Preface

Stochastic control theory is a relatively young branch of mathematics. The beginning of its intensive development falls in the late 1950s and early 1960s. During that period an extensive literature appeared on optimal stochastic control using the quadratic performance criterion (see references in Wonham [76]). At the same time, Girsanov [25] and Howard [26] made the first steps in constructing a general theory, based on Bellman's technique of dynamic programming, developed by him somewhat earlier [4].

Two types of engineering problems engendered two different parts of stochastic control theory. Problems of the first type are associated with multistep decision making in discrete time, and are treated in the theory of discrete stochastic dynamic programming. For more on this theory, we note in addition to the work of Howard and Bellman, mentioned above, the books by Derman [8], Mine and Osaki [55], and Dynkin and Yushkevich [12].

Another class of engineering problems which encouraged the development of the theory of stochastic control involves time continuous control of a dynamic system in the presence of random noise. The case where the system is described by a differential equation and the noise is modeled as a time continuous random process is the core of the optimal control theory of diffusion processes. This book deals with this latter theory.

The mathematical theory of the evolution of a system usually begins with a differential equation of the form

$$\dot{x}_t = f(t,x_t)$$

with respect to the vector of parameters x of such a system. If the function $f(t,x)$ can be measured or completely defined, no stochastic theory is needed. However, it is needed if $f(t,x)$ varies randomly in time or if the errors of measuring this vector cannot be neglected. In this case $f(t,x)$ is, as a rule, representable as $b(t,x) + \sigma(t,x)\dot{\xi}_t$ where b is a vector, σ is a matrix, and ξ_t is a random vector process. Then

$$\dot{x}_t = b(t,x_t) + \sigma(t,x_t)\dot{\xi}_t. \tag{1}$$

It is convenient to write the equation in the integral form

$$x_t = x_0 + \int_0^t b(t,x_s)\,ds + \int_0^t \sigma(s,x_s)\,d\xi_s, \tag{2}$$

where x_0 is the vector of the initial state of the system. We explain why Eq. (2) is preferable to Eq. (1). Usually, one tries to choose the vector of parameters x_t of the system in such a way that the knowledge of them at time t enables one to predict the probabilistic behavior of the system after time t with the same certainty (or uncertainty) to the same extent as would knowledge of the entire prior trajectory x_s $(s \leq t)$. Such a choice of parameters is convenient because the vector x_t contains all the essential information about the system. It turns out that if x_t has this property, it can be proved under rather general conditions that the process ξ_t in (2) can be taken to be a Brownian motion process or, in other words, a Wiener process $\mathbf{w}_t$. The derivative of ξ_t is then the so-called "white noise," but, strictly speaking, $\dot{\xi}_t$ unfortunately cannot be defined and, in addition, Eq. (1) has no immediate meaning. However, Eq. (2) does make sense, if the second integral in (2) is defined as an Ito stochastic integral.

It is common to say that the process x_t satisfying Eq. (2) is a diffusion process. If, in addition, the coefficients b, σ of Eq. (2) depend also on some control parameters, we have a "controlled diffusion process."

The main subject matter of the book having been outlined, we now indicate how some parts of optimal control theory are related to the contents of the book.

Formally, the theory of deterministic control systems can be viewed as a special case of the theory of stochastic control. However, it has its own unique characteristics, different from those of stochastic control, and is not considered here. We mention only a few books in the enormous literature on the theory of deterministic control systems: Pontryagin, Boltyansky, Gamkrelidze, and Mishchenko [60] and Krassovsky and Subbotin [27].

A considerable number of works on controlled diffusion processes deal with control problems of linear systems of type (2) with a quadratic performance criterion. Besides Wonham [76] mentioned above, we can also mention Astrom [2] and Bucy and Joseph [7] as well as the literature cited in those books. We note that the control of such systems necessitates the construction of the so-called Kalman–Bucy filters. For the problems of the application of filtering theory to control it is appropriate to mention Lipster and Shiryayev [51].

Since the theory of linear control systems with quadratic performance index is represented well in the literature, we shall not discuss it here.

Control techniques often involve rules for stopping the process. A general and rather sophisticated theory of optimal stopping rules for Markov chains and Markov processes, developed by many authors, is described by Shiryayev [69]. In our book, problems of optimal stopping also receive considerable attention. We consider such problems for controlled processes with the help of the method of randomized stopping. It must be admitted, however, that our theory is rather crude compared to the general theory presented in [69] because of the fact that in the special case of controlled diffusion processes, imposing on the system only simply verifiable (and therefore crude) restrictions, we attempt to obtain strong assertions on the validity of the Bellman equation for the payoff function.

Concluding the first part of the Preface, we emphasize that in general the main aim of the book is to prove the validity of the Bellman differential equations for payoff functions, as well as to develop (with the aid of such equations) rules for constructing control strategies which are close to optimal for controlled diffusion processes.

A few remarks on the structure of the book may be helpful. The literature cited so far is not directly relevant to our discussion. References to the literature of more direct relevance to the subject of the book are given in the course of the presentation of the material, and also in the notes at the end of each chapter.

We have discussed only the main features of the subject of our investigation. For more detail, we recommend Section 1, of Chapter 1, as well as the introductions to Chapters 1–6.

The text of the book includes theorems, lemmas, and definitions, numeration of which is carried out throughout according to a single system in each section. Thus, the invoking of Theorem 3.1.5 means the invoking of the assertions numbered 5 in Section 1 in Chapter 3. In Chapter 3, Theorem 3.1.5 is referred to as Theorem 1.5, and in Section 1, simply as Theorem 5. The formulas are numbered in a similar way.

The initial constants appearing in the assumptions are, as a rule, denoted by K_i, δ_i. The constants in the assertions and in the proofs are denoted by the letter N with or without numerical subscripts. In the latter case it is assumed that in each new formula this constant is generally speaking unique to the formula and is to be distinguished from the previous constants. If we write $N = N(K_i, \delta_i, \ldots)$, this means that N depends only on what is inside the parentheses. The discussion of the material in each section is carried out under the same assumptions listed at the start of the section. Occasionally, in order to avoid the cumbersome formulation of lemmas and theorems, additional assumptions are given prior to the lemmas and theorems rather than in them.

Reading the book requires familiarity with the fundamentals of stochastic integral theory. Some material on this theory is presented in Appendix 1. The Bellman equations which we shall investigate are related to nonlinear partial differential equations. We note in this connection that we do not assume the reader to be familiar with the results related to differential equation theory.

In conclusion, I wish to express my deep gratitude to A. N. Shiryayev and all participants of the seminar at the Department of Control Probability of the Interdepartmental Laboratory of Statistical Methods of the Moscow State University for their assistance in our work in this book, and for their useful criticism of the manuscript.

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