

Preface

In the physicist's picture of quantum theory the notion of a real valued observable has, like Brahma, three visible faces : (i) a spectral measure on the line; (ii) a selfadjoint operator in a Hilbert space; (iii) a unitary representation of the real line as an additive group. The equivalence of these three descriptions is a consequence of von Neumann's spectral theorem for a, not necessarily bounded, selfadjoint operator and Stone's theorem on the infinitesimal generator of a one parameter unitary group in a Hilbert space. Real valued random variables on a classical probability space (Ω, \mathcal{F}, P) when viewed as selfadjoint multiplication operators in the Hilbert space $L^2(P)$ are special examples in the quantum description. This suggests the possibility of developing a theory of quantum probability within the framework of operators and group representations in a Hilbert space. The first chapter of the present work is aimed at an exploration of this idea assuming only a knowledge of elementary functional analysis and classical probability.

Through the papers of R.F. Streater [132] in 1969, H. Araki [9] in 1970 and K.R. Parthasarathy and K. Schmidt [100] in 1972 there has emerged a string of ideas exhibiting the close relationship between the infinitely divisible probability distributions of Levy and Khinchin on the one hand and the projective unitary Weyl representation of the Euclidean group of a Hilbert space in the associated symmetric (boson) Fock space on the other. The Weyl representation is a neat summary of the Weyl commutation relations or, equivalently, the canonical commutation relations (CCR) and the second quantization homomorphism. In its infinitesimal form the Weyl representation leads naturally to the fundamental notions of creation, conservation and annihilation operators which obey the (extended) CCR and can therefore be interpreted as the source of Heisenberg's uncertainty principle. Classical stochastic processes with independent increments are then realised as distributions in the vacuum state of commuting families of observables which are, essentially, linear combinations of the creation, conservation and annihilation operators in the backdrop of a filtration determined by a time observable. Such a realisation depends very much on the theory of tensor products of Hilbert spaces and operators on them. The reader may find a fairly self-contained approach to this theme in the second chapter.

In classical probability theory, infinitely divisible distributions are realised from stochastic processes with independent increments of which the paradigm examples are the standard Brownian motion and the Poisson process of a given intensity. Starting with the pioneering efforts of N. Wiener, K. Ito, P. Levy and J.L. Doob there has now evolved during the last six decades an extraordinarily rich stochastic calculus around these and more general local semimartingales encompassing stochastic integration, Ito's formula, stochastic differential equations and a variety of applications which constitute the fruit bearing orchard of modern stochastic analysis. (See, for example, the book [66] of N. Ikeda and S. Watanabe). The developments in the second chapter indicate the possibility of developing a quantum stochastic calculus around the basic creation, conservation and annihilation

lation operators arising from the Weyl representation of the Euclidean group of a Hilbert space equipped with a continuous time observable in order to induce a filtration. The motivation for such an attempt lies in the intuitive feeling that the description of any dynamical phenomenon depends on the creation of some objects, their preservation and motion for a period of time followed by their annihilation and, finally, the repetition of such a process in cycles, all subject to the laws of chance. A theory of quantum stochastic differential equations based on the notions of creation, conservation and annihilation operator-valued processes and time may possibly help in examining the point of view expressed above. Starting with the preliminary ideas in [57], an attempt to develop a quantum stochastic calculus along the lines mentioned already was made in 1984 by R. L. Hudson and K.R. Parthasarathy [59]. The principal aim of the third chapter is to present such a calculus leading to quantum Ito's formula and highlight the following essential features : (i) the classical Ito's formula for Brownian motion and Poisson process can also be viewed as consequences of CCR and hence the Heisenberg's uncertainty principle; (ii) canonical anticommutation relations (CAR) and CCR can be derived from each other through stochastic integration; (iii) quantum dynamical semigroups describing irreversible evolutions result from averaging the solutions of a Heisenberg equation in the presence of noise, strengthening the belief that irreversible evolutions are most likely to be shadows of reversible evolutions in an enlarged universe; (iv) quantum stochastic flows (in the sense of Evans-Hudson) arise from quantum stochastic differential equations; (v) classical Markov chains can be realised in the quantum framework through stochastic difference and differential equations for observables with coefficients depending on appropriate group actions.

The list of topics omitted and reasons for their omission is too long to be enumerated and hence the number of friends and colleagues to whom I must apologise is very high. Primarily, the subject is very young and dynamic and the growth of literature in recent years is so explosive that I can hardly keep pace with it. Secondly, notions like conditional expectation, stop time, local time etc., are in the process of assuming a visible shape but the body of interesting physical as well as mathematical examples seems to require much more nourishment. There do exist several other approaches to quantum stochastic integration like the one based on kernels in Guichardet's symmetric measure space due to H. Maassen [83] and the other on C^* - and W^* -algebraic methods due to C. Barnett R.F. Streater and I.F. Wilde [16,17] and L. Accardi and F. Fagnola [5]. The interested reader may find access to the voluminous literature on all these recent developments in the Lecture Notes of proceedings of the workshops in quantum probability and its applications [1,2] edited by L. Accardi, W. von Waldenfels and others as well as the expositions of P.A. Meyer entitled "Eléments de probabilités quantiques" in the Séminaire de Probabilités [88]–[93].

The present work is a revised version of the mimeographed notes [111] which owes its existence, primarily, to the influence of R.L. Hudson with whom

the author had the pleasure of collaboration during the last nine years. Indeed, this monograph was first proposed to be written jointly with him but the occurrence of such a happy event was prevented by the difficulties involved in simultaneously occupying the states of being the chairman of a department in a British university during hard times and indulging in the luxury of organising already discovered results. To R.L. Hudson and his colleagues D. Applebaum and J.M. Lindsay at Nottingham, P.A. Meyer at Strasbourg and L. Accardi at Rome, I express my heartfelt gratitude for uncountable conversations and exchanges of ideas on the subject of quantum stochastic calculus as well as the nonmeasurable hospitality I enjoyed during my visits to their departments and homes; to R.F. Streater who showed and emphasised passionately that Fock space is a fertile soil for new developments in probability theory; to my friend and colleague K.B. Sinha for his enthusiastic collaboration and illuminating suggestions whenever analytical difficulties obstructed imagination; to my devoted audience of three consisting of S.K. Muthu, A. Mohari and B.V.R. Bhat during my year-long seminar at the Delhi Centre of the Indian Statistical Institute during 1988; and to V.S. Varadarajan, the guru who initiated me to the charms of the geometry of quantum theory through his marvellous lectures on G.W. Mackey's approach to the mathematical foundations of quantum mechanics during the year 1965 in Calcutta prior to his total exit from the Indian mathematical scene. I am deeply indebted to B.V.R. Bhat, J.M. Lindsay and S.K. Muthu in checking the mushroom growth of mistakes in my manuscript as well as suggesting ways for its improvement. Special thanks are due to V.P. Sharma for his patient and elegant preparation of the manuscript in \TeX . Finally, no amount of thanks would suffice for the cheerful and encouraging support I have received from Shyama on the domestic front where life in Delhi in the sweltering summer heat of forty plus, accompanied by aandi winds is an everyday battle right from facing an erratic power and water supply to procuring the daily necessities of life like bread, rice, vegetables and milk.



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