Preface

Function spaces have a long history. They play an important part in both classical and modern mathematics. Spaces whose elements are continuous, or differentiable, or p-integrable functions are of interest for their own sake. They are also useful tools for the study of ordinary and partial differential equations (although they have some shortcomings for that purpose). Since the thirties more sophisticated function spaces have been used in the theory of partial differential equations, in the first place the Hölder spaces and the Sobolev spaces. Later on (culminating in the fifties and in the sixties) many new spaces were created and investigated, e. g. Besov spaces (Lipschitz spaces), Bessel-potential spaces (Liouville spaces, Lebesgue spaces), Zygmund classes, Hardy spaces (real n-dimensional version) and the space BMO. In the sixties and seventies, deep new tools were discovered: interpolation theory and the methods of the socalled "hard analysis" (Fourier analysis, maximal inequalities, etc.). The main aim of this book is to present a unified theory of function spaces of the above-mentioned types from the standpoint of Fourier analysis. (In [I] we tried to give a corresponding representation of some of the above types of spaces on the basis of interpolation theory.) Although many results have been proved very recently, the theory seems now to be at a stage which justifies the writing of a book about this subject. The two books [F] and [S] of the author may be considered as forerunners. But they are more or less research reports written at a moment when some of the most important problems of the above theory were unsolved (in contrast to the present situation). However, we do incorporate some material from [F] and [S]. In this sense, this book also may be considered as a new and thoroughly revised edition of some parts of [F] and [S].

The heart of the book is Part I, in particular Chapter 2. In this chapter we give a systematic study of the spaces $B^s_{p,q}(R_n)$ and $F^s_{p,q}(R_n)$, where $s \in R_1$, $0<p\leqq\infty$, and $0<q\leqq\infty$, on the n-dimensional Euclidean space R_n, which is based on methods of Fourier analysis. These two scales $B^s_{p,q}(R_n)$ and $F^s_{p,q}(R_n)$ contain as special cases all the above-mentioned classical spaces (local versions in the case of the Hardy spaces and the space BMO). The preceding Chapter 1 deals with entire analytic functions on R_n and sequences of those functions which belong to $L_p(R_n)$ and $L_p(R_n, l_q)$, respectively. This theory is of interest in itself, but it is also useful for Chapter 2 (the technique of maximal functions is used from the very beginning). Chapter 3 concerns the theory of the spaces $B^s_{p,q}(\Omega)$ and $F^s_{p,q}(\Omega)$ on bounded smooth domains Ω in R_n. Finally, in Chapter 4 we extend the well-known theory of boundary value problems for regular elliptic differential equations in Hölder spaces and in L_p-spaces (where $1<p<\infty$) to the spaces $B^s_{p,q}(\Omega)$ and $F^s_{p,q}(\Omega)$ on domains. In Part II we discuss briefly further types of spaces: homogeneous spaces $\dot{B}^s_{p,q}(R_n)$ and $\dot{F}^s_{p,q}(R_n)$ with the Hardy spaces (n-dimensional real version) and the space BMO as special cases; weighted spaces of the above type on R_n and on domains; and corresponding spaces on the n-torus. We sketch a few applications to degenerate elliptic differential equations and to some problems for multiple trigonometric series. In connection with weighted spaces we give a brief introduction to the theory of the Beurling ultra-distributions.

The styles of Part I and Part II are different. In Part I we give detailed proofs of the main assertions, whereas we restrict ourselves in Part II to formulations, hints and references. In many cases we remark that with the help of the indicated modifications one can carry over corresponding proofs from Part I. The main assertions of Part I are those which are needed for the study of the spaces $B^s_{p,q}(\Omega)$ and $F^s_{p,q}(\Omega)$ on domains Ω and their applications to regular elliptic boundary value problems. In many cases proofs of other assertions (also in Part I) are only outlined or omitted. It is clear that Part II is chiefly directed to specialists who work in that field of research. On the other hand, we hope that Part I is not only of interest to specialists in the theory of function spaces, Fourier analysis, and partial differential equations, but also to mathematicians, physicists, and graduate students, who use function spaces as a tool or who wish to have some information about this subject.

Of course, it is not the aim of the book to give an exhaustive treatment of the theory of function spaces in the widest sense of the word. We are primarily interested in demonstrating the power of Fourier analysis (in particular of Fourier multiplier theorems and of the technique of maximal functions) in connection with the above-mentioned spaces. This branch of the theory of function spaces was established at the end of the sixties and the beginning of the seventies. Above all we have in mind the papers by C. Fefferman, E. M. Stein [2] and by J. Peetre [2, 4, 6] and Peetre's book [8] (of course, these papers have their own forerunners: in particular, maximal functions have a long history, cf. e. g., E. M. Stein [1]).

The reader is expected to have a working knowledge of functional analysis as presented in the classical textbooks (including the standard facts of the theory of distributions). A familiarity with the basic results of the spaces of differentiable functions, L_p-spaces and Sobolev spaces would be helpful. In this connection we refer to A. Kufner, O. John, S. Fučik [1].

In recent years several books have appeared which deal with function spaces of the above type from diverse points of view. Beside the classical book by S. L. Sobolev [4] and the cited books by E. M. Stein [1], J. Peetre [8], A. Kufner, O. John, S. Fučik [1] and the author [I, F, S], there are books by R. A. Adams [1], S. M. Nikol'skij [3], O. V. Besov, V. P. Il'jin, S. M. Nikol'skij [1], W. Mazja [1], and A. Kufner [1]. Furthermore, we refer to L. Hörmander [2], J.-L. Lions, E. Magenes [2] and E. M. Stein, G. Weiss [2].

The book is organized on the decimal system. "Theorem 2.2.2/1" refers to Theorem 1 in Subsection 2.2.2., "Theorem 2.2.3." means the theorem in 2.2.3., etc. All unimportant positive constants will be denoted by c (with additional indices if there are several c's in the same estimate). [I], [F], and [S] refer to the books of the author, cf. the list of references.

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