

GTM22: AN ALGEBRAIC INTRODUCTION TO MATHEMATICAL LOGIC

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Errata

Exercise 1.3.8, p. 8, is incorrect. The class must include all the quotient rings. The exercise should read:

S is a commutative ring with 1. Show that the class of commutative rings R with $1_R = 1_S$ and which contain S as a subring, together with all their quotients, is a variety.

The proof of Theorem 1.4.3 on p. 9 is incomplete. The following should be inserted after the second paragraph of the proof:

Suppose (u, v) is an n -variable law of V . We show that R satisfies (u, v) . Let $r_1, \dots, r_n \in R$. Choose any $y_1, \dots, y_n \in F$ such that $\eta y_i = r_i$. Now let $f : F \rightarrow A$ be any homomorphism of F into an algebra A in V . The map $x_i \mapsto y_i$ defines a homomorphism $g : F_n \rightarrow F$ and we have a homomorphism $f \circ g : F_n \rightarrow A$. Since A satisfies (u, v) , $f \circ g(u(x_1, \dots, x_n)) = f \circ g(v(x_1, \dots, x_n))$, that is $f(u(y_1, \dots, y_n)) = f(v(y_1, \dots, y_n))$. Thus $u(y_1, \dots, y_n) \sim v(y_1, \dots, y_n)$ and so $u(r_1, \dots, r_n) = v(r_1, \dots, r_n)$. It follows that $R \in V$.

Definition III.1.3 on p. 18 only makes sense for a logic which has a distinguished element F . It should be replaced by:

Definition 1.3 a logic $\mathcal{L}$ is *consistent* if there exists a proposition p of $\mathcal{L}$ which is not a theorem.

For the logics studied in this book, this is equivalent to:

Definition 1.3' a logic $\mathcal{L}$ is *consistent* if F of $\mathcal{L}$ is not a theorem.

Insert after the first sentence of Exercise III.3.14 on p. 24:

Replace Axiom 3 with $(\sim q \Rightarrow \sim p) \Rightarrow (p \Rightarrow q)$.

If simple translations of the axioms are used, the resulting logic is not adequate.

The last sentence of the first paragraph of P. 76 should be replaced with:

For each n , we have the empty n -ary relation ϕ_n . The statement that $\rho \in \mathcal{O}^1(S)$ is an n -ary relation on S can be formalised as

$$\tau_n(\rho) = (\rho = \phi_n) \vee (\exists x_1) \dots (\exists x_n) (\in^1(S, x_1) \wedge \dots \wedge \in^1(S, x_n) \wedge \in^n(\rho, x_1, \dots, x_n)).$$

On p. 106 in the statement of Theorem X.1.2, “interations” should be “iterations”.

Solution of Exercise IV.1.5

Parts (i) and (ii) of Exercise IV.1.5 on p. 28 do not provide sufficient indication of the method of proof of (iii). Here is a solution.

Part (c)(ii) of the definition of $\approx$ can be stated in a more clearly symmetrical manner “ $w_1 = (\forall x)a, w_2 = (\forall y)b$ and there exists $c(t)$ with $x, y \notin V(c(t))$ such that $c(x) \approx a$ and $c(y) \approx b$ ”. Given $c(x)$ as in (c)(ii), we may take any $t \notin V(c(x)) \cup y$ and $c(t)$ has this property, while starting from such a $c(t)$, we have that $c(x)$ is as required for (c)(ii).

Observe first that if $w_1 \approx w_2$, then $d(w_1) = d(w_2)$. By induction over the depth of the elements, we prove:

- (i) (Strengthened form of (i) in book.) If $w_1 = (\forall x)a \approx w_2 = (\forall y)b$ where $y \neq x$ and $z \notin V(w_1)$, then there exists $c(x) \approx a, c(y) \approx b$ and $y, z \notin V(c(x))$,
- (ii) If $u(x) \approx v(x)$ and $y \notin V(u(x)) \cup V(v(x))$, then $u(y) \approx v(y)$, and
- (iii) If $w_1 \approx w_2 \approx w_3$, then $w_1 \approx w_3$.

We prove these results by a simultaneous induction, that is, to prove one of them for elements of depth k , we assume all of them for elements of depth less than k .

Proof. (i) There exists $u(x)$ with $y \notin V(u(x))$, $u(x) \approx a$ and $u(y) \approx b$. If $d(w_1) = 1$, then $d(u(x)) = 0$, so $u(x) = a, u(y) = b$ and as $z \notin V(u(x))$, we can take $c(x) = u(x)$. Suppose $d(w_1) = k > 1$. If $z \notin V(u(x))$, we can take $c(x) = u(x)$, so suppose $u = u(x, z)$. We can take some $t \notin V(u(x, z)) \cup V(a)$. Now $d(u) < k$, so we can apply (ii) to $u(x, z) \approx a(x, z)$. Thus $u(x, t) \approx a(x, t)$. But $a(x, t) = a$ since $z \notin V(a)$. Take $c = u(x, t)$. $\square$

Proof. (ii) If $d(u(x)) = 0$, then $u(x) = v(x)$, so $u(y) = v(y)$. Suppose $d(u(x)) > 0$. If $u(x) = (u_1(x) \Rightarrow u_2(x))$, then $v(x) = (v_1(x) \Rightarrow v_2(x))$ with $u_1(x) \approx v_1(x)$ and $u_2(x) \approx v_2(x)$. If the result holds for u_1, v_1, u_2, v_2 , we have $u_1(y) \approx v_1(y)$ and $u_2(y) \approx v_2(y)$ and it follows that $u(y) \approx v(y)$. Thus we may suppose that $u(x) = (\forall s)u_1(s, x)$ and $v(x) = (\forall t)v_1(t, x)$. We separate cases according to possible coincidences among s, t, x . By assumption, $y \notin V(u(x)) \cup V(v(x))$, so is distinct from s, t, x .

Suppose $s = t = x$. Then $u(x) = (\forall x)u_1(x), v(x) = (\forall x)v_1(x)$ and $u_1(x) \approx v_1(x)$. By induction, $u_1(y) \approx v_1(y)$ and so $u(y) = (\forall y)u_1(y) \approx (\text{forally})v_1(y) = v(y)$.

Suppose $s = t \neq x$. Then $u(x) = (\forall s)u_1(s, x), v(x) = (\forall s)v_1(s, x)$ and $u_1(s, x) \approx v_1(s, x)$. By induction, $u_1(s, y) \approx v_1(s, y)$ and it follows that $u(y) \approx v(y)$.

Suppose $s = x \neq t$. We have $u(x) = (\forall x)u_1(x) \approx v(x) = (\forall t)v_1(x, t)$. We have already proved (i) for elements of depth k , so there exists $c(x)$ with $y, t \notin V(c(x))$ such that $c(x) \approx u_1(x)$ and $c(t) \approx v_1(x, t)$. (This implies that $x \notin V(v_1(x, t))$.) By induction, we have $v_1(y, t) \approx c(t)$ and $u_1(y) \approx c(y)$. Thus $u(y) = (\forall y)u_1(y) \approx (\forall t)v_1(y, t) = v(y)$.

Now suppose s, t, x are distinct. By (i), there exists $c(s, x)$ with $c(s, x) \approx u_1(s, x), c(t, x) \approx v_1(t, x)$ and $y, t \notin V(c(s, x))$. By induction, $c(s, y) \approx u_1(s, y), c(t, y) \approx v_1(t, y)$ and so $u(y) \approx v(y)$. $\square$

Proof. (iii) Suppose $w_1 \approx w_2 \approx w_3$. If $w_1 = (a_1 \Rightarrow b_1)$, then $w_2 = (a_2 \Rightarrow b_2), w_3 = (a_3 \Rightarrow b_3)$ and $a_1 \approx a_2 \approx a_3$ and $b_1 \approx b_2 \approx b_3$. If the result holds for the a_i and b_i , then $a_1 \approx a_3$ and $b_1 \approx b_3$, so $w_1 \approx w_3$. Thus we may suppose $w_1 = (\forall x_1)a_1, w_2 = (\forall x_2)a_2$ and $w_3 = (\forall x_3)a_3$. Again we divide into cases of possible coincidences among x_1, x_2, x_3 .

If $x_1 = x_2 = x_3$, then we have $a_1 \approx a_2 \approx a_3$. By induction, $a_1 \approx a_3$ and $w_1 \approx w_3$.

If $x_1 = x_2 \neq x_3$, then we have $c(x_2)$ such that $x_3 \notin V(c(x_2))$, $c(x_2) \approx a_2$ and $c(x_3) \approx a_3$. But $a_1 \approx a_2$, so by induction, $c(x_2) \approx a_1$. As $x_2 = x_1$, we have $w_1 \approx w_3$.

Suppose $x_1 = x_3 \neq x_2$. Then there exists $u_2(x_2)$ with $x_1 \notin V(u_2(x_2))$, $u_2(x_2) \approx a_2$ and $u_2(x_1) \approx a_1$. Also there exists $v(x_2)$ with $x_3 \notin V(v(x_2))$, $v(x_2) \approx a_2$ and $v(x_3) \approx a_3$. By induction, $u_2(x_2) \approx v(x_2)$. By (ii), $u_2(x_1) \approx v(x_1) = v(x_3)$. By induction, $a_1 \approx a_3$ and so $w_1 \approx w_3$.

Now suppose x_1, x_2, x_3 are distinct. Since $x_1, x_3 \notin V(w_2)$, by (i), there exists $u(x_2)$ with $x_1, x_3 \notin V(u(x_2))$, $u(x_1) \approx a_1$ and $u(x_2) \approx a_2$. Also there exists $v(x_2)$ with $x_1, x_3 \notin V(v(x_2))$, $v(x_2) \approx a_2$ and $v(x_3) \approx a_3$. Thus $u(x_2) \approx a_2 \approx v(x_2)$ and by induction, we have $u(x_2) \approx v(x_2)$. Since $x_3 \notin V(u(x_2)) \cup V(v(x_2))$, by (ii), we have $u(x_3) \approx v(x_3) \approx a_3$. By induction, $u(x_3) \approx a_3$ and $w_1 \approx w_3$. $\square$

The point of the use of $\approx$ is that we can avoid such troublesome expressions as $(\forall x)p(x) \Rightarrow (\forall x(q(x)))$ by replacing the inner dummy variable, obtaining $(\forall x)(p(x) \Rightarrow (\forall y)q(y))$ whose meaning is clear. We need never repeat a dummy variable in an expression. This should have been pointed out more clearly in the book.

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