

# Chapter 2

## The Spaces $B_{pq}^s$ and $F_{pq}^s$ : Definitions and Characterizations

### 2.1 Introduction

It was the aim of the first chapter to look at function spaces from a historical point of view and to convince the reader that the two scales  $B_{pq}^s$  and  $F_{pq}^s$  occupy the very heart of the theory of function spaces. In the present chapter and the following one we develop systematically the technical part of the theory of the spaces  $B_{pq}^s$  and  $F_{pq}^s$  on  $\mathbb{R}^n$  as it stands at the end of the eighties. The remaining chapters rest on these fundamentals. Our recent approach, compared with earlier ones, for example in [Triß], not to speak about [Triα], can be described as follows. We give rather general and, unfortunately, highly technical characterizations of the spaces  $F_{pq}^s$  and  $B_{pq}^s$  which more or less cover desirable concrete characterizations, for example, via differences or derivatives of functions, or via harmonic or thermic extensions, and which also provide the basis for later applications. This will be done in 2.4 for  $F_{pq}^s$  and in 2.5 for  $B_{pq}^s$ , always with a preference of the more complicated spaces  $F_{pq}^s$ . In this sense, 2.4 and 2.5 may be considered as the heart of the present chapter. Characterizations of some spaces  $F_{pq}^s$  and  $B_{pq}^s$  in terms of differences of functions, or as harmonic or thermic extensions are known, for example, we derived them in [Triß] by rather specific and sometimes quite tricky arguments. Now we return in 2.6 to this subject, but this time as consequences of the characterizations in 2.4 and 2.5.

### 2.2 Prerequisites

#### 2.2.1 Basic notations

For the convenience of the reader we collect in 2.2 some known assertions on maximal inequalities and entire analytic functions.

First we recall some standard notations. Let:

- $\mathbb{R}^n$  be the  $n$ -dimensional real euclidean space,
- $\mathbb{N}$  be the collection of all natural numbers, and
- $\mathbb{N}_0$  be the collection of all non-negative integers.

The general point in  $\mathbb{R}^n$  is denoted by  $x = (x_1, \dots, x_n)$ . Furthermore  $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$  has the usual meaning:  $\alpha = (\alpha_1, \dots, \alpha_n)$  is a multi-index,  $\alpha_j \in \mathbb{N}_0$  and  $|\alpha| = \sum_{j=1}^n \alpha_j$ .

Let  $S = S(\mathbb{R}^n)$  be the Schwartz space of all complex-valued rapidly decreasing  $C^\infty$  functions on  $\mathbb{R}^n$  equipped with the usual topology. We adopt here and in the sequel the following convention: If there is no danger of confusion we omit  $\mathbb{R}^n$  in  $S(\mathbb{R}^n)$  and in the other spaces below. Let  $S'$  be the collection of all tempered distributions on  $\mathbb{R}^n$ , i.e., the topological dual of  $S$ , equipped with the strong topology. If  $\varphi \in S$ , then

$$\hat{\varphi}(x) = F\varphi(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(\xi) d\xi, \quad x \in \mathbb{R}^n, \quad (1)$$

denotes the Fourier transform  $\hat{\varphi}$  or  $F\varphi$  of  $\varphi$ . Here  $x\xi = \sum_{j=1}^n x_j \xi_j$  is the scalar product in  $\mathbb{R}^n$  of  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  and  $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ . The inverse Fourier transform of  $\varphi$  is given by

$$\check{\varphi}(x) = F^{-1}\varphi(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} \varphi(\xi) d\xi, \quad x \in \mathbb{R}^n. \quad (2)$$

One extends  $F$  and  $F^{-1}$  in the usual way from  $S$  to  $S'$ . Recall that both  $F$  and  $F^{-1}$  yield isomorphic mappings from  $S$  onto itself and from  $S'$  onto itself.

The collection of all complex-valued  $C^\infty$  functions on  $\mathbb{R}^n$  with compact support is denoted by  $D$ ; and  $D'$  stands for the set of all complex distributions on  $\mathbb{R}^n$ .

Let  $A$  be a complex linear vector space, then  $\|a \mid A\|$  is said to be a quasi-norm if  $\|a \mid A\|$  satisfies the usual conditions of a norm with exception of the triangle inequality which is replaced by the assumption that there exists a constant  $c \geq 1$  such that

$$\|a_1 + a_2 \mid A\| \leq c\|a_1 \mid A\| + c\|a_2 \mid A\| \quad (3)$$

holds for all  $a_1 \in A$  and  $a_2 \in A$ . (If  $c = 1$  is admissible then  $A$  is a normed space). A quasi-normed space  $A$  is said to be a quasi-Banach space if it is complete (i.e., any fundamental sequence in  $A$  with respect to  $\|\cdot \mid A\|$  converges).

Let  $0 < q \leq \infty$ , then  $l_q$  is the set of all sequences  $b = (b_k)_{k \in \mathbb{N}_0}$  of complex numbers such that

$$\|b \mid l_q\| = \left( \sum_{k=0}^{\infty} |b_k|^q \right)^{1/q} < \infty \quad (4)$$

(modification if  $q = \infty$  by  $\sup_{k \in \mathbb{N}_0} |b_k|$ ). Of course,  $l_q$  is a quasi-Banach space (if  $q \geq 1$ , a Banach space). Let  $0 < p \leq \infty$  and  $0 < q \leq \infty$ , let  $f = (f_k(x))_{k \in \mathbb{N}_0}$  be a sequence of complex-valued Lebesgue measurable functions on  $\mathbb{R}^n$ , then

$$\|f \mid L_p(l_q)\| = \left( \int_{\mathbb{R}^n} \left( \sum_{k=0}^{\infty} |f_k(x)|^q \right)^{p/q} dx \right)^{1/p} \quad (5)$$

and

$$\|f \mid l_q(L_p)\| = \left( \sum_{k=0}^{\infty} \left( \int_{\mathbb{R}^n} |f_k(x)|^p dx \right)^{q/p} \right)^{1/q} \quad (6)$$

(modification if  $p = \infty$  by  $\text{ess sup}_{x \in \mathbb{R}^n}$  and/or  $q = \infty$  by  $\sup_k$ ). Let  $L_p(l_q) = L_p(\mathbb{R}^n, l_q)$  be the set of all sequences  $f$  such that  $\|f \mid L_p(l_q)\| < \infty$ , and let  $l_q(L_p) = l_q(L_p(\mathbb{R}^n))$  be the set of all sequences  $f$  such that  $\|f \mid l_q(L_p)\| < \infty$ . In the scalar case the corresponding space is denoted by  $L_p$ , quasi-normed by  $\|\cdot \mid L_p\| = (\int_{\mathbb{R}^n} |f(x)|^p dx)^{1/p}$  (modification if  $p = \infty$  by  $\text{ess sup}_{x \in \mathbb{R}^n} |f(x)|$ ). Again we omit  $\mathbb{R}^n$  if there is no danger of confusion.  $L_p(l_q)$ ,  $l_q(L_p)$  and the scalar case  $L_p$  are quasi-Banach spaces (Banach spaces if  $p \geq 1$  and  $q \geq 1$ ).

### 2.2.2 Some maximal inequalities

Let  $f(x)$  be a complex-valued locally Lebesgue-integrable function on  $\mathbb{R}^n$ , then

$$Mf(x) = \sup |B|^{-1} \int_B |f(y)| dy \quad (1)$$

is the Hardy–Littlewood maximal function, where the supremum is taken over all balls  $B$  centered at  $x$ . We have  $Mf(x) \geq |f(x)|$  almost everywhere.

**Theorem.** Let  $1 < p < \infty$  and  $1 < q \leq \infty$ , let  $f = (f_k(x))_{k \in \mathbb{N}_0}$  and  $Mf = (Mf_k(x))_{k \in \mathbb{N}_0}$ , then there exists a constant  $c$  such that

$$\|Mf \mid L_p(l_q)\| \leq c \|f \mid L_p(l_q)\| \quad (2)$$

holds for all  $f \in L_p(l_q)$ .

**Remark.** The scalar case is due to G.H. Hardy, J.E. Littlewood ( $n = 1$ ) and N. Wiener ( $n > 1$ ), see [HaL1, Wie] (1930/39). The above vector-valued version goes back to C. Fefferman, E.M. Stein, see [FeS1] (1971). Comments and further informations may be found in [Stel].

### 2.2.3 Entire analytic functions

Let  $\varphi \in S'$  and  $\text{supp } \hat{\varphi}$  be compact in  $\mathbb{R}^n$ , then  $\varphi = \varphi(x)$  is an entire analytic function in  $\mathbb{R}^n$ . This is a well-known fact and by the Paley–Wiener–Schwartz theorem there is a close connection between the size of  $\text{supp } \hat{\varphi}$  and the growth of the analytic extension  $\varphi(z)$  with  $z \in \mathbb{C}^n$ . We shall not need these assertions and refer to [Triß: 1.2.1] for details. More interesting for us is the following Fourier multiplier assertion.

Let  $a_+ = \max(a, 0)$  where  $a \in \mathbb{R}$ . Let  $0 < p \leq \infty$  then

$$\sigma_p = n \left( \frac{1}{p} - 1 \right)_+ . \quad (1)$$

Let  $H_2^s = H_2^s(\mathbb{R}^n)$  be the fractional Sobolev spaces from 1.3.2 including the classical Sobolev spaces  $W_2^k = H_2^k$  if  $k \in \mathbb{N}_0$ .

**Theorem.** Let  $\Omega$  be a compact subset of  $\mathbb{R}^n$ ,  $0 < p \leq \infty$  and  $s > \sigma_p + \frac{n}{2}$ , then there exists a constant  $c$  such that

$$\|(M\hat{f})^\vee \mid L_p\| \leq c \|M \mid H_2^s\| \|f \mid L_p\| \quad (2)$$

for all  $f \in L_p \cap S'$  with  $\text{supp } \hat{f} \subset \Omega$  and all  $M \in H_2^s$ .

**Remark.** A proof may be found in [Triß: 1.5.2]. In comparison with [Triß] the number  $\sigma_p$  has now a different meaning.

### 2.2.4 The spaces $L_p^\Omega(l_q)$ of entire analytic functions

We need an extension of Theorem 2.2.3 from the scalar case to the case of vector functions, including corresponding maximal inequalities. Let  $0 < p \leq \infty$ ,  $0 < q \leq \infty$  and let  $\Omega = (\Omega_k)_{k \in \mathbb{N}_0}$  be a collection of compact subsets of  $\mathbb{R}^n$ , then

$$L_p^\Omega(l_q) = \{f = (f_k)_{k \in \mathbb{N}_0} \subset S' \text{ with } \text{supp } \hat{f}_k \subset \Omega_k \quad (1)$$

$$\text{if } k \in \mathbb{N}_0 \text{ and } \|f \mid L_p(l_q)\| < \infty\},$$

see (2.2.1/5). We always assume that the diameters

$$d_k = \sup |x - y| > 0, \quad k \in \mathbb{N}_0, \quad (2)$$

where the supremum in (2) is taken over all  $x \in \Omega_k$  and  $y \in \Omega_k$ .

**Theorem.** Let  $0 < p < \infty$ ,  $0 < q \leq \infty$ , and let  $\Omega = (\Omega_k)_{k \in \mathbb{N}_0}$  be a collection of compact subsets of  $\mathbb{R}^n$  with  $d_k > 0$  if  $k \in \mathbb{N}_0$ .

(i) Let  $0 < r < \min(p, q)$ , then there exists a constant  $c$  such that

$$\left\| \left( \sum_{k=0}^{\infty} \left( \sup_{t \in \mathbb{R}^n} \frac{|f_k(\cdot - z)|}{1 + |d_k z|^{n/r}} \right)^q \right)^{1/q} \mid L_p \right\| \leq c \|f \mid L_p(l_q)\| \quad (3)$$

(modification if  $q = \infty$ ) holds for all  $f = (f_k)_{k \in \mathbb{N}_0} \in L_p^\Omega(l_q)$ .

(ii) Let  $\kappa > \frac{n}{2} + \frac{n}{\min(p, q)}$ , then there exists a constant  $c$  such that

$$\left\| \left( \sum_{k=0}^{\infty} |(M_k \hat{f}_k)^\vee|^q \right)^{1/q} \mid L_p \right\| \leq c \sup_{l \in \mathbb{N}_0} \|M_l(d_l \cdot) \mid H_2^\kappa\| \|f \mid L_p(l_q)\| \quad (4)$$

(modification if  $q = \infty$ ) holds for all  $f = (f_k)_{k \in \mathbb{N}_0} \in L_p^\Omega(l_q)$  and all  $(M_k(x))_{k \in \mathbb{N}_0} \subset H_2^\kappa$ .

**Remark 1.** A proof of the theorem may be found in [Triß: 1.6.1–1.6.3]. The spaces  $H_2^\kappa$  have the same meaning as in the preceding subsection. For our later purposes it is useful to remark that (4) follows from (3) and

$$\sup_{z \in \mathbb{R}^n} \frac{|(M_k \hat{f}_k)^\vee(x - z)|}{1 + |d_k z|^{n/r}} \leq c \sup_{z \in \mathbb{R}^n} \frac{|f_k(x - z)|}{1 + |d_k z|^{n/r}} \|M_k(d_k \cdot) \mid H_2^\kappa\|, \quad (5)$$

with  $0 < r < \min(p, q)$ ,  $\kappa > \frac{n}{2} + \frac{n}{r}$ , and where  $c$  is independent of  $x \in \mathbb{R}^n$ ,  $k \in \mathbb{N}_0$ ,  $M_k$ , and  $f_k$ . The maximal inequality (5) coincides with [Triß: (1.6.3/2)].

**Remark 2.** The restrictions for  $s$  in Theorem 2.2.3 are better than the scalar case of (4). However one can improve the restrictions for  $\kappa$  in (4). Let

$$\sigma_{pq} = n \left( \frac{1}{\min(p, q)} - 1 \right)_+ \quad (6)$$

be the counterpart of (2.2.3). Let  $0 < p < \infty$ ,  $0 < q < \infty$ , and

$$\kappa > \frac{n}{2} + \sigma_{pq}, \quad (7)$$

then (4) holds for all  $f \in L_p^\Omega(l_q)$  and all  $(M_k(x))_{k \in \mathbb{N}_0} \subset H_2^\kappa$ . A proof may be found in [Triß: 2.4.9]. It is based on part (ii) of the above theorem and a rather sophisticated complex interpolation of quasi-Banach spaces, whereas the proof of (ii) is straightforward. Furthermore, for technical reasons, we have  $q < \infty$ , although there is no doubt that the just-mentioned improvement is also correct if  $q = \infty$ .

## 2.3 Definitions and basic properties

### 2.3.1 Definitions

Let  $\varphi$  be a  $C^\infty$  function on  $\mathbb{R}^n$  with

$$\text{supp } \varphi \subset \{\xi \in \mathbb{R}^n: |\xi| \leq 2\}, \quad \varphi(\xi) = 1 \quad \text{if } |\xi| \leq 1. \quad (1)$$

Let  $j \in \mathbb{N}$  and

$$\varphi_j(\xi) = \varphi(2^{-j}\xi) - \varphi(2^{-j+1}\xi), \quad \xi \in \mathbb{R}^n. \quad (2)$$

Then we have

$$\text{supp } \varphi_j \subset \{\xi \in \mathbb{R}^n: 2^{j-1} \leq |\xi| \leq 2^{j+1}\}, \quad j \in \mathbb{N}, \quad (3)$$

and, with  $\varphi_0 = \varphi$ ,

$$\sum_{k=0}^{\infty} \varphi_k(\xi) = 1 \quad \text{if } \xi \in \mathbb{R}^n. \quad (4)$$

In other words,  $(\varphi_k)_{k \in \mathbb{N}_0}$  is a resolution of unity with (1.3). Let  $f \in S'$  then

$$\varphi_k(D)f(x) = (\varphi_k \hat{f})^\vee(x), \quad k \in \mathbb{N}_0, \quad x \in \mathbb{R}^n, \quad (5)$$

is an entire analytic function, see 2.2.3, and we have

$$f = \sum_{k=0}^{\infty} \varphi_k(D)f \quad (\text{convergence in } S'). \quad (6)$$

In other words, we decompose  $f$  in entire analytic functions, and we introduce spaces by checking the behaviour of these analytic functions with respect to  $x \in \mathbb{R}^n$  and  $k \in \mathbb{N}_0$ .

**Definition.** (i) Let  $0 < p \leq \infty$ ,  $0 < q \leq \infty$  and  $s \in \mathbb{R}$  then

$$B_{pq}^s = \left\{ f \in S': \|f\|_{B_{pq}^s} = \left( \sum_{k=0}^{\infty} 2^{ksq} \|\varphi_k(D)f\|_{L_p}^q \right)^{1/q} < \infty \right\} \quad (7)$$

(modification if  $q = \infty$ ).

(ii) Let  $0 < p < \infty$ ,  $0 < q \leq \infty$  and  $s \in \mathbb{R}$  then

$$F_{pq}^s = \left\{ f \in S': \|f\|_{F_{pq}^s} = \left\| \left( \sum_{k=0}^{\infty} 2^{ksq} |\varphi_k(D)f(\cdot)|^q \right)^{1/q} \mid L_p \right\| < \infty \right\} \quad (8)$$

(modification if  $q = \infty$ ).

**Remark.** The history of these spaces has been described in 1.3.4 and 1.5.1, including their connections with classical function spaces. We return to the latter point later on. An extension of (8) to  $p = \infty$  is not reasonable, see 1.5.2 where we discussed this point in some detail.

## 2.3.2 Basic properties

Let  $(\varphi_k)_{k \in \mathbb{N}_0}$  be the same system of functions as in 2.3.1 with the generating function  $\varphi = \varphi_0$ . We introduce the maximal functions

$$(\varphi_k^* f)_a(x) = \sup_{y \in \mathbb{R}^n} \frac{|\varphi_k(D)f(x-y)|}{1 + |2^k y|^a}, \quad f \in S', \quad a > 0, \quad (1)$$

where  $x \in \mathbb{R}^n$  and  $k \in \mathbb{N}_0$ .

**Theorem.** (i) Let  $0 < p \leq \infty$ ,  $0 < q \leq \infty$ ,  $a > \frac{n}{p}$ , and  $s \in \mathbb{R}$ , then

$$B_{pq}^s = \left\{ f \in S': \left( \sum_{k=0}^{\infty} 2^{ksq} \|(\varphi_k^* f)_a\|_{L_p}^q \right)^{1/q} < \infty \right\} \quad (2)$$

(modification if  $q = \infty$ ) in the sense of equivalent quasi-norms. Furthermore,  $B_{pq}^s$  is a quasi-Banach space (Banach space if  $p \geq 1$ ,  $q \geq 1$ ) and it is independent of the generating function  $\varphi$  (equivalent quasi-norms).

(ii) Let  $0 < p < \infty$ ,  $0 < q \leq \infty$ ,  $a > \frac{n}{\min(p,q)}$ , and  $s \in \mathbb{R}$ , then

$$F_{pq}^s = \left\{ f \in S': \left\| \left( \sum_{k=0}^{\infty} 2^{ksq} |(\varphi_k^* f)_a|^q \right)^{1/q} \mid L_p \right\| < \infty \right\} \quad (3)$$

(modification if  $q = \infty$ ) in the sense of equivalent quasi-norms. Furthermore,  $F_{pq}^s$  is a quasi-Banach space (Banach space if  $p \geq 1, q \geq 1$ ) and it is independent of the generating function  $\varphi$  (equivalent quasi-norms).

**Proof.** Step 1. We apply Theorem 2.2.4(i) with  $d_k = c2^k$  and  $\frac{n}{r} = a > \frac{n}{\min(p,q)}$ , see also (2.3.1/1,2,4). Then it follows that the quasi-norms in (3) can be estimated from above by the quasi-norm  $\|f\|_{F_{pq}^s} \|f\|_\varphi$  in (2.3.1/8), besides an unimportant factor. This proves (3). Let  $\psi$  be another generating function. Then we have  $\psi_k(x) = \chi_k(x)\psi_k(x)$  with

$$\chi_k(x) = \sum_{r=-2}^2 \varphi_{k+r}(x), \quad k \in \mathbb{N}_0, \quad \varphi_l = 0 \text{ if } l < 0, \quad (4)$$

where  $\psi_k$  is constructed in the same way as  $\varphi_k$  in 2.3.1. Now (2.2.4/4) and

$$\psi_k(D)f = \sum_{r=-2}^2 (\psi_k[\varphi_{k+r}(D)f]^\wedge)^\vee \quad (5)$$

prove

$$\|f\|_{F_{pq}^s} \|f\|_\psi \leq c \|f\|_{F_{pq}^s} \|f\|_\varphi. \quad (6)$$

Hence,  $F_{pq}^s$  is independent of  $\varphi$ . In the same way one proves the corresponding assertions for  $B_{pq}^s$ .

Step 2. By (2.3.1/6) it follows that  $\|f\|_{F_{pq}^s} \|f\|_\varphi$  from (2.3.1/8) is a quasi-norm (norm if  $p \geq 1, q \geq 1$ ). We prove that  $F_{pq}^s$  is complete, where we may assume  $s = 0$ , without restriction of generality. For that purpose we introduce the space  $L_p^*(l_q)$  which consists, by definition, of all sequences  $(f_k)_{k \in \mathbb{N}_0}$  of measurable functions with

$$\left\| \left( \sum_{k=0}^{\infty} \left( \sup_{y \in \mathbb{R}^n} \frac{|f_k(\cdot - y)|}{1 + |2^k y|^a} \right)^q \right)^{1/q} \mid L_p \right\| < \infty \quad (7)$$

for some fixed  $a > \frac{n}{\min(p,q)}$  (modification if  $q = \infty$ ). Then  $L_p^*(l_q)$  is a quasi-Banach space, in particular it is complete. Let  $\varphi_k(x)$  and  $\chi_k(x)$  be the same functions as above, see (4). We construct two linear operators. First, by (3) and (7),

$$A: f \in F_{pq}^0 \longrightarrow (\varphi_k(D)f)_{k \in \mathbb{N}_0} \quad (8)$$

is a continuous map from  $F_{pq}^0$  into  $L_p^*(l_q)$ . Secondly, we claim that

$$B: (f_k)_{k \in \mathbb{N}_0} \in L_p^*(l_q) \longrightarrow \sum_{k=0}^{\infty} \chi_k(D)f_k \quad (9)$$

is a continuous map from  $L_p^*(l_q)$  into  $F_{pq}^0$ . Let  $f_k^*(x)$  be the maximal function in (7), then we have

$$f_k^*(x) \leq c f_k^*(z)(1 + 2^k|x - z|)^a, \quad x \in \mathbb{R}^n, \quad z \in \mathbb{R}^n, \quad (10)$$

for some  $c > 0$ , which follows from

$$1 + |2^k u| \leq (1 + 2^k|v|)(1 + 2^k|u - v|), \quad u \in \mathbb{R}^n, \quad v \in \mathbb{R}^n.$$

In particular  $f_k^*(0) < \infty$  and

$$|f_k(y)| \leq c f_k^*(0)(1 + |2^k y|^a) \quad (11)$$

which shows that  $f_k(y)$  is of at most polynomial growth. Then  $f_k \in S'$  and  $\chi_k(D)f_k$  makes sense. Let us assume that the sum in (9) converges in  $S'$  to some  $f \in S'$ : we return to this point in Remark 1 below. At most 5 summands in (9) contribute to  $\varphi_j(D)f$ , that ones with  $|j - k| \leq 2$ ,

$$\varphi_j(D)f = \sum_{|j-k| \leq 2} \varphi_j(D) \circ \chi_k(D)f_k = \varphi_j(D)f_j + \cdots \quad (12)$$

where we used  $\varphi_j \chi_j = \varphi_j$ , and  $+\cdots$  stands for 4 similar terms. Let  $j \in \mathbb{N}$  then by (2.3.1/2) we have

$$\begin{aligned} \varphi_j(D)f_j(x) &= \int_{\mathbb{R}^n} \tilde{\varphi}_j(y)f_j(x - y)dy = 2^{jn} \int_{\mathbb{R}^n} \hat{\varphi}(2^j y)f_j(x + y)dy \\ &= \int_{\mathbb{R}^n} \hat{\varphi}(y)f_j(x + 2^{-j})dy + \end{aligned} \quad (13)$$

where “+” indicates a second term of the same type. It follows

$$|\varphi_j(D)f_j(x)| \leq c \int_{\mathbb{R}^n} |\hat{\varphi}(y)|(1 + |y|^a)dy \cdot f_j^*(x). \quad (14)$$

Now (14) and similar estimates for the 4 other terms in (12) prove  $f \in F_{pq}^0$ , including an estimate for the involved quasi-norms. Hence  $B$  is a continuous map from  $L_p^*(l_q)$  into  $F_{pq}^0$ . By  $\varphi_k \chi_k = \varphi_k$  we have

$$B \circ A = \text{id} \quad (\text{identity in } F_{pq}^0) \quad (15)$$

and  $\text{Im } A$ , the image of  $A$ , consists of those  $(f_k)_{k \in \mathbb{N}_0} \in L_p^*(l_q)$  with

$$A \circ B(f_k) = (f_k). \quad (16)$$

In particular,  $\text{Im } A$  is a closed subspace of the quasi-Banach space  $L_p^*(l_q)$  and  $A$  is an isomorphic map from  $F_{pq}^0$  onto  $\text{Im } A$ . Hence  $F_{pq}^0$  is complete. The proof of part (ii) of the theorem is complete. Similarly one proves part (i).

**Remark 1.** We prove the convergence in  $S'$  of the sum in (9). Let  $\eta \in S$ , then we have

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi_k(D) f_k(x) \eta(x) dx &= \int_{\mathbb{R}^n} (\varphi_k \hat{f}_k)(\xi) \check{\eta}(\xi) d\xi = \int_{\mathbb{R}^n} (\varphi_k \hat{f}_k)(\xi) \chi_k(\xi) \check{\eta}(\xi) d\xi \quad (17) \\ &= \int_{\mathbb{R}^n} \varphi_k(D) f_k(x) (\chi_k \check{\eta})^\wedge(x) dx, \end{aligned}$$

where  $\chi_k$  has the same meaning as in (4). Recall the Plancherel–Polya–Nikol’skij inequality for entire analytic functions

$$|\varphi_k(D) f_k(x)| \leq c 2^{k \frac{n}{p}} \|\varphi_k(D) f_k\|_{L_p}, \quad (18)$$

see [Triß: (1.3.2/5) and Theorem 1.4.1(ii)], where  $c$  is independent of  $k \in \mathbb{N}_0$ . Let  $C$  be the number on the left-hand side of (7), then (17), (18), and (14) yield

$$\left| \int_{\mathbb{R}^n} \varphi_k(D) f_k(x) \eta(x) dx \right| \leq c_1 C 2^{c_2 k} \sup_{y \in \mathbb{R}^n} (1 + |y|)^{c_3} |(\chi_k \check{\eta})^\wedge(y)|, \quad (19)$$

where  $c_1, c_2, c_3$  are positive numbers which are independent of  $k$ . Let  $\Delta$  be the Laplacian, then we have by standard arguments

$$\begin{aligned} 2^{c_2 k} (1 + |y|)^{c_3} |(\chi_k \check{\eta})^\wedge(y)| &\leq 2^{c_2 k} |(1 - \Delta)^{N_1} \chi_k \check{\eta}^\wedge(y)| \quad (20) \\ &\leq c 2^{c_2 k} \sup_{z \in \mathbb{R}^n} (1 + |z|)^{N_2} |(1 - \Delta)^{N_1} (\chi_k \check{\eta})(z)| \\ &\leq c' 2^{-k} \sup_{z \in \mathbb{R}^n} (1 + |z|)^{N_3} \sum_{|\alpha| \leq N_4} |D^\alpha \eta(z)|, \end{aligned}$$

for some natural numbers  $N_1, N_2, N_3, N_4$ . We insert (20) in (19) and note that we have a corresponding estimate with  $\chi_k$  instead of  $\varphi_k$ . But now it is clear that the sum in (9) converges in  $S'$ .

**Remark 2.** By the same arguments as in the last remark we get

$$S \subset B_{pq}^s \subset S' \quad \text{and} \quad S \subset F_{pq}^s \subset S', \quad (21)$$

for all admissible values of  $s, p, q$ , where “ $\subset$ ” always stands for topological embedding. We complement (21) by some further embedding assertions:

(i) Let  $s \in \mathbb{R}$ ,  $0 < p < \infty$  and  $0 < q \leq \infty$ , then

$$B_{pu}^s \subset F_{pq}^s \subset B_{pv}^s, \quad u = \min(p, q), \quad v = \max(p, q). \quad (22)$$

(ii) Let  $0 < p \leq \infty$  and  $0 < r \leq \infty$ ,  $0 < t \leq \infty$ , then

$$B_{pr}^s \subset B_{pt}^\sigma, \quad -\infty < \sigma < s < \infty. \quad (23)$$

These elementary embeddings follow easily from Definition 2.3.1. However, we are not going into details and refer to [Triß: 2.3.2, 2.3.3] which covers also (21), a different proof of the completeness of  $B_{pq}^s$  and  $F_{pq}^s$  and further elementary embeddings.

**Remark 3.** Let  $0 < q \leq \infty$ , then

$$B_{pq}^s \subset L_p \quad \text{if } 1 \leq p \leq \infty \quad \text{and } s > 0 \quad (24)$$

follows immediately from (2.3.1/6,7), including a corresponding assertion for  $F_{pq}^s$  (with  $p < \infty$ ). At first glance (24) seems also to be valid if  $0 < p < 1$ . But this is not correct and it sheds some light on a dangerous pitfall in the theory of function spaces with  $p < 1$ . If  $1 \leq p \leq \infty$  then the convergence of the right-hand side of (2.3.1/6) in  $L_p$  is also a convergence in  $S'$  to the same limit element. This is the basis in order to prove (24). If  $p < 1$  then this needs not to be the case. For example, let  $\varphi$  be the above function, then  $2^{jn}\hat{\varphi}(2^jx) \rightarrow \delta$  in  $S'$  if  $j \rightarrow \infty$  where  $\delta$  stands for the  $\delta$ -distribution, but

$$2^{jn}\hat{\varphi}(2^jx) \rightarrow 0 \quad \text{in } L_p \quad \text{if } j \rightarrow \infty, \quad \text{where } 0 < p < 1.$$

Rescue comes from the following embedding. Let  $0 < q \leq \infty$ , then

$$B_{pq}^s \subset L_1 \quad \text{if } 0 < p \leq 1 \quad \text{and } s > \sigma_p = n \left( \frac{1}{p} - 1 \right) \quad (25)$$

and a corresponding embedding for  $F_{pq}^s$ . This assertion follows from (2.3.1/7) and the Plancherel–Polya–Nikol'skij inequality

$$\|\varphi_k(D)f \mid L_1\| \leq c 2^{k\sigma_p} \|\varphi_k(D)f \mid L_p\|, \quad k \in \mathbb{N}, \quad (26)$$

see [Triß: (1.3.2/5) and Theorem 1.4.1(ii)].

### 2.3.3 Spaces with $s > \sigma_p$

Recall  $a_+ = \max(0, a)$  where  $a \in \mathbb{R}$  and

$$\sigma_p = n \left( \frac{1}{p} - 1 \right)_+ \quad \text{where } 0 < p \leq \infty, \quad (1)$$

see (2.2.3/1). By (2.3.2/24, 25) all elements  $f$  of  $B_{pq}^s$  or  $F_{pq}^s$  with  $s > \sigma_p$  are regular distributions, more precisely,  $f \in L_{\max(p,1)}$ . We wish to complement this observation. Let  $\varphi$  be the same  $C^\infty$  function as in 2.3.1, in particular we have (2.3.1/1). We

extend the definition of  $\varphi_j$  from (2.3.1/2) to all integers  $j$ . It should be noted that  $\varphi_0$  has now a different meaning as in 2.3.1.

**Theorem.** (i) Let  $0 < p \leq \infty$ ,  $0 < q \leq \infty$ , and  $s > \sigma_p$ , then

$$\|\varphi(D)f \mid L_p\| + \left( \sum_{j=-\infty}^{\infty} 2^{jsq} \|\varphi_j(D)f \mid L_p\|^q \right)^{1/q} \quad (2)$$

and

$$\|f \mid L_p\| + \left( \sum_{j=-\infty}^{\infty} 2^{jsq} \|\varphi_j(D)f \mid L_p\|^q \right)^{1/q} \quad (3)$$

(modification if  $q = \infty$ ) are equivalent quasi-norms in  $B_{pq}^s$ .

(ii) Let  $0 < p < \infty$ ,  $0 < q \leq \infty$ , and  $s > \sigma_p$ , then

$$\|\varphi(D)f \mid L_p\| + \left\| \left( \sum_{j=-\infty}^{\infty} 2^{jsq} |\varphi_j(D)f(\cdot)|^q \right)^{1/q} \mid L_p \right\| \quad (4)$$

and

$$\|f \mid L_p\| + \left\| \left( \sum_{j=-\infty}^{\infty} 2^{jsq} |\varphi_j(D)f(\cdot)|^q \right)^{1/q} \mid L_p \right\| \quad (5)$$

(modification if  $q = \infty$ ) are equivalent quasi-norms in  $F_{pq}^s$ .

**Proof.** Step 1. In order to prove that (2) and (4) are equivalent quasi-norms in  $B_{pq}^s$  and  $F_{pq}^s$ , respectively, it is sufficient to show that there exists a constant  $c > 0$  such that

$$\|\varphi_j(D)f \mid L_p\| \leq c 2^{-j\sigma_p} \|\varphi(D) \mid L_p\|, \quad -j \in \mathbb{N}, \quad (6)$$

holds. For those  $j$ 's we have  $\varphi_j(\xi) = \varphi_j(\xi)\varphi(\xi)$  and hence

$$\begin{aligned} \|\varphi_j(D)f \mid L_p\| &= \|(\varphi_j(\varphi(D)f)^\wedge)^\vee \mid L_p\| \\ &\leq c \|\check{\varphi}_j \mid L_r\| \|\varphi(D)f \mid L_p\|, \quad r = \min(1, p), \end{aligned} \quad (7)$$

where the equality comes from (2.3.1/5) and the inequality is a Fourier multiplier assertion for entire analytic functions proved in [Triß: Proposition 1.5.1]. By  $\check{\varphi}_j(x) = 2^{jn} \check{\varphi}_0(2^{jn}x)$  we obtain (6).

Step 2. We prove that (5) is an equivalent quasi-norm in  $F_{pq}^s$ . Because  $s > \sigma_p$  we may assume that (2.3.1/6) converges not only in  $S'$ , but also, say, almost everywhere in  $\mathbb{R}^n$ . Then we have

$$\|f \mid L_p\| \leq c\|\varphi(D)f \mid L_p\| + c\left(\sum_{j=1}^{\infty} \|\varphi_j(D)f \mid L_p\|^p\right)^{1/p} \quad (8)$$

if  $0 < p \leq 1$  and a corresponding estimate if  $1 < p < \infty$ . Now, (4) and (8) prove that (5) can be estimated from above by  $c\|f \mid F_{pq}^s\|$ . We prove the reverse inequality. Because  $f$  is a regular distribution we have, say, a.e.

$$\varphi(D)f(x) = f(x) + ((1 - \varphi(\cdot))\hat{f})^\vee(x) = f(x) + \sum_{j=0}^{\infty} ((1 - \varphi(\cdot))\varphi_j(\cdot)\hat{f})^\vee(x). \quad (9)$$

By the above-mentioned Fourier multiplier assertion we have

$$\|\varphi(D)f \mid L_p\| \leq c\|f \mid L_p\| + c\left(\sum_{j=0}^{\infty} \|\varphi_j(D)f \mid L_p\|^p\right)^{1/p} \quad (10)$$

if  $0 < p \leq 1$  and a corresponding estimate if  $1 < p < \infty$ . Now (4) and (10) prove that  $\|f \mid F_{pq}^s\|$  can be estimated from above by the quasi-norm in (5). In the same way one obtains that (3) is an equivalent quasi-norm in  $B_{pq}^s$ .

**Remark.** Later on, we shall see that any “discrete” quasi-norm of type (2–5) or also (2.3.1/7,8) has a “continuous” counterpart. To explain what this means we introduce

$$\rho(tD)f(x) = (\rho(t\cdot)\hat{f})^\vee(x), \quad t > 0, \quad x \in \mathbb{R}^n$$

with  $\rho(t\xi) = \varphi(t\xi) - \varphi(2t\xi)$ , see (2.3.1/2). Then the continuous counterparts of (3) and (5) read as follows:

(i) Let  $0 < p \leq \infty$ ,  $0 < q \leq \infty$ , and  $s > \sigma_p$ , then

$$\|f \mid L_p\| + \left(\int_0^\infty t^{-sq} \|\rho(tD)f \mid L_p\|^q \frac{dt}{t}\right)^{1/q} \quad (11)$$

(modification if  $q = \infty$ ) is an equivalent quasi-norm in  $B_{pq}^s$ .

(ii) Let  $0 < p < \infty$ ,  $0 < q \leq \infty$ , and  $s > \sigma_p$ , then

$$\|f \mid L_p\| + \left\| \left( \int_0^\infty t^{-sq} |\rho(tD)f(\cdot)|^q \frac{dt}{t} \right)^{1/q} \mid L_p \right\| \quad (12)$$

(modification if  $q = \infty$ ) is an equivalent quasi-norm in  $F_{pq}^s$ .

The quasi-norms (11) and (12) look more elegant than their discrete counterparts. Furthermore, both (11) and (12) is the sum of two homogeneous quasi-norms. Let  $\lambda > 0$ , then we have

$$\|f(\lambda \cdot) \mid L_p\| = \lambda^{-n/p} \|f \mid L_p\|, \quad (13)$$

and a corresponding assertion for the second terms in (11) and (12) with  $s - \frac{n}{p}$  instead of  $-\frac{n}{p}$ . The latter claim follows from

$$[\rho(tD)f(\lambda \cdot)](x) = ((\rho(\lambda t \cdot)\hat{f})^\vee)(\lambda x), \quad \lambda > 0. \quad (14)$$

## 2.4 General characterizations for $F_{pq}^s$

### 2.4.1 The main theorem

Let  $0 < p \leq \infty$  and  $0 < q \leq \infty$ , then we introduce the abbreviations

$$\sigma_p = n \left( \frac{1}{p} - 1 \right)_+ \quad \text{and} \quad \sigma_{pq} = n \left( \frac{1}{\min(p, q)} - 1 \right)_+ \quad (1)$$

which differ slightly from the corresponding notations in [Triß: 2.5.3]. Recall  $a_+ = \max(0, a)$  if  $a \in \mathbb{R}$ . Let  $h(x) \in S$  and  $H(x) \in S$  with

$$\text{supp } h \subset \{y \in \mathbb{R}^n: |y| \leq 2\}, \quad \text{supp } H \subset \left\{y \in \mathbb{R}^n: \frac{1}{4} \leq |y| \leq 4\right\}, \quad (2)$$

$$h(x) = 1 \quad \text{if } |x| \leq 1 \quad \text{and} \quad H(x) = 1 \quad \text{if } \frac{1}{2} \leq |x| \leq 2. \quad (3)$$

Recall our notation

$$\varphi(tD)f = (\varphi(t \cdot)\hat{f})^\vee, \quad t > 0. \quad (4)$$

In the theorem below  $\varphi$  need not be an element of  $S$ , and then it is not immediately clear what is meant by (4). It will be defined via limiting procedures, see Step 3 of the proof and the following subsection.

**Theorem.** Let  $0 < p < \infty$ ,  $0 < q \leq \infty$  and  $s \in \mathbb{R}$ . Let  $s_0$  and  $s_1$  be two real numbers with

$$s_0 + \sigma_{pq} < s < s_1 \quad \text{and} \quad s_1 > \sigma_p. \quad (5)$$

Let  $\varphi_0$  and  $\varphi$  be two complex-valued  $C^\infty$  functions on  $\mathbb{R}^n$  and  $\mathbb{R}^n \setminus \{0\}$ , respectively, which satisfy the Tauberian conditions

$$|\varphi_0(x)| > 0 \quad \text{if } |x| \leq 2 \quad (6)$$

and

$$|\varphi(x)| > 0 \quad \text{if } \frac{1}{2} \leq |x| \leq 2. \quad (7)$$

Let  $a > \frac{n}{\min(p,q)}$ ,

$$\int_{\mathbb{R}^n} \left| \left( \frac{\varphi(z)h(z)}{|z|^{s_1}} \right)^\vee(y) \right| (1+|y|)^a dy < \infty, \quad (8)$$

$$\sup_{m \in \mathbb{N}} 2^{-ms_0} \int_{\mathbb{R}^n} |(\varphi(2^m \cdot)H(\cdot))^\vee(y)| (1+|y|)^a dy < \infty, \quad (9)$$

and

$$\sup_{m \in \mathbb{N}} 2^{-ms_0} \int_{\mathbb{R}^n} |(\varphi_0(2^m \cdot)H(\cdot))^\vee(y)| (1+|y|)^a dy < \infty. \quad (10)$$

Let  $\varphi_j(x) = \varphi(2^{-j}x)$  if  $x \in \mathbb{R}^n \setminus \{0\}$  and  $j \in \mathbb{N}$ . Then

$$\left\| \left( \sum_{j=0}^{\infty} 2^{jsq} |\varphi_j(D)f(\cdot)|^q \right)^{1/q} \mid L_p \right\| \quad (11)$$

and

$$\|\varphi_0(D)f \mid L_p\| + \left\| \left( \int_0^1 t^{-sq} |\varphi(tD)f(\cdot)|^q \frac{dt}{t} \right)^{1/q} \mid L_p \right\| \quad (12)$$

(modification if  $q = \infty$ ) are equivalent quasi-norms in  $F_{pq}^s$ .

**Proof.** Step 1. Let  $f \in F_{pq}^s$ . In the first two steps we prove that the quasi-norms in (11) can be estimated from above by  $c\|f \mid F_{pq}^s\|$ . Let  $\{\rho_k(x)\}_{k \in \mathbb{N}_0}$  be a resolution of unity in the sense of (2.3.1/1-4) with  $\rho$  instead of  $\varphi$ . Then we have (2.3.1/8) with  $\rho$  instead of  $\varphi$ . Let  $\rho_l(x) = 0$  if  $-l \in \mathbb{N}$ . Let  $j \in \mathbb{N}_0$ , then we obtain

$$2^{js}(\varphi_j \hat{f})^\vee(x) = \sum_{l=-\infty}^{\infty} 2^{js}(\varphi_j \rho_{l+1} \hat{f})^\vee(x) = \sum_{l=-\infty}^K \cdots + \sum_{l=K+1}^{\infty} \cdots \quad (13)$$

where  $K \in \mathbb{N}$  will be chosen later on. Temporarily we take it for granted that  $(\varphi_j \hat{f})^\vee \in S'$  is a regular distribution and that (13) covers not only in  $S'$ , but also pointwise a.e. We estimate the first sum where we have no problems of convergence. Let

$$\tilde{\rho}_j(x) = |2^{-j}x|^{s_1} \rho_j(x) \quad \text{if } j \in \mathbb{N}_0. \quad (14)$$

Then we have

$$\left| \sum_{l=-\infty}^K 2^{js} (\varphi_j \rho_{l+j} \hat{f})^\vee(x) \right| \leq \sum_{l=-\infty}^K 2^{l(s_1-s)} \left| \left( \frac{\varphi_j(z)}{|2^{-j}z|^{s_1}} 2^{s(j+l)} \tilde{\rho}_{j+l}(z) \hat{f} \right)^\vee(x) \right|. \quad (15)$$

One can replace  $\varphi_j(z)$  on the right-hand side of (15) by  $\varphi_j(z)h(c2^{-j}z)$ , where  $c$  is an appropriate positive number which depends on  $K$ . Let  $j \in \mathbb{N}$  then we have  $\varphi_j(z) = \varphi(2^{-j}z)$  and  $|(\dots)^\vee(x)|$  on the right-hand side of (15) can be estimated from above by

$$\int_{\mathbb{R}^n} \left| \left( \frac{\varphi(2^{-j}z)}{|2^{-j}z|^{s_1}} h(c2^{-j}z) \right)^\vee(y) \right| |(2^{s(j+l)} \tilde{\rho}_{j+l} \hat{f})^\vee(x-y)| dy. \quad (16)$$

We apply  $(\lambda(2^{-j}\cdot))^\vee(y) = 2^{jn} \lambda^\vee(2^j y)$  to the first factor in (16) and replace afterwards  $2^j y$  by  $y$ . We use the maximal function from (2.3.2/1) with  $\rho$  instead of  $\varphi$  and obtain for  $l \leq K$

$$2^{s(j+l)} |(\tilde{\rho}_{j+l} \hat{f})^\vee(x - 2^{-j}y)| \leq c 2^{s(j+l)} (\tilde{\rho}_{j+l}^* f)_a(x) (1 + |y|)^a, \quad (17)$$

where  $c$  depends on  $K$ , but not on  $x, y, j$ , and  $l$ . By (8) it follows that (16) can be estimated from above by  $c 2^{s(j+l)} (\tilde{\rho}_{j+l}^* f)_a(x)$ . We return to (15) and have

$$\left| \sum_{l=-\infty}^K 2^{js} (\varphi_j \rho_{l+j} \hat{f})^\vee(x) \right| \leq c \sum_{l=-\infty}^K 2^{l(s_1-s)} 2^{s(j+l)} (\tilde{\rho}_{j+l}^* f)_a(x), \quad (18)$$

where  $j \in \mathbb{N}$ . We apply first the  $l_q$ -quasi-norm with respect to  $j$  and afterwards the  $L_p$ -quasi-norm with respect to  $x$ . Because  $s_1 > s$  we obtain

$$\left\| \left( \sum_{j=1}^{\infty} \left| \sum_{l=-\infty}^K 2^{js} (\varphi_j \rho_{l+j} \hat{f})^\vee(\cdot) \right|^q \right)^{1/q} \right\|_{L_p} \leq c \left\| \left( \sum_{m=0}^{\infty} 2^{msq} (\tilde{\rho}_m^* f)_a^q(x) \right)^{1/q} \right\|_{L_p}. \quad (19)$$

Because  $a > \frac{n}{\min(p,q)}$  we can use the vector-valued maximal inequality (2.2.4/3), only the term with  $\tilde{\rho}_0(x) = |x|^{s_1} \rho_0(x)$  is critical. We return to this point later on in Remark 1, then we prove

$$\|(\tilde{\rho}_0 \hat{f})^\vee \mid L_p\| \leq c \|(\rho_0 \hat{f})^\vee \mid L_p\|, \quad (20)$$

where we have to use  $s_1 > \sigma_p$ . Now the maximal inequality can be applied and (19) can be estimated from above by

$$c \left\| \left( \sum_{m=0}^{\infty} 2^{msq} |(\tilde{\rho}_m \hat{f})^\vee(\cdot)|^q \right)^{1/q} \mid L_p \right\|. \quad (21)$$

Finally, by (20) and (2.2.4/4), we can estimate (21), and consequently also (19), from above by

$$c \left\| \left( \sum_{m=0}^{\infty} 2^{msq} |(\rho_m \hat{f})^\vee(\cdot)|^q \right)^{1/q} \mid L_p \right\| = c \|f \mid F_{pq}^s\|_\rho. \quad (22)$$

The term with  $j = 0$  can be incorporated afterwards. In other words, we have

$$\left\| \left( \sum_{j=0}^{\infty} \left| \sum_{\rho=-\infty}^K 2^{js} (\varphi_j \rho_{l+j} \hat{f})^\vee(\cdot) \right|^q \right)^{1/q} \mid L_p \right\| \leq c \|f \mid F_{pq}^s\|, \quad (23)$$

where  $c$  depends on  $K$ .

Step 2. We estimate the second sum in (13) and we have to make sure now that (13) converges a.e. and also in some space  $L_r$  with  $1 \leq r \leq \infty$ . However the latter comes out as a by-product, see Step 3. Similar as in (14) we introduce

$$\rho'_j(x) = |2^{-j}x|^{s_0} \rho_j(x), \quad j \in \mathbb{N}. \quad (24)$$

Instead of (15) we have now

$$\begin{aligned} & \left| \sum_{l=K+1}^{\infty} 2^{js} (\varphi_j \rho_{l+j} \hat{f})^\vee(x) \right| \\ & \leq \sum_{l=K+1}^{\infty} 2^{l(s_0-s)} \left| \left( \frac{\varphi_j(z)}{|2^{-j}z|^{s_0}} H(2^{-j-l}z) 2^{s(j+l)} \rho'_{j+l}(z) \hat{f} \right)^\vee(x) \right|. \end{aligned} \quad (25)$$

We have obvious counterparts of (16) and (17), where we replace  $\tilde{\rho}$  and  $x - 2^{-j}y$  by  $\rho'$  and  $x - 2^{-j-l}y$ , respectively. The crucial integral looks like

$$\int_{\mathbb{R}^n} \left| \left( \frac{\varphi(2^l z)}{|2^l z|^{s_0}} H(z) \right)^\vee(y) \right| (1+|y|)^a dy, \quad (26)$$

which in turn can be estimated from above by

$$c 2^{-ls_0} \int_{\mathbb{R}^n} |(\varphi(2^l \cdot) H(\cdot))^\vee(y)| (1+|y|)^a dy, \quad (27)$$

see (2.2.3/2). Together with its  $\varphi_0$ -counterpart, (9) and (10), we arrive in the same way as in the first step at

$$\left| \sum_{l=K+1}^{\infty} 2^{js} (\varphi_j \rho_{l+j} \hat{f})^\vee(x) \right| \leq c \sum_{l=K+1}^{\infty} 2^{l(s_0-s)} 2^{(j+l)s} (\rho_{j+l}^* f)_a(x) \quad (28)$$

and

$$\left\| \left( \sum_{j=0}^{\infty} \left| \sum_{l=K+1}^{\infty} 2^{js} (\varphi_j \rho_{l+j} \hat{f})^\vee(x) \right|^q \right)^{1/q} \mid L_p \right\| \leq c 2^{K(s-s_0)} \|f \mid F_{pq}^s\|, \quad (29)$$

where  $c$  is independent of  $K$ . We used  $s_0 < s$ . Now (13), (23), and (29) prove that the quasi-norm in (11) can be estimated from above by  $c \|f \mid F_{pq}^s\|$ .

Step 3. We postponed the technical discussion of  $(\varphi_j \hat{f})^\vee$  with  $f \in F_{pq}^s$ . First we remark that there are no problems with  $(\varphi_j \rho_m \hat{f})^\vee(x)$  if  $j \in \mathbb{N}_0$  and  $m \in \mathbb{N}$ . Recall that  $\varphi$  need not be  $C^\infty$  near the origin, hence  $(\varphi_j \rho_0 \hat{f})^\vee$  with  $j \in \mathbb{N}$  must be treated separately. We have (18) term by term, in particular

$$|(\varphi_j \rho_0 \hat{f})^\vee(x)| \leq c(\tilde{\rho}_0^* f)_a(x). \quad (30)$$

Taking (20) for granted, see Remark 1 below, then the scalar case of Theorem 2.2.4 proves  $(\varphi_j \rho_0 \hat{f})^\vee \in L_p \cap S'$  in any case. (A more detailed discussion of  $L_p$  spaces of entire analytic functions, related Fourier multipliers and maximal functions has been given in [Triß: Chapter 1]). We have to care about the convergence on the right-hand side of (13), both pointwise a.e. and in  $S'$ . We can rewrite (28) as follows:

$$\left| \sum_{l=K}^L 2^{js} (\varphi_j \rho_{l+j} \hat{f})^\vee(x) \right| \leq c \sum_{l=K}^L 2^{l(s_0-s)} 2^{(j+l)s} (\rho_{j+l}^* f)_a(x) \quad (31)$$

with  $L > K$ . If  $K$  is large then the right-hand side of (31) can be estimated from above by

$$\varepsilon \left( \sum_{l=K}^{\infty} 2^{lsq} (\rho_l^* f)_a^q(x) \right)^{1/q}, \quad (32)$$

where  $\varepsilon > 0$  is given. By Theorem 2.3.2 and the above considerations the last expression is finite a.e. This proves the desired pointwise convergence. We prove the  $S'$ -convergence and assume  $0 < p < 1$ . Let  $\sigma = s - \sigma_p$ , then we have the embedding  $F_{pq}^s \subset F_{1q}^\sigma$ , see [Triß: 2.7.1] and (1). We reformulate (31) by

$$\left| \sum_{l=K}^L 2^{j\sigma} (\varphi_j \rho_{l+j} \hat{f})^\vee(x) \right| \leq c \sum_{l=K}^L 2^{l(s_0-\sigma)} 2^{(j+l)\sigma} (\rho_{j+l}^* f)_a(x). \quad (33)$$

By (5) we have  $\sigma = s - \sigma_p > s_0$ . Via the counterpart of (32) and the  $L_1$ -convergence, resulting from the indicated embedding, it follows that (33), and hence (13), converges in  $L_1$ . If  $1 \leq p < \infty$ , then we have by the same arguments a  $L_p$ -convergence. What has been done in the first two steps is now completely justified.

Step 4. Let again  $f \in F_{pq}^s$ . We prove that the quasi-norm in (12) can be estimated from above by  $c\|f\|_{F_{pq}^s}$ . We indicate the necessary modifications in the first two steps. Instead of  $\varphi_j(x) = \varphi(2^{-j}x)$  we have now  $\varphi(tx)$  with, say,  $2^{-j} \leq t \leq 2^{-j+1}$ . We use the same splitting as in (13). We follow (14–18), where we can replace now the left-hand side of (18) by

$$\sup_{2^{-j} \leq t \leq 2^{-j+1}} \left| \sum_{l=-\infty}^K 2^{js} (\varphi(t \cdot) \rho_{l+j} \hat{f})^\vee(x) \right| \quad (34)$$

with the corresponding counterpart in (19). Similarly in (28) and hence in (29). In other words, the estimates in the first two steps yield

$$\|\varphi_0(D)f\|_{L_p} + \left\| \left( \sum_{j=1}^{\infty} 2^{jsq} \sup_{2^{-j} \leq t \leq 2^{-j+1}} |\varphi(tD)f(\cdot)|^q \right)^{1/q} \right\|_{L_p} \leq c\|f\|_{F_{pq}^s} \quad (35)$$

which is even stronger than the desired estimate.

Step 5. Let  $f \in F_{pq}^s$ . We prove that  $\|f\|_{F_{pq}^s}$  from (22) can be estimated from above by the quasi-norm in (11). Let  $\psi \in S$  be a function with  $\text{supp } \psi \subset \{y \in \mathbb{R}^n: |y| \leq 2^{K+1}\}$  and  $\psi(x) = 1$  if  $|x| \leq 2^K$ , where we choose the natural number  $K$  later on. By (6,7) and the properties of  $\varphi_j$  and  $\rho_j(x)$  we have

$$\begin{aligned} |(\rho_j \hat{f})^\vee(x)| &= |(\rho_j \psi(2^{-j} \cdot) \hat{f})^\vee(x)| \\ &\leq c \int_{\mathbb{R}^n} \left| \left( \frac{\rho_j}{\varphi_j} \right)^\vee(y) (\varphi_j \psi(2^{-j} \cdot) \hat{f})^\vee(x-y) \right| dy, \quad j \in \mathbb{N}_0. \end{aligned} \quad (36)$$

For fixed  $x \in \mathbb{R}^n$  the Fourier transform of the  $y$ -function in the integral in (36) has a support contained in a ball of radius  $c2^{j+K}$ , where  $c$  is independent of  $x$ ,  $j$ , and  $K$ . Let  $0 < r < \min(1, p, q)$ . We use again an inequality of Plancherel–Polya–Nikol'skij type for entire analytic functions, see [Triß: (1.3.2/5)], and obtain

$$|(\rho_j \hat{f})^\vee(x)|^r \leq c 2^{(j+K)n(1-r)} \int_{\mathbb{R}^n} \left| \left( \frac{\rho_j}{\varphi_j} \right)^\vee(y) (\varphi_j \psi(2^{-j} \cdot) \hat{f})^\vee(x-y) \right|^r dy. \quad (37)$$

Let  $j \in \mathbb{N}$  then we have  $\varphi_j(x) = \varphi(2^{-j}x)$  and  $\rho_j(x) = \bar{\rho}(2^{-j}x)$  with  $\bar{\rho}(x) = \rho(x) - \rho(2x)$ , see (2.3.1/2), and

$$\left| \left( \frac{\rho_j}{\varphi_j} \right)^\vee(y) \right|^r = 2^{jnr} \left| \left( \frac{\bar{\rho}}{\varphi} \right)^\vee(2^j y) \right|^r \leq c 2^{jnr} (1 + |2^j y|)^{-b}, \quad (38)$$

where  $b > 0$  is at our disposal. A corresponding estimate holds for  $j = 0$ . We put (38) in (37) and obtain

$$\begin{aligned} & |(\rho_j \hat{f})^\vee(x)|^r \\ & \leq c 2^{(j+K)n(1-r)+jnr} \sum_{l=0}^{\infty} 2^{-ld} \int_{\{y \in \mathbb{R}^n: |y| < 2^{-j+l}\}} |(\varphi_j \psi(2^{-j}\cdot) \hat{f})^\vee(x-y)|^r dy, \end{aligned} \quad (39)$$

where  $d > 0$  is at our disposal. The integrals in (39) can be estimated from above by

$$c 2^{-jn+ln} (M|(\varphi_j \psi(2^{-j}\cdot) \hat{f})^\vee|^r)(x), \quad (40)$$

where  $M$  stands for the Hardy–Littlewood maximal function. We put (40) in (39), choose  $d > n$ , and arrive at

$$|(\rho_j \hat{f})^\vee(x)|^r \leq c 2^{Kn(1-r)} (M|(\varphi_j \psi(2^{-j}\cdot) \hat{f})^\vee|^r)(x). \quad (41)$$

Recall  $1 < \frac{p}{r} < \infty$  and  $1 < \frac{q}{r} \leq \infty$ . We multiply (41) with  $2^{jsr}$ , apply the  $l_{q/r}$ -norm with respect to  $j$  and afterwards the  $L_{p/r}$ -norm with respect to  $x$ , then we obtain by Theorem 2.2.2

$$\begin{aligned} & \left\| \left( \sum_{j=0}^{\infty} |2^{js} (\rho_j \hat{f})^\vee(\cdot)|^q \right)^{1/q} \mid L_p \right\|^r \leq \\ & c 2^{Kn(1-r)} \left\| \left[ \sum_{j=0}^{\infty} (M|2^{js} \varphi_j \psi(2^{-j}\cdot) \hat{f})^\vee|^r(\cdot))^{q/r} \right]^{r/q} \mid L_{p/r} \right\|^r \\ & \leq c' 2^{Kn(1-r)} \left\| \left( \sum_{j=0}^{\infty} |2^{js} (\varphi_j \psi(2^{-j}\cdot) \hat{f})^\vee|^q \right)^{1/q} \mid L_p \right\|^r, \end{aligned} \quad (42)$$

where  $c$  and  $c'$  are independent of  $K$ . Because

$$\varphi_j \psi(2^{-j}\cdot) = \varphi_j - \varphi_j(1 - \psi(2^{-j}\cdot)) \quad (43)$$

the right-hand side of (42) can be estimated from above by the  $r$ th power of the quasi-norm in (11) (this is just what we want) and the additional term

$$c 2^{Kn(1-r)} \left\| \left( \sum_{j=0}^{\infty} 2^{j n q} |(\varphi_j(1 - \psi(2^{-j}\cdot)) \hat{f})^\vee(\cdot)|^q \right)^{1/q} \mid L_p \right\|^r. \quad (44)$$

However this term can be treated in the same way as in the second step, in particular, we have a counterpart of the estimate in (29). Hence the term in (44) can be estimated from above by

$$c 2^{Kn(1-r)} 2^{-Kr(s-s_0)} \|f \mid F_{pq}^s\|^r. \quad (45)$$

By (5) we may assume  $n(\frac{1}{r} - 1) - (s - s_0) < 0$ . Recall that the natural number  $K$  is at our disposal. We choose  $K$  large. Then the term in (45) can be estimated from above, say, by  $\frac{1}{2}\|f\|_{F_{pq}^s}^r$ . Now (42) and the above splitting yield the desired result.

Step 6. Let  $f \in F_{pq}^s$ . We estimate  $\|f\|_{F_{pq}^s}$  from (22) from above by the quasi-norm in (12). We follow Step 5 and indicate the necessary changes. First we replace  $\varphi_j(\xi) = \varphi(2^{-j}\xi)$  in (36) with  $j \in \mathbb{N}$  by  $\varphi_j(\tau\xi) = \varphi(\tau 2^{-j}\xi)$  with  $1 \leq \tau \leq 2$ . This makes sense, maybe after immaterial changes of  $\rho_j$ . Then we have (39) with  $\varphi(\tau 2^{-j}\xi)$  instead of  $\varphi(2^{-j}\xi)$ . We integrate over  $\tau$ . Besides the factor  $c'2^{Kn(1-r)}$  and the term with  $j = 0$  the modified right-hand side of (42) looks as follows,

$$\left\| \left[ \sum_{j=1}^{\infty} \left( 2^{jsr} \int_1^2 |(\varphi(\tau 2^{-j}\cdot)\hat{f})^\vee(\cdot)|^r d\tau \right)^{q/r} \right]^{r/q} \right\|_{L_{p/r}}. \quad (46)$$

We use (43) with  $\varphi(\tau 2^{-j}\xi)$  instead of  $\varphi(2^{-j}\xi)$ . The remainder term can be treated as above, see (35). The main term, i.e., (46) without the factor  $\psi(2^{-j}\cdot)$ , can be estimated from above by

$$\begin{aligned} & \left\| \left[ \int_1^2 \sum_{j=1}^{\infty} 2^{jsq} |(\varphi(\tau 2^{-j}\cdot)\hat{f})^\vee(\cdot)|^q \right]^{r/q} \right\|_{L_{p/r}} \\ & \leq c \left\| \left( \int_0^1 t^{-sq} |(\varphi(t\cdot)\hat{f})^\vee(\cdot)|^q \right)^{1/q} \right\|_{L_p}, \end{aligned} \quad (47)$$

which is the desired result. We used  $r < q$ .

**Remark 1.** We prove (20). This problem can be reduced to

$$\|(|x|^{s_1}\sigma(x)\hat{g})^\vee\|_{L_p} \leq c\|g\|_{L_p} \quad (48)$$

where  $\sigma \in S$  with  $\sigma(x) = 1$  if  $|x| \leq 1$  and  $\sigma(x) = 0$  if  $|x| \geq 2$ , and  $g \in L_p \cap S'$  is an arbitrary function with  $\text{supp } \hat{g} \subset \{y \in \mathbb{R}^n: |y| \leq 1\}$ . By (2.2.3/2) this estimate is valid if

$$|x|^{s_1}\sigma(x) \in H_2^s \quad \text{with} \quad s > \sigma_p + \frac{n}{2}. \quad (49)$$

Because  $s_1 > \sigma_p$  we may assume  $s_1 + \frac{n}{2} > s > \sigma_p + \frac{n}{2}$ . Let  $\lambda(x)$  be a  $C^\infty$  function on  $\mathbb{R}^n$  with  $\lambda(x) = 1$  if  $|x| \geq 2$  and  $\lambda(x) = 0$  if  $|x| \leq 1$ . Then  $\{|x|^{s_1}\sigma(x)\lambda(2^j x)\}_{j=1}^\infty$  is a fundamental sequence in  $H_2^s$ . This follows by straightforward calculations if  $s \in \mathbb{N}$ . For fractional numbers  $s$  it is a matter of interpolation or of the so-called multiplicative inequalities for fractional Sobolev spaces. This completes the proof of (48) and hence of (20).

**Remark 2.** The following observation will be of some use later on. In Steps 1, 2, and 4 of the above proof, we used only  $s_0 < s < s_1$  and that (13) converges in  $S'$  and pointwise a.e. Under these assumptions the quasi-norms in (11) and (12) can be estimated from above by  $c\|f\|_{F_{pq}^s}$ .

It will be useful to give conditions (8–10) a more handsome reformulation. Recall that  $H_2^\sigma$  are the fractional Sobolev spaces on  $\mathbb{R}^n$  (Sobolev spaces if  $\sigma \in \mathbb{N}_0$ ).

**Corollary 1.** Let  $p, q, s, s_0, s_1$ , and  $a$  be the same numbers as in the Theorem. Let  $\sigma > a + \frac{n}{2}$ . Let  $\varphi_0$  and  $\varphi$  be two complex-valued  $C^\infty$  functions on  $\mathbb{R}^n$  and  $\mathbb{R}^n \setminus \{0\}$ , respectively, which satisfy the Tauberian conditions (6), (7), and

$$\left\| \frac{\varphi(x)h(x)}{|x|^{s_1}} \mid H_2^\sigma \right\| < \infty, \quad (50)$$

$$\sup_{m \in \mathbb{N}} 2^{-ms_0} \|\varphi(2^m \cdot)H(\cdot) \mid H_2^\sigma\| < \infty, \quad (51)$$

$$\sup_{m \in \mathbb{N}} 2^{-ms_0} \|\varphi_0(2^m \cdot)H(\cdot) \mid H_2^\sigma\| < \infty, \quad (52)$$

where  $h(x)$  and  $H(x)$  have the same meaning as in (2), (3). Let  $\varphi_j(x) = \varphi(2^{-j}x)$  if  $x \in \mathbb{R}^n \setminus \{0\}$  and  $j \in \mathbb{N}$ . Then (11) and (12) are equivalent quasi-norms in  $F_{pq}^s$ .

**Proof.** Let  $0 < r \leq 1$  and  $b \in \mathbb{R}$ , then

$$\|\hat{\lambda}(y)(1 + |y|)^b \mid L_r\| \leq c\|\lambda \mid H_2^\delta\| \quad (53)$$

with  $\delta > n(\frac{1}{r} - \frac{1}{2}) + b$ . This is a well-known estimate, a proof may be found, e.g., in [ScT: 1.7.5], see also [Triß: 1.5.2, 1.5.4] for further informations. However, (53) with  $r = 1$  and  $a = b$  shows that the terms (8–10) can be estimated from above by the corresponding terms in (50–52).

**Corollary 2.** Let  $0 < p < \infty$ ,  $0 < q \leq \infty$  and  $s \in \mathbb{R}$ . Let  $a > \frac{n}{\min(p,q)}$ . Let  $s_0$  and  $s_1$  be two real numbers with

$$s_0 + a < s < s_1 \quad \text{with} \quad s_1 > \sigma_p. \quad (54)$$

Let  $\varphi_0$  and  $\varphi$  be two complex-valued  $C^\infty$  functions on  $\mathbb{R}^n$  and  $\mathbb{R}^n \setminus \{0\}$ , respectively, which satisfy the Tauberian conditions (6), (7), and either (8–10) or (50–52). Let  $\varphi_j(x) = \varphi(2^{-j}x)$  and

$$(\varphi_k^* f)_a(x) = \sup_{y \in \mathbb{R}^n} \frac{|\varphi_k(D)f(x-y)|}{1 + |2^k y|^a}, \quad k \in \mathbb{N}_0, \quad (55)$$

be the corresponding maximal functions, see (2.3.2/1). Then

$$\left\| \left( \sum_{k=0}^{\infty} 2^{ksq} (\varphi_k^* f)_a(\cdot)^q \right)^{1/q} \mid L_p \right\| \quad (56)$$

(modification if  $q = \infty$ ) is an equivalent quasi-norm in  $F_{pq}^s$ .

**Proof.** One can follow the proof of the Theorem. We indicate the necessary modifications. We begin with (13) with  $x - 2^{-j}z$  instead of  $x$ . Then we use (17) with  $x - 2^{-j}y - 2^{-j}z$  instead of  $x - 2^{-j}y$  and the additional factor  $(1 + |z|)^a$  on the right-hand side. We divide both sides of the modified estimate (18) by  $(1 + |z|)^a$  and take afterwards the supremum with respect to  $z \in \mathbb{R}^n$ . This yields (23) with

$$\sup_{z \in \mathbb{R}^n} \frac{|(\varphi_j \rho_{l+j} \hat{f})^\vee(x - 2^{-j}z)|}{(1 + |z|)^a} \quad \text{instead of} \quad (\varphi_j \rho_{l+j} \hat{f})^\vee(x). \quad (57)$$

We modify the second step of the proof of the Theorem in the same way. We obtain (28) with  $x - 2^{-j}z$  on the left-hand side and the additional factor  $2^{la}(1 + |z|)^a$  on the right-hand side. Because now  $s_0 + a < s$  we obtain (29) with the same substitute as in (57). This proves that the quasi-norm in (56) can be estimated from above by  $c\|f\|_{F_{pq}^s} \| \cdot \|_\rho$ . The other direction follows from the Theorem because  $\sigma_{pq} < a$ .

**Remark 3.** In this subsection we followed closely [Tri16]. On the other hand, considerations of the above type are not new, both for the spaces  $F_{pq}^s$  and  $B_{pq}^s$ , and their homogeneous counterparts, see [Triß: 2.3.6, in particular, Remark 3] where we have given some references to preceding papers. Homogeneous spaces of type  $F_{pq}^s$  and  $B_{pq}^s$  will not be treated in this book, but in [Tri16] one finds detailed formulations. Conditions of type (5) or (54), and their even better counterparts for the spaces  $B_{pq}^s$ , see 2.5, improve essentially corresponding assertions in [Triß: 2.3.6]. Furthermore, the great service of the Tauberian conditions (6), (7) was simply overlooked in [Triß]. A first improvement was obtained in [Tri12], mostly for the spaces  $B_{pq}^s$ . Tauberian conditions have a long history. As far as the systematic use of ideas of the above type in the theory of function spaces and related problems in approximation theory is concerned, we refer to H.S. Shapiro, see [Sha1,2]. Furthermore, N.M. Rivi re and W.R. Madych developed in [Riv, MaR] this method in great detail in order to study H lder spaces. Some results in this connection for the homogeneous spaces of type  $B_{p\infty}^s$  with  $1 \leq p \leq \infty$  may also be found in [Pee6: Chapter 8], see also [Bom] for

further informations and references. Recently G.A. Kaljabin obtained in [Ka15, 6] characterizations of spaces of  $F_{pq}^s$  type with  $s > 0$ ,  $1 < p < \infty$ ,  $1 < q < \infty$  which are at least partly closely connected with the results in 2.4. But his method is rather different.

## 2.4.2 Characterizations

In Theorem 2.4.1 we described equivalent quasi-norms in  $F_{pq}^s$ . The question arises whether any  $f \in S'$  such that (2.4.1/11) or (2.4.1/12) is finite belongs to  $F_{pq}^s$ . This is not covered by our method because we used in Step 5 of the proof of Theorem 2.4.1 that  $f$  is an element of  $F_{pq}^s$ . Now we discuss this problem in some detail where we restrict ourselves to the quasi-norm in (2.4.1/11). But corresponding assertions can be obtained for the quasi-norm in (2.4.1/12). First we recall how to understand

$$\varphi_j(D)f(x) = (\varphi_j \hat{f})^\vee(x), \quad x \in \mathbb{R}^n, \quad j \in \mathbb{N}_0, \quad (1)$$

where the  $\varphi_j$ 's are the same functions as in Theorem 2.4.1 and  $f \in F_{pq}^s$ . Let again  $\{\rho_k(x)\}_{k \in \mathbb{N}_0}$  be a resolution of unity in the sense of (2.3.1/1–4) with  $\rho$  instead of  $\varphi$ . Then  $\|f\|_{F_{pq}^s}^\rho$  has the same meaning as in (2.3.1/8) with  $\rho$  instead of  $\varphi$ . As an abbreviation we introduce temporarily

$$\|f\|_{F_{pq}^s}^\varphi = \left\| \left( \sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \hat{f})^\vee(\cdot)|^q \right)^{1/q} \right\|_{L_p}. \quad (2)$$

In Step 3 of the proof of Theorem 2.4.1 we explained

$$(\varphi_j \hat{f})^\vee(x) = \sum_{l=0}^{\infty} (\varphi_j \rho_l \hat{f})^\vee(x), \quad f \in F_{pq}^s, \quad j \in \mathbb{N}_0, \quad (3)$$

where each summand on the right-hand side is well-defined, and (3) converges both pointwise a.e. and in  $L_{\bar{p}}$  with  $\bar{p} = \max(p, 1)$ . In particular, (3) converges in  $S'$ . We extend these considerations somewhat and assume

$$s_0 < \sigma = s - \frac{n}{p} \quad (4)$$

in addition to the assumptions from Theorem 2.4.1. Recall the embedding assertion

$$F_{pq}^s \subset B_{\infty\infty}^\sigma, \quad (5)$$

see [Triß: 2.7.1]. By the reasoning given in Step 3 of the proof of Theorem 2.4.1 it follows that (3) converges uniformly, in particular  $(\varphi_j \hat{f})^\vee \in L_\infty$ . If  $f \in S'$  then  $(\varphi_j \hat{f})^\vee \in L_\kappa$  with  $1 \leq \kappa \leq \infty$  must always be understood as convergence of (3) in  $L_\kappa$ . Finally let  $\psi \in S$  with

$$\text{supp } \psi \subset \{y \in \mathbb{R}^n, |y| < 2\} \quad \text{and} \quad \psi(x) = 1 \quad \text{if } |x| \leq 1. \quad (6)$$

If  $f \in S'$  then we put temporarily  $f^k = (\psi(2^{-k}\cdot)\hat{f})^\vee$ . We have

$$f^k \rightarrow f \quad \text{in } S' \quad \text{if } k \rightarrow \infty. \quad (7)$$

All other notations have the same meaning as in 2.4.1, in particular  $\sigma_p$ ,  $\sigma_{pq}$ , and  $a$ . Furthermore we assume in the theorem below that  $\varphi_0$  and  $\varphi$  are two complex-valued  $C^\infty$  functions on  $\mathbb{R}^n$  and  $\mathbb{R}^n \setminus \{0\}$ , respectively, which satisfy (2.4.1/6–10) (or (2.4.1/50–52) instead of (2.4.1/8–10), respectively) with the indicated values of  $s_0$  and  $s_1$ . The maximal function  $(\varphi_k^* f)_a$  has been defined in (2.4.1/55).

**Theorem.** (i) Let  $1 < p < \infty$ ,  $1 < q \leq \infty$  and  $s \in \mathbb{R}$ . Let  $s_0$  and  $s_1$  be two real numbers with

$$s_0 < s < s_1 \quad \text{and} \quad s_1 > \sigma_p, \quad (8)$$

then

$$F_{pq}^s = \{f \in S': \|f \mid F_{pq}^s\|^\varphi < \infty\} \quad (9)$$

(equivalent norm).

(ii) Let  $0 < p < \infty$ ,  $0 < q \leq \infty$  and  $s \in \mathbb{R}$ . Let  $s_0$  and  $s_1$  be two real numbers with

$$s_0 + \sigma_{pq} < s < s_1 \quad \text{and} \quad s_1 > \sigma_p, \quad (10)$$

let  $\bar{p} = \max(p, 1)$ , then

$$F_{pq}^s = \left\{ f \in S': (\varphi_j \hat{f})^\vee \in L_{\bar{p}} \quad \text{for } j \in \mathbb{N}_0 \quad \text{and} \quad \varliminf_{k \rightarrow \infty} \|f^k \mid F_{pq}^s\|^\varphi < \infty \right\}. \quad (11)$$

Furthermore, both

$$\varliminf_{k \rightarrow \infty} \|f^k \mid F_{pq}^s\|^\varphi \quad \text{and} \quad \|f \mid F_{pq}^s\|^\varphi \quad (12)$$

are equivalent quasi-norms in  $F_{pq}^s$ .

(iii) Let  $0 < p < \infty$ ,  $0 < q \leq \infty$  and  $s \in \mathbb{R}$ . Let  $s_0$  and  $s_1$  be two real numbers with

$$s_0 + a < s < s_1, \quad s_1 > \sigma_p, \quad (13)$$

then

$$F_{pq}^s = \left\{ f \in S': (\varphi_j \hat{f})^\vee \in L_\infty \quad \text{for } j \in \mathbb{N}_0 \quad \text{and} \right\} \quad (14)$$

$$\left\| \left( \sum_{k=0}^{\infty} 2^{ksq} (\varphi_k^* f)_a(\cdot)^q \right)^{1/q} \mid L_p \right\| < \infty \Big\}$$

(equivalent quasi-norm).

**Proof.** Step 1. Let  $f \in F_{pq}^s$ , then we have the desired convergence in (3), including  $(\varphi_j \hat{f})^\vee \in L_\infty$  under the hypotheses of (iii) because  $a > \frac{n}{p}$ . Furthermore we have  $f^j \in L_p$  with  $j \in \mathbb{N}_0$ , and

$$\|f^j \mid F_{pq}^s\|_\rho \leq c \|f \mid F_{pq}^s\|_\rho, \quad (15)$$

where the latter follows from the definition (2.3.1/8), with  $\rho$  instead of  $\varphi$ , and the Fourier multiplier assertion (2.2.4/4), see also [Triß: 2.3.7] for a more explicit version of a Fourier multiplier theorem in  $F_{pq}^s$ . The constant  $c$  in (15) is independent of  $j$ . Hence, we have

$$\varliminf_{k \rightarrow \infty} \|f^k \mid F_{pq}^s\|_\rho \leq c \|f \mid F_{pq}^s\|_\rho. \quad (16)$$

The reverse estimate follows from  $(\rho_l \hat{f}^k)^\vee(x) \rightarrow (\rho_l \hat{f})^\vee(x)$  if  $k \rightarrow \infty$  pointwise and Fatou's lemma. Hence the quasi-norms in (12) are equivalent. Consequently, if  $f \in F_{pq}^s$  then the right-hand sides of (9,11) and the equivalence in (12) are covered by Theorem 2.4.1 and the above observations, under the respective hypotheses for the involved parameters. Corollary 2.4.1/2 covers the right-hand side of (14).

Step 2. Let the hypotheses of part (i) be satisfied. Then  $\|f \mid F_{pq}^s\|^\varphi < \infty$  includes  $(\varphi_j \hat{f})^\vee \in L_p$  in the sense of the  $L_p$ -convergence of (3). Let  $\{\rho_k\}_{k \in \mathbb{N}_0}$  be the above system of functions. Similarly as in (2.4.1/36) we have

$$|(\rho_j \hat{f})^\vee(x)| \leq c \int_{\mathbb{R}^n} \left| \left( \frac{\rho_j}{\varphi_j} \right)^\vee(y) \right| |(\varphi_j \hat{f})^\vee(x-y)| dy, \quad j \in \mathbb{N}_0. \quad (17)$$

By the same reasoning as after (2.4.1/36) we arrive at (2.4.1/41) with  $r = 1$  and without the factor  $\psi(2^{-j} \cdot)$ . We apply Theorem 2.2.2 to this modified estimate and obtain

$$\left\| \left( \sum_{j=0}^{\infty} 2^{jsq} |(\rho_j \hat{f})^\vee(\cdot)|^q \right)^{1/q} \mid L_p \right\| \leq c \|f \mid F_{pq}^s\|^\varphi \quad (18)$$

(modification if  $q = \infty$ ), and, consequently,  $f \in F_{pq}^s$ , see (2.3.1/8). The proof of (i) is complete.

Step 3. We prove (ii). Let  $f \in S'$  such that the right-hand side of (11) holds. Then we have

$$(\varphi_j \hat{f}^k)^\vee(x) = (\psi(2^{-k} \cdot) \varphi_j(\cdot) \hat{f}(\cdot))^\vee(x) = 2^{kn} \int_{\mathbb{R}^n} \psi^\vee(2^k y) (\varphi_j \hat{f})^\vee(x - y) dy, \quad (19)$$

and, in particular,  $(\varphi_j \hat{f}^k)^\vee \in L_{\bar{p}}$ . Let  $j \geq k$ , then the same argument as in (2.4.1/37) yields

$$|(\rho_j \hat{f}^k)^\vee(x)|^r \leq c 2^{jn(1-r)} \int_{\mathbb{R}^n} \left| \left( \frac{\rho_j}{\varphi_j} \right)^\vee(y) (\varphi_j \hat{f}^k)^\vee(x - y) \right|^r dy \quad (20)$$

with  $0 < r < \min(1, p, q)$ , and we arrive at the counterpart of (2.4.1/41),

$$|(\rho_j \hat{f}^k)^\vee(x)|^r \leq c(M|(\varphi_j \hat{f}^k)^\vee|^r)(x). \quad (21)$$

This estimate can be extended to  $k < j$ , where  $c$  may depend on  $k$ . Now by our assumption it follows  $f^k \in F_{pq}^s$ , in the same way as after (2.4.1/41). Hence, Theorem 2.4.1 yields

$$\left\| \left( \sum_{j=0}^N 2^{jsq} |(\rho_j \hat{f}^k)^\vee(\cdot)|^q \right)^{1/q} \right\|_{L_p} \leq c \|f^k\|_{F_{pq}^s}^\varphi, \quad (22)$$

where  $c$  is independent of  $N$  and  $k$ . We used that both sides of (22) are equivalent quasi-norms in  $F_{pq}^s$  at least if  $N$  is large. By  $(\rho_j \hat{f}^k)^\vee(x) \rightarrow (\rho_j \hat{f})^\vee(x)$  and Fatou's lemma it follows  $f \in F_{pq}^s$ . Together with Step 1 this completes the proof of (ii).

Step 4. We prove (iii). Let  $f \in S'$  be such that the right-hand side of (14) holds. By (17) and (2.4.1/38) we have

$$|(\rho_j \hat{f})^\vee(x)| \leq c \int_{\mathbb{R}^n} \left| \left( \frac{\bar{\rho}}{\varphi} \right)^\vee(y) \right| |(\varphi_j \hat{f})^\vee(x - 2^{-j}y)| dy \quad (23)$$

where  $j \in \mathbb{N}$ . The second factor in the integral can be estimated from above by  $c(1 + |y|)^a (\varphi_j^* f)_a(x)$ . Then we obtain

$$|(\rho_j \hat{f})^\vee(x)| \leq c(\varphi_j^* f)_a(x), \quad x \in \mathbb{R}^n, \quad j \in \mathbb{N}_0. \quad (24)$$

By our assumption we have  $f \in F_{pq}^s$ . The proof is complete.

**Remark.** Parts (i) and (iii) of the theorem are quite satisfactory. Based on the following interpretation one can even omit the assumption  $(\varphi_j \hat{f})^\vee \in L_\infty$  for  $j \in \mathbb{N}_0$  in (14). We assume that  $(\varphi_j \hat{f})^\vee$  with  $j \in \mathbb{N}_0$  is a regular distribution and that the quasi-norm in (14) is finite. In particular we have  $(\varphi_j^* f)_a \in L_p$ . By definition (2.4.1/55) and the counterpart of (2.3.2/10) we have

$$(\varphi_j^* f)_a(x) \leq c(\varphi_j^* f)_a(y) \quad \text{if } |x - y| \leq 2^{-j}. \quad (25)$$

Let  $(\varphi_j \hat{f})^\vee(x)$  be unbounded, then there exists for any  $A \in \mathbb{R}$  a point  $x^A \in \mathbb{R}^n$  with

$$|(\varphi_j^* f)_a(y)| \geq A \quad \text{if} \quad |y - x^A| \leq 2^{-j}. \quad (26)$$

We used (2.4.1/55) and (25). However (26) contradicts  $(\varphi_j^* f)_a \in L_p$ . In other words, we have  $(\varphi_j \hat{f})^\vee \in L_\infty$  if  $j \in \mathbb{N}_0$ . Let  $p \geq 1$ , then we can omit the assumption  $(\varphi_j \hat{f})^\vee \in L_p$  for  $j \in \mathbb{N}_0$  in (11). In this case  $\|f^k \mid F_{pq}^s\|^\varphi < \infty$  yields  $(\varphi_j \hat{f}^k)^\vee \in L_p$  and it was just this assertion which we needed in Step 3 of the above proof. However the situation seems to be different if  $p < 1$ . Furthermore it would be desirable to replace  $\lim_{k \rightarrow \infty} \|f^k \mid F_{pq}^s\|^\varphi < \infty$  in (11) by  $\|f \mid F_{pq}^s\|^\varphi < \infty$ . In general such a possibility seems to be somewhat doubtful. Later on we discuss this point under further restrictions for  $\varphi$ , see 2.6.4 in connection with thermic and harmonic extensions.

### 2.4.3 Modifications of the main theorem

Some modifications of Theorem 2.4.1 will be useful later on. Our formulations will be given in terms of equivalent quasi-norms. Although possible, we do not try to rephrase some of these assertions in the sense of Theorem 2.4.2. To give a feeling of our intentions we begin with a somewhat heuristic discussion. Let  $\varphi(x) = (e^{i\gamma x} - 1)^M$ , where  $\gamma x$  stands for the scalar product of the variable  $x \in \mathbb{R}^n$  and  $\gamma \in \mathbb{R}^n$ , and  $M \in \mathbb{N}$ . Then  $(\varphi \hat{f})^\vee(x) = \Delta_\gamma^M f(x)$ , where  $\Delta_\gamma^M$  are the usual differences, see (1.2.2/3). Characterizations of some spaces of type  $F_{pq}^s$  and  $B_{pq}^s$  via differences  $\Delta_\gamma^M$  are very desirable. For appropriate numbers  $s_0$  and  $s_1$  hypotheses of type (2.4.1/8,9) for the above function  $\varphi(x)$  are fulfilled, but not the Tauberian condition (2.4.1/7). On the other hand if one does not deal with a single function  $(e^{i\gamma x} - 1)^M$  but with an appropriate finite set of these functions  $\{(e^{i\gamma^k x} - 1)^M\}_{k=1}^N$  or families of these functions of type  $\{(e^{i\gamma x} - 1)^M\}_{|\gamma|=1}$  or  $\{(e^{i\gamma x} - 1)^M\}_{1 \leq |\gamma| \leq 2}$  then the Tauberian condition (2.4.1/7) is satisfied in some sense and one can expect equivalent quasi-norms of type (2.4.1/11,12). In this subsection we describe the necessary modifications compared with 2.4.1. We are not interested in most general formulations but we restrict our attention to those functions  $\varphi$  which cover the examples which we have in mind. We use the same notations as in 2.4.1, in particular, the numbers  $\sigma_p$  and  $\sigma_{pq}$  are given by (2.4.1/1), and  $h(x) \in S$  and  $H(x) \in S$  are the same functions as at the beginning of 2.4.1 with (2.4.1/2,3). Furthermore,  $(\varphi(h \cdot) \hat{f})^\vee$  must be interpreted similarly as in 2.4.1 and 2.4.2.

**Theorem 1.** Let  $0 < p < \infty$ ,  $0 < q \leq \infty$ , and  $s \in \mathbb{R}$ . Let  $s_0$  and  $s_1$  be two real numbers with (2.4.1/5). Let  $\varphi_0(x)$  be the same function as in Theorem 2.4.1,

including (2.4.1/6,10) with  $a > \frac{n}{\min(p,q)}$ . Let  $\varphi(t)$  be a complex-valued  $C^\infty$  function on  $\mathbb{R} \setminus \{0\}$  which satisfies

$$|\varphi(t)| > 0 \quad \text{if} \quad \frac{1}{8} < t < 8 \quad (1)$$

(Tauberian condition) and

$$\sup_{1 \leq |\gamma| \leq 2} \int_{\mathbb{R}^n} \left| \left( \frac{\varphi(\gamma z) h(z)}{|z|^{s_1}} \right)^\vee(y) \right| (1 + |y|)^a dy < \infty, \quad (2)$$

$$\sup_{1 \leq |\gamma| \leq 2} \sup_{m \in \mathbb{N}} 2^{-ms_0} \int_{\mathbb{R}^n} |(\varphi(2^m \gamma z) H(z))^\vee(y)| (1 + |y|)^a dy < \infty. \quad (3)$$

Then

$$\|\varphi_0(D)f\|_{L_p} + \left\| \left( \int_{|h| \leq 1} |h|^{-sq} |(\varphi(h \cdot) \hat{f})^\vee(\cdot)|^q \frac{dh}{|h|^n} \right)^{1/q} \right\|_{L_p} \quad (4)$$

and

$$\|\varphi_0(D)f\|_{L_p} + \left\| \left( \int_0^1 t^{-sq} \sup_{\frac{t}{2} \leq |h| \leq t} |(\varphi(h \cdot) \hat{f})^\vee(\cdot)|^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p} \quad (5)$$

(modification if  $q = \infty$ ) are equivalent quasi-norms in  $F_{pq}^s$ .

**Proof.** Step 1. We follow Step 1 of the proof of Theorem 2.4.1 with  $\varphi(2^{-j}\gamma x)$  instead of  $\varphi_j(x)$ , where  $j \in \mathbb{N}$  and  $1 \leq |\gamma| \leq 2$ . Then we obtain (2.4.1/18). We take the supremum with respect to  $\gamma$  and proceed afterwards in the same way as in (2.4.1/19–23). We obtain

$$\left\| \left[ \sum_{j=1}^{\infty} \left( \sum_{l=-\infty}^K 2^{js} \sup_{1 \leq |\gamma| \leq 2} |(\varphi(2^{-j}\gamma z) \rho_{l+j}(z) \hat{f})^\vee(\cdot)| \right)^q \right]^{1/q} \right\|_{L_p} \leq c \|f\|_{F_{pq}^s} \quad (6)$$

and a corresponding estimate with  $\varphi_0$ . Next we use the same arguments as in Step 2 of the proof of Theorem 2.4.1. We arrive at (2.4.1/28) with  $\varphi(2^{-j}\gamma x)$  instead of  $\varphi_j(x)$ , where  $j \in \mathbb{N}$  and  $1 \leq |\gamma| \leq 2$ . We take the supremum with respect to  $\gamma$  and obtain the following counterpart of (2.4.1/29)

$$\left\| \left[ \sum_{j=1}^{\infty} \left( \sum_{l=K+1}^{\infty} 2^{js} \sup_{1 \leq |\gamma| \leq 2} |(\varphi(2^{-j}\gamma z) \rho_{l+j}(z) \hat{f})^\vee(\cdot)| \right)^q \right]^{1/q} \right\|_{L_p} \quad (7)$$

$$\leq c 2^{-K(s-s_0)} \|f\|_{F_{pq}^s}$$

and a corresponding estimate with  $\varphi_0$ . However the quasi-norm in (4) can be estimated from above by the quasi-norm in (5), which, in turn, can be estimated from above by the sum of the left-hand sides of (6) and (7) and corresponding terms with  $\varphi_0$ , and hence by  $c\|f\|_{F_{pq}^s}$ .

Step 2. In order to prove the reverse inequality we must modify Step 5 of the proof of Theorem 2.4.1. Let  $\gamma$  with  $1 \leq |\gamma| \leq 2$  be given. Then

$$\left\{x \in \mathbb{R}^n: \frac{1}{2} \leq |x| \leq 2, |\varphi(\gamma x)| > 0\right\}$$

covers a sectorial set

$$\Omega_\gamma = \left\{y \in \mathbb{R}^n: \frac{1}{2} \leq |y| \leq 2, \left|\frac{y}{|y|} - \frac{\gamma}{|\gamma|}\right| \leq b\right\}$$

for some  $b > 0$ . Let again  $\{\rho_k(x)\}_{k \in \mathbb{N}_0}$  be a resolution of unity in the sense of (2.3.1/1-4) with  $\rho$  instead of  $\varphi$ . Let  $\bar{\rho}(x) = \rho(x) - \rho(2x)$ ,  $x \in \mathbb{R}^n$ , see (2.3.1/2), then  $\rho_j(x) = \bar{\rho}(2^{-j}x)$  if  $j \in \mathbb{N}$ . We decompose the basic function  $\bar{\rho}(x)$  by  $\bar{\rho}(x) = \sum_{k=1}^N \rho^{(k)}(x)$  such that for any  $\gamma$  with  $1 \leq |\gamma| \leq 2$  we find a number  $k$  such that  $\text{supp } \rho^{(k)} \subset \Omega_\gamma$ . This is always possible if one chooses  $N$  large enough. Let  $k$  be given and let  $1 \leq |\gamma| \leq 2$  with  $\text{supp } \rho^{(k)} \subset \Omega_\gamma$ . We follow the arguments of Step 5 of the proof of Theorem 2.4.1 with  $\rho^{(k)}(2^{-j}x)$  and  $\varphi(2^{-j}\gamma x)$  instead of  $\rho_j$  and  $\varphi_j$ , respectively. We arrive at the counterparts of (2.4.1/37,38), where in the latter inequality we may assume that the corresponding right-hand side is independent of  $k$  and  $\gamma$ . We substitute this inequality in the just described modification of (2.4.1/37) and integrate over those  $\gamma$ 's which are connected with the given number  $k$  in the above sense. Afterwards this integration can be extended to all admissible  $\gamma$ 's. Then the corresponding right-hand sides are independent of  $k$ . Summation over  $k$  yields (2.4.1/39) with

$$\left(\int_{1 \leq |\gamma| \leq 2} |(\varphi(2^{-j}\gamma z)\psi(2^{-j}z)\hat{f})^\vee(x-y)|^q d\gamma\right)^{1/q} \quad (8)$$

instead of  $|(\varphi_j\psi(2^{-j}\cdot)\hat{f})^\vee(x-y)|$ , where we used that

$$\int_\gamma |\cdots|^r d\gamma \leq c \left(\int_\gamma |\cdots|^q d\gamma\right)^{r/q}. \quad (9)$$

We have the counterparts of (2.4.1/42,44). We use the same splitting (2.4.1/43). The substitute of  $|(\varphi_j\hat{f})^\vee|$  is

$$\left(\int_{1 \leq |\gamma| \leq 2} |(\varphi(2^{-j}\gamma\cdot)\hat{f})^\vee|^q d\gamma\right)^{1/q},$$

and the corresponding term yields (4). The remaining term, i.e., the counterpart of (2.4.1/44), can be estimated with the help of (7) in the same way as after (2.4.1/44). This shows that  $\|f\|_{F_{pq}^s, \rho}$  can be estimated from above by the quasi-norm (4) and, consequently, also by the quasi-norm (5). The proof is complete.

**Remark 1.** We followed closely [Tri16: 175–178]. Under our specific assumptions for  $\varphi(x)$  one can omit  $\sup_{1 \leq |\gamma| \leq 2}$  in (2) and (3), because it is sufficient to know corresponding estimates for a fixed  $\gamma \neq 0$ . However, if one replaces  $\varphi(\gamma x)$  by more general families of functions  $\varphi_\gamma(x)$ , then one needs formulations of type (2) and (3). Furthermore, in Step 1 we need only  $s_0 < s < s_1$  and that  $(\varphi(\gamma \cdot) \hat{f})^\vee$  is well-defined in the sense of the beginning of 2.4.2, see Remark 2.4.1/2. In other words, under these assumptions, the quasi-norms in (4) and (5) can be estimated from above by  $c\|f\|_{F_{pq}^s, \rho}$ .

A second modification of the main theorem in 2.4.1 will be useful later on, especially in connection with spaces on Riemannian manifolds. Let

$$l \in S'(\mathbb{R}) \quad \text{with} \quad \text{supp } l \subset [-1, 1]. \quad (10)$$

Then the (one-dimensional) Fourier transform  $\psi = F_1 l$  is an analytic function. Let

$$0 < |\psi(\lambda)| \leq c\lambda^m \quad \text{if} \quad 0 < \lambda < \delta \quad (11)$$

for some  $\delta > 0$  and some  $m \in \mathbb{N}$ . We introduce the (essentially one-dimensional) means

$$l(h, f)(x) = \int_{\mathbb{R}} l(\lambda) f(x + h\lambda) d\lambda, \quad x \in \mathbb{R}^n, \quad 0 \neq h \in \mathbb{R}^n, \quad (12)$$

with  $f \in S' = S'(\mathbb{R}^n)$ , as far as this expression makes sense: in the sequel this always will be ensured when these means are applied, maybe via limiting procedures, where one begins with smooth functions  $f$ . As in Theorem 1, we use the same notations as in 2.4.1, in particular the numbers  $\sigma_p$  and  $\sigma_{pq}$  are given by (2.4.1/1), and  $H(x) \in S$  is the same function as at the beginning of 2.4.1 with (2.4.1/2,3). We shall now not need the function  $h(x)$  from (2.4.1/2,3).

**Theorem 2.** Let  $0 < p < \infty$ ,  $0 < q \leq \infty$ , and  $s \in \mathbb{R}$ . Let  $s_0 \in \mathbb{R}$  and  $m \in \mathbb{N}$  with

$$s_0 + \sigma_{pq} < s < m, \quad m > \sigma_p, \quad (13)$$

and (11) for some  $\delta > 0$ . Let

$$\sup |h|^{-s_0} \int_{\mathbb{R}^n} |(\psi(hz)H(z))^\vee(y)|(1+|y|)^a dy < \infty \quad (14)$$

with  $a > \frac{n}{\min(p,q)}$ , where the supremum is taken over all  $h \in \mathbb{R}^n$  with  $|h| \geq 1$ . Let  $\varphi_0 \in S$  with  $\varphi_0(0) \neq 0$ , then

$$\|\varphi_0(D)f\|_{L_p} + \left\| \left( \int_{|h| \leq 1} |h|^{-sq} |l(h, f)(\cdot)|^q \frac{dh}{|h|^n} \right)^{1/q} \right\|_{L_p} \quad (15)$$

and

$$\|\varphi_0(D)f\|_{L_p} + \left\| \left( \int_0^1 t^{-sq} \sup_{\frac{t}{2} \leq |h| \leq t} |l(h, f)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p} \quad (16)$$

(modification if  $q = \infty$ ) are equivalent quasi-norms in  $F_{pq}^s$ .

**Proof.** The function  $\varphi_0 \in S$  satisfies the modified Tauberian condition (2.4.1/6), and also (2.4.1/10), see (2.4.1/53). Otherwise, the above theorem is a modification of Theorem 1. First we observe

$$(\psi(h \cdot) \hat{f})^\vee(x) = \int_{\mathbb{R}} (F_1^{-1} \psi)(\lambda) f(x - \lambda h) d\lambda = l(-h, f)(x). \quad (17)$$

Now we identify  $\psi(\lambda)$  with  $\varphi(\lambda)$  in Theorem 1. Then (11) modifies (1) which compensates the above modification of (2.4.1/6). Furthermore, (14) coincides with (3). We have to check (2). For this purpose we wish to replace  $|z|^{s_1}$  in (2) by  $(z\gamma)^m$ , where  $m$  has the above meaning. We substitute (2.4.1/14) by

$$\tilde{\rho}_j(x) = (2^{-j}\gamma x)^m \rho_j(x) \quad \text{if } j \in \mathbb{N}_0. \quad (18)$$

Because  $(\gamma x)^m$  is a polynomial one has no difficulty to follow the reasoning after (2.4.1/14) with  $\varphi(2^{-j}\gamma z)$  instead of  $\varphi_j(z)$  up to the end of Step 1 of the proof of Theorem 2.4.1. The rest remains unchanged. In other words we may replace  $|z|^{s_1}$  in (2) by  $(z\gamma)^m$ . However, under the above assumptions we have  $\varphi(\gamma z) = \psi(\gamma z)$  where  $\psi(\lambda)$  is an analytic function with (11). Hence (2) is satisfied. Now by (17) the quasi-norms in (4) and (5) coincide with the quasi-norms in (15) and (16), respectively. The proof is complete.

**Remark 2.** An interesting choice of the above function  $\psi(\lambda)$  is given by  $\psi(\lambda) = (e^{i\nu\lambda} - 1)^m$ , where  $\nu > 0$  is an appropriate number, e.g.,  $\nu = 1/m$ . Then (15) and (16) are quasi-norms in terms of differences, see the beginning of this subsection. Details will be given later on.

### 2.4.4 Complements

We complement the two theorems of the last subsection. First we ask whether  $\sup_{\frac{t}{2} \leq |h| \leq t}$  in (2.4.3/5,16) can be replaced by  $\sup_{0 < |h| \leq t}$ . This is of interest in connection with classical descriptions.

**Proposition 1.** (i) Let all the hypotheses of Theorem 2.4.3/1 be fulfilled and let in addition  $s > 0$ . Then

$$\|\varphi_0(D)f \mid L_p\| + \left\| \left( \int_0^1 t^{-sq} \sup_{0 < |h| \leq t} |(\varphi(h \cdot) \hat{f})^\vee(\cdot)|^q \frac{dt}{t} \right)^{1/q} \mid L_p \right\| \quad (1)$$

(modification if  $q = \infty$ ) is an equivalent quasi-norm in  $F_{pq}^s$ .

(ii) Let all the hypotheses of Theorem 2.4.3/2 be fulfilled and let in addition  $s > 0$ . Then

$$\|\varphi_0(D)f \mid L_p\| + \left\| \left( \int_0^1 t^{-sq} \sup_{0 < |h| \leq t} |l(h, f)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \mid L_p \right\| \quad (2)$$

(modification if  $q = \infty$ ) is an equivalent quasi-norm in  $F_{pq}^s$ .

**Proof.** We have to prove that the quasi-norm in (1) can be estimated from above by the quasi-norms in (2.4.3/5). For this purpose we estimate  $\sup_{0 < |h| \leq t}$  from above by

$$\sum_{j=0}^{\infty} \sup_{2^{-j-1}t \leq |h| \leq 2^{-j}t}.$$

Then we have

$$\begin{aligned} & \int_0^1 t^{-sq} \sup_{0 < |h| \leq t} |(\varphi(h \cdot) \hat{f})^\vee(\cdot)|^q \frac{dt}{t} \\ & \leq \sum_{j=0}^{\infty} \int_0^1 t^{-sq} \sup_{2^{-j-1}t \leq |h| \leq 2^{-j}t} |(\varphi(h \cdot) \hat{f})^\vee(\cdot)|^q \frac{dt}{t} \\ & \leq \left( \sum_{j=0}^{\infty} 2^{-jsq} \right) \int_0^1 t^{-sq} \sup_{\frac{t}{2} \leq |h| \leq t} |(\varphi(h \cdot) \hat{f})^\vee(\cdot)|^q \frac{dt}{t}. \end{aligned} \quad (3)$$

This is just the desired estimate. In the same way one estimates the quasi-norm in (2) from above by the quasi-norm in (2.4.3/16).

**Proposition 2.** Let  $0 < p < \infty$ ,  $0 < q \leq \infty$  and  $s \in \mathbb{R}$ . Let  $s_0$  and  $s_1$  be two real numbers with (2.4.1/5). Let  $\varphi_0(x)$  be the same function as in Theorem 2.4.1, including (2.4.1/6,10), where  $a > \frac{n}{\min(p,q)}$ . Let  $\varphi^1(x), \dots, \varphi^N(x)$  be  $N$  complex-valued  $C^\infty$  functions on  $\mathbb{R}^n \setminus \{0\}$  which satisfy the Tauberian condition

$$\sum_{k=1}^N |\varphi^k(x)| > 0 \quad \text{if} \quad \frac{1}{2} \leq |x| \leq 2 \quad (4)$$

and (2.4.1/8,9) with  $\varphi^k$  instead of  $\varphi$ . Then

$$\|\varphi_0(D)f\|_{L_p} + \sum_{k=1}^N \left\| \left( \int_0^1 t^{-sq} |\varphi^k(tD)f(\cdot)|^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p} \quad (5)$$

and

$$\|\varphi_0(D)f\|_{L_p} + \sum_{k=1}^N \left\| \left( \int_0^1 t^{-sq} \sup_{\frac{t}{2} \leq \tau \leq t} |\varphi^k(\tau D)f(\cdot)|^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p} \quad (6)$$

(modification if  $q = \infty$ ) are equivalent quasi-norms in  $F_{pq}^s$ . If, in addition,  $s > 0$  then

$$\|\varphi_0(D)f\|_{L_p} + \sum_{k=1}^N \left\| \left( \int_0^1 t^{-sq} \sup_{0 < \tau \leq t} |\varphi^k(\tau D)f(\cdot)|^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p} \quad (7)$$

(modification if  $q = \infty$ ) is also an equivalent quasi-norm in  $F_{pq}^s$ .

**Proof.** This proposition is simply the discrete version of Theorem 2.4.3/1. The proof is the same. Furthermore, (7) is covered by the same estimate as in (3).

## 2.4.5 Characterizations via Lusin functions

In 1.9.3 we gave a brief description of tent spaces and their connections with  $F_{pq}^s$  spaces via Lusin functions. Now we return to this subject in the above context. Descriptions of  $F_{pq}^s$  spaces in terms of Lusin functions go back to L. Pavarinta [Pai2] (1982), see also [CaT] and [Tri: 2.12.1]. Recently, H.G. Feichtinger and K.H. Grochenig gave a group-theoretical interpretation of atomic representations in function spaces, where they used in a decisive way characterizations of  $F_{pq}^s$  spaces in terms of Lusin functions, see Remark 1.9.4/3 for references and some details. In other words, a detailed description of this possibility seems to be of some interest. We follow [Tri16: 179/180]. Let

$$\gamma(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < t < 1\} \quad (1)$$

be a truncated cone of aperture 1 whose vertex is  $x \in \mathbb{R}^n$ . Let  $\varphi$  be the same function as in Theorem 2.4.1. We use (2.4.1/4) and introduce

$$\varphi_q^s f(x) = \left( \int_{\gamma(x)} t^{-sq} |\varphi(tD)f(y)|^q \frac{dy dt}{t^{n+1}} \right)^{1/q}, \quad x \in \mathbb{R}^n, \quad (2)$$

where  $0 < q \leq \infty$  (modification if  $q = \infty$ ) and  $s \in \mathbb{R}$ . We have

$$\varphi_q^s f(x) = c \left( \int_0^1 t^{-sq} \frac{1}{|B(x, t)|} \int_{B(x, t)} |\varphi(tD)f(y)|^q dy \frac{dt}{t} \right)^{1/q} \quad (3)$$

for some  $c > 0$ , where  $B(x, t)$  stands for a ball of radius  $t$  in  $\mathbb{R}^n$  centered at  $x \in \mathbb{R}^n$  with  $|B(x, t)|$  as its volume. However (3) is a function of Lusin type, see [Päi1: p. 123] and [Triß: 2.12.1].

**Theorem.** Let  $0 < p < \infty$ ,  $0 < q \leq \infty$ , and  $s \in \mathbb{R}$ . Let  $a > \frac{n}{\min(p, q)}$  and let  $s_0$  and  $s_1$  be two real numbers with (2.4.1/54). Let  $\varphi_0$  and  $\varphi$  be the same two functions as in Theorem 2.4.1 with (2.4.1/6–10). Then

$$\|\varphi_0(D)f\|_{L_p} + \|\varphi_q^s f\|_{L_p} \quad (4)$$

(modification if  $q = \infty$ ) is an equivalent quasi-norms in  $F_{pq}^s$ .

**Proof.** For fixed  $x \in \mathbb{R}^n$  and, say,  $t = 2^{-j}$  with  $j \in \mathbb{N}$  we have

$$|\varphi(tD)f(x + y)| \leq c(\varphi_j^* f)_a(x), \quad |y| \leq t, \quad (5)$$

where  $(\varphi_j^* f)_a(x)$  is the maximal function introduced in (2.4.1/55). If, say,  $2^{-j} \leq t \leq 2^{-j+1}$  with  $j \in \mathbb{N}$ , then one has an obvious counterpart of (5) and also of the equivalent quasi-norm (2.4.1/56) with  $\int_0^1 \dots \frac{dt}{t}$  instead of  $\sum_{k=1}^\infty \dots$ . To justify the latter claim one has to modify the proof of Corollary 2.4.1/2 slightly. We insert (5) in (3), then the just indicated modification of (2.4.1/56) shows that the quasi-norm in (4) can be estimated from above by  $\|f\|_{F_{pq}^s}$ . In order to prove the reverse estimate we use (2.4.1/37) with  $y + z$  instead of  $y$  (integration over  $y$ ), where  $|z| \leq t \sim 2^{-j}$ . We have (2.4.1/38) with  $y + z$  on the left-hand side and  $y$  on the right-hand side. We put this estimate in the just-described modified version of (2.4.1/37) and take afterwards the  $(\int_{|z| \leq t} |\dots|^{q/r} dz)^{r/q}$ -norm on both sides. Now the rest is the same as at the end of Step 2 of the proof of Theorem 2.4.3/1.

### 2.4.6 Local means

Let  $B = \{y \in \mathbb{R}^n : |y| < 1\}$  be the unit ball in  $\mathbb{R}^n$ , and let  $k$  be a  $C^\infty$  function in  $\mathbb{R}^n$  with  $\text{supp } k \subset B$ , then we introduce the local means

$$k(t, f)(x) = \int_{\mathbb{R}^n} k(y) f(x + ty) dy, \quad x \in \mathbb{R}^n, \quad t > 0, \quad (1)$$

which makes sense for any  $f \in S'$  (appropriately interpreted). Let  $k_0$  and  $k^0$  be two  $C^\infty$  functions in  $\mathbb{R}^n$  with

$$\text{supp } k_0 \subset B, \quad \text{supp } k^0 \subset B, \quad (2)$$

and

$$\hat{k}_0(0) \neq 0, \quad \hat{k}^0(0) \neq 0. \quad (3)$$

Let

$$k(y) = \Delta^N k^0(y) = \left( \sum_{j=1}^n \frac{\partial^2}{\partial y_j^2} \right)^N k^0(y), \quad y \in \mathbb{R}^n, \quad (4)$$

where  $N \in \mathbb{N}$  is at our disposal. Let again  $\sigma_p = n \left( \frac{1}{p} - 1 \right)_+$ , see (2.4.1/1).

**Theorem.** Let  $0 < p < \infty$ ,  $0 < q \leq \infty$ , and  $s \in \mathbb{R}$ . Let  $N \in \mathbb{N}$  with

$$2N > \max(s, \sigma_p), \quad (5)$$

then

$$\|k_0(1, f) \mid L_p\| + \left\| \left( \sum_{j=0}^{\infty} 2^{jsq} |k(2^{-j}, f)(\cdot)|^q \right)^{1/q} \mid L_p \right\|, \quad (6)$$

$$\|k_0(1, f) \mid L_p\| + \left\| \left( \int_0^1 t^{-sq} |k(t, f)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \mid L_p \right\|, \quad (7)$$

and

$$\|k_0(1, f) \mid L_p\| + \left\| \left( \int_0^1 t^{-sq} \sup_{\frac{t}{2} \leq \tau \leq t} |k(\tau, f)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \mid L_p \right\| \quad (8)$$

(modification if  $q = \infty$ ) are equivalent quasi-norms in  $F_{pq}^s$ . If, in addition,  $s > 0$  then

$$\|k_0(1, f) \mid L_p\| + \left\| \left( \int_0^1 t^{-sq} \sup_{0 < \tau \leq t} |k(\tau, f)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \mid L_p \right\| \quad (9)$$

(modification if  $q = \infty$ ) is also an equivalent quasi-norms in  $F_{pq}^s$ .

**Proof.** We identify  $k_0^\vee$  and  $k^\vee$  with  $\varphi_0$  and  $\varphi$  in Theorem 2.4.1, respectively. Then (3) and  $k^\vee(x) = |x|^{2N} k^{0\vee}$  fulfill immaterial modifications of the Tauberian conditions (2.4.1/6,7). We choose  $s_1 = 2N$ , then we have (2.4.1/8–10), see also (2.4.1/53). Hence, (2.4.1/11,12) are equivalent quasi-norms in  $F_{pq}^s$ . We have

$$\varphi(tD)f(x) = (\varphi(t\cdot)\hat{f})^\vee(x) = c \int_{\mathbb{R}^n} (\varphi(t\cdot))^\vee(y) f(x-y) dy. \quad (10)$$

We observe

$$(\varphi(t\cdot))^\vee(y) = t^{-n} \varphi^\vee\left(\frac{y}{t}\right) = t^{-n} \hat{\varphi}\left(-\frac{y}{t}\right) = t^{-n} k\left(-\frac{y}{t}\right),$$

which shows that (10) coincides with (1). Hence, (6) and (7) are equivalent quasi-norms in  $F_{pq}^s$ . The corresponding assertion for (8) follows from the arguments in the proof of Theorem 2.4.3/1. Finally, as far as (9) is concerned we refer to the proof of Proposition 2.4.4/1.

**Remark.** The above theorem covers in particular Theorem 1.8.4(ii), see Remark 1.8.4/1 for some references. The advantage of (1) compared, for instance, with  $\varphi_k(D)$  from Definition 2.3.1 is its strictly local nature: In order to calculate  $k(t, f)(x)$  in a given point  $x \in \mathbb{R}^n$  one needs only a knowledge of  $f(z)$  in a ball of radius  $t$  centered at  $x$ . This observation will be of great service for us in the sequel.

**Proposition.** Let  $k_0$  and  $k^0$  be the above functions. Let  $0 < p < \infty$ ,  $0 < q \leq \infty$ , and  $s \in \mathbb{R}$ . Let  $M \in \mathbb{N}$  with

$$M > \max(s, \sigma_p), \quad (11)$$

then the counterpart of (7),

$$\|k_0(1, f) \mid L_p\| + \sum_{|\alpha|=M} \left\| \left( \int_0^1 t^{(M-s)q} \mid k^0(t, D^\alpha f)(\cdot) \mid^q \frac{dt}{t} \right)^{1/q} \mid L_p \right\| \quad (12)$$

(modification if  $q = \infty$ ) and the corresponding counterparts of (6,8), and, if  $s > 0$ , of (9), are equivalent quasi-norms in  $F_{pq}^s$ .

**Proof.** We have

$$k^0(t, D^\alpha f)(x) = c t^{-M} \int_{\mathbb{R}^n} D^\alpha k^0(y) f(x + ty) dy. \quad (13)$$

We wish to apply Proposition 2.4.4/2 and put  $k_0^\vee = \varphi_0$  and  $(D^\alpha k^0)^\vee = \varphi^\alpha$  with  $|\alpha| = M$ . Recall  $\varphi^\alpha(x) = x^\alpha (k^0)^\vee(x)$ . By (3), obvious counterparts of (2.4.1/6) and (2.4.4/4) are fulfilled, similarly for (2.4.1/9,10). As for (2.4.1/8) we recall the slight modification described in the proof of Theorem 2.4.3/2 which shows that we can replace  $\frac{\varphi(z)}{|z|^{s+1}} h(z)$  in (2.4.1/8) by  $\frac{\varphi^\alpha(z)}{z^\alpha} h(z)$  and hence by  $(k^0)^\vee(z) h(z)$ .

## 2.4.7 Localization principle

Recall that  $F_{pq}^s$  and  $B_{pq}^s$  have been introduced in Definition 2.3.1. Let  $\psi$  be a compactly supported  $C^\infty$  function in  $\mathbb{R}^n$  and let

$$\psi_k(y) = \psi(y - k), \quad k \in \mathbb{Z}^n, \quad x \in \mathbb{R}^n, \quad (1)$$

where  $\mathbb{Z}^n$  stands for the lattice of all points  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  with integer-valued components  $x_j$ . We assume that

$$\sum_{k \in \mathbb{Z}^n} \psi_k(x) = 1, \quad x \in \mathbb{R}^n, \quad (2)$$

is a resolution of unity.

**Theorem.** (i) Let  $0 < p < \infty$ ,  $0 < q \leq \infty$ , and  $s \in \mathbb{R}$ , then

$$\left( \sum_{k \in \mathbb{Z}^n} \|\psi_k f\|_{F_{pq}^s}^p \right)^{1/p} \quad (3)$$

is an equivalent quasi-norm in  $F_{pq}^s$ .

(ii) Let  $0 < p \leq \infty$ ,  $0 < q \leq \infty$ , and  $s \in \mathbb{R}$ , then

$$\left( \sum_{k \in \mathbb{Z}^n} \|\psi_k f\|_{B_{pq}^s}^r \right)^{1/r} \quad (4)$$

with  $0 < r \leq \infty$  (modification if  $r = \infty$ ) is an equivalent quasi-norm in  $B_{pq}^s$  if and only if  $p = q = r$ .

**Proof.** Step 1. Let  $0 < p < \infty$ ,  $0 < q \leq \infty$ , and  $s \in \mathbb{R}$ . By the equivalent quasi-norm (2.4.6/7), based on (2.4.6/1), and

$$k(t, f)(x) = \sum_{m \in \mathbb{Z}^n} k(t, \psi_m f)(x), \quad x \in \mathbb{R}^n, \quad (5)$$

it follows that  $\|f\|_{F_{pq}^s}$  can be estimated from above by the quasi-norm in (3).

Step 2. Let  $0 < p < \infty$ ,  $0 < q \leq \infty$ , and  $s > n/p$ . We prove that the quasi-norm in (3) can be estimated from above by  $\|f\|_{F_{pq}^s}$ . We use again (2.4.6/7). Let  $M \in \mathbb{N}$  with  $M > s$ . Then we have

$$\begin{aligned} k(t, \psi_m f)(x) &= \int_{\mathbb{R}^n} k(y) \psi_m(x + ty) f(x + ty) dy \\ &= \sum_{|\alpha| \leq M-1} \frac{1}{\alpha!} D^\alpha \psi_m(x) t^{|\alpha|} \int_{\mathbb{R}^n} k(y) y^\alpha f(x + ty) dy \\ &\quad + \sum_{|\alpha|=M} t^M \int_{\mathbb{R}^n} k(y) y^\alpha R_\alpha(x, y) f(x + ty) dy \end{aligned} \quad (6)$$

where  $R_\alpha(x, y)$  are remainder terms of the type  $D^\alpha \psi_m(x + \vartheta^\alpha ty)$  with  $0 \leq \vartheta^\alpha \leq 1$ . By (2.4.2/4,5) and  $s > n/p$  it follows that  $f \in F_{pq}^s$  is a bounded continuous function. In Remark 1 below we shall modify this assertion by

$$\| \sup_{|x-y| \leq c} |f(y)| \| L_p \| \leq c \| f \|_{F_{pq}^s} \quad (7)$$

where  $c > 0$  is an arbitrary number and the integration in (7) is taken with respect to  $x$ . Let  $k_\alpha(y) = y^\alpha k(y)$ , then we have by (6) and  $M > s$

$$\begin{aligned} \left( \int_0^1 t^{-sq} |k(t, \psi_m f)(x)|^q \frac{dt}{t} \right)^{1/q} &\leq c_1 \sum_{|\alpha| \leq M-1} \left( \int_0^1 t^{-sq} |k_\alpha(t, f)(x)|^q \frac{dt}{t} \right)^{1/q} \\ &\quad + c_1 \sup_{|x-y| \leq c_2} |f(y)| \end{aligned} \quad (8)$$

if  $|x - m| \leq c_2$  and

$$k(t, \psi_m f)(x) = 0 \quad \text{for } |x - m| > c_2 \quad \text{and } 0 < t \leq 1, \quad (9)$$

for some positive numbers  $c_1$  and  $c_2$ . We use (2.4.6/7) with  $\psi_m f$  instead of  $f$ , where the corresponding first term in (2.4.6/7) can be estimated by the last term on the right-hand side of (8) with a counterpart of (9). We take the  $p$ th power of (8) (and its counterpart with  $k_0$ ), sum over  $m \in \mathbb{Z}^n$ , integrate over  $\mathbb{R}^n$ , and obtain

$$\begin{aligned} \sum_{m \in \mathbb{Z}^n} \|\psi_m f \mid F_{pq}^s\|^p &\leq c_1 \sum_{|\alpha| \leq M-1} \left\| \left( \int_0^1 t^{-sq} |k_\alpha(t, f)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \mid L_p \right\|^p \\ &+ c_1 \left\| \sup_{|x-y| \leq c_2} |f(y)| \mid L_p \right\|^p \end{aligned} \quad (10)$$

for some numbers  $c_1$  and  $c_2$ . We choose  $N$  in (2.4.6/5) large and have

$$k_\alpha^\vee(x) = (y^\alpha k(y))^\vee(x) = |x|^{2K} g_\alpha(x), \quad x \in \mathbb{R}^n, \quad (11)$$

with  $0 \leq |\alpha| \leq M-1$ ,  $K \in \mathbb{N}$ ,  $2K > \max(s, \sigma_p)$ , and  $g_\alpha \in S$ , see the beginning of the proof of Theorem 2.4.6. Then the corresponding counterparts of (2.4.1/8,9) are fulfilled. This is sufficient to estimate (2.4.1/12), and consequently the first terms on the right-hand side of (10), from above by  $\|f \mid F_{pq}^s\|$  (Tauberian conditions are only needed for the reverse estimate). Together with (7) we arrive at the desired estimate

$$\sum_{m \in \mathbb{Z}^n} \|\psi_m f \mid F_{pq}^s\|^p \leq c \|f \mid F_{pq}^s\|^p. \quad (12)$$

Step 3. Let  $0 < p < \infty$  and  $0 < q \leq \infty$ . Let (12) be valid for all  $s$  with  $s \geq \sigma$ . We wish to prove that (12) holds also for  $\sigma-1$  instead of  $s$ . Let  $f \in F_{pq}^{\sigma-1}$ , then we have

$$f = (\Delta - id)g, \quad g \in F_{pq}^{\sigma+1} \quad \text{and} \quad \|f \mid F_{pq}^{\sigma-1}\| \sim \|g \mid F_{pq}^{\sigma+1}\| \quad (13)$$

where  $\Delta$  stands for the Laplacian and  $id$  is the identity. (13) is a lifting property for  $F_{pq}^s$  spaces, see [Triß: 2.3.8], however, it is also a simple consequence of  $\hat{f} = -(1 + |x|^2)\hat{g}$ , Definition 2.3.1(ii), and the same arguments as in Step 1 of the proof of Theorem 2.3.2. We have

$$\psi_m f = (\Delta - id)(\psi_m g) - (\Delta \psi_m)g - 2 \sum_{j=1}^n \frac{\partial \psi_m}{\partial x_j} \frac{\partial g}{\partial x_j}, \quad (14)$$

and consequently

$$\|\psi_m f \mid F_{pq}^{\sigma-1}\| \leq c \|\psi_m g \mid F_{pq}^{\sigma+1}\| + c \|(\Delta \psi_m)g \mid F_{pq}^\sigma\| + c \sum_{j=1}^n \left\| \frac{\partial \psi_m}{\partial x_j} \frac{\partial g}{\partial x_j} \mid F_{pq}^\sigma \right\| \quad (15)$$

where we used the equivalence in (13) and the monotonicity properties (2.3.2/22,23). By assumption we have (12) with  $s = \sigma$  and  $s = \sigma+1$ , where we can replace  $\psi_m$  by  $\Delta \psi_m$  or  $\frac{\partial \psi_m}{\partial x_j}$ . We apply this estimate to (15) and obtain

$$\left( \sum_{m \in \mathbb{Z}^n} \|\psi_m f \mid F_{pq}^{\sigma-1}\|^p \right)^{1/p} \leq c \|g \mid F_{pq}^{\sigma+1}\| + c \|g \mid F_{pq}^\sigma\| + c \sum_{j=1}^n \left\| \frac{\partial g}{\partial x_j} \mid F_{pq}^\sigma \right\| \quad (16)$$

$$\leq c' \|g \mid F_{pq}^{\sigma+1}\| \sim c' \|f \mid F_{pq}^{\sigma+1}\|$$

where we used that  $\|\frac{\partial g}{\partial x_j} \mid F_{pq}^\sigma\|$  can be estimated from above by  $c\|g \mid F_{pq}^{\sigma+1}\|$ . This follows from  $(\frac{\partial g}{\partial x_m})^\wedge(\xi) = i\xi_m \hat{g}(\xi)$  and the estimates in Step 1 of the proof of Theorem 2.3.2, see also [Triß: 2.3.8]. Now (16), Step 2, and mathematical induction complete the proof of part (i) of the theorem.

Step 4. We prove (ii). By (i) it follows that (4) with  $r = q = p < \infty$  is an equivalent quasi-norm in  $B_{pp}^s = F_{pp}^s$ . The arguments from the Steps 1–3 can be extended to the case  $r = q = p = \infty$ , hence to the spaces  $B_{\infty\infty}^s$ , see Theorem 2.5.3.

Step 5. We assume that (4) is an equivalent quasi-norm in  $B_{pq}^s$ . Let  $s = 0$ , otherwise some immaterial modifications have to be made. Let  $\chi$  be a non-vanishing  $C^\infty$  function on  $\mathbb{R}^n$  with  $\text{supp } \hat{\chi} \subset B$ , where the latter stands for the unit ball. Let

$$f(x) = \sum_{m=1}^N \sum_{l=1}^N c_{ml} e^{ix\sigma_m} \chi(x - x_l), \quad (17)$$

where  $c_{ml}$  are arbitrary complex numbers,  $x_l \in \mathbb{R}^n$ ,  $\sigma_m \in \mathbb{R}^n$  with  $|\sigma_m| = 2^m$ . We wish to use (2.3.1/7). First we observe that  $(e^{i\sigma_m x} \chi(x - x_l))^\wedge$  is supported near  $\sigma_m$ . Hence we may assume

$$\varphi_m(D)f(x) = (\varphi_m \hat{f})^\vee(x) = \sum_{l=1}^N c_{ml} e^{ix\sigma_m} \chi(x - x_l). \quad (18)$$

We may choose  $x_l$  such that

$$\frac{1}{2} \|\varphi_m(D)f \mid L_p\|^p \leq \|\chi \mid L_p\|^p \sum_{l=1}^N |c_{ml}|^p \leq 2 \|\varphi_m(D)f \mid L_p\|^p. \quad (19)$$

By (2.3.1/7) we have

$$\|f \mid B_{pq}^0\| \sim \left[ \sum_{m=1}^N \left( \sum_{l=1}^N |c_{ml}|^p \right)^{q/p} \right]^{1/q} \quad (20)$$

where “ $\sim$ ” is independent of  $N \in \mathbb{N}$  and the numbers  $c_{ml}$ . Let  $x_j = k \in \mathbb{Z}^n$ , then

$$\psi_k(x)f(x) = \sum_{m=1}^N \sum_{l=1}^N c_{ml} e^{ix\sigma_m} \psi(x - x_j) \chi(x - x_l). \quad (21)$$

We choose  $x_l$  in an appropriate way and obtain by the same arguments as above

$$\|\psi_k f \mid B_{pq}^0\| \sim \left( \sum_{m=1}^N |c_{mj}|^q \right)^{1/q} \quad (22)$$

where “ $\sim$ ” is independent of  $N \in \mathbb{N}$ ,  $k \in \mathbb{Z}^n$ , and the numbers  $c_{mj}$ . In other words, if (4) with  $s = 0$  is an equivalent quasi-norm in  $B_{pq}^0$  then (20) and (22) yield

$$\left[ \sum_{m=1}^N \left( \sum_{l=1}^N |c_{ml}|^p \right)^{q/p} \right]^{1/q} \sim \left[ \sum_{l=1}^N \left( \sum_{m=1}^N |c_{ml}|^q \right)^{r/q} \right]^{1/r} \quad (23)$$

where “ $\sim$ ” is independent of  $N \in \mathbb{N}$  and the numbers  $c_{ml}$ . Let  $c_{ml} = a_m b_l$ , then it follows  $r = p$ . Afterwards the choice  $c_{ml} = \delta_{ml} a_l$  yields  $p = q$ . Hence, if (23) holds then we have  $p = q = r$ . The proof is complete.

**Remark 1.** Let  $0 < p < \infty$ ,  $0 < q \leq \infty$ , and  $s > n/p$ . We prove (7). Let  $f$  be given by (2.3.1/6), then we have

$$\left\| \sup_{|x-y| \leq c} |f(y)| \mid L_p \right\| \leq c \sum_{k=0}^{\infty} 2^{k\varepsilon} \left\| \sup_{|x-y| \leq c} |\varphi_k(D)f(y)| \mid L_p \right\| \quad (24)$$

where  $\varepsilon > 0$  is an arbitrary number. Now we use the scalar case of (2.2.4/3) with

$$\sup_{|x-y| \leq c} |\varphi_k(D)f(y)| \leq c' 2^{ka} \sup_{z \in \mathbb{R}^n} \frac{|\varphi_k(D)f(x-z)|}{1 + |2^k z|^a} \quad (25)$$

and  $a > n/p$ . Then the left-hand side of (24) can be estimated from above by

$$c \sum_{k=0}^{\infty} 2^{(a+\varepsilon)k} \|\varphi_k(D)f \mid L_p\| = c \|f \mid B_{p1}^{a+\varepsilon}\|, \quad (26)$$

where we used (2.3.1/7). Since  $a$  and  $\varepsilon$  are arbitrary numbers with  $a > n/p$  and  $\varepsilon > 0$ , we can estimate the right-hand side of (26) from above by  $c \|f \mid B_{pq}^s\|$ , see (2.3.2/23). The proof is complete.

**Remark 2.** We denote part (i) of the theorem as the localization principle for the spaces  $F_{pq}^s$ . Part (ii) indicates that there is no counterpart for the spaces  $B_{pq}^s$ . The proof shows that we did not use the special structure of the functions  $\psi_k(x) = \psi(x - k)$ , it is sufficient to have a good control of the supports of  $\psi_k$  and of some of its derivatives (in dependence on  $s, p, q$ ). The localization principle for  $F_{pq}^s$ , combined with the local means from the preceding subsections, are the basis for our considerations in the following chapters.

**Remark 3.** Instead of an “additive” localization based on  $\psi_k(x) = \psi(x - k)$  one can ask for a corresponding “multiplicative” localization based on  $\varphi_k(x) = \varphi(2^k x)$ , where  $\varphi$  is a suitable function. This problem has been studied in [Bou2, You1,2],

where the authors prefer the homogeneous counterparts of  $F_{pq}^s$ , see 1.4.5 as far as homogeneous spaces are concerned.

**Remark 4.** Actually, (3) is a characterization: If  $f \in S'$  such that (3) is finite, then  $f \in F_{pq}^s$ , see Proposition 7.2.2.

## 2.4.8 Fourier multipliers and maximal inequalities

In connection with pseudodifferential operators we need some Fourier multiplier assertions and maximal inequalities for  $F_{pq}^s$  spaces. Although the needed properties are more or less covered by previous considerations we formulate them separately in this subsection. Let  $a(\eta, \xi)$  be a complex-valued function defined for  $\eta \in \mathbb{R}^n$  and  $\xi \in \mathbb{R}^n$  such that  $\xi \rightarrow a(\eta, \xi)$  for any fixed  $\eta \in \mathbb{R}^n$  is a  $C^\infty$  function in  $\mathbb{R}^n$  with

$$\sup_{\eta \in \mathbb{R}^n} (1 + |\xi|)^k \sum_{|\alpha|=k} |D_\xi^\alpha a(\eta, \xi)| < \infty, \quad k \in \mathbb{N}_0. \quad (1)$$

We extend the notation (2.3.1/5) to

$$a(\eta, D)f(x) = (a(\eta, \cdot)\hat{f}(\cdot))^\vee(x), \quad x \in \mathbb{R}^n, \quad (2)$$

where  $\eta$  must be considered as a parameter. It is convenient for us to write temporarily  $F_{\infty\infty}^s = B_{\infty\infty}^s$  and to assume that Theorem 2.4.6 can be extended to  $p = q = \infty$ . This will be justified later on, see 2.5.3. Let  $k(t, f)$  be the same means as in (2.4.6/1) and let  $k_0$ ,  $k^0$ , and  $k$  be the same functions as in (2.4.6/2,4), where  $N \in \mathbb{N}$  is at our disposal. The Tauberian conditions (2.4.6/3) need not be satisfied now.

**Proposition.** (i) Let either  $0 < p < \infty$ ,  $0 < q \leq \infty$  or  $p = q = \infty$ , let  $s \in \mathbb{R}$ . Let  $N \in \mathbb{N}$  with  $2N > \frac{n}{2} + \frac{n}{\min(p,q)} + \max(s, \sigma_p)$ , then there exists a positive number  $c$  such that

$$\left\| \sup_{\eta \in \mathbb{R}^n} |a(\eta, D)k_0(1, f)| \mid L_p \right\| \quad (3)$$

$$+ \left\| \left( \int_0^1 t^{-sq} \sup_{\eta \in \mathbb{R}^n} |a(\eta, D)k(t, f)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \mid L_p \right\| \leq c \|f\| \mid F_{pq}^s \|$$

(modification if  $q = \infty$ ) for all  $f \in F_{pq}^s$ .

(ii) Let either  $0 < p < \infty$ ,  $0 < q \leq \infty$  or  $p = q = \infty$ , let  $s > n/p$  and  $b > 0$ , then there exists a positive number  $c$  such that

$$\left\| \sup_{|x-y| \leq b} \sup_{\eta \in \mathbb{R}^n} |a(\eta, D)f(y)| \right\|_{L_p} \leq c \|f\|_{F_{pq}^s} \quad (4)$$

for all  $f \in F_{pq}^s$ .

**Proof.** Step 1. We prove (i) by combining the proof of Theorem 2.4.6 and the two first steps of the proof of Theorem 2.4.1, where we assume  $0 < p < \infty$ ,  $0 < q \leq \infty$ . The case  $p = q = \infty$  can be treated in the same way based on Theorem 2.5.3. By (2) and the arguments from the proof of Theorem 2.4.6 we have

$$a(\eta, D)k(t, f)(x) = (a(\eta, \xi)t^{2N}|\xi|^{2N}k^{0V}(t\xi)\hat{f}(\xi))^\vee(x) \quad (5)$$

and similarly for  $a(\eta, D)k_0(1, f)$ . Now we follow the arguments from Step 1 of the proof of Theorem 2.4.1 and replace  $\varphi_j(\xi) = \varphi(2^{-j}\xi)$  by  $a(\eta, \xi)t^{2N}|\xi|^{2N}k^{0V}(t\xi)$  with  $t = 2^{-j}$ . We have the counterparts of (2.4.1/13,15,16,18). The critical first factor in the integral in (2.4.1/16) led to the hypotheses (2.4.1/8) or better (2.4.1/50) which now must be replaced by

$$\left\| a(\eta, 2^j x) |x|^{2N} \frac{k^{0V}(x)h(x)}{|x|^{s_1}} \right\|_{H_2^\sigma} < \infty \quad (6)$$

with  $\sigma > \frac{n}{2} + a$ . Let  $\sigma \in \mathbb{N}$ , then we can estimate the factors connected with  $a(\eta, 2^j x)$  by

$$|D_x^\alpha a(\eta, 2^j x)| \leq c 2^{j|\alpha|} (1 + 2^j |x|)^{-|\alpha|} \leq \frac{c}{|x|^{|\alpha|}}, \quad 0 \leq |\alpha| \leq \sigma, \quad (7)$$

where  $c$  is independent of  $\eta \in \mathbb{R}^n$  and  $j \in \mathbb{N}$  (and, of course, of  $x \in \mathbb{R}^n$ ). In other words, if  $2N$  is larger than  $\sigma + s_1$  with  $\sigma > \frac{n}{2} + \frac{n}{\min(p,q)}$  and  $s_1 > \max(s, \sigma_p)$ , then (6) is fulfilled uniformly with respect to  $\eta$  and  $j$ . Hence under our assumptions for  $N$  we obtain the counterpart of (2.4.1/18), where the corresponding constant  $c$  is independent of  $\eta \in \mathbb{R}^n$  and  $j \in \mathbb{N}$ . We take the supremum with respect to  $\eta$  and proceed afterwards as in Step 1 of the proof of Theorem 2.4.1. Next we follow Step 2 of the proof of Theorem 2.4.1 with the same replacement of  $\varphi_j$  as above. The crucial integral is the counterpart of (2.4.1/26) which led to the hypotheses (2.4.1/9,10) or better (2.4.1/51,52), which look similar as (6). In (7) we have now  $|x| \sim 1$ , which shows that the left-hand side of (7) can be estimated by a constant uniformly with respect to  $\eta \in \mathbb{R}^n$  and  $j \in \mathbb{N}_0$ . We arrive at the counterpart of (2.4.1/28), take the supremum with respect to  $\eta \in \mathbb{R}^n$  and obtain finally the discrete version of (3). However by the same arguments as in Step 4 of the proof of Theorem 2.4.1 we have also a corresponding estimate for the continuous version. The proof of (i) is complete.

Step 2. We prove (ii) by modifying the arguments from Remark 2.4.7/1. The counterparts of (2.4.7/24,25) read as follows,

$$\left\| \sup_{|x-y| \leq b} \sup_{\eta \in \mathbb{R}^n} |a(\eta, D)f(y)| \right\|_{L_p} \quad (8)$$

$$\leq c \sum_{k=0}^{\infty} 2^{k\varepsilon} \left\| \sup_{|x-y| \leq b} \sup_{\eta \in \mathbb{R}^n} |a(\eta, D)\varphi_k(D)f(y)| \right\|_{L_p}$$

and

$$\sup_{|x-y| \leq b} \sup_{\eta \in \mathbb{R}^n} |a(\eta, D)\varphi_k(D)f(y)| \leq c 2^{ka} \sup_{z \in \mathbb{R}^n, \eta \in \mathbb{R}^n} \frac{|a(\eta, D)\varphi_k(D)f(x-z)|}{1 + |2^k z|^a} \quad (9)$$

with  $a > n/p$ . Next we remark that the proof of (2.2.4/4) is based on

$$\sup_{z \in \mathbb{R}^n} \frac{|(M_k \hat{f}_k)^\vee(x-z)|}{1 + |d_k z|^a} \leq c \|M_k(d_k \cdot)\|_{H_2^\kappa} \sup_{z \in \mathbb{R}^n} \frac{|f_k(x-z)|}{1 + |d_k z|^a} \quad (10)$$

with  $\kappa > a + n/2$ , see [Triß: (1.6.3/2)]. We estimate the right-hand side of (9) by using (10) with  $d_k \sim 2^k$ ,  $M_k(\xi) = a(\eta, \xi)\varphi_k(\xi)$  and with  $\sum_{r=-2}^2 \varphi_{k+r}(D)f$  instead of  $f$  in (9), see (2.3.2/4) as far as the latter replacement is concerned. By (1) the first factor on the right-hand side of (10) can be estimated from above independently of  $k \in \mathbb{N}_0$  and  $\eta \in \mathbb{R}^n$ . Taking the supremum over  $\eta \in \mathbb{R}^n$  the indicated modification of (10) shows that the right-hand side of (9) can be estimated from above by

$$c 2^{ka} \sup_{z \in \mathbb{R}^n} \sum_{r=-2}^2 \frac{|\varphi_{k+r}(D)f(x-z)|}{1 + |2^k z|^a}. \quad (11)$$

The rest is the same as in Remark 2.4.7/1 and the proof of (ii) is complete.

## 2.5 General characterizations for $B_{pq}^s$

### 2.5.1 The main theorem

In 2.3 we introduced and studied the spaces  $B_{pq}^s$  and  $F_{pq}^s$  simultaneously. This will also be done later on. Sections 2.4 and 2.5 are exceptions: we treated the more complicated spaces  $F_{pq}^s$  in 2.4 and collect now in 2.5 the respective counterparts for the simpler spaces  $B_{pq}^s$ , where we refer mostly to 2.4 and indicate the necessary changes in the proofs. Recall

$$\sigma_p = n \left( \frac{1}{p} - 1 \right)_+, \quad 0 < p \leq \infty, \quad (1)$$

see (2.4.1/1). Furthermore the functions  $h$  and  $H$  as well as  $\varphi(tD)f$  have the same meaning as at the beginning of 2.4.1.

**Theorem.** Let  $0 < p \leq \infty$ ,  $0 < q \leq \infty$ , and  $s \in \mathbb{R}$ . Let  $s_0$  and  $s_1$  be two real numbers with

$$s_0 + \sigma_p < s < s_1 \quad \text{and} \quad s_1 > \sigma_p. \quad (2)$$

Let  $\varphi_0$  and  $\varphi$  be two complex-valued  $C^\infty$  functions on  $\mathbb{R}^n$  and  $\mathbb{R}^n \setminus \{0\}$ , respectively, which satisfy the Tauberian conditions

$$|\varphi_0(x)| > 0 \quad \text{if} \quad |x| \leq 2 \quad (3)$$

and

$$|\varphi(x)| > 0 \quad \text{if} \quad \frac{1}{2} \leq |x| \leq 2.$$

Let  $\bar{p} = \min(1, p)$  and let

$$\int_{\mathbb{R}^n} \left| \left( \frac{\varphi(z)h(z)}{|z|^{s_1}} \right)^\vee (y) \right|^{\bar{p}} dy < \infty, \quad (4)$$

$$\sup_{m \in \mathbb{N}} 2^{-ms_0\bar{p}} \int_{\mathbb{R}^n} |(\varphi(2^m \cdot)H(\cdot))^\vee(y)|^{\bar{p}} dy < \infty \quad (5)$$

and

$$\sup_{m \in \mathbb{N}} 2^{-ms_0\bar{p}} \int_{\mathbb{R}^n} |(\varphi_0(2^m \cdot)H(\cdot))^\vee(y)|^{\bar{p}} dy < \infty. \quad (6)$$

Let  $\varphi_j(x) = \varphi(2^{-j}x)$  if  $x \in \mathbb{R}^n \setminus \{0\}$  and  $j \in \mathbb{N}$ . Then

$$\left( \sum_{j=0}^{\infty} 2^{jsq} \|\varphi_j(D)f\|_{L_p}^q \right)^{1/q} \quad (7)$$

and

$$\|\varphi_0(D)f\|_{L_p} + \left( \int_0^1 t^{sq} \|\varphi(tD)f\|_{L_p}^q \frac{dt}{t} \right)^{1/q} \quad (8)$$

(modification if  $q = \infty$ ) are equivalent quasi-norms in  $B_{pq}^s$ .

**Proof.** This is the counterpart of Theorem 2.4.1 and we modify its proof. We have again the splitting (2.4.1/13), the estimate (2.4.1/15), and the expression (2.4.1/16). Let  $1 \leq p \leq \infty$ . Then we apply the  $L_p$ -norm to (2.4.1/15,16), use (4) with  $\bar{p} = 1$ , and apply afterwards the  $l_q$ -quasi-norm. Then we obtain the following counterpart of (2.4.1/19)

$$\left( \sum_{j=1}^{\infty} 2^{jsq} \left\| \sum_{l=-\infty}^K 2^{js} (\varphi_j \rho_{l+j} \hat{f})^\vee \mid L_p \right\|^q \right)^{1/q} \leq c \left( \sum_{m=0}^{\infty} 2^{msq} \|(\tilde{\rho}_m \hat{f})^\vee \mid L_p\|^q \right)^{1/q}. \quad (9)$$

Let  $0 < p < 1$ . We use again an inequality of Plancherel–Polyá–Nikol’skij type, see [Triß: (1.3.2/5)] and estimate the integral in (2.4.1/16) from above by

$$c 2^{(j+K)\sigma_p} \left( \int_{\mathbb{R}^n} \left| \left( \frac{\varphi(2^{-j}z)}{|2^{-j}z|^{s_1}} h(c 2^{-j}z) \right)^\vee(y) | (2^{s(j+l)} \tilde{\rho}_{j+l} \hat{f})^\vee(x-y) \right|^p dy \right)^{1/p} \quad (10)$$

where  $c$  is independent of  $j$ . We put this estimate in (2.4.1/15), apply the  $L_p$ -quasi-norm, use (4) with  $\bar{p} = p$ , apply the  $l_q$ -quasi-norm and obtain (9). Recall that  $(\Sigma b_k)^p \leq \Sigma b_k^p$  for non-negative  $b_k$ ’s. As in the first step of the proof of Theorem 2.4.1 we arrive at the following counterpart of (2.4.1/23)

$$\left( \sum_{j=0}^{\infty} 2^{jsq} \left\| \sum_{l=-\infty}^K (\varphi_j \rho_{l+j} \hat{f})^\vee \mid L_p \right\|^q \right)^{1/q} \leq c \|f \mid B_{pq}^s\|. \quad (11)$$

In precisely the same way the second step of the proof of Theorem 2.4.1 can be modified. We have to use (5) and (6), and obtain

$$\left( \sum_{j=0}^{\infty} 2^{jsq} \left\| \sum_{l=K+1}^{\infty} (\varphi_j \rho_{l+j} \hat{f})^\vee \mid L_p \right\|^q \right)^{1/q} \leq c 2^{-K(s-s_0)} \|f \mid B_{pq}^s\| \quad (12)$$

as the counterpart of (2.4.1/29). The constant  $c$  is independent of  $K$ . However (11,12) prove that the quasi-norm in (7) can be estimated from above by  $c \|f \mid B_{pq}^s\|$ . One can carry over the technical remarks from Step 3 of the proof of Theorem 2.4.1 in a slightly modified way. This can also be done with the arguments from Step 4 of the proof of Theorem 2.4.1 which proves that the quasi-norm in (8) can be estimated from above by  $c \|f \mid B_{pq}^s\|$ . In order to prove the reverse inequalities we modify Step 5 of the proof of Theorem 2.4.1. We have (2.4.1/41) where now  $0 < r < \bar{p}$  is sufficient. We use the usual (scalar) Hardy–Littlewood maximal inequality with respect to the  $L_{p/r}$ -norm and apply afterwards the  $l_{q/r}$ -quasi-norm. By the same arguments as in Step 5 of the proof of Theorem 2.4.1 we obtain that  $\|f \mid B_{pq}^s\|$  can be estimated from above by the quasi-norm in (7). Finally we modify Step 6 of the proof of Theorem 2.4.1 which shows that  $\|f \mid B_{pq}^s\|$  can also be estimated from above by the quasi-norm in (8).

**Remark 1.** We followed closely [Tri16: pp. 180–182]. In contrast to Theorem 2.4.1 we avoided the technique of maximal functions completely. The advantages of

(4–6) compared with (2.4.1/8–10) will be clear later on when we discuss equivalent quasi-norms where differences  $\Delta_h^M$  are involved.

**Corollary 1.** Let  $p, q, s, s_0$ , and  $s_1$  be the same numbers as in the Theorem. Let  $\sigma > \sigma_p + \frac{n}{2}$ , see (1). Let  $\varphi_0$  and  $\varphi$  be two complex-valued  $C^\infty$  functions on  $\mathbb{R}^n$  and  $\mathbb{R}^n \setminus \{0\}$ , respectively, which satisfy the Tauberian conditions (3), (3'), and

$$\left\| \frac{\varphi(x)h(x)}{|x|^{s_1}} \mid H_2^\sigma \right\| < \infty, \quad (13)$$

$$\sup_{m \in \mathbb{N}} 2^{-ms_0} \|\varphi(2^m \cdot)H(\cdot) \mid H_2^\sigma\| < \infty, \quad (14)$$

$$\sup_{m \in \mathbb{N}} 2^{-ms_0} \|\varphi_0(2^m \cdot)H(\cdot) \mid H_2^\sigma\| < \infty, \quad (15)$$

where  $h$  and  $H$  have the same meaning as at the beginning of 2.4.1, and  $H_2^\sigma$  are the fractional Sobolev spaces. Then (7) and (8) are equivalent quasi-norms in  $B_{pq}^s$ , where  $\varphi_j(x) = \varphi(2^{-j}x)$  and  $\varphi(tD)f$  have the above meaning.

**Proof.** The assertion is an immediate consequence of the Theorem and (2.4.1/53).

**Corollary 2.** Let  $0 < p \leq \infty$ ,  $0 < q \leq \infty$ , and  $s \in \mathbb{R}^n$ . Let  $a > n/p$ . Let  $s_0$  and  $s_1$  be two real numbers with

$$s_0 + a < s < s_1 \quad \text{and} \quad s_1 > \sigma_p. \quad (16)$$

Let  $\varphi_0$  and  $\varphi$  be two complex-valued  $C^\infty$  functions on  $\mathbb{R}^n$  and  $\mathbb{R}^n \setminus \{0\}$ , respectively, which satisfy the Tauberian conditions (3), (3'), and either (2.4.1/8–10) with the above number  $a$  of (13–15) with  $\sigma > a + n/2$ . Then

$$\left( \sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j^* f)_a \mid L_p\|^q \right)^{1/q} \quad (17)$$

(modification if  $q = \infty$ ), with the maximal functions from (2.4.1/55), is an equivalent quasi-norm in  $B_{pq}^s$ .

**Proof.** Let (2.4.1/8–10) with  $a > n/p$  be fulfilled, then one can follow the proof of Theorem 2.4.1 with the modifications described in Corollary 2.4.1/2. The replacement of (2.4.1/8–10) by (13–15) with  $\sigma > a + \frac{n}{2}$  is covered by (2.4.1/53).

**Remark 2.** The above Theorem and the Corollaries 1 and 2 are the  $B_{pq}^s$ -counterparts of Theorem 2.4.1 and the Corollaries 2.4.1/1 and 2.4.1/2. We remark that the conditions (4–6) are less restrictive than (2.4.1/8–10) with  $a > n/p$ . This is clear if  $p \geq 1$  and it follows from Hölder's inequality if  $p < 1$ . Furthermore the characterizations of  $F_{pq}^s$  described in 2.4.2 can be carried over to  $B_{pq}^s$  spaces.

## 2.5.2 Modifications of the main theorem

We describe the counterparts of the two theorems in 2.4.3 for the spaces  $B_{pq}^s$ . We use the same notations as in 2.4.1 and 2.5.1, in particular  $\sigma_p$  is given by (2.5.1/1), and  $h(x) \in S$  and  $H(x) \in S$  are the same functions as at the beginning of 2.4.1 with (2.4.1/2,3). Furthermore,  $(\varphi(h \cdot) \hat{f})^\vee$  must be interpreted similarly as in 2.4.1 and 2.4.2.

**Theorem 1.** Let  $0 < p \leq \infty$ ,  $0 < q \leq \infty$ , and  $s \in \mathbb{R}$ . Let  $s_0$  and  $s_1$  be two real numbers with (2.5.1/2). Let  $\varphi_0(x)$  be the same function as in Theorem 2.5.1, including (2.5.1/3,6), where again  $\bar{p} = \min(1, p)$ . Let  $\varphi(t)$  be a complex-valued  $C^\infty$  function on  $\mathbb{R} \setminus \{0\}$  which satisfies

$$|\varphi(t)| > 0 \quad \text{if} \quad \frac{1}{8} < t < 8 \quad (1)$$

(Tauberian condition) and

$$\sup_{1 \leq |\gamma| \leq 2} \int_{\mathbb{R}^n} \left| \left( \frac{\varphi(\gamma z) h(z)}{|z|^{s_1}} \right)^\vee (y) \right|^{\bar{p}} dy < \infty, \quad (2)$$

$$\sup_{1 \leq |\gamma| \leq 2} \sup_{m \in \mathbb{N}} 2^{-ms_0 \bar{p}} \int_{\mathbb{R}^n} |(\varphi(2^m \gamma z) H(z))^\vee (y)|^{\bar{p}} dy < \infty. \quad (3)$$

Then

$$\|\varphi_0(D)f\|_{L_p} + \left( \int_{|h| \leq 1} |h|^{-sq} \|(\varphi(h \cdot) \hat{f})^\vee\|_{L_p}^q \frac{dh}{|h|^n} \right)^{1/q} \quad (4)$$

and

$$\|\varphi_0(D)f\|_{L_p} + \left( \int_0^1 t^{-sq} \sup_{\frac{t}{2} \leq |h| \leq t} \|(\varphi(h \cdot) \hat{f})^\vee\|_{L_p}^q \frac{dt}{t} \right)^{1/q} \quad (5)$$

(modification if  $q = \infty$ ) are equivalent quasi-norms in  $B_{pq}^s$ . If, in addition,  $s > 0$  then

$$\|\varphi_0(D)f \mid L_p\| + \left( \int_0^1 t^{-sq} \sup_{0 < |h| \leq t} \|(\varphi(h \cdot) \hat{f})^\vee \mid L_p\|^q \frac{dt}{t} \right)^{1/q} \quad (6)$$

(modification if  $q = \infty$ ) is also an equivalent quasi-norm in  $B_{pq}^s$ .

**Proof.** Recall that the above assertion, at least as far as (4) and (5) are concerned, is the counterpart of Theorem 2.4.3/1, which we proved by modifying the proof of Theorem 2.4.1. Now one can do the same with the above assertions starting with Theorem 2.5.1 and using the arguments from the proof of Theorem 2.4.3/1. We omit the details. Then we obtain that (4) and (5) are equivalent quasi-norms in  $B_{pq}^s$ . If, in addition,  $s > 0$ , then we can use the arguments from the proof of Proposition 2.4.4/1 in order to show that (6) is also an equivalent quasi-norm.

**Remark.** As we mentioned the above theorem is the counterpart both of Theorem 2.4.3/1 and Proposition 2.4.4/1(i). There is no problem to prove a corresponding counterpart of Proposition 2.4.4/2, where the latter one was of great service for us in connection with Proposition 2.4.6. We fix this observation.

**Proposition.** Let  $0 < p \leq \infty$ ,  $0 < q \leq \infty$  and  $s \in \mathbb{R}$ . Let  $s_0$  and  $s_1$  be two real numbers with (2.5.1/2). Let  $\varphi_0(x)$  be the same function as in Theorem 2.5.1, including (2.5.1/3,6), where again  $\bar{p} = \min(1, p)$ . Let  $\varphi^1(x), \dots, \varphi^N(x)$  be  $N$  complex-valued  $C^\infty$  functions on  $\mathbb{R}^n \setminus \{0\}$  which satisfy the Tauberian condition (2.4.4/4), and also (2.5.1/4,5) with  $\varphi^k$  instead of  $\varphi$ . Then

$$\|\varphi_0(D)f \mid L_p\| + \sum_{k=1}^N \left( \int_0^1 t^{-sq} \|\varphi^k(tD)f \mid L_p\|^q \frac{dt}{t} \right)^{1/q} \quad (7)$$

and

$$\|\varphi_0(D)f \mid L_p\| + \sum_{k=1}^N \left( \int_0^1 t^{-sq} \sup_{\frac{t}{2} \leq \tau \leq t} \|\varphi^k(\tau D)f \mid L_p\|^q \frac{dt}{t} \right)^{1/q} \quad (8)$$

(modification if  $q = \infty$ ) are equivalent quasi-norms in  $B_{pq}^s$ . If, in addition,  $s > 0$ , then

$$\|\varphi_0(D)f \mid L_p\| + \sum_{k=1}^N \left( \int_0^1 t^{-sq} \sup_{0 < \tau \leq t} \|\varphi^k(\tau D)f \mid L_p\|^q \frac{dt}{t} \right)^{1/q} \quad (9)$$

(modification if  $q = \infty$ ) is also an equivalent quasi-norm in  $B_{pq}^s$ .

Our next task is to describe the counterparts of Theorem 2.4.3/2 and Proposition 2.4.4/1(ii). Let again  $l$  be a distribution on the real line satisfying (2.4.3/10,11). Let  $l(h, f)(x)$  be given by (2.4.3/12) if it makes sense. Let  $\sigma_p$  be given by (2.5.1/1), and let  $H(x) \in S$  be the same function as at the beginning of 2.4.1 with (2.4.1/2,3). Again the function  $h(x)$  from (2.4.1/2,3) will not be needed now.

**Theorem 2.** Let  $0 < p \leq \infty$ ,  $0 < q \leq \infty$ , and  $s \in \mathbb{R}$ . Let  $s_0 \in \mathbb{R}$  and  $m \in \mathbb{N}$  with

$$s_0 + \sigma_p < s < m, \quad m > \sigma_p, \quad (10)$$

and (2.4.3/11) for some  $\delta > 0$ . Let

$$\sup |h|^{-s_0} \int_{\mathbb{R}^n} |(\psi(hz)H(z))^\vee(y)|^{\bar{p}} dy < \infty \quad (11)$$

with  $\bar{p} = \min(1, p)$ , where the supremum is taken over all  $h \in \mathbb{R}^n$  with  $|h| \geq 1$ . Let  $\varphi_0 \in S$  with  $\varphi_0(0) \neq 0$ , then

$$\|\varphi_0(D)f \mid L_p\| + \left( \int_{|h| \leq 1} |h|^{-sq} \|l(h, f) \mid L_p\|^q \frac{dh}{|h|^n} \right)^{1/q} \quad (12)$$

and

$$\|\varphi_0(D)f \mid L_p\| + \left( \int_0^1 t^{-sq} \sup_{\frac{t}{2} \leq |h| \leq t} \|l(h, f) \mid L_p\|^q \frac{dt}{t} \right)^{1/q} \quad (13)$$

(modification if  $q = \infty$ ) are equivalent quasi-norms in  $B_{pq}^s$ . If, in addition,  $s > 0$ , then

$$\|\varphi_0(D)f \mid L_p\| + \left( \int_0^1 t^{-sq} \sup_{0 < |h| \leq t} \|l(h, f) \mid L_p\|^q \frac{dt}{t} \right)^{1/q} \quad (14)$$

(modification if  $q = \infty$ ) is an equivalent quasi-norm in  $B_{pq}^s$ .

**Proof.** The proof is the same as the proof of Theorem 2.4.3/2, at least as far as the equivalent quasi-norms in (12) and (13) are concerned. It is reduced to Theorem 1 by the same scheme as Theorem 2.4.3/2 is reduced to Theorem 2.4.3/1. As far as (14) is concerned we refer to the proof of Proposition 2.4.4/1.

### 2.5.3 Local means

We formulate the counterparts of Theorem 2.4.6 and Proposition 2.4.6. Let  $k(t, f)$  be the same means as in 2.4.6 and let  $k_0$ ,  $k^0$ , and  $k$  be the same functions as in (2.4.6/2-4). Let again  $\sigma_p = n(\frac{1}{p} - 1)_+$ , see (2.4.1/1).

**Theorem.** Let  $0 < p \leq \infty$ ,  $0 < q \leq \infty$ , and  $s \in \mathbb{R}$ . Let  $N \in \mathbb{N}$  with

$$2N > \max(s, \sigma_p), \quad (1)$$

then

$$\|k_0(1, f) \mid L_p\| + \left( \sum_{j=0}^{\infty} 2^{jsq} \|k(2^{-j}, f) \mid L_p\|^q \right)^{1/q}, \quad (2)$$

$$\|k_0(1, f) \mid L_p\| + \left( \int_0^1 t^{-sq} \|k(t, f) \mid L_p\|^q \frac{dt}{t} \right)^{1/q}, \quad (3)$$

and

$$\|k_0(1, f) \mid L_p\| + \left( \int_0^1 t^{-sq} \sup_{\frac{t}{2} \leq \tau \leq t} \|k(\tau, f) \mid L_p\|^q \frac{dt}{t} \right)^{1/q} \quad (4)$$

(modification if  $q = \infty$ ) are equivalent quasi-norms in  $B_{pq}^s$ . If, in addition,  $s > 0$ , then

$$\|k_0(1, f) \mid L_p\| + \left( \int_0^1 t^{-sq} \sup_{0 < \tau \leq t} \|k(\tau, f) \mid L_p\|^q \frac{dt}{t} \right)^{1/q} \quad (5)$$

(modification if  $q = \infty$ ) is also an equivalent quasi-norm in  $B_{pq}^s$ .

**Proof.** We use precisely the same arguments as in the proof of Theorem 2.4.6, now with Theorems 2.5.1 and 2.5.2/1 as basis.

**Proposition.** Let  $k_0$  and  $k^0$  be the above functions. Let  $0 < p \leq \infty$ ,  $0 < q \leq \infty$ , and  $s \in \mathbb{R}$ . Let  $M \in \mathbb{N}$  with

$$M > \max(s, \sigma_p), \quad (6)$$

then the counterpart of (3), i.e.,

$$\|k_0(1, f) \mid L_p\| + \sum_{|\alpha|=M} \left( \int_0^1 t^{(M-s)q} \|k^0(t, D^\alpha f) \mid L_p\|^q \frac{dt}{t} \right)^{1/q}, \quad (7)$$

(modification if  $q = \infty$ ) and the corresponding counterparts of (2,4) and, if  $s > 0$ , of (5) are equivalent quasi-norms in  $B_{pq}^s$ .

**Proof.** We use precisely the same arguments as in the proof of Proposition 2.4.6, now with Proposition 2.5.2 as basis.

**Remark.** If one compares 2.4 with 2.5 then it is quite clear that we prefer  $F_{pq}^s$  to  $B_{pq}^s$ . By rule of thumb, assertions for  $B_{pq}^s$  can be proved easier than corresponding assertions for  $F_{pq}^s$ . The theorems, corollaries, and propositions in 2.5 can serve as examples, but this is not so clear by the way we came. The above claim is better reflected by the proofs given in 2.3. More important, occasionally, conditions in theorems for  $B_{pq}^s$  spaces have final character compared with corresponding conditions in related theorems for  $F_{pq}^s$  spaces. A typical example is given by the two main theorems if one compares (2.5.1/4–6) with (2.4.1/8–10). Technically speaking, this difference is caused by the fact that we had been forced in 2.4.1 to use the technique of maximal functions in contrast to the proof of Theorem 2.5.1. As a consequence, in some applications we obtain immediately final results for the spaces  $B_{pq}^s$ , whereas for the spaces  $F_{pq}^s$  few minor additional considerations are needed. On the other hand, there is a crucial exception of the above rule of thumb: the localization principle from Theorem 2.4.7 is correct only for the spaces  $F_{pq}^s$ . This observation will be of decisive importance in connection with the introduction of spaces of type  $F_{pq}^s$  and  $B_{pq}^s$  on general structures, such as Riemannian manifolds and Lie groups. We return to this subject in detail in Chapter 7.

## 2.6 Concrete characterizations

### 2.6.1 Differences and derivatives: the spaces $B_{pq}^s$

In 2.6 we shall describe concrete characterizations both for  $B_{pq}^s$  and  $F_{pq}^s$  which follow by specialization from 2.4 and 2.5. First we discuss equivalent quasi-norms in  $B_{pq}^s$  connected with derivatives  $D^\alpha$  and differences  $\Delta_h^M$ . This is a model case and for

some assertions we give two independent proofs which show clearly both the power and the shortcomings of the general characterizations from 2.4 and 2.5. We recall some notations. If  $\alpha = (\alpha_1, \dots, \alpha_n)$  is a multi-index, i.e.,  $\alpha_j \in \mathbb{N}_0$ , then

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \quad \text{with} \quad |\alpha| = \alpha_1 + \dots + \alpha_n$$

stands for the derivatives. Let

$$\Delta_h^M f(x) = \sum_{j=0}^M (-1)^{M-j} \binom{M}{j} f(x + hj), \quad M \in \mathbb{N}, \quad h \in \mathbb{R}^n, \quad x \in \mathbb{R}^n, \quad (1)$$

where  $\binom{M}{j}$  are the binomial coefficients, see also (1.2.2/4). If  $h = (t, 0, \dots, 0)$  with  $t \in \mathbb{R}$  then we write  $\Delta_h^M = \Delta_{t,1}^M$  for the differences with respect to the first direction of the coordinates. Similarly  $\Delta_{t,k}^M$  with  $k = 2, \dots, n$ . Recall  $\sigma_p = n(\frac{1}{p} - 1)_+$ .

**Theorem.** Let  $0 < p \leq \infty$ ,  $0 < q \leq \infty$ , and  $\sigma_p < s < M$  where  $M \in \mathbb{N}$ . Then

$$\|f\|_{L_p} + \left( \int_{|h| \leq 1} |h|^{-sq} \|\Delta_h^M f\|_{L_p}^q \frac{dh}{|h|^n} \right)^{1/q}, \quad (2)$$

$$\|f\|_{L_p} + \left( \int_0^1 t^{-sq} \sup_{0 < |h| \leq t} \|\Delta_h^M f\|_{L_p}^q \frac{dt}{t} \right)^{1/q}, \quad (3)$$

$$\|f\|_{L_p} + \sum_{k=1}^n \left( \int_0^1 t^{-sq} \|\Delta_{t,k}^M f\|_{L_p}^q \frac{dt}{t} \right)^{1/q}, \quad (4)$$

and

$$\|f\|_{L_p} + \sum_{k=1}^n \left( \int_0^1 t^{-sq} \sup_{0 < \tau \leq t} \|\Delta_{\tau,k}^M f\|_{L_p}^q \frac{dt}{t} \right)^{1/q} \quad (5)$$

(modification if  $q = \infty$ ) are equivalent quasi-norms in  $B_{pq}^s$ .

**Proof.** Step 1. We use Theorem 2.5.2/1 with  $\varphi_0(x) = 1$  and  $\varphi(t) = (e^{it} - 1)^M$ . Then we have (2.5.1/3,6) with  $s_0 = 0$ , and an immaterial modification of (2.5.2/1). Recall  $[(e^{i\gamma z} - 1)^M g]^\vee = \Delta_\gamma^M g^\vee$ , then it follows that (2.5.2/3) with  $s_0 = 0$  is fulfilled. Finally we fix  $s_1$  with  $s_1 > s$ . If  $M$  is large then (2.5.2/2) is fulfilled, see also

(2.4.1/53). Now (2.5.2/4,6) coincides with (2,3). This result can be extended to any  $M \in \mathbb{N}$  with  $M > s_1$ , what will be done in Remark 2 below.

Step 2. We give a second proof that the expressions in (2,3) are equivalent quasi-norms in  $B_{pq}^s$ . This time we use Theorem 2.5.2/2. Under the above restrictions for  $s$  the function  $\varphi_0 \in S$ ,  $\varphi_0(0) \neq 0$ , in this theorem can be replaced by the above choice  $\varphi_0(x) = 1$ . Let  $\psi(\lambda) = (e^{i\lambda} - 1)^M$ , then we have (2.4.3/11) and (2.5.2/10) with  $m = M$ . Furthermore (2.5.2/11) with  $s_0 = 0$  is fulfilled, see Step 1. Finally we have an immaterial modification of (2.4.3/10) with  $l = F_1^{-1}\psi$ , and  $l(h, f)(x) = c\Delta_h^M f(x)$ , where  $c$  is an unimportant number. Then (2.5.2/12,14) coincides with (2,3).

Step 3. In order to prove that (4) and (5) are equivalent quasi-norms in  $B_{pq}^s$  we choose  $\varphi_0(x) = 1$  and  $\varphi^k(x) = (e^{ix_k} - 1)^M$  with  $x = (x_1, \dots, x_n)$  and  $k = 1, \dots, n$ . We apply Proposition 2.5.2 where we assume that  $M$  is large. Then it follows by the same arguments as in Step 1 that the expressions in (2.5.2/7,9), and hence in (4) and (5), are equivalent quasi-norms in  $B_{pq}^s$ . The fact that even  $M > s_1$  is sufficient follows from Remark 2 below.

**Remark 1.** We gave two proofs that the expressions in (2) and (3) are equivalent quasi-norms in  $B_{pq}^s$  which show advantages and disadvantages of the general characterizations from 2.5. In Step 1 we used Theorem 2.5.2/1 and obtained the desired result for large values of  $M$ . In Step 2 we used the rather sophisticated Theorem 2.5.2/2 which gave a final result. This discussion shows that the general characterizations are strong enough to cover a lot of concrete characterizations, but that one has to add some minor additional arguments from case to case.

**Remark 2.** By the Steps 1 and 3 we know that the expressions in (2–5) are equivalent quasi-norms in  $B_{pq}^s$  if  $M$  is large. We have to extend this assertion to all  $M \in \mathbb{N}$  with  $M > s$ . In Step 2 we discussed one possibility. We describe a second possibility, where we restrict ourselves to (2). First we recall the identity

$$(\Delta_h^M f)(x) = 2^{-M}(\Delta_{2h}^M f)(x) + \Delta_h^{M+1}(\Sigma a_l f(x + lh)), \quad (6)$$

where  $\Sigma$  stands for a finite sum and  $a_l$  are real numbers, which is well-known and which may be found, e.g., in [Triß: (2.5.9/45)]. We denote the expression in (2) by  $\|f\|_{B_{pq}^s|_M}$ . Then we obtain

$$\begin{aligned} \|f\|_{B_{pq}^s|_M} &\leq c\|f\|_{B_{pq}^s|_{M+1}} + 2^{-M} \left( \int_{|h| \leq 1} |h|^{-sq} \|\Delta_{2h}^M f\|_{L_p}^q \frac{dh}{|h|^n} \right)^{1/q} \\ &\leq c\|f\|_{B_{pq}^s|_{M+1}} + c'\|f\|_{L_p} + 2^{-M+s} \left( \int_{|h| \leq 1} |h|^{-sq} \|\Delta_h^M f\|_{L_p}^q \frac{dh}{|h|^n} \right)^{1/q}, \end{aligned} \quad (7)$$

see also Remark 4 below. Because  $M > s$  we obtain

$$\|f \mid B_{pq}^s\|_M \leq c \|f \mid B_{pq}^s\|_{M+1}. \quad (8)$$

The reverse inequality is obvious. Hence,  $\|f \mid B_{pq}^s\|_M$  and  $\|f \mid B_{pq}^s\|_{M+1}$  are equivalent. Starting with large values of  $M$  we obtain that the expression in (2) is an equivalent quasi-norm for all  $M \in \mathbb{N}$  with  $M > s$ . By the same argument one proves a corresponding assertion for the expressions in (3–5).

We prefer formulations in terms of equivalent quasi-norms. In 2.4.2 we discussed the problem under which conditions quasi-norms characterize the spaces  $F_{pq}^s$ . One can do the same for the spaces  $B_{pq}^s$  on the abstract level of 2.4.2. However we restrict ourselves at the moment to an example connected with the above concrete situation. Let again  $\|f \mid B_{pq}^s\|_M$  be the quasi-norm in (2).

**Corollary 1.** Let  $1 \leq p \leq \infty$ ,  $0 < q \leq \infty$ , and  $0 < s < M$  where  $M \in \mathbb{N}$ , then

$$B_{pq}^s = \{f \in S': \|f \mid B_{pq}^s\|_M < \infty\}. \quad (9)$$

**Proof.** We return to the abstract situation where (2.4.2/23) can be taken as a starting point. In the case of  $B_{pq}^s$  one can apply first the  $L_p$ -norm and then the  $l_q$ -quasi-norm which yields the desired estimate under the respective hypothesis for  $\varphi$ . Immaterial modifications show that the arguments can also be applied to the situation described in Theorem 2.4.3/1 or, as far as the  $B_{pq}^s$  spaces are concerned, in Theorem 2.5.2/1. But (9) is simply a special case of these estimates.

**Remark 3.** If  $p < 1$  then the above abstract argument does not work. However one can use rather special tools from approximation theory in order to prove the following assertion: Let  $0 < p \leq \infty$ ,  $0 < q \leq \infty$ , and  $\sigma_p < s < M$  where  $M \in \mathbb{N}$ , then

$$B_{pq}^s = \{f \in L_p: \|f \mid B_{pq}^s\|_M < \infty\}. \quad (10)$$

See [Triß: 2.5.12, in particular, Remark 2.5.12/3] where we gave details and references to related papers by E.A. Storoženko, P. Oswald and coworkers.

**Remark 4.** We add a technical observation which we used occasionally, for example in (7). But it seems to be useful to formulate it explicitly. Let  $\nu > 0$  then one can

replace the integration over  $|h| \leq 1$  in (2) by  $|h| \leq \nu$  or by  $\mathbb{R}^n$ . This is an immediate consequence of  $\|\Delta_h^M f \mid L_p\| \leq c\|f \mid L_p\|$ . Similarly one can replace the integration over the interval  $(0, 1)$  in (3–5) by an integration over the interval  $(0, \nu)$  or over  $\mathbb{R}$ .

**Corollary 2.** Let  $0 < p \leq \infty$ ,  $0 < q \leq \infty$ , and  $\sigma_p < s - L < M$ , where  $L \in \mathbb{N}$  and  $M \in \mathbb{N}$ , then

$$\|f \mid L_p\| + \sum_{|\alpha|=L} \left( \int_{|h| \leq 1} |h|^{-(s-L)q} \|\Delta_h^M D^\alpha f \mid L_p\|^q \frac{dh}{|h|^n} \right)^{1/q} \quad (11)$$

and

$$\|f \mid L_p\| + \sum_{k=1}^n \left( \int_0^1 t^{-(s-L)q} \left\| \Delta_{t,k}^M \frac{\partial^L f}{\partial x_k^L} \mid L_p \right\|^q \frac{dt}{t} \right)^{1/q} \quad (12)$$

(modification if  $q = \infty$ ) are equivalent quasi-norms in  $B_{pq}^s$ .

**Proof.** Step 1. We apply Proposition 2.5.2 with  $\varphi_0(x) = 1$  and  $\varphi^k(x) = x_k^L (e^{ix_k} - 1)^M$  where  $k = 1, \dots, n$ . Let  $s_0 = L$  and  $s_1 = L + M$ , then (2.5.1/2,3,6) are fulfilled. Furthermore we have an immaterial modification of (2.4.4/4); and (2.5.1/5) with  $\varphi^k$  instead of  $\varphi$  holds. As far as (2.5.1/4) is concerned we use the replacement described in connection with Theorems 2.4.3/2 and 2.5.2/2: the above functions  $\varphi^k$  take over the role of  $\psi$  in these theorems. Now (12) coincides with (2.5.2/7) and after these modifications it follows that (12) is an equivalent quasi-norm in  $B_{pq}^s$ .

Step 2. Now we complement the above functions  $\varphi^k(x)$  by all functions  $x_1^{L_1} \cdots x_n^{L_n} (e^{ix_k} - 1)^M$  with  $L_1 + \cdots + L_n = L$ . By the same arguments as above it follows that (12) with

$$\sum_{|\alpha|=L} \|\Delta_{t,k}^M D^\alpha f \mid L_p\| \quad \text{instead of} \quad \sum_{k=1}^n \left\| \Delta_{t,k}^M \frac{\partial^L f}{\partial x_k^L} \mid L_p \right\| \quad (13)$$

is also an equivalent quasi-norm in  $B_{pq}^s$ . We apply the above theorem to  $B_{pq}^{s-L}$ . Then it follows that the just indicated modifications of (12) and (11) are equivalent quasi-norms.

## 2.6.2 Differences and derivatives: the spaces $F_{pq}^s$

We formulate the  $F_{pq}^s$ -counterpart of Theorem 2.6.1, where the differences  $\Delta_h^M$  have the same meaning as in (2.6.1/1).

**Theorem.** Let  $0 < p < \infty$ ,  $0 < q \leq \infty$ , and  $\frac{n}{\min(p,q)} < s < M$  where  $M \in \mathbb{N}$ . Then

$$\|f\|_{L_p} + \left\| \left( \int_{|h| \leq 1} |h|^{-sq} |\Delta_h^M f(\cdot)|^q \frac{dh}{|h|^n} \right)^{1/q} \right\|_{L_p}, \quad (1)$$

$$\|f\|_{L_p} + \left\| \left( \int_0^1 t^{-sq} \sup_{0 < |h| \leq t} |\Delta_h^M f(\cdot)|^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p}, \quad (2)$$

$$\|f\|_{L_p} + \sum_{k=1}^n \left\| \left( \int_0^1 t^{-sq} |\Delta_{t,k}^M f(\cdot)|^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p}, \quad (3)$$

and

$$\|f\|_{L_p} + \sum_{k=1}^n \left\| \left( \int_0^1 t^{-sq} \sup_{0 < \tau \leq t} |\Delta_{\tau,k}^M f(\cdot)|^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p} \quad (4)$$

(modification if  $q = \infty$ ) are equivalent quasi-norms in  $F_{pq}^s$ .

**Proof.** We apply Theorem 2.4.3/1. We use the same functions  $\varphi_0(x) = 1$  and  $\varphi(t) = (e^{it} - 1)^M$  as in the proof of Theorem 2.6.1. Let, in addition,

$$\sigma_{pq} + \frac{n}{\min(p,q)} < s. \quad (5)$$

We choose  $s_0 = a > \frac{n}{\min(p,q)}$  with  $s_0 + \sigma_{pq} < s$  and  $s_1 = M$ . Then (2.4.1/5,6,10) are fulfilled. Furthermore an immaterial modification of (2.4.3/1) holds. (2.4.3/3) is satisfied where we need now  $s_0 = a$ . If  $M$  is large then we have also (2.4.3/2), see (2.4.1/53). However one can modify this argument by using the modification described in the proof of Theorem 2.4.3/2. Then  $M > s$  is sufficient, see also Step 2 of the proof of Theorem 2.6.1. Now the quasi-norms in (2.4.3/4) and (2.4.4/1) coincide with the quasi-norms in (1,2). As far as the quasi-norms in (3,4) are concerned, we argue in the same way as in Step 3 of the proof of Theorem 2.6.1 with Proposition 2.4.4/2 as basis. Then we obtain that the expressions in (3,4) are equivalent quasi-norms in  $F_{pq}^s$  at least if  $M$  is large. Again by the modification described in the proof of Theorem 2.4.3/2 it follows that  $M > s$  is sufficient. This completes the proof under the additional assumption (5). If we have only  $s > \frac{n}{\min(p,q)}$  and  $a = s_0 < s$ , then it follows by Remark 2.4.3/1 and the above modifications that the quasi-norms in (1–4) can be estimated from above by  $c\|f\|_{F_{pq}^s}$ . We postpone the converse inequality to Remark 2.6.3/3.

**Remark 1.** We follow closely in 2.6.1 and also in this subsection [Tri16]. However the results themselves are known and proved by different reasoning in [Triß: 2.5.10, 2.5.12] based on rather specific arguments directly connected with differences  $\Delta_h^M$ . Our aim in 2.6 is different. We try to convince the reader that the known concrete characterizations of  $B_{pq}^s$  and  $F_{pq}^s$  grow out in a natural way from general abstract characterizations. However as the proofs in 2.6.1 and also the above proof show some minor specific complements will be necessary from case to case.

We formulate the counterpart of Corollary 2.6.1/1 and denote the quasi-norm in (1) by  $\|f\|_{F_{pq}^s}$ .

**Corollary 1.** Let  $1 < p < \infty$ ,  $1 < q \leq \infty$ , and  $\frac{n}{\min(p,q)} < s < M$  where  $M \in \mathbb{N}$  then

$$F_{pq}^s = \{f \in S': \|f\|_{F_{pq}^s} < \infty\}. \quad (6)$$

**Proof.** The assertion follows from Theorem 2.4.2(i) or, even better, from Step 2 of the proof of that theorem.

**Remark 2.** The restrictions for  $s$  in Theorem 2.6.1 and Corollary 2.6.1/1 are natural. If  $q \geq p$  then we have also natural restrictions for  $s$  in the above theorem and the above corollary. For example, equivalent quasi-norms of type (2) cannot be expected if  $s < n/p$ , because in that case  $F_{pq}^s$  contains essentially unbounded functions for which the corresponding expression is infinite.

**Remark 3.** We prove the counterpart of Remark 2.6.1/4. Let  $\nu > 0$  then one can replace the integration over  $|h| \leq 1$  in (1) by  $|h| \leq \nu$  or by  $\mathbb{R}^n$ . For this purpose we generalize (2.4.7/7) as follows. Let  $\frac{n}{p} < a < s$ , then there exists a positive number  $c$  such that

$$\sup_{t>1} t^{-a} \left\| \sup_{|x-y|\leq t} |f(y)| \right\|_{L_p} \leq c \|f\|_{F_{pq}^s}. \quad (7)$$

We fix  $t > 1$  and follow the proof given in Remark 2.4.7/1, where (2.4.7/25) can be strengthened by

$$t^{-a} \sup_{|x-y|\leq t} |\varphi_k(D)f(y)| \leq c 2^{ka} \sup_{z \in \mathbb{R}^n} \frac{|\varphi_k(D)f(x-z)|}{1 + |2^k z|^a}, \quad (8)$$

where  $c$  is independent of  $t$ . Let  $|h| = t > 1$ , then

$$|h|^{-s} |\Delta_h^M f(x)| \leq c t^{-s} \sup_{|x-y| \leq Mt} |f(y)|. \quad (9)$$

This inequality and (7) show that (1), with  $\mathbb{R}^n$  instead of  $|h| \leq 1$ , can be estimated from above by  $c \|f\|_{F_{pq}^s}$ . Conversely, if  $\nu > 0$  is a small number then (1) with  $|h| \leq \nu$  instead of  $|h| \leq 1$  is an equivalent quasi-norm in  $F_{pq}^s$ . This claim is covered by the proof of the theorem. Hence we obtain the desired assertion as far as (1) is concerned. Similarly one proves corresponding assertions for (2–4).

**Corollary 2.** Let  $0 < p < \infty$ ,  $0 < q \leq \infty$ , and  $\frac{n}{\min(p,q)} < s - L < M$  where  $L \in \mathbb{N}$  and  $M \in \mathbb{N}$ , then

$$\|f\|_{L_p} + \sum_{|\alpha|=L} \left\| \left( \int_{|h| \leq 1} |h|^{-(s-L)q} |\Delta_h^M D^\alpha f(\cdot)|^q \frac{dh}{|h|^n} \right)^{1/q} \right\|_{L_p} \quad (10)$$

and

$$\|f\|_{L_p} + \sum_{k=1}^n \left\| \left( \int_0^1 t^{-(s-L)q} |\Delta_{t,k}^M \frac{\partial^L f}{\partial x_k^L}(\cdot)|^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p} \quad (11)$$

(modification if  $q = \infty$ ) are equivalent quasi-norms in  $F_{pq}^s$ .

**Proof.** We apply Proposition 2.4.4/2 and use the same arguments as in the proof of Corollary 2.6.1/2 and in the proof of the above theorem, see also Remark 2.6.3/3.

### 2.6.3 Weighted means of differences

The equivalent quasi-norms for  $B_{pq}^s$  and  $F_{pq}^s$  in the Theorems 2.6.1 and 2.6.2 are of interest for their own sake and from a historical point of view. This is the main reason why we included these results in this book. However for our later purposes we need some substantial modifications which we discuss now and which may also be considered as a modification of the local means introduced in 2.4.6, see also 2.5.3.

Let  $K \in S$  be non-negative and rotation-invariant (i.e.,  $K(y)$  depends only on  $|y|$ ) with a compact support. We assume that  $K$  does not vanish identically. Let  $M \in \mathbb{N}$ , then

$$K_M(t, f)(x) = \int_{\mathbb{R}^n} K(h) \Delta_{th}^M f(x) dh, \quad x \in \mathbb{R}^n, \quad t > 0, \quad (1)$$

may be interpreted as weighted means of differences. Let  $\sigma_p$  and  $\sigma_{pq}$  be the same numbers as above, see e.g., (2.4.1/1).

**Theorem.** (i) Let  $0 < p < \infty$ ,  $0 < q \leq \infty$ , and  $\sigma_{pq} < s < M$  with  $M \in \mathbb{N}$ . Let  $0 < \nu \leq \infty$ , then

$$\|f\|_{L_p} + \left\| \left( \int_0^\nu t^{-sq} |K_M(t, f)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \right\|_{L_p} \quad (2)$$

(modification if  $q = \infty$ ) is an equivalent quasi-norm in  $F_{pq}^s$ .

(ii) Let  $0 < p \leq \infty$ ,  $0 < q \leq \infty$ , and  $\sigma_p < s < M$  with  $M \in \mathbb{N}$ . Let  $0 < \nu \leq \infty$ , then

$$\|f\|_{L_p} + \left( \int_0^\nu t^{-sq} \|K_M(t, f)\|_{L_p}^q \frac{dt}{t} \right)^{1/q} \quad (3)$$

(modification if  $q = \infty$ ) is an equivalent quasi-norm in  $B_{pq}^s$ .

**Proof.** Step 1. We prove (i) with  $\nu > 0$  small. We apply Theorem 2.4.1 with  $s_0 = 0$ ,  $s_1 = M$  if  $M$  is even and  $s_1 = M + 1$  if  $M$  is odd,  $\varphi_0(x) = 1$ , and

$$\varphi(x) = \int_{\mathbb{R}^n} K(h)(e^{i\nu xh} - 1)^M dh, \quad x \in \mathbb{R}^n, \quad (4)$$

where  $xh$  stands for the scalar product of  $x \in \mathbb{R}^n$  and  $h \in \mathbb{R}^n$ . Then (2.4.1/5,6,10) are satisfied. If  $M$  is even then we have

$$\varphi(x) = \int_{\mathbb{R}^n} K(h)(i\nu xh)^M (1 + o(1)) dh \neq 0, \quad 0 < |x| \leq 2, \quad (5)$$

if  $M$  is odd then we have (5) with  $M + 1$ . In both cases we used that  $K(h)$  is rotation-invariant and  $\int_{\mathbb{R}^n} K(h) dh = \hat{K}(0) > 0$ . Hence, (2.4.1/7) is fulfilled. We check (2.4.1/8). First we remark that  $\varphi(x)$  is an analytic rotation-invariant function. If  $M$  is even then (5) shows that  $\frac{\varphi(x)}{|x|^M}$  is also an analytic function. If  $M$  is odd then  $\frac{\varphi(x)}{|x|^{M+1}}$  is an analytic function. In both cases we have (2.4.1/8) with the above choice of  $s_1$ . Finally we have to check (2.4.1/9) with  $s_0 = 0$ . First we remark that

$$\varphi(x) = (-1)^M \int_{\mathbb{R}^n} K(h) dh + \sum_{k=1}^M a_k \hat{K}(\nu k x) \quad (6)$$

holds, where  $a_k$  are appropriate numbers. We put (6) in (2.4.1/9) with  $s_0 = 0$ . The typical term we have to estimate reads as follows,

$$\int_{\mathbb{R}^n} (1 + |y|)^a |(\hat{K}(2^m \cdot) H(\cdot))^\vee(y)| dy. \quad (7)$$

We use (2.4.1/53) which shows that this term can be estimated from above by a constant, uniformly with respect to  $m \in \mathbb{N}$ . Finally we remark  $\varphi(tD)f = K_M(\nu t, f)$ . Hence the quasi-norms in (2.4.1/12) and (2) coincide. The proof of (i) is complete provided that  $\nu > 0$  is small.

Step 2. We extend (i) to arbitrary values of  $\nu > 0$  and to  $\infty$ . By Remark 2.3.3 we have

$$\|f(\lambda \cdot) \mid F_{pq}^s\| \leq c \lambda^{s-n/p} \|f \mid F_{pq}^s\|, \quad \lambda \geq 1, \quad (8)$$

where  $c$  is independent of  $\lambda$ . Furthermore we observe

$$K_M(t, f(\lambda \cdot))(x) = K_M(\lambda t, f)(\lambda x). \quad (9)$$

The second term of (2) with  $\nu > 0$  small and with  $f(\lambda x)$  instead of  $f(x)$  can be estimated from above by  $\|f(\lambda \cdot) \mid F_{pq}^s\|$  where  $\lambda \geq 1$ . Then (9) and (8) prove

$$\left\| \left( \int_0^{\lambda \nu} t^{-sq} |K_M(t, f)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \mid L_p \right\| \leq c \|f \mid F_{pq}^s\| \quad (10)$$

where  $c$  is independent of  $\lambda \geq 1$ . Now  $\lambda \rightarrow \infty$  yields the desired result.

Step 3. The proof of (ii) is the same. Now we have to use Theorem 2.5.1 and again Remark 2.3.3. The proof is complete.

**Remark 1.** Part (i) of the theorem will be of some use for us in connection with spaces of type  $F_{pq}^s$  on Riemannian manifolds. For this purpose it will be useful to reformulate (1) somewhat. Let  $l$  and  $l(h, f)$  be given by (2.4.3/10–12) and let  $K(y)$  be the above function. Then we introduce the means

$$\begin{aligned} K^l(t, f)(x) &= \int_{\mathbb{R}^n} K(h) l(th, f) dh \\ &= \varphi^l(tD)f(x) = (\varphi^l(t \cdot) \hat{f})^\vee(x), \quad t > 0, \quad x \in \mathbb{R}^n, \end{aligned} \quad (11)$$

with

$$\varphi^l(x) = \int_{\mathbb{R}^n} K(h) \psi(hx) dh, \quad (12)$$

see (2.4.3/17). This is the generalization of  $\varphi(tD)f = K_M(\nu t, f)$ , where  $K_M$  and  $\varphi$  are given by (1) and (4), respectively. One can try to replace  $l(h, f)$  in Theorem 2.4.3/2 by the means (11) and to generalize the above theorem. However for our purpose the above theorem and the just given reformulation will be sufficient.

**Remark 2.** The proof shows that it is not necessary to assume  $K \in S$ . However we shall not discuss the differentiability properties for  $K$  which are really needed. In [Triß: 2.5.11] we proved via rather specific means that even the characteristic function of the unit ball is an admissible choice of  $K$ , at least for  $F_{pq}^s$ , but this can be extended to  $B_{pq}^s$ . In other words, one can replace  $K_M(t, f)$  in (2) and (3) by

$$\int_{|h| \leq 1} \Delta_{th}^M f(x) dh.$$

See also 3.5.3 for further characterizations via differences.

**Remark 3.** We return to Theorem 2.6.2 and fill the gap at the end of the proof which simultaneously finishes the proof of Corollary 2.6.2/2. Recall that we have to prove that  $\|f \mid F_{pq}^s\|$  can be estimated from above by each of the quasi-norms in (2.6.2/1–4). We restrict ourselves to (2.6.2/1,2) with  $\int_{|h| \leq \nu}$  and  $\int_0^\nu$  instead of  $\int_{|h| \leq 1}$  and  $\int_0^1$ , respectively, where  $0 < \nu \leq \infty$ . The proof for the remaining quasi-norms is the same. Let  $0 < p < \infty$ ,  $0 < q \leq \infty$ , and  $\frac{n}{\min(p,q)} < s < M$  with  $M \in \mathbb{N}$ . We use the equivalent quasi-norm (2). We have

$$|K_M(t, f)(x)| \leq c \sum_{l=0}^{\infty} 2^{-rl} \sup_{0 < |h| \leq t2^l} |\Delta_h^M f(x)| \quad (13)$$

where  $r > 0$  is at our disposal. Then (2) with  $r = \infty$  yields

$$\|f \mid F_{pq}^s\| \leq c \|f \mid L_p\| + c \sum_{l=0}^{\infty} 2^{-r'l} \left\| \left( \int_0^\infty t^{-sq} \sup_{0 < |h| \leq t2^l} |\Delta_h^M f(x)|^q \frac{dt}{t} \right)^{1/q} \mid L_p \right\|, \quad (14)$$

where  $r' > 0$  is at our disposal. We substitute  $\tau = t2^l$  in the respective terms on the right-hand side of (14). Then we have an additional factor  $2^{ls}$ . We choose  $r' > s$ . It follows that  $\|f \mid F_{pq}^s\|$  can be estimated from above by the quasi-norm in (2.6.2/2) with  $\int_0^\infty$  instead of  $\int_0^1$ , and the proof is complete as far as this special case is concerned. Next we extend the proof to the quasi-norm (2.6.2/2) with  $\int_0^\nu$  instead of  $\int_0^1$  where  $0 < \nu < \infty$ . By Remark 2.6.2/3 we have

$$\left\| \left( \int_\nu^\infty t^{-sq} \sup_{0 < |h| \leq t} |\Delta_h^M f(\cdot)|^q \frac{dt}{t} \right)^{1/q} \mid L_p \right\| \leq c \|f \mid F_{pq}^s\| \quad (15)$$

with  $n/p < \overline{s} < s$ . However the right-hand side can be estimated from above by  $\varepsilon \|f \mid F_{pq}^s\| + c_\varepsilon \|f \mid L_p\|$  where  $\varepsilon > 0$  is at our disposal, see Remark 4 below. Now it follows that  $\|f \mid F_{pq}^s\|$  can be estimated from above by the quasi-norm in (2.6.2/2) with  $\int_0^\nu$  instead of  $\int_0^1$ . It remains to show that  $\|f \mid F_{pq}^s\|$  can be estimated from

above by the quasi-norm in (2.6.2/1) with  $\int_{|h| \leq \nu}$  instead of  $\int_{|h| \leq 1}$ . We may assume  $\nu = \infty$  because the cases with  $\nu < \infty$  can be treated afterwards on the basis of (15). We use (2) with  $\nu = \infty$ . Let  $1 \leq q \leq \infty$ , then we have by Hölder's inequality (modification if  $q = \infty$ )

$$|K_M(t, f)(x)|^q \leq c \int_{\mathbb{R}^n} K(h) |\Delta_{th}^M f(x)|^q dh = ct^{-n} \int_{\mathbb{R}^n} K\left(\frac{h}{t}\right) |\Delta_h^M f(x)|^q dh \quad (16)$$

and

$$\int_0^\infty t^{-sq} |K_M(t, f)(x)|^q \frac{dt}{t} \leq c \int_{\mathbb{R}^n} |h|^{-sq} |\Delta_h^M f(x)|^q \frac{dh}{|h|^n}. \quad (17)$$

We put (17) in (2) and obtain the desired estimate in the case  $1 \leq q \leq \infty$ . Let  $0 < q < 1$ . Then we modify (13) by

$$|K_M(t, f)(x)| \leq c \sum_{l=0}^\infty 2^{-rl} \sup_{0 < |h| \leq t2^l} |\Delta_h^M f(x)|^{1-q} \int_{\mathbb{R}^n} \overline{K}(h) |\Delta_{th}^M f(x)|^q dh, \quad (18)$$

where  $\overline{K}(h)$  is a non-negative function on  $\mathbb{R}^n$  with  $\sup_{h \in \mathbb{R}^n} |h|^N \overline{K}(h) < \infty$  for any  $N \in \mathbb{N}$ . We take the  $q$ th power in (18) term by term, multiply afterwards with  $t^{-sq-1}$ , and integrate over  $0 < t < \infty$ . Then we apply Hölder's inequality, based on  $q + (1 - q) = 1$ , and obtain

$$\begin{aligned} \int_0^\infty t^{-sq} |K_M(t, f)(x)|^q \frac{dt}{t} &\leq c \sum_{l=0}^\infty 2^{-rlq} \left( \int_0^\infty t^{-sq} \sup_{0 < |h| \leq t2^l} |\Delta_h^M f(x)|^q \frac{dt}{t} \right)^{1-q} \\ &\quad \times \left( \int_0^\infty t^{-sq} \int_{\mathbb{R}^n} \overline{K}(h) |\Delta_{th}^M f(x)|^q dh \frac{dt}{t} \right)^q. \end{aligned} \quad (19)$$

The second factors are independent of  $l$  and they can be estimated in the same way as in (16) and (17). The first factors can be treated as above, i.e., we substitute  $\tau = 2^l t$  and choose  $r > s$ . Now we apply the  $L_p$ -quasi-norm and we use again Hölder's inequality with respect to  $q + (1 - q) = 1$ . By (2) with  $\int_0^\infty$  instead of  $\int_0^\nu$  and by (2.6.2/2) with  $\int_0^\infty$  instead of  $\int_0^1$  we arrive at

$$\begin{aligned} \|f\|_{F_{pq}^s} &\leq c \|f\|_{F_{pq}^s}^{1-q} \left( \|f\|_{L_p} \right. \\ &\quad \left. + \left\| \left( \int_{\mathbb{R}^n} |h|^{-sq} |\Delta_h^M f(\cdot)|^q \frac{dh}{|h|^n} \right)^{1/q} \right\|_{L_p} \right)^q \end{aligned} \quad (20)$$

which yields finally the desired estimate in the case  $0 < q < 1$ .

**Remark 4.** Let  $\sigma_p < \overline{s} < s$  and let  $\varepsilon > 0$ , then there exists a number  $c_\varepsilon$  such that

$$\|f \mid B_{p1}^s\| \leq c_\varepsilon \|f \mid L_p\| + \varepsilon \|f \mid B_{p1}^s\|. \quad (21)$$

This estimate covers what we needed after (15), see the elementary embeddings mentioned in 2.3.2. We use (2.3.1/7) where  $\varphi$  and  $\varphi_k$  have the same meaning as in 2.3.1. Let  $K \in \mathbb{N}$  and  $k \in \mathbb{N}_0$  with  $k \leq N-1$ , then a refined version of (2.2.3/2), see [Triß: (1.5.2/13)], yields

$$\begin{aligned} \|(\varphi_k \hat{f})^\vee \mid L_p\| &= \|(\varphi_k \varphi(2^{-K} \cdot) \hat{f})^\vee \mid L_p\| \leq c 2^{(K-k)\bar{s}} \|(\varphi(2^{-K} \cdot) \hat{f})^\vee \mid L_p\| \\ &\leq c 2^{(K-k)\bar{s}} \|f \mid L_p\| + c' 2^{(K-k)\bar{s}} \|(1 - \varphi(2^{-K} \cdot)) \hat{f}\|^\vee \mid L_p\|, \end{aligned} \quad (22)$$

where  $c$  and  $c'$  are independent of  $K$  and  $k$ . It follows

$$2^{k\bar{s}} \|(\varphi_k \hat{f})^\vee \mid L_p\| \leq c 2^{K\bar{s}} \|f \mid L_p\| + c 2^{-K\delta} \|f \mid B_{p1}^s\| \quad (23)$$

for some  $\delta > 0$ , where  $c$  is independent of  $K$ . We sum over  $k = 0, \dots, K-1$ , choose  $K$  large and estimate the remaining terms with  $k \geq K$  in an obvious way. Then we obtain (21).

## 2.6.4 Harmonic and thermic extensions

In 1.8.1 we discussed harmonic and thermic extensions from a historical point of view, and described in 1.8.3 more recent results, including relevant references. Now we prove that Theorem 1.8.3 is more or less a special case of the above general characterizations. In order to be self-contained we recall some basic notations. Let  $x \in \mathbb{R}^n$  and  $t > 0$ , then

$$W(t)f(x) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y) dy \quad (1)$$

is the Gauss–Weierstrass semi-group and

$$P(t)f(x) = c_n \int_{\mathbb{R}^n} \frac{t}{(|x-y|^2 + t^2)^{(n+1)/2}} f(y) dy \quad (2)$$

is the Cauchy–Poisson semi-group with  $c_n \|(1+|x|^2)^{-(n+1)/2} \mid L_1\| = 1$ , complemented by  $W(0) = P(0) = id$ . One can rewrite (1) and (2) by

$$u(x, t) = W(t)f(x) = (e^{-t|\xi|^2} \hat{f})^\vee, \quad x \in \mathbb{R}^n, \quad t \geq 0, \quad (3)$$

and

$$v(x, t) = P(t)f(x) = (e^{-t|\xi|} \hat{f})^\vee, \quad x \in \mathbb{R}^n, \quad t \geq 0. \quad (4)$$

We mention that  $u(x, t)$  is a solution of the heat equation in  $\mathbb{R}_+^{n+1}$  whereas  $v(x, t)$  is a harmonic function in  $\mathbb{R}_+^{n+1}$ . We have  $u(x, 0) = v(x, 0) = f(x)$  what explains to call  $u(x, t)$  a thermic and  $v(x, t)$  a harmonic extension of  $f$ . See 1.6.5 and, in particular, 1.8.1 for further details and references. The classical background, including proofs of (3), (4), and further references may also be found in [Triα: 2.5.2, 2.5.3]. Recall  $\sigma_p = n(\frac{1}{p} - 1)_+$ .

**Theorem.** Let  $s \in \mathbb{R}$ ,  $0 < q \leq \infty$ , and let  $\varphi_0$  be a  $C^\infty$  function with a compact support and  $\varphi_0(0) \neq 0$ .

(i) Let  $0 < p \leq \infty$ . Let  $m \in \mathbb{N}_0$  and  $k \in \mathbb{N}_0$  with

$$m > \frac{s}{2} \quad \text{and} \quad k > \sigma_p + \max(s, \sigma_p), \quad (5)$$

then

$$\|f \mid B_{pq}^s\|_W = \|\varphi_0(D)f \mid L_p\| + \left( \int_0^1 t^{(m-\frac{s}{2})q} \left\| \frac{\partial^m W(t)f}{\partial t^m} \mid L_p \right\|^q \frac{dt}{t} \right)^{1/q} \quad (6)$$

and

$$\|f \mid B_{pq}^s\|_P = \|\varphi_0(D)f \mid L_p\| + \left( \int_0^1 t^{(k-s)q} \left\| \frac{\partial^k P(t)f}{\partial t^k} \mid L_p \right\|^q \frac{dt}{t} \right)^{1/q} \quad (7)$$

(modification if  $q = \infty$ ) are equivalent quasi-norms in  $B_{pq}^s$ . Furthermore

$$B_{pq}^s = \{f \in S': \|f \mid B_{pq}^s\|_W < \infty\} = \{f \in S': \|f \mid B_{pq}^s\|_P < \infty\}. \quad (8)$$

If  $s > \sigma_p$  then  $\|\varphi_0(D)f \mid L_p\|$  in (6–8) can be replaced by  $\|f \mid L_p\|$ .

(ii) Let  $0 < p < \infty$ . Let  $m \in \mathbb{N}_0$  and  $k \in \mathbb{N}_0$  with

$$m > \frac{s}{2} \quad \text{and} \quad k > \frac{n}{\min(p, q)} + \max(s, \sigma_p) \quad (9)$$

then

$$\|f \mid F_{pq}^s\|_W = \|\varphi_0(D)f \mid L_p\| + \left\| \left( \int_0^1 t^{(m-\frac{s}{2})q} \left| \frac{\partial^m W(t)f}{\partial t^m}(\cdot) \right|^q \frac{dt}{t} \right)^{1/q} \mid L_p \right\| \quad (10)$$

and

$$\|f \mid F_{pq}^s\|_P = \|\varphi_0(D)f \mid L_p\| + \left\| \left( \int_0^1 t^{(k-s)q} \left| \frac{\partial^k P(t)f}{\partial t^k}(\cdot) \right|^q \frac{dt}{t} \right)^{1/q} \mid L_p \right\| \quad (11)$$

(modification if  $q = \infty$ ) are equivalent quasi-norms in  $F_{pq}^s$ . Furthermore

$$F_{pq}^s = \{f \in S': \|f\|_{F_{pq}^s} < \infty\} = \{f \in S': \|f\|_{F_{pq}^s} < \infty\}. \quad (12)$$

If  $s > \sigma_p$  then  $\|\varphi_0(D)f\|_{L_p}$  in (10–12) can be replaced by  $\|f\|_{L_p}$ .

**Proof.** Step 1. We prove that (10,11) are equivalent quasi-norms in  $F_{pq}^s$ . We use Theorem 2.4.1 with the above function  $\varphi_0$  and  $\varphi(x) = |x|^k e^{-|x|}$ . Then immaterial modifications of (2.4.1/6,7) are satisfied. Furthermore (2.4.1/9,10), or better their substitutes (2.4.1/51,52), are fulfilled for any  $s_0$ . Finally (2.4.1/50) can be reduced to the question whether  $e^{-|x|}|x|^{k-s_1}$  with  $s_1 > \max(s, \sigma_p)$  belongs to  $H_2^\sigma$  with  $\sigma > \frac{n}{\min(p,q)} + \frac{n}{2}$ . By the arguments in Remark 2.4.1/1 this property holds if  $k - s_1 + \frac{n}{2} > \sigma$ . By (9) we may assume that this is satisfied. Hence we can apply Theorem 2.4.1. We have

$$\begin{aligned} \varphi(tD)f &= (\varphi(t\cdot)\hat{f})^\vee = t^k(|y|^k e^{-|y|}\hat{f}(y))^\vee \\ &= t^k \frac{\partial^k}{\partial t^k} (e^{-|y|}\hat{f})^\vee = t^k \frac{\partial^k}{\partial t^k} P(t)f, \end{aligned} \quad (13)$$

see (4), which proves that  $\|f\|_{F_{pq}^s}$  in (11) is an equivalent quasi-norm in  $F_{pq}^s$ . In order to prove the corresponding assertion for  $\|f\|_{F_{pq}^s}$  we use again Theorem 2.4.1 with  $\varphi_0$  and  $\varphi(x) = |x|^{2m} e^{-|x|^2}$ . Then all conditions, including (2.4.1/50–52) with  $s_1 = 2m > s$ , are satisfied, also (2.4.1/49) which shows that the assumption  $s_1 > \sigma_p$  in Theorem 2.4.1 is not necessary now. The counterpart of (13) reads as follows,

$$\begin{aligned} \varphi(\sqrt{t}D)f &= (\varphi(\sqrt{t}\cdot)\hat{f})^\vee = t^m(|y|^{2m} e^{-|y|^2}\hat{f})^\vee \\ &= t^m \frac{\partial^m}{\partial t^m} (e^{-|y|^2}\hat{f})^\vee = t^m \frac{\partial^m}{\partial t^m} W(t)f, \end{aligned} \quad (14)$$

see (3). Then it follows that  $\|f\|_{F_{pq}^s}$  is an equivalent quasi-norm in  $F_{pq}^s$ , where one has to take into consideration that we substituted  $t$  by  $\sqrt{t}$ . Let  $s > \sigma_p$ . Then we have (2.3.3/8,10) which prove that we can replace  $\|\varphi_0(D)f\|_{L_p}$  in (10,11) by  $\|f\|_{L_p}$ .

Step 2. Let  $1 < p < \infty$  and  $1 < q \leq \infty$ , then (12) with the discrete versions of (10,11) follows from Theorem 2.4.2(i). However the argument in Step 2 of the proof of Theorem 2.4.2 are based on (2.4.2/17), which can easily be modified in order to replace the discrete version by the continuous one. Then we obtain (12) with  $1 < p < \infty$  and  $1 < q \leq \infty$ . To extend this assertion to arbitrary values of  $p$  and  $q$  we need some specific tools. Recall that  $P(t)f(x)$  and hence also  $v(x, t) = \frac{\partial^k P(t)f(x)}{\partial t^k}$  are harmonic functions in  $\mathbb{R}_+^{n+1}$ . We use the sub-mean value property

$$|v(x, t)|^r \leq c_r (\text{vol } \omega)^{-1} \int_\omega |v(y, \tau)|^r dy d\tau, \quad 0 < r < \infty, \quad (15)$$

where  $\omega$  is an arbitrary ball in  $\mathbb{R}_+^{n+1}$  centered at  $(x, t)$ . A proof of (15) may be found in [FeS2: Lemma 2 on p. 172] or in a more general version in [HiK]. We replace  $t$  in (15) by  $2t$  and  $\omega$  by  $B \times [t, 3t]$ , where  $B$  is a ball in  $\mathbb{R}^n$  of radius  $t$ , centered at  $x$ . Afterwards by (13) we can substitute  $v(x, 2t)$  by  $\varphi(2tD)f(x)$ . We arrive at

$$|\varphi(2tD)f(x)|^r \leq c(\text{vol } B)^{-1} \int_B \int_{-1}^1 |\varphi((2t + \tau t)D)f(z)|^r d\tau dz. \quad (16)$$

We use (2.4.2/17) with, say,  $2t = 2^{-j}$  and  $j \in \mathbb{N}$ . We obtain the following counterpart of (2.4.1/39)

$$|(\rho_j \hat{f})^\vee(x)| \leq c 2^{jn} \sum_{l=0}^{\infty} 2^{-ld} \int_{\{y \in \mathbb{R}^n: |y| < 2^{-j+l}\}} |\varphi(2tD)f(x-y)| dy. \quad (17)$$

Let  $0 < r < \min(1, p, q)$ . We use (16) with  $x-y$  instead of  $x$  where  $|y| < 2^{-j+l}$ . We replace the ball  $B$  of radius  $2^{-j+1}$  and centered at  $x-y$  by a ball of radius  $c2^{-j+l}$  centered at  $x$ . Then we obtain

$$|\varphi(2tD)f(x-y)|^r \leq c 2^{jn} 2^{(l-j)n} M \left( \int_{-1}^1 |\varphi((2t + \tau t)D)f|^r d\tau \right) (x), \quad (18)$$

where  $M$  stands again for the Hardy–Littlewood maximal function. We put (18) in (17), choose  $d > n(1 + \frac{1}{r})$  and arrive at

$$|(\rho_j \hat{f})^\vee(x)|^r \leq c M \left( \int_{-1}^1 |\varphi((2t + \tau t)D)f|^r d\tau \right) (x). \quad (19)$$

Now we are in the same position as in (2.4.1/41). We apply Theorem 2.2.2 and obtain

$$\begin{aligned} \|f \mid F_{pq}^s\| &\leq \|\varphi_0(D)f \mid L_p\| \\ &+ c \left\| \left( \sum_{j=1}^{\infty} 2^{jsq} \left( \int_{-1}^1 |\varphi((2^{-j+1} + \tau 2^{-j})D)f|^r d\tau \right)^{q/r} \right)^{1/q} \mid L_p \right\|. \end{aligned} \quad (20)$$

Because  $q/r \geq 1$  one can change the order of summation with respect to  $j$  and of integration with respect to  $\tau$ . This shows that the right-hand side of (20) can be estimated from above by  $c\|f \mid F_{pq}^s\|_P$ . The proof of the second assertion in (12) is complete. As far as the first assertion is concerned we use the following counterpart of (15). Let  $u(x, t)$  be a solution of the heat equation in  $\mathbb{R}_+^{n+1}$  and let

$$\omega = \left\{ (y, \tau) \in \mathbb{R}_+^{n+1}: |y_j - x_j| \leq \frac{\lambda}{2}, \quad t - \lambda^2 \leq \tau \leq t \right\} \subset \mathbb{R}_+^{n+1},$$

then

$$|u(x, t)|^r \leq c_r (\text{vol } \omega)^{-1} \int_{\omega} |u(y, \tau)|^r dy d\tau, \quad 0 < r < \infty. \quad (21)$$

This inequality is a consequence of a more subtle sub-mean value property for temperatures due to H.-Q. Bui, see [Bui2: Lemma 2]. Now one obtains the first assertion in (12) in the same way as above. Finally let  $s > \sigma_p$  and let  $f \in S \cap L_p$  such that the second term on the right-hand side of (10) is finite. Let  $\varphi_0$  be the above function with  $\varphi_0(x) = 1$  near the origin. Then it follows by the above arguments  $((1 - \varphi_0)\hat{f})^\vee \in F_{pq}^s$  and by the discussion given in Step 2 of the proof of Theorem 2.3.3

$$(\varphi_0 \hat{f})^\vee = f + ((1 - \varphi_0)\hat{f})^\vee \in L_p.$$

Hence  $f \in F_{pq}^s$ . By the end of Step 1 we can replace  $\|\varphi_0(D)f \mid L_p\|$  in (12) by  $\|f \mid L_p\|$ . The proof of (ii) is complete.

Step 3. The proof of (i) is the same. Instead of Theorem 2.4.1 we have to use Theorem 2.5.1 and Corollary 2.5.1/1.

**Remark.** In connection with Theorem 1.8.1 the question is of interest whether  $\int_0^1$  in (6,7,10,11) can be replaced by  $\int_0^\nu$  with  $0 < \nu \leq \infty$ . Corresponding assertions in connection with differences and means of differences may be found in Remarks 2.6.1/4 and 2.6.2/3 and Theorem 2.6.3. By the proof and the fact that  $\varphi(x) \neq 0$  if  $x \neq 0$  in both cases, see Step 1, it follows that  $\int_0^1$  in (6,7,10,11) can be replaced by  $\int_0^\nu$  with  $0 < \nu < \infty$ . Let  $s > \sigma_p$  then we can argue in the same way as in Step 2 of the proof of Theorem 2.6.3 based on corresponding counterparts of (2.6.3/9). In other words, under the additional restriction  $s > \sigma_p$  one can replace  $\int_0^1$  in (6-8, 10-12) by  $\int_0^\infty$ .

## 2.6.5 The classical spaces

In Chapter 1 we introduced several classical function spaces on  $\mathbb{R}^n$ , discussed some properties, and described their connections with the spaces  $B_{pq}^s$  and  $F_{pq}^s$ , see Theorem 1.5.1 as far as the latter aspect is concerned. Now we return briefly to this subject and comment previous assertions on the basis of the results obtained in this chapter.

**Hölder–Zygmund and Besov spaces.** In 1.2.2 we defined the Hölder spaces  $C^s$  with  $0 < s \neq \text{integer}$  and the Zygmund spaces  $C^s$  with  $s > 0$ . We have the classical assertions (1.2.2/6,7), where the norm  $\|f \mid C^s\|_{0,m}$  is of special interest now. The Besov spaces  $B_{pq}^s$  with  $s > 0$ ,  $1 < p < \infty$ ,  $1 \leq q \leq \infty$  had been introduced in 1.2.5. Here we have the classical assertion (1.2.5/3) where again the norm  $\|f \mid B_{pq}^s\|_{0,m}$  is of interest now. On the other hand, we have Corollary 2.6.1/1 with the quasi-norm

(2.6.1/2). By Remark 2.6.1/4 we can replace  $\int_{|h|\leq 1}$  in (2.6.1/2) by  $\int_{\mathbb{R}^n}$ . Then this modified quasi-norm coincides with the above norms  $\|f \mid \mathcal{C}^s\|_{0,m} = \|f \mid B_{\infty\infty}^s\|_{0,m}$  and  $\|f \mid B_{pq}^s\|_{0,m}$ . In other words, Corollary 2.6.1/1 proves that the classical Besov spaces  $B_{pq}^s$  from 1.2.5 coincide with the spaces  $B_{pq}^s$  introduced in (2.3.1/7), and that

$$\mathcal{C}^s = B_{\infty\infty}^s, \quad s > 0, \quad (1)$$

holds. These assertions cover Theorem 1.5.1(i,ii).

**Fractional Sobolev spaces.** Although we have nothing new to say we recall for the sake of completeness, the relevant assertions for fractional Sobolev spaces. The Sobolev spaces  $W_p^k$  with  $1 < p < \infty$  and  $k \in \mathbb{N}_0$  have been introduced in 1.2.3. They are special cases of the fractional Sobolev spaces  $H_p^s$  from Definition 1.3.2, where  $1 < p < \infty$  and  $s \in \mathbb{R}$ . The relevant assertion (1.3.2/4) is classical, it is based on  $L_p$ -Fourier multipliers which we shall not discuss in this book again. The connection of  $H_p^s$  with  $F_{p2}^s$  is based on Littlewood–Paley theorems for  $L_p$  spaces, see 1.3.3, in particular (1.3.3/13). Compared with (2.3.1/8) one arrives at

$$H_p^s = F_{p2}^s, \quad s \in \mathbb{R}, \quad 1 < p < \infty, \quad (2)$$

what coincides with Theorem 1.5.1(iii).

**Hardy spaces.** We discussed in 1.4 Hardy spaces  $H_p$  and  $h_p$ ,  $0 < p < \infty$ , rather extensively, see in particular (1.4.4/1). In Theorem 1.4.4 we formulated a Littlewood–Paley theorem which results in

$$h_p = F_{p2}^0, \quad 0 < p < \infty, \quad (3)$$

which, in turn, generalizes (2) with  $s = 0$ . In particular  $h_p = L_p$  if  $1 < p < \infty$ .

**Remark.** Only the above assertions on Hölder–Zygmund spaces and classical Besov spaces are covered by the results of this chapter. We did not prove any Littlewood–Paley assertions and mentioned (2,3) for the sake of completeness. More detailed discussions and proofs may be found in [Triß: 2.5.6–2.5.8, 2.5.12] and in the corresponding references given in the first chapter of this book.

**Local methods.** In contrast to [Triα: Triß], this book is characterized by the systematic use of local assertions and local methods. In Chapter 1 we described

these new tools in 1.8.4 and in Proposition 1.11.3. These crucial assertions are now fully covered by 2.4.6 and Theorem 2.4.7.

**Harmonic and thermic extensions.** Characterizations of function spaces on  $\mathbb{R}^n$  as trace spaces of certain spaces of harmonic and thermic functions on  $\mathbb{R}_+^{n+1}$  have a long history. Classical results for Besov spaces may be found in 1.8.1. Corresponding results for Hardy spaces have been formulated in 1.4.2. In 1.8.3 we collected more recent results, including some references. Theorem 1.8.3 is completely covered by Theorem 2.6.4 where the latter one has a rather final character. In Theorem 2.6.4 we assumed that the function  $\varphi_0$  has a compact support. This was helpful in connection with the characterizations (2.6.4/8,12). On the other hand, if one looks only for equivalent quasi-norms then such a restriction is not necessary.



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