Preface

This book is an elaboration of ideas of Irving Kaplansky introduced in his book *Rings of operators* ([**52**], [**54**]).

The subject of Baer $*$-rings has its roots in von Neumann's theory of 'rings of operators' (now called von Neumann algebras), that is, $*$-algebras of operators on a Hilbert space, containing the identity operator, that are closed in the weak operator topology (hence also the name W^*-algebra). Von Neumann algebras are blessed with an excess of structure—algebraic, geometric, topological—so much, that one can easily obscure, through proof by overkill, what makes a particular theorem work.

The urge to axiomatize at least portions of the theory of von Neumann algebras surfaced early, notably in work of S. W. P. Steen [**84**], I. M. Gel'fand and M. A. Naĭmark [**30**], C. E. Rickart [**74**], and von Neumann himself [**53**]. A culmination was reached in Kaplansky's AW^*-algebras [**47**], proposed as a largely algebraic setting for the intrinsic (nonspatial) theory of von Neumann algebras (i. e., the parts of the theory that do not refer to the action of the elements of the algebra on the vectors of a Hilbert space).

Other, more algebraic developments had occurred in lattice theory and ring theory. Von Neumann's study of the projection lattices of certain operator algebras led him to introduce continuous geometries (a kind of lattice) and regular rings (which he used to 'coordinatize' certain continuous geometries, in a manner analogous to the introduction of division ring coordinates in projective geometry).

Kaplansky observed [**47**] that the projection lattice of every 'finite' AW^*-algebra is a continuous geometry. Subsequently [**51**], he showed that certain abstract lattices were also continuous geometries, employing 'complete $*$-regular rings' as a basic tool. A similar style of ring theory—emphasizing $*$-rings, idempotents and projections, and annihilating ideals—underlies both enterprises.

Baer $*$-rings, introduced by Kaplansky in 1955 lecture notes [**52**], are a common generalization of AW^*-algebras and complete $*$-regular rings. The definition is simple: A Baer $*$-ring is a ring with involution in which the right annihilator of every subset is a principal right ideal generated by a projection. The AW^*-algebras are precisely the Baer

$*$-rings that happen to be C^*-algebras; the complete $*$-regular rings are the Baer $*$-rings that happen to be regular in the sense of von Neumann.

Although Baer $*$-rings provided a common setting for the study of (1) certain parts of the algebraic theory of von Neumann algebras, and (2) certain lattices, the two themes were not yet fully merged. In AW^*-algebras, one is interested in '$*$-equivalence' of projections; in complete $*$-regular rings, 'algebraic equivalence'. The finishing touch of unification came in the revised edition of Kaplansky's notes [**54**]: one considers Baer $*$-rings with a postulated equivalence relation (thereby covering $*$-equivalence and algebraic equivalence simultaneously).

"Operator algebra" would have been a conceivable subtitle for the present book, alluding to the roots of the subject in the theory of operator algebras and to the fact that the subject is a style of argument as well as a coherent body of theorems; the book falls short of earning the subtitle because large areas of the algebraic theory of operator algebras are omitted (for example, general linear groups and unitary groups, module theory, derivations and automorphisms, projection lattice isomorphisms) and because the theory elaborated here—$*$-equivalence in Baer $*$-rings—does not develop Kaplansky's theory in its full generality. My reason for limiting the scope of the book to $*$-equivalence in Baer $*$-rings is that the reduced subject is more fully developed and is more attuned to the present state of the theory of Hilbert space operator algebras; the more general theories (as far as they go) are beautifully exposed in Kaplansky's book, and need no re-exposition here.

Perhaps the most important thing to be explained in the Preface is the status of functional analysis in the exposition that follows. The subject of Baer $*$-rings is essentially pure algebra, with historic roots in operator algebras and lattice theory. Accordingly, the exposition is written with two principles in mind: (1) if all the functional analysis is stripped away (by hands more brutal than mine), what remains should stand firmly as a substantial piece of algebra, completely accessible through algebraic avenues; (2) it is not very likely that the typical reader of this book will be unacquainted with, or uninterested in, Banach algebras.

Interspersed with the main development are examples and applications pertaining to C^*-algebras, AW^*-algebras and von Neumann algebras. In principle, the reader can skip over all such matters. One possible exception is the theory of commutative AW^*-algebras (Section 7). The situation is as follows. Associated with every Baer $*$-ring there is a complete Boolean algebra (the set of central projections in the ring); the Stone representation space of a complete Boolean algebra is an extremally disconnected, compact topological space (briefly, a Stonian space); Stonian spaces are precisely the compact spaces $\mathscr{X}$ for which the algebra $C(\mathscr{X})$ of continuous, complex-valued functions on $\mathscr{X}$ is a commutative AW^*-algebra. These algebras play an important role in the dimension theory and reduction theory of finite rings (Chapters 6 and 7). They can be approached either through the theory of commutative Banach algebras (as in the text) or from general topology. The choice is mainly one of order of development; give or take some terminology, commutative AW^*-algebras are essentially a topic in general topology.

The reader can avoid topological considerations altogether by restricting attention to factors, i.e., rings in which 0 and 1 are the only central projections (this amounts to restricting $\mathscr{X}$ to be a singleton). However, the chapter on reduction theory (Chapter 7) then disappears, the objects under study (finite factors) being already irreducible. There is ample precedent for limiting attention to the factorial case the first time through; this is in fact how von Neumann wrote out the theory of continuous geometries [**71**], and the factorial case dominates the early literature of rings of operators.

Baer $*$-rings are a compromise between operator algebras and lattice theory. Both the operator-theorist ("but this is too general!") and the lattice-theorist ("but this can be generalized!") will be unhappy with the compromise, since neither has any need to feel that the middle ground makes his own subject easier to understand; but uncommitted algebraists may find them enjoyable. I personally believe that Baer $*$-rings have the didactic virtue just mentioned, but the issue is really marginal; the test that counts is the test of intrinsic appeal. The subject will flourish if and only if students find its achievements exciting and its problems provocative.

Exercises are graded A–D according to the following mnemonics:

A ("Above"): can be solved using preceding material.

B ("Below"): can be solved using subsequent material.

C ("Complements"): can be solved using outside references.

D ("Discovery"): open questions.

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