Corrections to: A.S. Troelstra (editor), Metamathematical Investigation of Intuitionistic Arithmetic and Analysis June 22, 2009

This text contains corrections and additions to "Metamathematical Investigation of Intuitionistic Arithmetic and Analysis" which appeared in 1973 as number 344 in the "Springer Lecture Notes in Mathematics" series.

The original edition ran out of print many years ago. A small run of a corrected edition has been produced in 1993 as a report (X-93-05) of the Institute of Logic, Language and Information (ILLC) of the University of Amsterdam, but this is now also out of print.

In the mean time Springer Verlag has decided to make the volumes of the series Lecture Notes in Mathematics which are out of print available as "publish-on-demand" editions.

The text has been typeset in Latex. Wavy underlining in the original text is now interpreted as boldface, underlining as italics. Double wavy underlining has been interpreted by a sans serif fount. However, we have retained double underlining and did not replace it by Fraktur.

A first list of Errata appeared in 1974 as a report of the Mathematical Institute of the University of Amsterdam; many more errata have been discovered since then. In particular I should like to thank Marc Bezem, Susumu Hayashi, Jane Bridge Kister, Jaap van Oosten and Jeffery Zucker.

The counting of lines includes the lines in displayed formulas; for indications, e.g. a name or a number for a group of displayed lines, which are between lines so to speak, an ad hoc indication will be chosen.

Underlining in the original text has been rendered as italics in these correction; double underlining has been rendered as such, but a double wavy underlining corresponds to a sans serif letter in these corrections.

XIII Add below the summary of §6:

§7 *Applications: proof theoretic closure properties* 258
List of rules (3.7.1) — closure under ED, DP, CR_0, ECR_0, ED$'$, ACR, IPR'^{ω} (3.7.2–5) — closure under CR (3.7.6) — closure under ECR_1 (3.7.7–8) — extensions to analysis (3.7.9)

6 In 1.1.7, interchange "$\vee \mathrm{I_r}$" and "$\vee \mathrm{I_l}$".

7_{13} Read " $\exists$E " for " $\exists$I ".

8^5 Read "essentially".

8 In the first proof tree, "$B \to \curlywedge$" should be "$B \to \curlywedge$ (2)" and "(2)" should be repeated at the lowest horizontal line.

16^{13} Read " A " for " F ".

18 In 1.3.3 the axiom

$$x_i = x_i' \to \phi(x_1, \ldots, x_i, \ldots, x_n) = \phi(x_1, \ldots, x_i', \ldots, x_n)$$

(for any n-ary function constant ϕ, $1 \leq i \leq n$) can be replaced by the corresponding axiom for S only:

$$x = y \rightarrow Sx = Sy,$$

since the general case can be established by induction (since all ϕ except S are introduced by schemas for primitive recursive functions).

18_{11} Read "*Defining*" for "*Definining*".

19 Add at the bottom of the page rules expressing the functional character of the F_k:

$$\frac{F_k t_1 \ldots t_{n-1} t_n \qquad F_k t_1 \ldots t_{n-1} t'_n}{t_n = t'_n}.$$

20^8 Replace "F'_{k_m}" by "F_{k_m}".

25 Addition to second paragraph of (D) : "Canonical" essentially means that the arithmetization provably satisfies the "same" inductive closure conditions as the predicate itself.

26^3 Read "$\ulcorner t \urcorner$" for "t".

26^5 Read "that $\ulcorner A(\bar{x}_1, \ldots, \bar{x}_n) \urcorner$ stands for ...".

27^8 Read "$\simeq$" for "$=$".

27 Add at the bottom of the page a paragraph:

We follow *Kleene 1952* and use $\Lambda x.t$, t a p-term, to indicate a gödelnumber for t as partial recursive function of x; if t contains, besides the free variables $x, x_1, \ldots, x_n$, $\Lambda x.t$ is a (primitive) recursive function of $x_1, \ldots, x_n$.

29_2 Read "y_0" for "y".

29_1 Read "y_1" for "y".

33_{10} Read "We put $\ulcorner \xi t_1 \ldots t_n \urcorner = \ldots$".

41_2 Read "$t' \not\equiv x^\sigma$" for "$t' \neq x^\sigma$".

44^{17} Replace "slightly ... is" by "seemingly stronger (but in fact equivalent) variant is".

44^{19-21} Delete "In ... EXT-R′."

44_6–45^5 Replace these lines by the following:

The following two propositions are due to M. Bezem (Equivalence of Bar Recursors in the Theory of Functionals of Finite Type, *Archive for Mathematical Logic* 27 (1988), 149–160).

PROPOSITION. The rule EXT-R′ is derivable in qf-$\mathbf{WE}$-$\mathbf{HA}^\omega$.

PROOF. Assume EXT-R, and let $\vdash P \to s_1 = s_2$, $\vdash Q[x/t_1]$ Here $s_1 = s_2$ as usual is shorthand for an equation between terms of type 0 $s_1 x_1 x_2 \ldots x_n = s_2 x_1 x_2 \ldots x_n$, where $x_1, x_2, \ldots, x_n$ are variables not free in P, s_1, s_2. Without loss of generality we can assume $P \equiv (t_1 = 0)$, $Q[x] \equiv (t[x] = 0)$ (x not free in P, s_1, s_2. Below we shall abbreviate $t[x/s]$, for arbitrary s, as $t[s]$. So we have

$$\text{(1)} \qquad \vdash t_1 = 0 \to s_1 = s_2, \quad \vdash t[s_1] = 0.$$

Define

$$s_i' := \mathbf{R}_\sigma s_i 0^{(\sigma)(0)\sigma}, \quad s_i \in \sigma.$$

Then, with $x \notin \mathrm{FV}(s_i), i = 1, 2$:

$$\vdash x = 0 \to s_i' x = s_i, \qquad \vdash x \neq 0 \to s_i' x = 0^\sigma.$$

Applying EXT-R to $s_i'0 = s_i$ yields $\vdash t[s_i] = t[s_i'0]$. By replacement (i.e. $x = y \to t[x] = t[y]$) we obtain

$$\vdash t_1 = 0 \to t[s_i'0] = t[s_i' t_1].$$

Since also (1) holds, and $t_1 = 0$ is decidable ,$\vdash s_1' t_1 = s_2' t_1$,, so again using EXT-R

$$\vdash t[s_1' t_1] = t[s_2' t_1],$$

hence

$$\vdash t_1 = 0 \to t[s_2] = t[s_2'0] = t[s_2' t_1] = t[s_1' t_1] = t[s_1'0] = t[s_1] = 0.$$

Q.e.d.

PROPOSITION. The deduction theorem holds for qf-$\mathbf{WE}$-$\mathbf{HA}^\omega$ + EXT-R$'$, hence also for qf-$\mathbf{WE}$-$\mathbf{HA}^\omega$.

PROOF. It suffices to prove the deduction theorem for the system with EXT-R$'$, and in this case the deduction theorem is easy.

$45_{16,17}$ Delete these lines.

55_1 Read " $z < x$ " for " $z < z$ ".

56_{12} Read " $Q(x, \underline{v})$ " for " $Q(0, \underline{v})$ ".

56_6 Read " T " for " T ".

58_{12} Read " T " for " T ".

59 Add at the bottom: "Cf. also *Luckhardt 73*, pp. 66–67."

63^{10} Delete the first equation.

67_{11} At end of line we add "assumed to be provably linear in **HA**".

71^{15} Read " $\sigma((Q_2 V_n)A) \equiv (Q_1 v_{m+n+1})\sigma(A), \ldots$ ".

74 In comparing section 1.9.14 with more recent literature (such as A.S. Troelstra, D. van Dalen, *Constructivism in Mathematics*, Amsterdam 1988), it is to be noted that definedness of a term containing functions and numbers with partial application is here supposed to be defined in the sense of *Kleene* 1969, that is to say a function applied to an argument is defined if we can sufficiently many values of the function to find its value at the argument; this convention does not agree with the logic of partial terms with its strictness condition.

75^4 Read " $U(\mathrm{lth}(u)) = y$ " for " $U(\mathrm{lth}(u) = y)$ ".

75_5 Read " x " for " α ".

80_{13} Addition to 1.9.23: "Cf. also *Kreisel 1967*, page 249, where the role of generalized bar induction in proving the continuous functionals to be a model of bar recursion is mentioned."

$83_{18,13}$ Read " 0_σ " for " 0^σ ".

83_{10} Read " $B_\sigma yzu(c * \hat{v}))c$ " for " $B_\sigma yzuc(c * \hat{v}))c$ ".

91^{12} Add " In *Friedman B* it is shown that for r.e. axiomatizable extensions of **HA**, DP implies ED. ".

$94_{4,5}$ Replace by: "In *Luckhardt A* it is shown that the principle is equivalent to M. "

95 Add at the end of 1.11.6:

It has been noted by C.A. Smorynski that, for theories with decidable prime formulas, IP + M together amount to the principle of the excluded third. E.g. for **HA**, $\mathbf{HA} + \mathrm{IP} + \mathrm{M} = \mathbf{HA}^c$, which is seen as follows. Assume $A \vee \neg A$ to be proved already in $\mathbf{HA} + \mathrm{IP} + \mathrm{M}$, and consider $\exists x Ax$. By M, $\forall x(Ax \vee \neg Ax) \& \neg\neg\exists x Ax \to \exists x Ax$; by the induction hypothesis and IP, this implies $\exists x Ax \vee \neg\exists x Ax$. Application of propositional operators preserves decidability, and $\forall x Ax \leftrightarrow \neg\exists\neg Ax$ by the decidability of A, hence $(\forall x Ax \vee \neg\forall x Ax) \leftrightarrow (\neg\exists x \neg Ax \vee \neg\neg\exists x \neg Ax) \leftrightarrow \neg\exists x \neg Ax \vee \exists x \neg Ax$, hence $\forall x Ax \vee \neg\forall x Ax$.

95_4 Replace " $\alpha x = Uz$ " by " $\alpha y = Uz$ ".

98^6 Read " $x_1^{(\sigma)\tau}$ " for " $x^{(\sigma)\tau}$ ".

111^9 Read " $t_1 \equiv t_1'$ " for " $t_1 \equiv t'$ ".

113_4 Read " $\mathbf{H} \vdash t = \bar{n}$ ".

$114^{21,22}$ Delete the sentence beginning "For yet another ...".

117_6 Read "and t is" for "and τ is".

119^4 Read "…representing αn).".

125_1 Read " $(\sigma)(\tau)\sigma$ " for " $(\sigma)(\tau), \sigma$ ".

126^{10} Read "of" for "if".

$128^{12,13}$ The open problem has been solved by M. Bezem, in the sense that the two structures are isomorphic: J.S.L. 50 (1985), pp. 359–371.

129 Add between lines 6 and 7:

If we replace in the right hand side of this equivalence A by a predicate letter X, we have the inductive condition $B(X, x, y)$ characterizing A.

$132^{18,20}$ Replace " $\mathbf{I}$-$\mathbf{HA}^{\omega}$ " by " $\mathbf{I}$-$\mathbf{HA}^{\omega} + \mathrm{IE}_0$ ".

133^4 Delete ", CTM′, CTNF′ ".

$133^{10,11}$ Replace final comma by stop in line 10 and delete " CTM′, CTNF′ " in line 11.

$133_{5,4}$ Read " CTNF′ " for " CTNF ".

141^{10} Read " W^1_σ " for " I^1_σ ".

144^8 Read " (4) " for " (1) ".

147_1,148^1 Read " p_q " for " p_k ".

148^1 Insert before comma "& $\{z\}(p_q) = \{z\}(y)$ (since $z \in E(V)$, $y \in V^*$) ".

158^{13} Read " $t =_{\mathrm{e}} s$ " for " $t = s$ ".

158^{16} Read " CTNF " for " CTNF′ ".

158_{11} Read " $x^2[\Sigma(\Pi x^2)\Pi]$ ".

158_9 Read ", that $\Sigma(\Pi x^2_1)\Pi$ and ".

158_8 Read " $\Sigma(\Pi x^2_2)\Pi$ in the model ".

159^{12} Read "... which $x^1\bar{n}_i$ contr $\overline{\alpha n_i}$ has".

159_{14} Read " $t' \in 2$ " for " $t' \in z$ ".

$159_{1,2}$ – 160^{1-4} Replace these lines by:

$$t^2\alpha = \begin{cases} 0 \text{ if } \exists i(1 \leq i \leq k \;\&\; \bar{\alpha}_1(k+1) = \bar{\alpha}(k+1) \\ m+1 \text{ otherwise, where } m = \max\{\alpha_i(y) \,|\, 1 \leq i \leq k,\ 0 \leq y \leq k\}. \end{cases}$$

Now $Ft^2 \neq 0$; for, $\overline{(\pi_{0,0}0)}(k+1)$, $\ldots$, $\overline{(\pi_{0,0}k)}(k+1)$ are all distinct, hence one of them, say $\overline{(\pi_{0,0}k_0)}(k+1)$ $(0 \leq k_0 \leq k)$ is distinct from all $\bar{\alpha}_1(k+1)$, $\ldots$, $\bar{\alpha}_k(k+1)$, and therefore $t^2(\pi_{0,0}k_0) = m+1$; but then $\overline{(\lambda x^0.t^2(\pi_{0,0}x))}(k+1)$ differs from all $\bar{\alpha}_1(k+1)$, $\ldots$, $\bar{\alpha}_k(k+1)$, and thus Ft^2 takes the value $m+1$.

160_1 Read " $=$ " for " $\equiv$ ".

161^{12} Delete “not”.

164 Remark to be added at the end of 2.8.5: J.M.E. Hyland showed in his thesis that Scarpellini's model coincides with the model ECF ”.

167_7 Read “establishing” for “establish”.

173 Remark to be added in 2.9.10:

If a coding by functions is given for the elements of $\sigma^{\smile}$, such that there are continuous Φ_0, Φ with $\Phi_0\xi$ the length of the sequence coded by ξ, $\Phi(n,\xi)$ the n-th component extracted from ξ, then one can construct a bijection between two codings of this kind.”

179^{17} Read “ $\mathbf{HA}^{\mathrm{c}} + \mathrm{G}_1$.

180^6 Read “ $|F$ holds ” for “ $|$ holds ”.

180_{13} read “ P ” for “ D ”.

183_2 Read “ PCA). ” for “ P A). ”.

184^{17} Add after “ terms ”: “satisfying $(t \in V$ and $t' = t \Rightarrow t' \in V)$ ”.

188^{17} Read “mathematical”.

189^{12} Read “ $\& P(B(j_1 x))$ ” for “ $\& P(A(j_1 x))$ ”.

190_{17} Delete “, and $P(F_1^*), \ldots, P(F_n^*)$ ”.

190_{12} Delete “ Also $\forall x P(Bx)$ ”.

190_4 Delete “It follows that ... hence $P(C)$.”, and replace “Also” by “Then”.

192^1 Add after “hence”: “ $!t \,\&\, t \,\mathsf{r}_P\, A$ is an abbreviation for $(\exists x(t \simeq x \,\&\, x \,\mathsf{r}_P\, A).)$”.

192_{1-7} The argument given in the first edition is not correct. The result is a consequence of the unprovability in **HA** of the DP, which has been proved by J. Myhill (A note on indicator functions, Proc. Amer. math. Soc. 29 (1973), 181–183) and by Friedman in a stronger form (On the derivability of instantiation properties, J.S.L. 42 (1977), 506–514).

194_{10} Insert “ $\mathbf{HA} \vdash$ ” between “ (ii) ” and “ $A(\underline{\underline{a}}) \rightarrow\, !\psi(\underline{\underline{a}})\& \ldots$ ”.

194_9 Replace this line by

Proof. The “only if” part is established as follows. Assume $\vdash A\underline{\underline{a}} \leftrightarrow B\underline{\underline{a}}$, B almost negative. Then there is a recursive ϕ such that $\vdash \forall u(u \,\mathsf{r} A\underline{\underline{a}} \rightarrow\, !\{j_1\phi\underline{\underline{a}}\}(u)\ \&\ \{j_1\phi\underline{\underline{a}}\}(u) \,\mathsf{r} B\underline{\underline{a}})$, and $\vdash \forall u(u \,\mathsf{r} B\underline{\underline{a}} \rightarrow\, !\{j_2\phi\underline{\underline{a}}\}(u)\ \&\ \{j_2\phi\underline{\underline{a}}\}(u) \,\mathsf{r} A\underline{\underline{a}})$, which together with 3.2.11 for B readily yields the desired conclusion.

194_2 Read “ $Uv \,\mathsf{r}\, A\underline{\underline{a}}$ ” for “ $v \,\mathsf{r}\, A\underline{\underline{a}}$ ”.

198 Add after 3.2.22:

Remark. In the writings of the Russian constructivist school (cf. e.g. Dragalin 1969) one finds the following extension of CT_0:

$$\mathrm{CT}' \qquad \forall x(\neg Ax \to \exists y Bxy) \to \exists u \forall x(\neg Ax \to \exists v(Tuxv \,\&\, B(x, Uv))).$$

However, in the presence of M this is equivalent to ECT_0, i.e.

$$\mathbf{HA} + \mathrm{ECT}_0 + M = \mathbf{HA} + \mathrm{CT}' + M.$$

To see this, let us first assume CT', M, and let Ax be almost negative. then by M $Ax \leftrightarrow A'x$, A' negative, and hence $\neg\neg A'x \leftrightarrow Ax$ (1.10.8); thus an instance of ECT_0 can be interpreted as an instance of CT'.

Conversely, if ECT_0 and M are assumed, and we let $\forall x(\neg Ax \to \exists y Bxy)$, then by ECT_0, 3.2.8 $\neg Ax \leftrightarrow \exists z(z \,\mathsf{r}\, \neg Ax) \leftrightarrow \forall z(z \,\mathsf{r}\, \neg Ax) \leftrightarrow 0 \,\mathsf{r}\, \neg Ax$; $0 \,\mathsf{r}\, \neg Ax$ is almost negative. Replacing $\neg Ax$ by $0 \,\mathsf{r}\, \neg Ax$ we have $\forall x(0 \,\mathsf{r}\, \neg Ax \to \exists y Bxy)$ to which we can apply ECT_0 etc.

200–201 The claim of 3.2.26 is correct, but the proof given is incorrect. The attempted proof in list of Errata from 1974 is also flawed, but J. van Oosten presented me with a proof that IP_0 is derivable from ECT_0 and MP.

Here follows the proof: Assume

$$\forall x(Ax \vee \neg Ax), \;\; \forall x\, Ax \to \exists y\, B.$$

By ECT_0 there is a number n such that

$$\forall x(\{n\}x = 0 \leftrightarrow Ax), \;\text{ hence }\; \forall x(\{n\}x = 0) \to \exists y\, B.$$

Again by ECT_0 there is a number m such that

$$\forall x(\{n\}x = 0) \to !\{m\}0 \wedge B(\{m\}0).$$

Let k be a number such that

$$\{k\}j(a, b) = \min_x[\{a\}x \neq 0 \vee Tb0x.]$$

Since

$$\begin{aligned}
&\neg\neg(\exists x(\{n\}x \neq 0) \vee \forall x(\{n\}x = 0)), \;\text{ it follows that}\\
&\neg\neg(\exists x(\{n\}x \neq 0) \vee !\{m\}0, \;\text{ therefore}\\
&\neg\neg!\{k\}j(n, m); \;\text{ hence with MP }\; !\{k\}j(n, m).
\end{aligned}$$

From this we see, that by the definition of k

$$\begin{aligned}
&\{n\}(\{k\}j(n, m)) \neq 0 \to \neg\forall x\, Ax,\\
&T(m, 0, \{k\}j(n, m)) \to (\forall x\, Ax \to B(U(\{k\}j(n, m))).
\end{aligned}$$

Hence

$$\exists y(\forall x\, Ax \to By).$$

202^{6-13} The alternative argument for the non-realizability of formula (1) is erroneous.

203 Add after 3.2.29: "Friedman has shown (*Friedman B*) how to extend q-realizability by a similar trick.".

203_4 Read "*Cellucci*".

214_5 Replace "negative" by " $\exists$-free (i.e. not containing $\vee, \exists$) ".

$214_{4,3}$ Delete "on the convention ... omitted,".

215^1 Replace "negative" by " $\exists$-free".

215^9 Add " $\mathbf{N}$-$\mathbf{HA}^\omega$," after " $\mathbf{HA}^\omega$, ".

215^{11} Read "... sequence $\underline{t}$ of ... ".

215_{15} Replace "negative" by " $\exists$-free".

216^{20} Read " $\underline{\underline{y}}\,\mathsf{mr}_P\, A$ " for " $y\,\mathsf{mr}_P\, A$ ".

$217^{1,2,17}$ Read "$\underline{\underline{T}}$" for "$T$" (4 times).

217^{13} Replace this line by:

$$\mathrm{IP}^- \qquad (A \to \exists y^\sigma B) \to \exists y^\sigma(\neg A \to B)$$

(y^σ not free in A, A $\exists$-free, i.e. not containing $\vee, \exists$)

$217_{16,15}$ Delete ", taking for ... into account". Add after 3.4.7:

Remark. The schema

$$\mathrm{IP}^\omega \qquad (\neg A \to \exists y^\sigma B) \to \exists y^\sigma(\neg A \to B) \ \ (y^\sigma \text{ not free in } B).$$

is readily seen to be modified-realizable, hence $\mathbf{H} + \mathrm{IP}^- + AC \vdash \mathrm{IP}^\omega$. Since in systems with decidable prime formulae negative and $\exists$-free formulas coincide, and for negative A $\neg\neg A \leftrightarrow A$, we have in such cases also that IP^ω implies IP^-.

$217_{10,9,4,2,1}$ Replace "IP^ω " by " IP^- ".

217_{10} Read " $\mathbf{H}$ + " for " $\mathbf{H} \vdash$ ".

217_8 Add after line: "For $\mathbf{H} = \mathbf{HA}^\omega$, $\mathbf{I}$-$\mathbf{HA}^\omega$, $\mathbf{HRO}^-$, $\mathbf{E}$-$\mathbf{HA}^\omega$, IP^- may be replaced by IP^ω.".

217_5 Read "$\exists$-free" for "negative".

221_7 Read " M_{PR} " for " $\mathrm{MP}_{\mathrm{PR}}$ ".

221_2 Read "3.4.14" for "3.4.4".

222 Add after 3.4.14:

Remark. V.A. Lifschitz has shown (Proceedings of the American Mathematical Society 73 (1979), 101–106) that also $\mathbf{HA} + \mathrm{CT}_0! \nvdash \mathrm{CT}_0$, where

$\mathrm{CT}_0!$ $\quad \forall x \exists ! y A(x, y) \to \exists u \forall x \exists v (Tuxv \,\&\, A(x, Uv))$. ".

222_2 Read " $\forall\alpha\neg\neg\exists x$ " for " $\forall\alpha\neg\neg\exists z$.

223^1 Read "... was suggested by results contained in".

224_1,225^1 Read " IP^- " for " IP^ω ".

226_{16} Replace " for " by " . For ".

226_8 Insert after "... numbers" "(provably linear in **HA**)".

227 Add "($\prec$ provably linear in **HA**)".

228_{16} Read " $U^1_{j(n,i)}x$ " for " $U^1_{j(n,i)}$ ".

$228_{7,6}$ These lines must read respectively " $\ldots \equiv \forall X^* \forall D_X (x \,\mathsf{mr}\, A(X))$ " and " $\ldots \equiv \exists X^* \exists D_X (x \,\mathsf{mr}\, A(X))$ ".

229_{11} Read " of s^1 in HRO ".

230^5 Read "eliminating" for elementary".

233_5 Read " Π^0_2 " for " Π^0_z ".

$238^{4,6,7}$ Delete " $]^{\mathrm{D}}$ ".

239_4 Read " $(\underline{\underline{x}}, \underline{\underline{v}}, \underline{\underline{Zv}})$ " for " $(\underline{\underline{x}}, \underline{\underline{vZ}}, \underline{\underline{v}})$ ".

240^3 Read " $\underline{\underline{y}}$ " for " $\underline{\underline{Yx}}$ ".

242^{12} Read " $\vdash F^{\mathrm{D}}$ " for " $+F^{\mathrm{D}}$ ".

242^{17} Read "now" for "not".

$244_{7,6}$ Replace these lines by:

If we take everywhere X to be identically 1, we obtain the Dialectica interpretation.

245^{10} "Shoenfield" should be underlined.

251_4 Delete " $\mathbf{N\text{-}HA}^\omega$, " and add " ; $(\mathbf{N\text{-}HA}^\omega + \mathrm{IP}^- + \mathrm{AC}) \cap \Gamma_1 = \mathbf{N\text{-}HA}^\omega \cap \Gamma_1$ " (cf. the corrections to page 217).

255_6 Read " $\ldots \,\&\, A^*y)]$ ".

264^{16} Read " **HA** " for " HA ".

264_8 Read " $\to \exists x^\sigma A$ " for " $\to \neg\exists x^\sigma A$ ".

265^1 Read " $(z \neq 0 \to \neg A)$ " for " $(z \neq 0 \to A)$ ".

267_2 Read " $\exists y(y\,\mathsf{p}$ " for " $\exists(y\,\mathsf{p}$ ".

273_{17} Read " 3.9.13 " for " 3.9.11 ".

274^1 Read " 3.9.14 " for " 3.9.12 ".

274_4 Read " 3.9.15 " for " 3.9.13 ".

275_6 Delete " (".

278 Second proof tree under 4), read " Π " for the highest " Π_i ".

279 Replace in the first four proof trees exhibited the occurrences of "A" (but not the A in "A_1", "A_2", "Aa" or "$\exists x Ax$") by "B" (7 occurrences).

280 Replace under " 13) " " $A0$ " by " Aa ".

282^7 Read " $\Pi' \succ_1 \Pi''$, (without ... ".

282^{23} Add " of $\curlywedge_{\mathrm{I}}$ " at the end.

282 In the display at the bottom of the page, the first two lines should be

$$\begin{array}{l}\Pi\\ \underline{A \qquad 0=0}\end{array} \qquad\qquad \begin{array}{l}\Pi\\ \underline{A \qquad sa=sa}\end{array}$$

283 In the displayed prooftree read " $\&_1\mathrm{E}$ " for " $\&\mathrm{E}$ ".

284^1 read "form (*Prawitz*" for "(from *Prawitz*".

285 Immediately above the paragraph starting with "This makes it ..." read

$$\begin{array}{c}\Pi'\\ [A]\\ \Pi\\ B\end{array} \quad \text{for} \quad \begin{array}{c}\Pi\\ [A]\\ \Pi\\ B\end{array}$$

286 Replace last paragraph of 4.1.7. by:

For applications, we need only a normalization theorem (not a strong normalization theorem) relative to $\mathcal{R}_{\mathrm{c}\curlywedge}$; so if the reader wishes, he may use the preceding remark and delete everything in the proof below referring to $\curlywedge$-contractions.

287^{16} Read " $\mathrm{Prd}_1(\Pi), \mathrm{Prd}_2(\Pi), \ldots$ ".

287_5 Read " $\mathrm{SV}(\mathrm{Sub}(A, \Pi, \mathrm{Prd}(\Pi), \mathrm{Ass}(\Pi)))$ ".

287_2 Read " Π " for " Π_1 ".

288^9 Insert ", $\mathrm{Ass}(\Pi)$ " after " $\mathrm{Prd}_2(\Pi)$ ".

288 Directly above footnote, read

$$\frac{\Pi_i'}{A} \quad \text{for} \quad \frac{\Pi_i}{A}.$$

290^{17} Read " $\Pi' \in \mathrm{PRD}(\Pi)$ ".

290 In the first displayed prooftree, replace " At " by " At' ".

291 The second displayed prooftree should read:

$$\frac{\Pi_1' \ \ldots \ \Pi_n'}{A}.$$

292^{1} Read "condition IV " for " condition II ".

293 Second line of paragraph starting with "Condition IV for Π'…", read "for Π " instead of "for $\exists$ ".

294 Replace in the second display " Π_3 " by " Π_4 ", and in the line under this display, replace "Π_4" by "Π_5".

294 Replace in the third display "Π_3" by "Π_4". In the line under the third display, insert after "reduces to": "the left subdeduction of".

295^{6} Read " Π_6 " for " Π_3 ".

295 In the third display, place in the second proof tree " Π_0 " above " $\exists x Bx$ ".

295_{12} Insert between "condition" and "I": "IV, and hence".

295 In the line below the third display, insert before "is SV":
"and also

$$\begin{array}{c} \Pi_4 \\ [B_1 t'] \\ \Pi_1'(t') \\ [Dt] \\ \Pi_1(t) \\ A \end{array}$$

"

297^{17} Read " 2.2.25 " for " 2.2.31 ".

299^{15} Add " (major) " at the end.

300^{6} Add after the comma "which may be empty,".

$300^{9,10}$ Replace "preceding" by "succeeding".

301^{2} Read "were" for "would be".

301^{10} Add before ") ": " ; also, A_i cannot be discharged by IND, since no application of IND lies below A_1 ".

301_6 read " 4.2.8 " for " 2.8 ".

302^5 Read "normal" for "formal".

302^{14} Add " σ " at the end.

304^3 Insert "(by 4.2.7)" before " ; ".

304^{10} Read "were" for "would be".

304^{11} Read "or IND-application occurring" for "occurring".

304^{14} Replace " IND-application " by " begin with a conclusion of an IND-application".

304_{19} Read "Let Φ denote the ".

304 The proof in subsection 4.2.16 in the first edition contains a gap. Much simpler is the following argument:

PROOF. Note that Ψ is equivalent to a set of Harrop formulas: if $\exists x Px \in \Psi$, then we may replace this formula by $P\bar{n}$ for some $\bar{n}$ such that $\mathbf{HA} \vdash P\bar{n}$. Then we can apply 4.2.12.

305_1 delete " , or A_1 is a basic axiom ".

305^5 Read " $A_1 \in \Psi$, i.e. B is prime ".

305 Remark at 4.2.17: instead of referring to 3.6.7(ii), it suffices to note that only true closed Σ^0_1-formulae are provable in $\mathbf{HA}$ and $\mathbf{HA}^c$.

306, proof of 4.2.18. This proof is incorrect as it stands, since the conclusion of an IND-application is not necessarily atomic, only quantifier-free. The proof is correct if we replace in the statement of the theorem $\mathbf{H}$, qf-$\mathbf{HA}$ by the corresponding systems with induction for atomic formulas only.

To establish the theorem as stated, we can e.g. proceed as follows: define a path of order 0 to be a path $A_1, \dots, A_n$ with A_n conclusion of the deduction, and define a path of order $m+1$ to be a path $A_1, \dots, A_n$ such that either A_n is minor premiss of an $\to$E-application the major premiss of which belongs to a path of order m, or premiss of an IND-application the conclusion of which belongs to a path of order m.

In a strictly normal derivation, every formula occurrence belongs to some path of order m, for suitable m (since redundant applications of $\vee$E, $\exists$E have been removed). Then one readily proves, by induction on m, that for a strictly normal derivation of a quantifier-free formula in $\mathbf{H}$ all formula occurrences on a path of order m are quantifier-free. (Note that here normalization also w.r.t. permutative reductions is necessary, in contrast to other applications. This could have been avoided by reduction of qf-$\mathbf{HA}$ to a logic-free calculus, which is not a very elegant solution, however.)

306_{16} Read " 2.5.7 " for " 2.5.6 ".

307, line 2 below second display. Read " $\mathcal{R}_{\mathrm{c}}$ " for " $\mathcal{R}_{\mathrm{C}}$ ".

307_2 Read " 4.1.16 " for " 4.1.15 ".

308 Last line of first display, replace in proof tree Π'' " $\neg A \to Bx$ " by " $\neg A \to \exists x Bx$ ".

309 Replace second sentence of the statement of 4.3.5 by:

Then a spine of Π not ending with an introduction does not contain IP-applications, and ends with an application of a basic rule or an atomic instance of $\curlywedge_{\mathrm{I}}$.

Delete the third sentence.

The proof of 4.3.5. should be reformulated as follows:

Let $A_1, \dots, A_n$ be a spine of Π not ending with an introduction. Then, by 4.3.4(iii) there are two cases:
(1°) A_1 is a basic axiom. So the spine coincides with its minimum part, hence A_n is atomic.
(2°) A_1 is of the form $\neg B$, to be discharged by $\to$I, followed by IP. this case is excluded, for the sort of inference following A_1 can be (not IP, or $\to$I + IP, but) $\to$E only, leaving us with $A_2 \equiv \curlywedge$, and a minimum part $A_2, \dots, A_n$.

311^8 read "any one".

311^{17} Read "Red_1" for "Red".

311_{19} Read "of" for "from".

311_{12} Read " $\vee\mathrm{I_r}$ " for " $\vee\mathrm{I_l}$ ".

311_{11} Read " $\vee\mathrm{I_l}$ " for " $\vee\mathrm{I_r}$ ".

311_7 Read " $\mathrm{SV_{d-1}(Subst(Param(\Pi)},x,\mathrm{Prd}_1(\Pi)))]$ ".

313^1 Read " (ii) " for " (iii) ".

313_{12} Read " 4.4.3 " for " 4.4.2 ".

313_{10} Insert "(1.5.6)" before " : ".

314^4 Read " Proof_n " for " Proof ".

314_5 Read " 4.4.3 " for " 4.4.2 ".

314_3 Read " $\mathbf{HA} \vdash \forall x \exists yz(\mathrm{Proof}_n(y, \ulcorner A(\bar{x},\bar{z})\urcorner \,\&\, A(x,z))$ ".

321 As observed by S.Hayashi, (On derived rules of intuitionistic second order arithmetic, Commentarii Mathematici Universitatis Sancti Pauli 26 (1977), 77–103), the proof of 4.5.8 indicated in the text of the first edition establishes a result which is too weak, e.g.

$$\forall n \forall A \in \mathrm{Fm}^{(n)}(\vdash \mathrm{Sat}^{(n)}(X, \ulcorner \forall x A x \urcorner) \leftrightarrow \forall x \mathrm{Sat}^{(n)}(X, \ulcorner A(\bar{x})\urcorner))$$

instead of

$$\forall n(\vdash \forall A \in \mathrm{Fm}^{(n)}(\mathrm{Sat}^{(n)}(X, \ulcorner \forall x A x \urcorner) \leftrightarrow \forall x \mathrm{Sat}^{(n)}(X, \ulcorner A(\bar{x})\urcorner))).$$

Following Hayashi, the desired stronger conclusion can be established as follows. We first define the notion of a formation sequence of a formula A in $\mathrm{Fm}^{(n)}$.

DEFINITION. A *formation sequence* (fs) of $A \in \mathrm{Fm}^{(n)}$ is a finite sequence of quadruples $\langle a_0, b_0, c_0, t_0\rangle, \ldots, \langle a_m, b_m, c_m, t_m\rangle$ such that

(1) $t_m = \ulcorner A \urcorner$; t_0, and c_i for $1 \leq i \leq m$ are codes of formulas of complexity $\leq n$.
(2) $a_i \in \mathbb{N}$ for $0 \leq i \leq m$, $a_{i+1} \leq i$ for $0 \leq i < m$.
(3) $b_i, c_i \in \mathbb{N}$ for $0 \leq i \leq m$; t_{i+1} is the code of the term which is the result of substituting the term with code $t_{a_{i+1}}$ for the second-order variable $V^p_{b_{i+1}}$ in the formula (with index) c_{i+1} and logical complexity $\leq n$, where p is the number of free variables in $t_{a_{i+1}}$ (end of definition).

Now $\mathrm{Sat}_n(X, \ulcorner A \urcorner)$ is constructed as before. Let f, g, h range over formation sequences. We then define, similar to $\mathrm{Sat}^{(n)}(X, \ulcorner A \urcorner)$ of the text, and with help of Sat_n, the formula $\mathrm{Sat}^{(n)}_f(X, \ulcorner A \urcorner)$, where f is an fs for A with $t_m = \ulcorner A \urcorner$, and $\mathrm{Sat}^{(n)}_f$ is constructed parallel to the substitutions of f. Then one proves

LEMMA. In **HAS**

(i) $\forall f \forall A, B \in \mathrm{Fm}^{(n)} \exists g, h \forall X(\mathrm{Sat}^{(n)}_f(X, \ulcorner A \circ B \urcorner)$
$\leftrightarrow \mathrm{Sat}^{(n)}_g(X, \ulcorner A \urcorner) \circ \mathrm{Sat}^{(n)}_h(X, \ulcorner B \urcorner))$
for $\circ \in \{\rightarrow, \&, \vee\}$.
(ii) $\forall f \forall A \in \mathrm{Fm}^{(n)} \exists g \forall X(\mathrm{Sat}^{(n)}_f(X, \ulcorner Q v_i A(v_i)\urcorner) \leftrightarrow (Q v_i)\mathrm{Sat}^{(n)}_g(X, \ulcorner A(\bar{v}_i)\urcorner))$
for $Q \in \{\forall_1, \exists_1\}$.
(iii) $\forall f \forall A \in \mathrm{Fm}^{(n)} \exists g \forall X(\mathrm{Sat}^{(n)}_f(X, \ulcorner Q V^p_i A(V^p_i)\urcorner) \leftrightarrow$
$(QY^p)\forall Z^1(\forall y_1, y_2(j(y_1, y_2) \neq j(p, i) \rightarrow Z^1_{(y_1,y_2)} = X_{(y_1,y_2)}) \wedge Z^1_{(p,i)} = Y \rightarrow$
$\mathrm{Sat}^{(n)}_g(Z, \ulcorner A(V^p_i)\urcorner)$
for $Q \in \{\forall_2, \exists_2\}$.
(iv) $\forall X, f, g, n(\mathrm{FS}(f, n) \wedge \mathrm{FS}(g, n) \rightarrow \mathrm{Sat}^{(n)}_f(X, n) \leftrightarrow \mathrm{Sat}^{(n)}_g(X, n))$, where $\mathrm{FS}(f, n)$ expresses "f is a formation sequence of a formula A with $\ulcorner A \urcorner = n$".

Proof. The proof of (i)–(iii) by induction on the length of f; the proof of (iv) uses (i)–(iii) and induction on n.

We may then put

$$\mathrm{Sat}^{(n)}(X, \ulcorner A \urcorner) \leftrightarrow \exists f \mathrm{Sat}^{(n)}_f(X, \ulcorner A \urcorner)$$

and can then establish a stronger version of 4.5.8, namely

$$\forall n(\mathbf{HAS} \vdash \forall A \in \mathrm{Fm}^{(n)}(\mathrm{Sat}^{(n)}(X, \ulcorner \forall x A x \urcorner) \leftrightarrow \forall x \mathrm{Sat}^{(n)}(X, \ulcorner A \bar{x} \urcorner)))$$

etc. etc.

322^{13} read "choice" for "hoice".

375_{17} Read "$\alpha > \alpha_0$ &" for "$\alpha > \alpha_0$".

389 Subsection 5.7.3: more information about Kripke models for second-order intuitionistic arithmetic may be found in: D.H.J. de Jongh, C.A. Smoryński, Kripke models and the intuitionistic theory of species, Annals of Mathematical Logic 9 (1977), 157–186.

391_3 read "($\alpha x \neq 0$)" for "($\alpha x \neq 0$".

391_1 Add after ")" "in the presence of continuity axioms".

398_{12} Read "$\mathcal{L}[x]$" for "$\mathcal{L}$".

414_{11} Read "$S_2 f \in Y$" for "$S_1 f \in Y$".

422_{10} Read "$\mathbf{ID}_\nu^{\mathrm{c}}$" for "$\mathbf{ID}_\nu^{\mathrm{C}}$".

435^{10} Read "$|\mathbf{ID}_\nu^{\mathrm{c}}|$" for "$|\mathbf{ID}^{\mathrm{c}}|$".

438^5 Read "$P_1(Xn, n)]$" for "$P_1(Xn, x)]$".

439_{16} Read "type-0-valued".

440_6 Read "$\mathrm{Max}_1(\alpha, 0) = \alpha$" for "$\mathrm{Max}_1(\alpha, 0)$".

$448_{13,14}$ The equality in (7) of 6.9.1 was proved for all recursive ν by 1977, independently by Buchholz, Pohlers and Sieg, using various sophisticated proof-theoretic techniques (see W. Buchholz, S. Feferman, W. Pohlers, and W. Sieg, Iterated Inductive Definitions and Subsystems of Analysis: Recent Proof-Theoretical Studies. Springer Verlag, Berlin 1977). Hence the equalities

$$|\mathbf{ID}_2^{\mathrm{c}}| = |\mathbf{ID}_2| = |\mathbf{T}_2|$$

hold (end of 6.8.9). Hence also the equalities (5) and (6) of 6.9.1 are true.

451^{12-17} See the remark to page 448.

457_{14} Read "$(\lambda n.X_1 \dots X_k)$" for "$(\lambda X_1 \dots X_k)$".

462^1 Insert before the second line "Corrections in the bibliography consist sometimes in replacements, sometimes in added information between square brackets."

$462^{8,9}$ Replace by "LMPS IV: P.Suppes, L.Henkin, Gr.C. Moisil, A. Joja (eds.), North-Holland Publ. Co., Amsterdam 1973."

462^{17} Read "Cambridge Summer School in Mathematical Logic"

462^{18} Read: H.Rogers, A.R.D. Mathias (eds.), 1973.

462^{19} Add at the end ", 1973".

462^{20} Delete.

462_{15} Add "[Zeitschrift für mathematische Logik und Grundlagen der Mathematik 20 (1974), 289–306.]"

462_{14} Add: "[cf. H.P. Barendregt, Combinatory logic and the axiom of choice, Indagationes Mathematicae 35(1973), 203–221.]"

$462_{10,11}$ Replace by: " Theoretical Computer Science 3 (1977), 225–242.

462_5 Add "[J.S.L. 41 (1976), 328– 336]".

462_3 Add "[J.S.L. 40 (1975), 321– 346]".

462_1 Add "[J.S.L. 41 (1976), 18–24]".

463^1 Read " Cellucci".

463^{15} Read "Cambr. Proc. 1–94."

463_{25} Add "[Did not appear]"

463_{21} Read "Archiv für mathematische Logik 16 (1974), 49–66."

464^5 Read "Cambr. Proc. 113–170."

$464_2 3$ Read "Schliessen."

464_1 Add ", 232–252".

465^{21} Read "Cambr. Proc. 274–298."

465^{24} Read "J.S.L. 38 (1973), 453–459."

465^{26} Add "[Did not appear]"

465^{28} Add "[Did not appear]"

465^{30} Read "J.S.L. 41 (1976), 574–582."

466^{26} Read "Section VI" for "Section IV".

466_{14} Add: "[Appeared in: J.P.Seldin, J.R.Hindley (eds.), To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism. Academic Press, New York 1980, 480–490.]"

467^{22} Add: "[Never published]"

467^{24} Add: "[Appeared in: A.S. Troelstra, D. van Dalen (eds.), The L.E.J Brouwer Centenary Symposium. North-Holland Publ. Co., Amsterdam 1982, 51–64.]"

468^8 Read "Philosophica".

469^9 Read "in" for "in:".

469^{12} Read "Zentralblatt".

469^{26} Read "IPT" for Oslo Proc."

469_{24} Read " Π^1_1" for "Π".

469_8 Read "1" for "I".

470_{12} Add: "[Cf. paper under this title in: S. Kanger (ed.), Proceedings of the 3rd Scandinavian Logic Symposium, North-Holland Publ. Co., Amsterdam 1975, 81–109.]"

471^{10} Read: "Compositio Mathematica 26 (1973), 261–275."

472^5 Insert before stop: ", 225–250".

472_1 Replace by "Archiv für Mathematische Logik 16 (1974), 147–158."

473^{11} Add "[Unpublished]".

474_{11} Read "1970" for "170".

475^{13} Read "Cambr. Proc. 171–205."

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