NON-SEPARABLE AND PLANAR GRAPHS*

BY

HASSLER WHITNEY

Introduction. In this paper the structure of graphs is studied by purely combinatorial methods. The concepts of rank and nullity are fundamental. The first part is devoted to a general study of non-separable graphs. Conditions that a graph be non-separable are given; the decomposition of a separable graph into its non-separable parts is studied; by means of theorems on circuits of graphs, a method for the construction of non-separable graphs is found, which is useful in proving theorems on such graphs by mathematical induction. In the second part, a dual of a graph is defined by combinatorial means, and the paper ends with the theorem that a necessary and sufficient condition that a graph be planar is that it have a dual.

The results of this paper are fundamental in papers by the author on *Congruent graphs and the connectivity of graphs*† and on *The coloring of graphs*.‡

I. Non-separable graphs

1. **Definitions.**§ A *graph* G consists of two sets of symbols, finite in number: *vertices*, $a, b, c, \cdots, f$, and *arcs*, $\alpha(ab), \beta(ac), \cdots, \delta(cf)$. If an arc $\alpha(ab)$ is present in a graph, its *end vertices* a, b are also present. We may write an arc $\alpha(ab)$ or $\alpha(ba)$ at will; we may write it also ab or ba if no confusion arises,—if there is but a single arc joining a and b in G. We say the vertices a and b are *on* the arc $\alpha(ab)$, and the arc $\alpha(ab)$ is *on* the vertices a and b. The *null graph* is the graph containing no arcs or vertices.

The obvious geometrical interpretation of such a graph, or *abstract graph*, is a *topological graph*, let us say. Corresponding to each vertex of the abstract graph, we select a point in 3-space, a vertex of the topological graph. Corresponding to each arc $\alpha(ab)$ of the abstract graph, we select an arc joining the corresponding vertices of the topological graph. An arc is here a set of points in (1, 1) correspondence with the unit interval, its end vertices corresponding

* Presented to the Society, October 25, 1930; received by the editors February 2, 1931. An outline of this paper will be ound in the Proceedings of the National Academy of Sciences, vol. 17 (1931), pp. 125–127.

† American Journal of Mathematics, vol. 54 (1932), pp. 150–168.

‡ An outline will be found in the Proceedings of the National Academy of Sciences, vol. 17 (1931), pp. 122–125.

§ Compare Ste. Laguë, *Les Rêseaux*, Mémorial des Sciences Mathématiques, fascicule 18, Paris, 1926.

with the ends of the interval. Moreover, we let no arc pass through other vertices or intersect other arcs. We shall consider topological graphs no further till we come to the section on planar graphs.

An *isolated vertex* is a vertex which is not on any arc. A *chain* is a set of one or more distinct arcs which can be ordered thus: ab, bc, cd, $\cdots$, ef, where vertices in different positions are distinct, i.e. the chain may not intersect itself. A *suspended chain* is a chain containing two or more arcs such that no vertex of the chain other than the first and last is on other arcs, and these two vertices are each on at least two other arcs. A *circuit* is a set of one or more distinct arcs which can be put in cyclic order, ab, bc, $\cdots$, ef, fa, vertices being distinct as in the case of the chain. A *k-circuit* is a circuit containing k arcs. Thus, the arc $\alpha(aa)$ is a 1-circuit; the two arcs $\alpha(ab)$, $\beta(ab)$ form a 2-circuit.

A graph is *connected* if any two of its vertices are joined by a chain. Obviously, if a and b are joined by a chain, and b and c are joined by a chain, then a and c are joined by a chain. Any graph consists of a certain number of *connected pieces* (one, if the graph is connected). In particular, an isolated vertex is one of the connected pieces of a graph. A graph is called *cyclicly connected* if any two of its vertices are contained in a circuit. If G_1, G_2, $\cdots$, G_m are a set of graphs, no two of which have a common vertex (or arc, therefore), we say the graph G, formed of the arcs and vertices of all these graphs, is the *sum* of these graphs. Thus, a graph is the sum of its connected pieces. A *forest* is a graph containing no circuit. A *tree* is a connected forest. A *subgraph H* of G is a graph containing a subset (in particular, all or none), of the arcs of G, and those vertices of G which are on these arcs.

2. **Rank and nullity.*** Given a graph G which contains V vertices, E arcs, and P connected pieces, we define its *rank R*, and its *nullity* (or *cyclomatic number* or first Betti number) N, by the equations

$$R = V - P,$$

$$N = E - R = E - V + P.$$

If G contains the single arc ab, it is of rank 1, nullity 0, while if it contains the single arc aa, it is of rank 0, nullity 1.

The first two theorems follow immediately from the definitions of rank and nullity:

THEOREM 1. *If isolated vertices be added to or subtracted from a graph, the rank and nullity remain unchanged.*

* These are just the rank and nullity of the matrix H_1 of Poincaré. See Veblen's Colloquium Lectures, *Analysis Situs*.

THEOREM 2. *Let the graph G' be formed from the graph G by adding the arc ab. Then*

(1) *if a and b are in the same connected piece in G, then*

$$R' = R, \quad N' = N + 1;$$

(2) *if a and b are in different connected pieces in G, then*

$$R' = R + 1, \quad N' = N.$$

THEOREM 3. *In any graph G,*

$$R \geqq 0, \quad N \geqq 0.$$

For let G_1 be the graph containing the vertices of G but no arcs. Then if R_1 and N_1 are its rank and nullity,

$$R_1 = N_1 = 0.$$

We build up G from G_1 by adding the arcs one at a time. The theorem now follows from Theorem 2.

THEOREM 4. *A forest G is a graph of nullity* 0, *and conversely.*

Suppose first G contained a circuit P. We shall show that the nullity of G is >0. We build up G arc by arc, adding first the arcs of the circuit P. In adding the last arc of the circuit, the nullity is increased by 1, as this arc joins two vertices already connected. (This argument holds even if the circuit is a 1-circuit.) But in adding the rest of the arcs, the nullity is never decreased, by Theorem 2. Thus the nullity of G is >0.

Now suppose G is a forest, and therefore contains no circuit. Build up G arc by arc. Each arc we add joins two vertices formerly not connected. For otherwise, this arc, together with the arcs of a chain connecting the two vertices, would form a circuit. Therefore, by Theorem 2, the nullity remains always the same, and is thus 0.

3. **Theorems on non-separable graphs.** We introduce the following

Definitions. Let H_1, which contains the vertex a_1, and H_2, which contains the vertex a_2, be two graphs without common vertices. Let us rename a_1 a, and rename the arcs of H_1 on a_1 accordingly; that is, if a_1b is an arc on a_1, we rename it ab. Rename also a_2 a, and rename the arcs of H_2 accordingly. H_1 and H_2 have now the vertex a in common; they form the graph G, say. We say G is formed by letting the vertex a_1 of H_1 *coalesce* with the vertex a_2 of H_2, or, by joining H_1 and H_2 at a vertex. Geometrically, we pull the vertices a_1 and a_2 together to form the single vertex a.

Let G be a connected graph such that there exist no two graphs H_1 and

H_2, each containing at least one arc, which form G if they are joined at a vertex. Then G is called *non-separable.* Geometrically, a connected graph is non-separable if we cannot break it at a single vertex into two graphs, each containing an arc. For example, the graph consisting of the two arcs ab, bc is separable, as is the graph consisting of the two arcs $\alpha(aa)$, $\beta(aa)$. A graph containing but a single arc is non-separable, as is the graph containing only the arcs $\alpha(ab)$, $\beta(ab)$.

If G is not non-separable, we say G is *separable.* Thus, a graph that is not connected is separable. Suppose some connected piece G_1 of G is separable. If H_1 and H_2 joined at the vertex a form G_1, we say a is a *cut vertex* of G. We have consequently

THEOREM 5. *A necessary and sufficient condition that a connected graph be non-separable is that it have no cut vertex.*

THEOREM 6. *Let G be a connected graph containing no* 1*-circuit. A necessary and sufficient condition that the vertex a be a cut vertex of G is that there exist two vertices b, c in G, each distinct from a, such that every chain from b to c passes through a.*

First suppose a is a cut vertex of G. Then, by definition, H_1 and H_2, each containing at least one arc which is not a 1-circuit, form G if they are joined at a. Let b be a vertex of H_1 and c a vertex of H_2, each distinct from a. As a is the only vertex in both H_1 and H_2, every chain from b to c in G passes through a.

Suppose now every chain from b to c in G passes through a. Remove the vertex a and all the arcs on a. The resulting graph G' is not connected, b and c being in different connected pieces. Let H_1' be that connected piece of G' containing b, and let H_2' be the rest of G'. Replace a by the two vertices a_1 and a_2. Now put back the arcs we removed, letting them touch a_1 if their other end vertices are in H_1', and letting them touch a_2 otherwise. Let H_1 and H_2 be the resulting graphs. Then H_1 and H_2 each contain at least one arc, and they form G if the two vertices a_1, a_2 are made to coalesce. Hence, by definition, a is a cut vertex of G.

THEOREM 7. *Let G be a graph containing no* 1*-circuit and containing at least two arcs. A necessary and sufficient condition that G be non-separable is that it be cyclicly connected.**

If G is not connected, the theorem is obvious. Assume therefore G is connected.

* A similar theorem has been proved for more general continuous curves by G. T. Whyburn, Bulletin of the American Mathematical Society, vol. 37 (1931), pp. 429–433.

Suppose first G is separable. Then, by Theorem 5, G has a cut vertex a, and by Theorem 6, there are two vertices b, c in G such that every chain from b to c passes through a. Hence there is no circuit in G containing b and c.

Suppose now there exist two vertices b, c in G which are contained in no circuit. Let $bd, de, \cdots, gc$ be some chain from b to c.

Case 1. There exists a circuit containing b and d. In this case, let a be the last vertex of the chain which is contained in a circuit passing also through b. Let f be the next vertex of the chain. Then every chain from f to b passes through a. For suppose the contrary. Let C be a chain from f to b not passing through a. Let P be a circuit containing b and a. Follow C from f till we first reach a vertex of P. Follow the circuit P now as far as b if b was not the vertex we reached, and continue along P till we reach a. Passing from a to f along the arc af completes a circuit containing both b and f, contrary to hypothesis. Hence, by Theorem 6, a is a cut vertex of G, and therefore G is separable.

Case 2. There exists no circuit containing b and d. Then there is but a single arc joining b and d, and they are joined by no other chain. As G is connected and contains at least two arcs, there is either another arc on b or another arc on d, say the first. The other case is exactly similar. If we add a vertex b' and replace the arc bd by the arc $b'd$, b and d are no longer joined by a chain, and hence the resulting graph G' is not connected. Let H_1 be that part of G' containing the arc $b'd$, and let H_2 be the rest of G'. As there is still an arc on b, H_2 contains at least one arc. Letting the vertices b and b' coalesce forms G, and hence G is separable. The proof is now complete.

THEOREM 8. *A non-separable graph G containing at least two arcs contains no* 1-*circuit and is of nullity* >0. *Each vertex is on at least two arcs.*

Suppose G contained a 1-circuit. Call it H_1. Let H_2 be the rest of the graph. Then H_1 and H_2 have but a single vertex in common, and thus G is separable.

Next, by Theorem 7, G is cyclicly connected. As G contains no 1-circuit, G contains at least two vertices. Containing these there is a circuit. Therefore, by Theorem 4, the nullity of G is >0.

Finally, if there were a vertex on no arcs, G would not be connected. If there were a vertex a on the single arc ab, b would be a cut vertex of G.

THEOREM 9. *Let G be a graph of nullity* 1 *containing no isolated vertices, such that the removal of any arc reduces the nullity to* 0. *Then G is a circuit.*

By Theorem 4, G contains a circuit. Suppose G contained other arcs besides. Removing one of these, the nullity remains 1, as the circuit is still present, contrary to hypothesis. There are no other vertices in G, as G contains no isolated vertices. Hence G is just this circuit.

THEOREM 10. *A non-separable graph G of nullity* 1 *is a circuit.*

If G contains but a single arc, it is a 1-circuit, being of nullity 1. Suppose G contains at least two arcs. By Theorem 8, it contains no 1-circuit. By Theorem 7, it is cyclicly connected. Remove any arc ab from G; a and b are still connected, and therefore, by Theorem 2, the nullity of G is reduced to 0. Hence, by Theorem 9, G is a circuit.

The converses of the last two theorems are obviously true.

4. **Decomposition of separable graphs.** If the graph G contains a connected piece which is separable, we may separate that piece into two graphs, these graphs having formerly but a single vertex in common. We may continue in this manner until every resulting piece of G is non-separable. We say G is separated into its *components*.

LEMMA. *Let the connected separable graph G be decomposed into the two pieces H_1 and H_2 which had only the vertex a in common in G. Then every non-separable subgraph of G is contained wholly in either H_1 or H_2.*

Suppose the contrary. Then some non-separable subgraph I of G is not contained wholly in either H_1 or H_2. Let I_1 be that part of I in H_1, and I_2 that part in H_2; I_1 and I_2 have at most the vertex a in common. I_1 and I_2 each contain at least one arc. For otherwise, if I_1, say, contained no arc, as it contains a vertex distinct from a, it would not be connected. Thus I is separable into the pieces I_1 and I_2, a contradiction again.

THEOREM 11. *Every non-separable subgraph of G is contained wholly in one of the components of G.*

This follows upon repeated application of the above lemma.

THEOREM 12. *A graph G may be decomposed into its components in a unique manner.*

Suppose we could decompose G into the components $H_1, H_2, \cdots, H_m$, and also into the components $H_1', H_2', \cdots, H_n'$. We shall show that these sets are identical. Take any H_i. It is a non-separable subgraph of G, and thus is contained in some component H_j', by Theorem 11. Similarly, H_j' is contained in some component H_k. Thus H_i is contained in H_k, and they are therefore identical. Hence H_i and H_j' are identical. In this manner we show that each H_k is identical with some H_l', and each H_l' is identical with some H_k, proving the theorem.

THEOREM 13. *Let $H_1, H_2, \cdots, H_m$ be the components of G. Let $R_1, R_2, \cdots, R_m$, and $N_1, N_2, \cdots, N_m$ be their ranks and nullities. Then*

$$R = R_1 + R_2 + \cdots + R_m,$$

$$N = N_1 + N_2 + \cdots + N_m.$$

Let G' be G separated into its components, and let R' be the rank of G'. G is formed from G' by letting vertices of different components coalesce. Each time we join two pieces, the number of vertices and the number of connected pieces are each reduced by 1, so that the rank remains the same. Thus

$$R = R'.$$

Now

$$V' = V_1 + V_2 + \cdots + V_m,$$

$$P' = P_1 + P_2 + \cdots + P_m$$

(where each $P_i = 1$). Subtracting,

$$R = R' = R_1 + R_2 + \cdots + R_m.$$

As also

$$E = E_1 + E_2 + \cdots + E_m,$$

it follows that

$$N = N_1 + N_2 + \cdots + N_m.$$

For a converse of this theorem, see Theorem 17.

THEOREM 14. *Divide the arcs of the non-separable graph G into two groups, each containing at least one arc, forming the subgraphs H_1 and H_2, of ranks R_1 and R_2. Then*

$$R_1 + R_2 > R.$$

Let the connected pieces of H_1 be $H_{11}, \cdots, H_{1m}$ (there may be but one piece, H_{11}), and let those of H_2 be $H_{21}, \cdots, H_{2n}$. Then obviously

$$R_1 = R_{11} + \cdots + R_{1m},$$

$$R_2 = R_{21} + \cdots + R_{2n},$$

whence

$$R_1 + R_2 = R_{11} + \cdots + R_{1m} + R_{21} + \cdots + R_{2n}.$$

Let G' be the sum of the graphs $H_{11}, \cdots, H_{2n}$. Then G' is of rank $R_{11} + \cdots + R_{2n}$. We form G from G' by letting vertices of the graphs $H_{11}, \cdots, H_{2n}$ coalesce. Each time we let vertices of different connected pieces coalesce, the rank is unaltered. Each time we let vertices in the same connected piece coalesce, the rank is reduced by 1. This latter operation happens at least once. For otherwise, let a_1 and a_2 be the last two vertices we let coalesce. Then a_1 and a_2 were formerly in two different pieces, I_1 and I_2. Thus

I_1 and I_2 joined at a vertex form G, and G is separable, contrary to hypothesis. Thus the rank of G is less than the rank of G', that is,

$$R < R_{11} + \cdots + R_{2n}.$$

Hence

$$R_1 + R_2 > R.^*$$

Theorems 13 and 14 give

THEOREM 15. *A necessary and sufficient condition that a graph be non-separable is that there exist no division of its arcs into two groups H_1 and H_2, each containing at least one arc, so that*

$$R = R_1 + R_2.$$

5. **Circuits of graphs.** We shall say two non-separable graphs, each containing at least one arc, form a *circuit of graphs,* if they have at least two common vertices. (They may also have common arcs.) Thus the two graphs G_1: $\alpha(ab)$ and G_2: $\alpha(ab)$ (which are the same graph) form a circuit of graphs. However, the two graphs G_1: $\alpha(aa)$ and G_2: $\beta(aa)$, having but one common vertex, do not form a circuit of graphs. We shall say three or more non-separable graphs form a *circuit of graphs* if we can name them $G_1, G_2, \cdots, G_m$ in such a way that G_1 and G_2 have just the vertex a_1 in common, G_2 and G_3 have just the vertex a_2 in common, $\cdots$, G_m and G_1 have just the vertex a_m in common, these vertices are all distinct, and no other two of these graphs have a common vertex. Thus the three graphs $G_1{:}ab$, $G_2{:}bc$, $G_3{:}ca$ form a circuit of graphs.

We note that there can be no 1-circuit in a circuit of graphs; also, no subset of the graphs in a circuit of graphs form a circuit of graphs. We may think of a circuit of graphs as forming a single graph.

THEOREM 16. *A circuit of graphs G is a non-separable graph.*

First suppose there are but two graphs, G_1 and G_2, present. Suppose G were separable. Then it is separable into at least two components $H_1, H_2, \cdots, H_k$. By Theorem 11, G_1 and G_2 are each contained wholly in one of these components. As G_1 and G_2 together form G, there are just two components, and they are G_1 and G_2. These, when joined at a vertex, form G. But this is contrary to the hypothesis that G_1 and G_2 have at least two vertices in common.

Next suppose there are more than two graphs present. Let C_1 be a chain in G_1 joining a_m and a_1, let C_2 be a chain in G_2 joining a_1 and $a_2, \cdots$, let C_m be a chain in G_m joining a_{m-1} and a_m. These chains taken together form a cir-

* This theorem may also be proved easily from Theorem 17.

cuit P passing through all the graphs. Now separate G into its components. By Theorem 11 (see the converse of Theorem 10), P is contained in one of these components. The same is true of each of the graphs $G_1, G_2, \cdots, G_m$, and hence these graphs are all contained in the same component. Thus G is itself this component, that is, G is non-separable.

THEOREM 17. *Let $G_1, \cdots, G_m$ be a set of non-separable graphs, each containing at least one arc, and let G be formed by letting vertices and arcs of different graphs coalesce. Then the following four statements are all equivalent:*

(1) *$G_1, \cdots, G_m$ are the components of G.*

(2) *No two of the graphs $G_1, \cdots, G_m$ have an arc in common, and there is no circuit in G containing arcs of more than one of these graphs.*

(3) *No subset of these graphs form a circuit of graphs.*

(4) *If $R, R_1, \cdots, R_m$ are the ranks of $G, G_1, \cdots, G_m$ respectively, then*

$$R = R_1 + \cdots + R_m.$$

We note that we cannot replace the word rank by the word nullity in (4). For let G be the graph containing the arcs $\alpha(ab)$, $\beta(ab)$, $\gamma(ab)$. Let G_1 contain α and β, and G_2, β and γ. Then the nullity of G is the sum of the nullities of G_1 and G_2, but G_1 and G_2 are not the components of G. We shall prove

(a) if (1) holds, (2) holds,

(b) if (2) holds, (3) holds,

(c) if (3) holds, (1) holds, establishing the equivalence of (1), (2) and (3);

(d) if (1) holds, (4) holds, and finally

(e) if (4) holds, (3) holds, establishing the equivalence of (4) and the other statements.

(a) If (1) holds, (2) holds. For first, in forming G from its components $G_1, \cdots, G_m$, we let vertices alone coalesce, and thus no two of the graphs have an arc in common. Also, there is no circuit in G containing arcs of more than one of the graphs; for each circuit, being a non-separable graph, is contained entirely in one of the components of G, by Theorem 11.

(b) If (2) holds, (3) holds. For suppose the contrary. If, first, some two graphs, say G_1 and G_2, form a circuit of graphs, they have at least two vertices in common, say a and b. Join a and b by a chain C in G_1 and by a chain D in G_2. By hypothesis, G_1 and G_2 have no arcs in common, and thus the arcs of C and D are distinct. From a follow along C till we first reach a vertex d of D. From d follow along D till we get back to a. We have formed thus a circuit containing arcs of both G_1 and G_2, contrary to hypothesis.

Now suppose the graphs $G_1, \cdots, G_k$, $k>2$, formed a circuit of graphs. In the proof of Theorem 16 we found a circuit passing through all the graphs of such a circuit of graphs, again contrary to hypothesis.

(c) If (3) holds, (1) holds. Assuming that no subset of the graphs $G_1, \cdots, G_m$ forms a circuit of graphs, we will show first that some one of these graphs has at most a single vertex in common with other of the graphs. For suppose each graph had at least two vertices in common with other graphs. Then G_1 has a vertex a_1 in common with some graph, say G_2. As G_2 has at least two vertices in common with other graphs, it has a vertex a_2, distinct from a_1, in common with another graph, say G_3. If we continue in this manner, we must at some point get back to a graph we have already considered.

Now starting with G_1, consider the graphs in order, and let G_i be the first one which has a vertex in common with one of the preceding graphs other than the vertex a_{i-1}, which we know already it has in common with G_{i-1}. Now of the graphs $G_{i-1}, G_{i-2}, \cdots, G_1$, let G_j be the first with which G_i has a common vertex, other than the vertex a_{i-1}. First suppose G_j is G_{i-1}. Then G_i and G_{i-1} have at least two vertices in common, and they form therefore a circuit of graphs, contrary to hypothesis. Next suppose G_j is not G_{i-1}. Then on account of the choice of G_i and G_j, G_j and G_{j+1} have just one common vertex a_j, G_{j+1} and G_{j+2} have just one common vertex $a_{j+1}, \cdots$, G_i and G_j have just one common vertex a_i (for otherwise G_i and G_j would form a circuit of graphs), and no other two of these graphs have a vertex in common. These vertices $a_j, a_{j+1}, \cdots, a_i$ are all distinct. For, on account of the construction of the chain of graphs, two succeeding vertices a_k and a_{k+1} are distinct. a_i and a_j are distinct, for otherwise G_i and G_{j+1} would have a common vertex, etc. These graphs $G_j, G_{j+1}, \cdots, G_i$ form therefore a circuit of graphs, contrary to hypothesis.

Some graph therefore, say G_1, has at most a single vertex in common with the other graphs. Thus either it is separated from them, or we can separate it at a single vertex. Now among the graphs $G_2, \cdots, G_m$, there is also no circuit of graphs, so again we can separate one of them, say G_2. Continuing, we have finally separated G into its components $G_1, G_2, \cdots, G_m$.

(d) If (1) holds, (4) holds. This is just Theorem 13.

(e) If (4) holds, (3) holds. Let G' be the sum of the graphs $G_1, \cdots, G_m$. We form G from G' by letting vertices and arcs of different graphs coalesce. Each time we let two vertices coalesce, either (α) the two vertices were formerly in different connected pieces, in which case the rank is unchanged, or (β) the two vertices were in the same connected piece, in which case the rank is reduced by 1. Letting arcs alone coalesce (their end vertices having already coalesced) does not alter the rank. Thus in any case, the rank is never increased. To begin with, the rank of G' is $G_1 + \cdots + G_m$, and by hypothesis, the rank of G is $G_1 + \cdots + G_m$. Thus the rank is never altered, and (β) never occurs. Hence, obviously, no circuit of graphs is formed in forming G from G'. This completes the proof of the theorem.

6. **Construction of non-separable graphs.** We prove the following theorem:

THEOREM 18. *If G is a non-separable graph of nullity $N>1$, we can remove an arc or suspended chain from G, leaving a non-separable graph G' of nullity $N-1$.*

Assume the theorem is true for all graphs of nullity $2, 3, \cdots, N-1$. We shall prove it for any graph of nullity N (including the case where $N=2$). This will establish the theorem in general.

Take any non-separable graph G of nullity $N>1$. It contains at least two arcs, and therefore, by Theorem 8, it contains no 1-circuit. Remove from G any arc ab, forming the graph G_1. If G_1 is non-separable, we are through. Suppose therefore G_1 is separable, and let its components be $H_1, H_2, \cdots, H_{m-1}$. G_1 is connected, for between any two vertices c, d there exists a circuit in G by Theorem 7, and therefore there is a chain joining them in G_1.

Let H_m consist of the arc ab. By Theorem 17, no subset of the graphs $H_1, \cdots, H_{m-1}$ form a circuit of graphs, while some subset of the graphs $H_1, \cdots, H_m$ form a circuit of graphs. We shall show that the whole set of graphs $H_1, \cdots, H_m$ form a circuit of graphs. Otherwise, some proper subset, which includes H_m, form a circuit of graphs.

Let H be the graph formed from this circuit of graphs by dropping out H_m. By Theorem 16, the circuit of graphs is a non-separable graph; hence H is connected. All the arcs in G_1 not in the circuit of graphs, form a graph I. Let I_1 be a connected piece of I. Then I_1 has at most a single vertex in common with the rest of G. For suppose I_1 had the two vertices c and d in common with H. From c follow along some chain towards d in H till we first reach a vertex e in I_1. From e follow back along some chain in I_1 to c. We have formed thus a circuit containing arcs of both H and I_1. But as H consists of a certain subset of the components of G_1, this circuit contains arcs of at least two components of G_1, contrary to Theorem 17. Thus I_1 has at most a single vertex in common with the rest of G, and hence G is separable, contrary to hypothesis. Thus $H_1, \cdots, H_m$ form a circuit of graphs, that is, G is formed of a circuit of graphs.

As we assumed G_1 was separable, $m \geqq 3$. Therefore we can order the graphs so that H_1 and H_2 have just the vertex a_1 in common, $\cdots$, H_{m-1} and H_m have just the vertex $a_{m-1}=b$ in common, and H_m and H_1 have just the vertex $a_m=a$ in common. Moreover, these vertices are all distinct, and no other two of the graphs $H_1, \cdots, H_m$ have a common vertex.

As the nullity of G was >1, the nullity of G_1 is >0. By Theorem 13, this is the sum of the nullities of $H_1, \cdots, H_{m-1}$. Therefore the nullity of some one of these graphs, say H_i, is >0.

Suppose first the nullity of H_i is 1. Then, by Theorem 10, H_i is a circuit, consisting of two chains joining a_{i-1} and a_i. Remove one of these chains from G. This leaves a graph G', which again is a circuit of graphs. For the graph H_i we replace by an ordered set of non-separable graphs, each consisting of one of the arcs of the chain we have left in H_i.

Suppose next the nullity of H_i is >1. It is less than N, as H_i is contained in G_1, whose nullity is $N-1$. Therefore, by induction, we can remove an arc or a suspended chain, leaving a non-separable graph H_i' of nullity one less. If neither a_{i-1} nor a_i has thus been removed, we again have a circuit of graphs. Suppose a_i but not a_{i-1} was removed. Replace that part of the chain we removed joining a_i and a vertex of H_i distinct from a_{i-1}. Here again we have a circuit of graphs, H_i being replaced by H_i' and a set of arcs. The case is the same if a_{i-1} but not a_i was removed. If finally, both a_i and a_{i-1} were in the chain we removed, we put back all of the chain but that part between these two vertices. Here again, the resulting graph G' is a circuit of graphs.

Thus in all cases we can drop out from G an arc or suspended chain, leaving a circuit of graphs. By Theorem 16, the resulting graph G' is non-separable. As also the nullity of G' is one less than the nullity of G, the theorem is now proved.

As a consequence of this theorem, Theorem 8, and Theorem 10, we have

THEOREM 19. *We can build up any non-separable graph containing at at least two arcs by taking first a circuit, then adding successively arcs or suspended chains, so that at any stage of the construction we have a non-separable graph.*

It is easily seen that, conversely, any graph built up in this manner is non-separable. For each time we add an arc or suspended chain, these arcs, each considered as a graph, together with the non-separable graph already present, form a circuit of graphs.

II. DUALS, PLANAR GRAPHS

7. Congruent graphs. We introduce the following

Definitions. Given two graphs G and G', if we can rename the vertices and arcs of one, giving distinct vertices and distinct arcs different names, so that it becomes identical with the other, we say the two graphs are *congruent*.* (We used formerly the word "homeomorphic.")

* See the author's American Journal paper, cited in the introduction.

The geometrical interpretation is that we can bring the two graphs into complete coincidence by a (1, 1) continuous transformation.

Two graphs are called *equivalent* if, upon being decomposed into their components, they become congruent, except possibly for isolated vertices.

8. **Duals.** Given a graph G, if H_1 is a subgraph of G, and H_2 is that subgraph of G containing those arcs not in H_1, we say H_2 is the *complement* of H_1 in G.

Throughout this section, R, R', r, r', etc., will stand for the ranks of G, G', H, H', etc., respectively, with similar definitions for V, E, P, N.

Definition. Suppose there is a (1, 1) correspondence between the arcs of the graphs G and G', such that if H is any subgraph of G and H' is the complement of the corresponding subgraph of G', then

$$r' = R' - n.$$

We say then that G' is a *dual* of G.*

Thus, if the nullity of H is n, then H' (including all the vertices of G') is in n more connected pieces than G'.

THEOREM 20. *Let G' be a dual of G. Then*

$$R' = N,$$
$$N' = R.$$

For let H be that subgraph of G consisting of G itself. Then

$$n = N.$$

If H' is the complement of the corresponding subgraph of G', H' contains no arcs, and is the null graph. Thus

$$r' = 0.$$

But as G' is a dual of G,

$$r' = R' - n.$$

These equations give

$$R' = N.$$

The other equation follows when we note that $E' = E$.

THEOREM 21. *If G' is a dual of G, then G is a dual of G'.*

Let H' be any subgraph of G', and let H be the complement of the corresponding subgraph of G. Then, as G' is a dual of G,

* While this definition agrees with the ordinary one for graphs lying on a plane or sphere, a graph on a surface of higher connectivity, such as the torus, has in general no dual. (See Theorems 29 and 30.)

$$r' = R' - n.$$

By Theorem 20,

$$R' = N.$$

We note also,

$$e + e' = E.$$

These equations give

$$\begin{aligned} r &= e - n = e - (R' - r') = e - N + (e' - n') \\ &= E - N - n' = R - n'. \end{aligned}$$

Thus G is a dual of G'.

Whenever we have shown that one graph is a dual of another graph, we may now call the graphs "dual graphs."

LEMMA. *If a graph G is decomposed into its components, the rank and nullity of any subgraph H is left unchanged.*

For each time we separate G at a vertex, H is either unchanged or is separated at a vertex. Hence neither its rank nor its nullity is altered. (See the proof of Theorem 13.)

THEOREM 22. *If G' and G'' are equivalent and G' is a dual of G, then G'' is a dual of G.*

Let H be any subgraph of G, and let H' be the complement of the corresponding subgraph of G'. Let G_1' and G_1'' be G' and G'' decomposed into their components. Then G_1' and G_1'' are congruent. H' turns into a subgraph H_1' of G'. Let H_1'' be the corresponding subgraph of G_1'', and H'' the same subgraph in G''. Then

$$r_1' = r_1''.$$

But by the above lemma,

$$r' = r_1', \qquad r'' = r_1''.$$

Hence

$$r' = r''.$$

As a special case of this equation, letting H' be the whole of G', we have

$$R' = R''.$$

As G' is a dual of G,

$$r' = R' - n.$$

Therefore

$$r'' = R'' - n,$$

and G'' is a dual of G.

The converse of this theorem is not true. For define the three graphs

G: $\alpha(ab)$, $\beta(ab)$, $\gamma(ac)$, $\delta(cb)$, $\epsilon(ad)$, $\zeta(db)$;

$G': \alpha'(a'b'), \beta'(c'd'), \gamma'(a'd'), \delta'(a'd'), \epsilon'(b'c'), \zeta'(b'c');$
$G'': \alpha''(a''b''), \beta''(b''c''), \gamma''(a''d''), \delta''(a''d''), \epsilon''(c''d''), \zeta''(c''d'').$
G' and G'' are both duals of G, but they are not congruent.*

THEOREM 23. *Let $G_1, \cdots, G_m$ and $G_1', \cdots, G_m'$ be the components of G and G' respectively, and let G_i' be a dual of G_i, $i=1, \cdots, m$. Then G' is a dual of G.*

Let H be any subgraph of G, and let the parts of H in $G_1, \cdots, G_m$ be $H_1, \cdots, H_m$. Let H_i' be the complement of the subgraph corresponding to H_i in G_i', $i=1, \cdots, m$, and let H' be the union of $H_1', \cdots, H_m'$ in G'. Then H' is the complement of the subgraph in G' corresponding to H in G. Using the proof of Theorem 13, we find that

$$r' = r_1' + \cdots + r_m',$$

and

$$n = n_1 + \cdots + n_m.$$

As also

$$R' = R_1' + \cdots + R_m'$$

and

$$r_i' = R_i' - n_i \qquad (i = 1, \cdots, m),$$

adding these last equations gives

$$r' = R' - n,$$

and hence G' is a dual of G.

THEOREM 24. *Let $G_1, \cdots, G_m$ and $G_1', \cdots, G_m'$ be the components of the dual graphs G and G', and let the correspondence between these two graphs be such that arcs in G_i correspond to arcs in G_i', $i=1, \cdots, m$. Then G_i and G_i' are duals, $i=1, \cdots, m$.*

Let H_1 be any subgraph of G_1, let H' be the complement of the corresponding subgraph in G', and let H_1' be the complement in G'. Then $H_1', G_2', \cdots, G_m'$ form H'. By Theorem 13, we find

$$R' = R_1' + R_2' + \cdots + R_m'$$

and

$$r' = r_1' + R_2' + \cdots + R_m'.$$

Now

$$r' = R' - n_1,$$

hence

$$r_1' = R_1' - n_1,$$

and G_1' is a dual of G_1. Similarly for $G_2', \cdots, G_m'$.

* See the author's American Journal paper, however.

THEOREM 25. *Let G and G' be dual graphs, and let $H_1, \cdots, H_m$ be the components of G. Let $H_1', \cdots, H_m'$ be the corresponding subgraphs of G'. Then $H_1', \cdots, H_m'$ are the components of G', and H_i' is a dual of H_i, $i=1, \cdots, m$.*

H_1 is the subgraph of G corresponding to H_1' in G'. Its complement is I_1, the graph formed of the arcs of $H_2, \cdots, H_m$. Obviously $H_2, \cdots, H_m$ are the components of I_1. Hence, by Theorem 13, the nullity of I_1 is $n_2+n_3+\cdots+n_m$. Thus, as G' is a dual of G,

$$r_1' = R' - (n_2 + n_3 + \cdots + n_m).*$$

Similarly,

$$r_2' = R' - (n_1 + n_3 + \cdots + n_m),$$

$$\cdots\cdots\cdots\cdots$$

$$r_m' = R' - (n_1 + n_2 + \cdots + n_{m-1}).$$

Adding these equations gives

$$r_1' + r_2' + \cdots + r_m' = mR' - (m-1)(n_1 + n_2 + \cdots + n_m).$$

As $H_1, H_2, \cdots, H_m$ are the components of G,

$$N = n_1 + n_2 + \cdots + n_m.$$

Also, as G and G' are duals, by Theorem 20,

$$R' = N.$$

Hence

$$\begin{aligned} r_1' + r_2' + \cdots + r_m' &= mR' - (m-1)R' \\ &= R'. \end{aligned}$$

Let now $H_{11}', \cdots, H_{1k_1}'$ be the components of H_1' (there may be but one) and similarly for $H_2', \cdots, H_m'$. Then, by Theorem 13,

$$r_1' = r_{11}' + \cdots + r_{1k_1}',$$

$$\cdots\cdots\cdots\cdots$$

$$r_m' = r_{m1}' + \cdots + r_{mk_m}'.$$

Adding these equations gives

$$\sum_{i,j} r_{ij}' = r_1' + \cdots + r_m' = R'.$$

As the graphs $H_{11}', \cdots, H_{mk_m}'$ are non-separable, Theorem 17 tells us that they are the components of G'. Hence G' has at least as many components as

* Which equals n_1.

G. Similarly, G has at least as many components as G'. They have therefore the same number, m, of components.

There are therefore m graphs in the set $H_{11}', \cdots, H_{mk_m}'$. But there is at least one such graph in each graph $H_1', \cdots, H_m'$, and there is therefore exactly one in each. Hence each graph H_{i1}' fills out the graph H_i', and the two sets of graphs $H_{11}', \cdots, H_{mk_m}'$ and $H_1', \cdots, H_m'$ are identical, that is, $H_1', \cdots, H_m'$ are the components of G'.

The rest of the theorem follows from Theorem 24.

As a special case of this theorem, we have

THEOREM 26. *A dual of a non-separable graph is non-separable.*

9. **Planar graphs.** Up till now, we have been considering abstract graphs alone. However, the definition of a planar graph is topological in character. This section may be considered as an application of the theory of abstract graphs to the theory of topological graphs.

Definitions. A topological graph is called *planar* if it can be mapped in a (1, 1) continuous manner on a sphere (or a plane). For the present, we shall say that an abstract graph is *planar* if the corresponding topological graph is planar. Having proved Theorem 29, we shall be justified in using the following purely combinatorial definition: *A graph is planar if it has a dual.*

We shall henceforth talk about "graphs" simply, the terms applying equally well to either abstract or topological graphs.

LEMMA. *If a graph can be mapped on a sphere, it can be mapped on a plane, and conversely.*

Suppose we have a graph mapped on a sphere. We let the sphere lie on the plane, and rotate it so that the new north pole is not a point of the graph. By stereographic projection from this pole, the graph is mapped on the plane. The inverse of this projection maps any graph on the plane onto the sphere.

By the *regions* of a graph lying on a sphere or in a plane is meant the regions into which the sphere or plane is thereby divided. A given region of the graph is characterized by those arcs of the graph which form its boundary. If the graph is in a plane, the outside region is the unbounded region.

LEMMA. *A planar graph may be mapped on a plane so that any desired region is the outside region.*

We map the graph on a sphere, and rotate it so that the north pole lies inside the given region. By stereographic projection, the graph is mapped onto the plane so that the given region is the outside region.

We return now to the work in hand.

THEOREM 27. *If the components of a graph G are planar, G is planar.*

Suppose the graphs G_1 and G_2 are planar, and G' is formed by letting the vertices a_1 and a_2 of G_1 and G_2 coalesce. We shall show that G' is planar. Map G_1 on a sphere, and map G_2 on a plane so that one of the regions adjacent to the vertex a_2 is the outside region. Shrink the portion of the plane containing G_2 so it will fit into one of the regions of G_1 adjacent to a_1. Drawing a_1 and a_2 together, we have mapped G' on the sphere.* The theorem follows as a repeated application of this process.

THEOREM 28. *Let G and G' be dual graphs, and let $\alpha(ab)$, $\alpha'(a'b')$ be two corresponding arcs. Form G_1 from G by dropping out the arc $\alpha(ab)$, and form G_1' from G' by dropping out the arc $\alpha'(a'b')$, and letting the vertices a' and b' coalesce if they are not already the same vertex. Then G_1 and G_1' are duals, preserving the correspondence between their arcs.*

Let H_1 be any subgraph of G_1 and let H_1' be the complement of the corresponding subgraph of G_1'.

Case 1. Suppose the vertices a' and b' were distinct in G'. Let H be the subgraph of G identical with H_1. Then

$$n = n_1.$$

Let H' be the complement in G' of the subgraph corresponding to H. Then

$$r' = R' - n.$$

Now H' is the subgraph in G' corresponding to H_1' in G_1', except that H' contains the arc $\alpha'(a'b')$, which is not in H_1'. Thus if we drop out $\alpha'(a'b')$ from H' and let a' and b' coalesce, we form H_1'. In this operation, the number of connected pieces is unchanged, while the number of vertices is decreased by 1. Hence

$$r_1' = r' - 1.$$

As a special case of this equation, if H' contains all the arcs of G', we find

$$R_1' = R' - 1.$$

These equations give

$$r_1' = R_1' - n_1.$$

Thus G_1' is a dual of G_1.

Case 2. Suppose a' and b' are the same vertex in G'. In this case, defining H and H' as before, we form H_1' from H' by dropping out the arc $\alpha'(a'a')$. This leaves the number of vertices and the number of connected pieces un-

* Here and in a few other places we are using point-set theorems which, however, are geometrically evident.

changed. Thus two of the equations in Case 1 are replaced by the equations

$$r_1' = r_1, \quad R_1' = R_1.$$

The other equations are as before, so we find again that G_1' is a dual of G_1. The theorem is now proved.

THEOREM 29. *A necessary and sufficient condition that a graph be planar is that it have a dual.*

We shall prove first the necessity of the condition. Given any planar graph G, we map it onto the surface of a sphere. If the nullity of G is N, it divides the sphere into $N+1$ regions. For let us construct G arc by arc Each time we add an arc joining two separate pieces, the nullity and the number of regions remain the same. Each time we add an arc joining two vertices in the same connected pieces, the nullity and the number of regions are each increased by 1. To begin with, the nullity was 0 and the number of regions was 1. Therefore, at the end, the number of regions is $N+1$.

We construct G' as follows: In each region of the graph G we place a point, a vertex of G'. Therefore G' contains $V'=N+1$ vertices. Crossing each arc of G we place an arc, joining the vertices of G' lying in the two regions the arc of G separates (which may in particular be the same region, in which case this arc of G' is a 1-circuit). The arcs of G and G' are now in (1, 1) correspondence.

G' is the dual of G in the ordinary sense of the word. We must show it is the dual as we have defined the term.

Let us build up G arc by arc, removing the corresponding arc of G' each time we add an arc to G. To begin with, G contains no arcs and G' contains all its arcs, and at the end of the process, G contains all its arcs and G' contains no arcs. We shall show

(1) each time the nullity of G is increased by 1 upon adding an arc, the number of connected pieces in G' is reduced by 1 in removing the corresponding arc, and

(2) each time the nullity of G remains the same, the number of connected pieces in G' remains the same.

To prove (1) we note that the nullity of G is increased by 1 only when the arc we add joins two vertices in the same connected piece. Let ab be such an arc. As a and b were already connected by a chain, this chain together with ab forms a circuit P. Let $a'b'$ be the arc of G' corresponding to ab. Before we removed it, a' and b' were connected. Removing it, however, disconnects them. For suppose there were still a chain C' joining them. As a' and b' are on opposite sides of the circuit P, C' must cross P, by the Jordan Theorem, that is, an arc of C' must cross an arc of P. But we removed this arc of C' when we put in the arc of P it crosses. (1) is now proved.

The total increase in the nullity of G during the process is of course just N. Therefore the increase in the number of connected pieces in G' must be at least N. But G' was originally in at least one connected piece, and is at the end of the process in $V=N+1$ connected pieces. Thus the increase in the number of connected pieces in G' is just N (hence, in particular, G' itself is connected) and therefore this number increases only when the nullity of G increases, which proves (2).

Let now H be any subgraph of G, let H' be the complement of the corresponding subgraph of G', and let H' include all the vertices of G'. We build up H arc by arc, at the same time removing the corresponding arcs of G'. Thus when H is formed, H' also is formed. By (1) and (2), the increase in the number of connected pieces in forming H' from G' equals the nullity of H, that is,

$$p' - P' = n.$$

But

$$r' = V' - p', \quad R' = V' - P',$$

as G' and H' contain the same vertices. Therefore

$$r' = R' - n,$$

that is, G' is a dual of G.

To prove the sufficiency of the condition, we must show that if a graph has a dual, it is planar. It is enough to show this for non-separable graphs. For if the separable graph G has a dual, its components have duals, by Theorem 25, hence its components are planar, and hence G is planar, by Theorem 27. This part of the theorem is therefore a consequence of the following theorem:

THEOREM 30. *Let the non-separable graph G have a dual G'. Then we can map G and G' together on the surface of a sphere so that*

(1) *corresponding arcs in G and G' cross each other, and no other pair of arcs cross each other, and*

(2) *inside each region of one graph there is just one vertex of the other graph.*

The theorem is obviously true if G contains a single arc. (The dual of an arc ab is an arc $a'a'$, and the dual of an arc aa is an arc $a'b'$.) We shall assume it to be true if G contains fewer than E arcs, and shall prove it for any graph G containing E arcs. By Theorem 8, each vertex of G is on at least two arcs.

Case 1. G contains a vertex b on but two arcs, ab and bc. As G is non-separable, there is a circuit containing these arcs. Thus dropping out one of them will not alter the rank, while dropping out both reduces the rank by 1. As G' is a dual of G, the arcs corresponding to these two arcs are each of nullity 0, while the two arcs taken together are of nullity 1. They are thus of the form $\alpha'(a'b')$, $\beta'(a'b')$, the first corresponding to ab, and the second, to bc.

Form G_1 from G by dropping out the arc bc and letting the vertices b and c coalesce, and form G_1' from G' by dropping out the arc $\beta'(a'b')$. By Theorem 28, G_1 and G_1' are duals, preserving the correspondence between the arcs. As these graphs contain fewer than E arcs,* we can, by hypothesis, map them together on a sphere so that (1) and (2) hold; in particular, $\alpha'(a'b')$ crosses ac. Mark a point on the arc ac of G_1 lying between the vertex c and the point where the arc $\alpha'(a'b')$ of G' crosses it. Let this be the vertex b, dividing the arc ac into the two arcs ab and bc. Draw the arc $\beta'(a'b')$ crossing the arc bc. We have now reconstructed G and G', and they are mapped on a sphere so that (1) and (2) hold.

Case 2. Each vertex of G is on at least three arcs. As then G contains no suspended chain, and G is not a circuit and therefore is of nullity $N>1$, we can, by Theorem 18, drop out an arc ab so that the resulting graph G_1 is non-separable. G' is non-separable, by Theorem 26, and hence the arc $a'b'$ corresponding to ab in G is not a 1-circuit. Drop it out and let the vertices a', b' coalesce into the vertex a_1', forming the graph G_1'. By Theorem 28, G_1 and G_1' are duals, and thus G_1' also is non-separable.

Consider the arcs of G' on a'. If we drop them out, the resulting graph G'' has a rank one less than that of G'. For if its rank were still less, G'' would be in at least three connected pieces, one of them being the vertex a'. Let c and d be vertices in two other connected pieces of G''. They are joined by no chain in G'', and hence every chain joining them in G' must pass through a', which contradicts Theorem 6. If we put back any arc, the rank is brought back to its original value, as a' is then joined to the rest of the graph. Hence, G' being a dual of G, the arcs of G corresponding to these arcs are together of nullity 1, while dropping out one of them reduces the nullity to 0. Therefore, by Theorem 9, these arcs form a circuit P. One of these arcs is the arc ab. The remaining arcs form a chain C. Similarly, the arcs of G corresponding to the arcs of G' on b' form a circuit Q, and this circuit minus the arc ab forms a chain D. C and D have the vertices a and b as end vertices. Also, the arcs of G_1 corresponding to the arcs of G_1' on a_1' form a circuit R. These arcs of G_1' are the arcs of G' on either a' or b', except for the arc $a'b'$ we dropped out. Thus the arcs of G_1 forming the circuit R are the arcs of the chains C and D.

As G_1 and G_1' contain fewer than E arcs, we can map them together on a

* Obviously G_1 is non-separable.

sphere so that properties (1) and (2) hold. a_1' lies on one side of the circuit R, which we call the inside. Each arc of R is crossed by an arc on a_1', and thus there are no other arcs of G_1' crossing R. There is no part of G_1' lying inside R other than a_1', for it could have only this vertex in common with the rest of G_1', and G_1' would be separable. Also, there is no part of G_1 lying inside R, for any arc would have to be crossed by an arc of G_1', and any vertex would have to be joined to the rest of G_1 by an arc, as G_1 is non-separable.

Let us now replace a_1' by the two vertices a' and b', and let those arcs abutting on a_1' that were formerly on a' be now on a', and those formerly on b', now on b'. As the first set of arcs all cross the chain C, and the second set all cross the chain D, we can do this in such a way that no two of the arcs cross each other. We may now join a and b by the arc ab, crossing none of these arcs. This divides the inside of R into two parts, in one of which a' lies, and in the other of which b' lies. We may therefore join a' and b' by the arc $a'b'$, crossing the arc ab. G and G' are now reconstructed, and are mapped on the sphere as required. This completes the proof of the theorem, and therefore of Theorem 29.

THEOREM 31. *A necessary and sufficient condition that a graph be planar is that it contain neither of the two following graphs as subgraphs*:

G_1. *This graph is formed by taking five vertices a, b, c, d, e, and joining each pair by an arc or suspended chain.*

G_2. *This graph is formed by taking two sets of three vertices, a, b, c, and d, e, f, and joining each vertex in one set to each vertex in the other set by an arc or suspended chain.*

This theorem has been proved by Kuratowski.* It would be of interest to show the equivalence of the conditions of the theorem and Theorem 29 directly, by combinatorial methods. We shall do part of this here, in the following theorem:†

THEOREM 32. *Neither of the graphs G_1 and G_2 has a dual.*

Suppose the graph G_1 had a dual. By Theorem 28, if G_1 contains a suspended chain, we can drop out one of its arcs and let the two end vertices coalesce, and the resulting graph will have a dual. Continuing, we see that the graph G_3, in which each pair of vertices of the set a, b, c, d, e are joined by an arc, must have a dual. Similarly, if G_2 has a dual, then the graph G_4, in which each vertex of the set a, b, c is joined to each vertex of the set d, e, f by an arc, must have a dual. Both of these are impossible.

* Fundamenta Mathematicae, vol. 15 (1930), pp. 271–283.

† The other half has recently been proved by the author. See Bulletin of the American Mathematical Society, abstract (38-1-39). (Note added in proof.)

(a) *The graph* G_3. To avoid subscripts, let us call it G. Suppose it had a dual, G'. Then

$$R = N' = 4,$$
$$N = R' = 6,$$
$$E = E' = 10.$$

If G' has isolated vertices, we drop them out, which does not alter its relation to G.

(1) There are no 1-circuits, 2-circuits or triangles in G'. For if there were, dropping out the corresponding arcs of G would have to reduce the rank of G. But we cannot reduce its rank without dropping out at least four arcs.

(2) G' contains at least five quadrilaterals. For if we drop out the four arcs on any vertex of G, the rank is reduced by 1, and if we put back any of these arcs, the rank is brought back to its original value; Theorem 9 now applies.

(3) At least two of these quadrilaterals have an arc in common, as there are but ten arcs in G'.

There are just two ways of forming two quadrilaterals out of fewer than eight arcs without forming any 2-circuits or triangles. One of these graphs, I_1', contains the arcs $a'b'$, $b'e'$, $a'c'$, $c'e'$, $a'd'$, $d'e'$. The other, I_2', contains the arcs $a'e'$, $e'f'$, $f'b'$, $b'a'$, $e'c'$, $c'd'$, $d'f'$. But there is no subgraph of the type I_1' in G', for this subgraph is of rank 4 and nullity 2, and there would have to be a subgraph of G of rank 2 and nullity 2, and such a graph contains a 1- or a 2-circuit, of which there are none in G. Hence G' contains a subgraph I_2'.

(4) Each vertex of G' is on at least three arcs, as there are no 1- or 2-circuits in G.

Each of the vertices a', b', c', d' of I_2' is on but two arcs. Hence there must be another arc on each of these vertices. As I_2' contains seven arcs, and G' contains but ten, one of the three arcs left must join two of these vertices. But if we add an arc $a'b'$ or $c'd'$, we would form a 2-circuit; if we add an arc $a'c'$ or $b'd'$, we would form a triangle; if we add an arc $a'd'$ or $b'c'$, we would form a graph of the type I_1'. As G' contains none of these graphs, we have a contradiction.

(b) *The graph* G_4. Let us call it G. If it has a dual G', then

$$R = N' = 5,$$
$$N = R' = 4,$$
$$E = E' = 9.$$

We proceed exactly as for the graph G_3. In outline:

(1) G' contains no 1- or 2-circuits.

(2) There is no subgraph of G' containing four vertices, each pair being joined by an arc. For this graph is of rank 3 and nullity 3, and G would have to contain a subgraph of rank 2 and nullity 1, that is, a 2-circuit.

(3) There are at least nine subgraphs of G' of rank 3 and nullity 2, and hence of the form $a'b'$, $a'c'$, $b'c'$, $b'd'$, $c'd'$, as there are nine quadrilaterals in G.

(4) As G' contains but nine arcs, two of these subgraphs have an arc in common. There is therefore a subgraph of one of the forms I_1' : $a'e'$, $a'b'$, $b'e'$, $a'c'$, $c'e'$, $a'd'$, $d'e'$, or I_2' : $a'e'$, $a'b'$, $b'e'$, $b'c'$, $c'e'$, $c'd'$, $d'e'$.

(5) Each vertex of G' is on at least four arcs.

Now each of the graphs I_1', I_2' contains seven arcs. We have but two arcs left which we must place so that each vertex of I_1' or I_2' is on at least four arcs. This cannot be done. The theorem is now proved.

Theorem 31 together with this theorem gives an alternative proof of the second part of Theorem 29. For suppose a graph G had a dual. Then it contains neither the graph G_1 nor G_2. For if it did, dropping out all the arcs of G but those forming one of these graphs, Theorem 28 tells us that this graph has a dual. But we have just seen that this is not so. Hence, by Theorem 31, G is planar.

Euler's formula. Map any connected planar graph G on a sphere, and construct its connected dual G' as described in the proof of Theorem 29. Then in each region of G there is a vertex of G'. Let F be the number of regions (or faces) in G. Then

$$R' = N,$$
$$R = V - 1,$$
$$R' = V' - 1,$$
$$V' = F,$$

and hence

$$V - E + F = R + 1 - E + N + 1 = 2,$$

which is Euler's formula.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.

Reprinted from
Trans. Amer. Math. Soc. **34** (1932), 339–362

Springer

http://www.springer.com/978-0-8176-4841-1

Classic Papers in Combinatorics
Gessel, I.; Rota, G.-C. (Eds.)
1987, X, 492 p. 10 illus., Softcover
ISBN: 978-0-8176-4841-1
A product of Birkhäuser Basel