PC (Proposition 3), it follows from Proposition 7 that A has GC. ∎

Corollary 2. *If A is a Baer $*$-ring such that* $\mathrm{LP}(x) \sim \mathrm{RP}(x)$ *for all x in A, then A has* GC *and satisfies the parallelogram law* (P).

Proof. Since A satisfies (P) [§ 13, Prop. 2], by Proposition 7 it suffices to show that A has PC. Suppose e, f are projections such that $eAf \neq 0$, say $x = eaf \neq 0$; then $e_0 = \mathrm{LP}(x)$, $f_0 = \mathrm{RP}(x)$ are nonzero subprojections of e, f such that $e_0 \sim f_0$. ∎

The parallelogram law is not the most natural of hypotheses. Some ways of achieving it were shown in Section 13; an application (see also Exercise 5):

Theorem 1. *If A is a Baer $*$-ring satisfying the* (EP)*-axiom and the* (UPSR)*-axiom, then A has* GC *and satisfies the parallelogram law* (P).

Proof. A satisfies (P) [§ 13, Th. 4] and has PC (Proposition 3), therefore A has GC by Proposition 7. ∎

Incidentally, Theorem 1 provides a second proof of the AW^* case (Corollary 1 of Proposition 7).

We close the section with two items for later application. The first is for application in Section 17 [§ 17, Th. 2]:

Proposition 8. *Let A be a Rickart $*$-ring with* GC, *satisfying the parallelogram law* (P). *If e, f is any pair of projections in A, there exists a central projection h such that*

$$he \precsim hf,$$
$$(1-h)(1-e) \precsim (1-h)(1-f).$$

Proof. Apply GC to the pair $e \cap (1-f)$, $(1-e) \cap f$: there exists a central projection h such that

$$h[e \cap (1-f)] \precsim h[(1-e) \cap f], \tag{1}$$
$$(1-h)[(1-e) \cap f] \precsim (1-h)[e \cap (1-f)]. \tag{2}$$

It follows from the parallelogram law (see [§ 13, Prop. 1]) that

$$e - e \cap (1-f) \sim f - (1-e) \cap f$$

and (replacing e, f by $1-e, 1-f$)

$$(1-e) - (1-e) \cap f \sim (1-f) - e \cap (1-f),$$

therefore

$$h[e - e \cap (1-f)] \sim h[f - (1-e) \cap f], \tag{3}$$
$$(1-h)[(1-e) - (1-e) \cap f] \sim (1-h)[(1-f) - e \cap (1-f)]. \tag{4}$$

Adding (1) and (3) yields $he \precsim hf$, while (2) and (4) yield $(1-h)(1-e) \precsim (1-h)(1-f)$. ∎

The final proposition is for application in [§ 18, Prop. 5]:

Proposition 9. *Let A be a Baer $*$-ring with* PC, *and suppose $(e_\iota)_{\iota \in I}$ is a family of projections in A with the following property: for every nonzero central projection h, the set of indices*

$$\{\iota \in I : he_\iota \neq 0\}$$

is infinite; in other words, there exists no direct summand of A (other than 0) on which all but finitely many of the e_ι vanish.

Then, given any positive integer n, there exist n distinct indices $\iota_1, \ldots, \iota_n$, and nonzero projections $g_\nu \leq e_{\iota_\nu}$ $(\nu = 1, \ldots, n)$, such that

$$g_1 \sim g_2 \sim \cdots \sim g_n.$$

Proof. The proof is by induction on n. The case $n=1$ is trivial: the set $\{\iota: 1e_\iota \neq 0\}$ is infinite, and any of its members will serve as ι_1, with $g_1 = e_{\iota_1}$.

Assume inductively that all is well with $n-1$, and consider n. By assumption, there exist distinct indices $\iota_1, \dots, \iota_{n-1}$ and nonzero projections $f_1, \dots, f_{n-1}$ such that $f_\nu \le e_{\iota_\nu}$ $(\nu = 1, \dots, n-1)$ and $f_1 \sim \cdots \sim f_{n-1}$.

Since $C(f_1) \neq 0$, it is clear from the hypothesis that there exists an index ι_n distinct from $\iota_1, \dots, \iota_{n-1}$ such that $C(f_1)e_{\iota_n} \neq 0$. Then $C(f_1)C(e_{\iota_n}) \neq 0$, thus $f_1 A e_{\iota_n} \neq 0$ [§ 6, Cor. 1 of Prop. 3]; citing PC, there exist nonzero subprojections $g_1 \le f_1$ and $g_n \le e_{\iota_n}$ such that $g_1 \sim g_n$. For $\nu = 2, \dots, n-1$, the equivalence $f_1 \sim f_\nu$ transforms g_1 into a subprojection $g_\nu \le f_\nu$ with $g_1 \sim g_\nu$. Thus $g_n \sim g_1 \sim g_\nu$ $(\nu = 2, \dots, n-1)$.

{The proof shows that the indices for n may be obtained by augmenting the indices for $n-1$; but as n increases, the projections g_ν will in general shrink.} ∎

Exercises

1A. A Baer $*$-ring with orthogonal GC, but without PC (hence without GC): the ring of all 2×2 matrices over the field of three elements [§1, Exer. 17].

2B. A Baer $*$-ring A has GC if and only if (i) A has PC, and (ii) equivalence in A is additive.

3A. In a Baer $*$-ring with finitely many elements, PC and GC are equivalent.

4B. In a properly infinite Baer $*$-ring [§15, Def. 3], PC and GC are equivalent.

5C. If A is a Baer $*$-ring satisfying the (EP)-axiom and the (SR)-axiom, then A has GC and satisfies the parallelogram law (P). (This generalizes Theorem 1.)

6A. (i) If A is a $*$-ring with GC and if g is any projection in A, then gAg has GC.
(ii) If A is a Baer $*$-ring, if g is a projection in A, and if e, f are projections in gAg that are generalized comparable in gAg, then e, f are generalized comparable in A.

7A. If e, f are partially comparable projections in a Baer $*$-ring, then $C(e)C(f) \neq 0$.

8A. If A is a Rickart $*$-ring satisfying the parallelogram law (P), and if e, f are projections in A such that $ef \neq 0$, then e, f are partially comparable.

9A. Let A be a Baer $*$-ring satisfying the parallelogram law (P). If A satisfies any of the following conditions, then A has GC:
(1) For every projection e, $C(e) = \sup\{e' : e' \sim e\}$ [cf. §6, Exer. 7].
(2) If e, f are projections such that $eAf \neq 0$, then there exists a unitary u such that $euf \neq 0$.
(3) If e, f are projections such that $eAf \neq 0$, then there exists a projection g such that $e(2g)f \neq 0$.

10A. The following conditions on a $*$-ring A are equivalent: (a) A has GC; (b) A has orthogonal GC and, for every pair of projections e, f, there exist orthogonal decompositions $e = e' + e''$, $f = f' + f''$ with $e' \sim f'$ and $e''f'' = f''e''$.

11A. Let A be a Rickart $*$-ring in which every sequence of orthogonal projections has a supremum. As in [§12, Exer. 1], write $[e] = \{f : f \sim e\}$ and define $[e] \le [f]$

iff $e \precsim f$. If A has GC then the set of equivalence classes is a lattice with respect to this ordering.

12A. Let A be a Baer $*$-ring satisfying the (EP)-axiom and the (SR)-axiom (or let A be a Rickart C^*-algebra with GC). If e, f is any pair of projections in A, there exist orthogonal decompositions $e = e_1 + e_2$, $f = f_1 + f_2$ such that e_1 and e_2 are exchangeable by a symmetry and e_2, f_2 are very orthogonal.

13B. If A is a Baer $*$-ring satisfying the (WEP)-axiom, then the following conditions are equivalent: (a) A has GC; (b) $\mathrm{LP}(x) \sim \mathrm{RP}(x)$ for all $x \in A$; (c) A satisfies the parallelogram law (P).

14B. Let A be a Baer $*$-ring with GC, and let e, f be any pair of projections in A. Either (1) $f \precsim e$, or (2) there exists a central projection h with the following property: for a central projection k, $ke \precsim kf$ iff $k \le h$. In case (2), such a projection h is unique, $h \ge 1 - C(e)$, and $(1-h)f \precsim (1-h)e$.

15A. If A is a Baer $*$-ring with PC, the following conditions on a pair of projections e, f imply one another: (a) $C(e) \le C(f)$; (b) $e = \sup e_\iota$ with (e_ι) an orthogonal family of projections such that $e_\iota \precsim f$ for all ι; (c) $e = \sup e_\iota$ with (e_ι) a family of projections such that $e_\iota \precsim f$ for all ι.

16A. Let A be a Rickart $*$-ring with GC, let n be a positive integer, and suppose that the $n \times n$ matrix ring A_n is a Rickart $*$-ring satisfying the parallelogram law (P). Then A_n has GC.

17C. Let A be a Rickart $*$-ring with orthogonal GC (e.g., let A be a Baer $*$-ring with PC) and let e be a projection in A. The following conditions on e are equivalent: (a) e is central in A; (b) e commutes with every projection in A (that is, e is central in the reduced ring A°); (c) e has a unique complement.

18A. (i) If A is a Baer $*$-ring with PC, then a projection in A is central iff it commutes with every projection of A (thus a projection is central in A iff it is central in the reduced ring A° [§3, Exer. 18]).

(ii) The converse of (i) is false: there exists a Baer $*$-ring A such that $A^\circ = A$ but A does not have PC.

19A. Let A be a $*$-ring with unity. A partial isometry u in A is said to be *extremal* if the projections $1 - u^*u$ and $1 - uu^*$ are very orthogonal in the sense of Definition 2. {The terminology is motivated by the fact that if A is an AW^*-algebra, then the closed unit ball of A is a convex set whose extremal points are precisely the extremal partial isometries.} For example, if u is an isometry ($u^*u = 1$) or a co-isometry ($uu^* = 1$) then u is an extremal partial isometry; when A is factorial, there are no others [§6, Def. 3].

If A has GC and if w is any partial isometry in A, then there exists an extremal partial isometry u that 'extends' w, in the sense that $u(w^*w) = w$.

20D. *Problem:* If A is a Baer $*$-ring with PC, does it follow that A has GC?

21D. *Problem:* If A is a Baer $*$-ring satisfying the parallelogram law (P), does it follow that A has PC?

22D. *Problem:* If A is a Baer $*$-ring with PC, does it follow that A satisfies the parallelogram law (P)?

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