Chapter II

Observables and States in Tensor Products of Hilbert Spaces

15 Positive definite kernels and tensor products of Hilbert spaces

Suppose $(\Omega_i, \mathscr{F}_i)$, $1 \leq i \leq n$ are sample spaces describing the elementary outcomes and events concerning n different statistical systems in classical probability. To integrate them into a unified picture under the umbrella of a single sample space one takes their cartesian product $(\Omega, \mathscr{F})$ where $\Omega = \Omega_1 \times \cdots \times \Omega_n$, $\mathscr{F} = \mathscr{F}_1 \times \cdots \times \mathscr{F}_n$, the smallest σ-algebra containing all rectangles of the form $F_1 \times F_2 \times \cdots \times F_n$, $F_j \in \mathscr{F}_j$ for each j. Now we wish to search for an analogue of this description in quantum probability when we have n systems where the events concerning the j-th system are described by the set $\mathcal{P}(\mathscr{H}_j)$ of all projections in a Hilbert space $\mathscr{H}_j$, $j = 1, 2, \ldots, n$. Such an attempt leads us to consider tensor products of Hilbert spaces. We shall present a somewhat statistically oriented approach to the definition of tensor products which is at the same time coordinate free in character. To this end we introduce the notion of a positive definite kernel.

Let $\mathscr{X}$ be any set and let $K : \mathscr{X} \times \mathscr{X} \to \mathbb{C}$ be a map satisfying the following:

$$\sum_{i,j} \overline{\alpha}_i \alpha_j K(x_i, x_j) \geq 0$$

for all $\alpha_i \in \mathbb{C}$, $x_i \in \mathscr{X}$, $i = 1, 2, \ldots, n$. Such a map K is called a *positive definite kernel* or simply a *kernel* on $\mathscr{X}$. We denote by $\mathcal{K}(\mathscr{X})$ the set of all such kernels on $\mathscr{X}$.

If $\mathscr{X} = \{1, 2, \ldots, n\}$, a kernel on $\mathscr{X}$ is just a positive (semi) definite matrix. If $\mathscr{H}$ is a Hilbert space the scalar product $K(x, y) = \langle x, y \rangle$ is a kernel on $\mathscr{H}$. If G is a group and $g \to U_g$ is a homomorphism from G into the unitary group $\mathscr{U}(\mathscr{H})$ of a Hilbert space $\mathscr{H}$ then $K(g, h) = \langle u, U_{g^{-1}h} u \rangle$ is a kernel on G for every u in $\mathscr{H}$. If ρ is a state in $\mathscr{H}$ then $K(X, Y) = \operatorname{tr} \rho X^* Y$ is a kernel on $\mathscr{B}(\mathscr{H})$.

Proposition 15.1: Let $((a_{ij})), ((b_{ij})), 1 \leq i, j \leq n$ be two positive definite matrices. Then $((a_{ij} b_{ij}))$ is positive definite.

Proof: Let $A = ((a_{ij}))$. Choose any matrix C of order n such that $CC^* = A$. Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ be any $2n$ independent and identically distributed standard Gaussian (normal) random variables. Write $\gamma_j = 2^{-\frac{1}{2}}(\alpha_j + i\beta_j)$, $\underline{\xi} = C\underline{\gamma}$ where $\underline{\gamma}$ denotes the column vector with j-th entry γ_j. Then $\underline{\xi}$ is a complex valued Gaussian random vector satisfying $\mathbb{E}\underline{\xi} = \underline{0}$, $\mathbb{E}\underline{\xi}\underline{\xi}^* = CC^* = A$.

Using the procedure described above select a pair of independent complex Gaussian random vectors $\underline{\xi}, \underline{\eta}$ such that $\mathbb{E}\xi_j = \mathbb{E}\eta_j = 0$, $\mathbb{E}\overline{\xi}_i \xi_j = a_{ij}$, $\mathbb{E}\overline{\eta}_i \eta_j = b_{ij}$

K.R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus*,
DOI 10.1007/978-3-0348-0566-7_2, © Springer Basel 1992

for $1 \leq i,\ j \leq n$. Let $\zeta_j = \xi_j \eta_j$. Then $\mathbb{E}\zeta_j = 0$, $\mathbb{E}\overline{\zeta}_i \zeta_j = a_{ij} b_{ij}$. Thus $((a_{ij} b_{ij}))$ is the covariance matrix of the complex random vector $\underline{\zeta}$ and hence positive definite. ■

Corollary 15.2: The space $K(\mathscr{X})$ of all kernels on $\mathscr{X}$ is closed under pointwise multiplication.

Proof: This follows immediately from Proposition 15.1 and the fact that K is a kernel on $\mathscr{X}$ if and only if for any finite set $\{x_1, x_2, \ldots, x_n\} \subset \mathscr{X}$ the matrix $((a_{ij}))$ where $a_{ij} = K(x_i, x_j)$ is positive definite. ■

Corollary 15.3: Let $\mathscr{X}_i$, $1 \leq i \leq n$ be sets and let $K_i \in K(\mathscr{X}_i)$ for each i. Define $\mathscr{X} = \mathscr{X}_1 \times \cdots \times \mathscr{X}_n$ and

$$K(\underline{x}, \underline{y}) = \Pi_{i=1}^n K_i(x_i, y_i),\ \underline{x} = (x_1, \ldots, x_n),\ \underline{y} = (y_1, \ldots, y_n)$$

where $x_i, y_i \in \mathscr{X}_i$. Then $K \in K(\mathscr{X})$.

Proof: Let $\underline{x}^{(r)} = (x_{1r},\ x_{2r}, \ldots, x_{nr}) \in \mathscr{X}$, $1 \leq r \leq m$. Putting $a^i_{rs} = K_i(x_{ir}, x_{is})$ we observe that

$$K(\underline{x}^{(r)}, \underline{x}^{(s)}) = \Pi_{i=1}^n a^i_{rs}.$$

Since $((a^i_{rs}))$, $1 \leq r,\ s \leq m$ is positive definite for each fixed i the required result follows from Proposition 15.1. ■

Proposition 15.4: Let $\mathscr{X}$ be any set and let $K \in K(\mathscr{X})$. Then there exists a (not necessarily separable) Hilbert space $\mathscr{H}$ and a map $\lambda : \mathscr{X} \to \mathscr{H}$ satisfying the following: (i) the set $\{\lambda(x), x \in \mathscr{X}\}$ is total in $\mathscr{H}$; (ii) $K(x, y) = \langle \lambda(x), \lambda(y) \rangle$ for all x, y in $\mathscr{X}$.

If $\mathscr{H}'$ is another Hilbert space and $\lambda' : \mathscr{X} \to \mathscr{H}'$ is another map satisfying (i) and (ii) with $\mathscr{H}, \lambda$ replaced by $\mathscr{H}', \lambda'$ respectively then there is a unitary isomorphism $U : \mathscr{H} \to \mathscr{H}'$ such that $U\lambda(x) = \lambda'(x)$ for all x in $\mathscr{X}$.

Proof: For any finite set $F = \{x_1, x_2, \ldots, x_n\} \subset \mathscr{X}$ it follows from the argument in the proof of Proposition 15.1 that there exists a complex Gaussian random vector $(\zeta^F_1, \zeta^F_2, \ldots, \zeta^F_n)$ such that

$$\mathbb{E}\overline{\zeta^F_i} \zeta^F_j = K(x_i, x_j),\ \mathbb{E}\zeta^F_i = 0, \quad 1 \leq i,\ j \leq n.$$

If $G = \{x_1, x_2, \ldots, x_n, x_{n+1}\} \supset F$ then the marginal distribution of $(\zeta^G_1, \ldots, \zeta^G_n)$ derived from that of $(\zeta^G_1, \ldots, \zeta^G_n, \zeta^G_{n+1})$ is the same as the distribution of $(\zeta^F_1, \ldots, \zeta^F_n)$. Hence by Kolmogorov's Consistency Theorem there exists a Gaussian family $\{\zeta_x, x \in \mathscr{X}\}$ of complex valued random variables on a probability space $(\Omega, \mathscr{F}, P)$ such that

$$\mathbb{E}\zeta_x = 0,\ \mathbb{E}\overline{\zeta}_x \zeta_y = K(x, y) \text{ for all } x, y \in \mathscr{X}.$$

If $\mathscr{H}$ is the closed linear span of $\{\zeta_x, x \in \mathscr{X}\}$ in $L^2(P)$ and $\lambda(x) = \zeta_x$ then the first part of the proposition holds.

To prove the second part consider $S = \{\lambda(x) | x \in \mathscr{X}\}$, $S' = \{\lambda'(x) | x \in \mathscr{X}\}$ and the map $U : \lambda(x) \to \lambda'(x)$ from S onto S'. Then U is scalar product preserving and S and S' are total in $\mathscr{H}$ and $\mathscr{H}'$ respectively. By the obvious generalisation of Proposition 7.2 for not necessarily separable Hilbert spaces, U extends uniquely to a unitary isomorphism from $\mathscr{H}$ onto $\mathscr{H}'$. ■

The pair $(\mathscr{H}, \lambda)$ determined uniquely up to a unitary isomorphism by the kernel K on $\mathscr{X}$ is called a *Gelfand pair* associated with K.

We are now ready to introduce the notion of tensor products of Hilbert spaces using Proposition 15.4. Let $\mathscr{H}_i$, $1 \leq i \leq n$ be Hilbert spaces and let $\mathscr{X} = \mathscr{H}_1, \times \cdots \times \mathscr{H}_n$ be their cartesian product as a set. Then the function $K_i(u, v) = \langle u, v \rangle$, $u, v \in \mathscr{H}_i$ is a kernel on $\mathscr{H}_i$ for each i. By Corollary 15.3 the function

$$K(\underline{u}, \underline{v}) = \Pi_{i=1}^n \langle u_i, v_i \rangle,\ \underline{u} = (u_1, \ldots, u_n),\ \underline{v} = (v_1, \ldots, v_n)$$

where $u_i, v_i \in \mathscr{H}_i$ for each i, is a kernel on $\mathscr{X}$. Consider any Gelfand pair $(\mathscr{H}, \lambda)$ associated with K and satisfying (i) and (ii) of Proposition 15.4. Then $\mathscr{H}$ is called a *tensor product* of $\mathscr{H}_i$, $i = 1, 2, \ldots, n$. We write

$$\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \cdots \otimes \mathscr{H}_n = \bigotimes_{i=1}^n \mathscr{H}_i, \tag{15.1}$$

$$\lambda(\underline{u}) = u_1 \otimes u_2 \otimes \cdots \otimes u_n = \bigotimes_{i=1}^n u_i \tag{15.2}$$

and call $\lambda(\underline{u})$ the *tensor product* of the vectors $u_i, 1 \leq i \leq n$. If $\mathscr{H}_i = h$ for all i then $\mathscr{H}$ is called the *n-fold tensor product* of h and denoted by $h^{\otimes^n}$. If, in addition, $u_i = u$ for all i in (15.2) then $\lambda(\underline{u})$ is denoted by $u^{\otimes^n}$ and called the *n-th power* of u. (Since $(\mathscr{H}, \lambda)$ is determined uniquely upto a Hilbert space isomorphism we take the liberty of calling $\mathscr{H}$ the *tensor product* of $\mathscr{H}_i$, $1 \leq i \leq n$ in (15.1)).

Proposition 15.5: The map $(u_1, u_2, \ldots, u_n) \to u_1 \otimes u_2 \otimes \cdots \otimes u_n$ from $\mathscr{H}_1 \times \mathscr{H}_2 \times \cdots \times \mathscr{H}_n$ into $\mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \cdots \otimes \mathscr{H}_n$ defined by (15.1) and (15.2) is multilinear: for all scalars α, β

$$u_1 \otimes \cdots \otimes u_{i-1} \otimes (\alpha u_i + \beta v_i) \otimes u_{i+1} \otimes \cdots \otimes u_n \tag{15.3}$$

$$= \alpha u_1 \otimes \cdots \otimes u_n + \beta u_1 \otimes \cdots \otimes u_{i-1} \otimes v_i \otimes u_{i+1} \otimes \cdots \otimes u_n \tag{15.4}$$

Furthermore

$$\langle \bigotimes_{i=1}^n u_i, \bigotimes_{i=1}^n v_i \rangle = \Pi_{i=1}^n \langle u_i, v_i \rangle. \tag{15.5}$$

The set $\{\bigotimes_{i=1}^n u_i | u_i \in \mathscr{H}_i, i = 1, 2, \ldots, n\}$ is total in $\bigotimes_{i=1}^n \mathscr{H}_i$.

Proof: Only (15.3) remains to be proved. Straightforward computation using (15.4) and the sesquilinearity of scalar products show that for each fixed i

$$\begin{aligned}
&\|u_1 \otimes \cdots \otimes u_{i-1} \otimes (\alpha u_i + \beta v_i) \otimes u_{i+1} \otimes \cdots \otimes u_n - \alpha \bigotimes_{j=1}^{n} u_j \\
&\quad -\beta u_1 \otimes \cdots \otimes u_{i-1} \otimes v_i \otimes u_{i+1} \otimes \cdots \otimes u_n\|^2 \\
&\quad = \Pi_{j \neq i} \|u_j\|^2 \{\|\alpha u_i + \beta v_i\|^2 + |\alpha|^2 \|u_i\|^2 + |\beta|^2 \|v_i\|^2 \\
&\quad\quad -2\,\mathrm{Re}(\langle \alpha u_i + \beta v_i, \alpha u_i \rangle + \langle \alpha u_i + \beta v_i, \beta v_i \rangle + \langle \alpha u_i, \beta v_i \rangle)\} \\
&\quad = 0.
\end{aligned}$$

■

For any $u_i \in \mathscr{H}_i, 1 \leq i \leq n$ the product vector $\bigotimes_{i=1}^{n} u_i$ may be interpreted as the multi-antilinear functional

$$(\bigotimes_{i=1}^{n} u_i)(v_1, v_2, \ldots, v_n) = \Pi_{i=1}^{n} \langle v_i, u_i \rangle.$$

Such multi-antilinear functionals generate a linear manifold M to which the scalar product (15.4) can be extended by sesquilinearity to make it a pre-Hilbert space. M is the usual algebraic tensor product of the vector spaces $\mathscr{H}_i, 1 \leq i \leq n$ and $\bigotimes_{i=1}^{n} \mathscr{H}_i$ is its completion.

Exercise 15.6: (i) Let K be a kernel on $\mathscr{X}$. A bijective map $g : \mathscr{X} \to \mathscr{X}$ is said to leave K *invariant* if $K(g(x), g(y)) = K(x, y)$ for all x, y in $\mathscr{X}$. Let G_K denote the group of all such bijective transformations of $\mathscr{X}$ leaving K invariant and let $(\mathscr{H}, \lambda)$ be a Gelfand pair associated with K. Then there exists a unique homomorphism $g \to U_g$ from G_K into the unitary group $\mathscr{U}(\mathscr{H})$ of $\mathscr{H}$ satisfying the relation

$$U_g \lambda(x) = \lambda(g(x)) \text{ for all } x \in \mathscr{X}, g \in G_K.$$

If $(\mathscr{H}', \lambda')$ is another Gelfand pair associated with $K, V : \mathscr{H} \to \mathscr{H}'$ is the unitary isomorphism satisfying $V\lambda(x) = \lambda'(x)$ for all x and $g \to U'_g$ is the homomorphism from G_K into $\mathcal{U}(\mathscr{H}')$ satisfying $U'_g \lambda'(x) = \lambda'(g(x))$ for all $x \in \mathscr{X}$, $g \in G_K$ then $VU_gV^{-1} = U'_g$ for all g.

(ii) In (i) let $\mathscr{X}$ be a separable metric space and let K be continuous on $\mathscr{X} \times \mathscr{X}$. Suppose $G_0 \subset G_K$ is a subgroup which is a topological group acting continuously on $\mathscr{X} \times \mathscr{X}$. Then in any Gelfand pair $(\mathscr{H}, \lambda)$ the map $\lambda : \mathscr{X} \to \mathscr{H}$ is continuous, $\mathscr{H}$ is separable and the homomorphism $g \to U_g$ restricted to G_0 is continuous.

(Hint: Use Proposition 7.2 for (i) and examine $\|\lambda(x) - \lambda(y)\|$ for (ii)).

Exercise 15.7: (i) Let $\mathscr{H}_i, 1 \leq i \leq n$ be Hilbert spaces and let $S_i \subset \mathscr{H}_i$ be a total subset for each i. Then the set $\{\bigotimes_{i=1}^{n} u_i | u_i \in S_i \text{ for each } i\}$ is total in $\bigotimes_{i=1}^{n} \mathscr{H}_i$. $\mathscr{H}$ is separable if each $\mathscr{H}_i$ is separable.

(ii) There exist unitary isomorphisms

$$U_{12,3} : (\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes \mathcal{H}_3 \to \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3,$$

$$U_{1,23} : \mathcal{H}_1 \otimes (\mathcal{H}_2 \otimes \mathcal{H}_3) \to \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3,$$

such that

$$U_{12,3}(u_1 \otimes u_2) \otimes u_3 = U_{1,23} u_1 \otimes (u_2 \otimes u_3) = u_1 \otimes u_2 \otimes u_3$$

for all $i \in \mathcal{H}_i, i = 1, 2, 3$. (Hint: Use Proposition 7.2.)

Exercise 15.8: Let $\{e_{ij} | j = 1, 2, \ldots\}$ be an orthonormal basis in $\mathcal{H}_i, i = 1, 2, \ldots, n$ respectively. Then the set

$$\{e_{1j_1} \otimes e_{2j_2} \otimes \cdots \otimes e_{nj_n} | j_1 = 1, 2, \ldots, j_2 = 1, 2, \ldots, j_n = 1, 2, \ldots\}$$

is an orthonormal basis for $\bigotimes_{i=1}^n \mathcal{H}_i$. (Note that when $\dim \mathcal{H}_i = m_i < \infty$, $j_i = 1, 2, \ldots, m_i$). In particular,

$$\bigotimes_{i=1}^n u_i = \sum_{j_1, j_2, \ldots, j_n} \{\Pi_{i=1}^n \langle e_{ij_i}, u_i \rangle\} \bigotimes_{i=1}^n e_{ij_i}$$

where the right hand side is a strongly convergent sum in $\bigotimes_{i=1}^n \mathcal{H}_i$. If $\dim \mathcal{H}_i = m_i < \infty$ for every i then

$$\dim \bigotimes_{i=1}^n \mathcal{H}_i = m_1 m_2 \cdots m_n.$$

Exercise 15.9: Let $\mathcal{H}_i = L^2(\Omega_i, \mathcal{F}_i, \mu_i)$, $1 \leq i \leq n$ where $(\Omega_i, \mathcal{F}_i, \mu_i)$ is a σ-finite measure space for each i. If $(\Omega, \mathcal{F}, \mu) = \Pi_{i=1}^n (\Omega_i, \mathcal{F}_i, \mu_i)$ is the cartesian product of these measure spaces there exists a unitary isomorphism $U : \bigotimes_{i=1}^n \mathcal{H}_i \to L^2(\Omega, \mathcal{F}, \mu)$ such that

$$(U u_1 \otimes \cdots \otimes u_n)(\omega_1, \ldots, \omega_n) = \Pi_{i=1}^n u_i(\omega_i) \text{ a.e. } \mu.$$

Exercise 15.10: Let $\mathcal{H}_n$, $n = 1, 2, \ldots$ be a sequence of Hilbert spaces and let $\{\phi_n\}$ be a sequence of unit vectors where $\phi_n \in \mathcal{H}_n$ for each n. Suppose

$$M = \{\underline{u} | \underline{u} = (u_1, u_2, \ldots),\ u_j \in \mathcal{H}_j,\ u_n = \phi_n \text{ for all large } n\}.$$

Define

$$K(\underline{u}, \underline{v}) = \Pi_{j=1}^\infty \langle u_j, v_j \rangle, \quad \underline{u}, \underline{v} \in M.$$

Then K is a kernel on M. The Hilbert space $\mathcal{H}$ in a Gelfand pair $(\mathcal{H}, \lambda)$ associated with K is called the *countable tensor product* of the sequence $\{\mathcal{H}_n\}$ with respect to the *stabilizing sequence* $\{\phi_n\}$. We write

$$\lambda(\underline{u}) = u_1 \otimes u_2 \otimes \cdots \quad \text{for} \quad \underline{u} \in M.$$

Suppose $\{e_{n0}, e_{n1}, \ldots\} = S_n$ is an orthonormal basis in $\mathcal{H}_n$ such that $e_{n0} = \phi_n$ for each n. Then the set $\{\lambda(\underline{u}) | \underline{u} \in M,\ u_j \in S_j \text{ for each } j\}$ is an orthonormal basis in $\mathcal{H}$.

Exercise 15.11: Let $\mathscr{H}_n = L^2(\Omega_n, \mathscr{F}_n, \mu_n)$, $n = 1, 2, \ldots$, $\mathscr{H} = L^2(\Omega, \mathscr{F}, \mu)$ where $(\Omega_n, \mathscr{F}_n, \mu_n)$ is a probability space for each n and $(\Omega, \mathscr{F}, \mu) = \Pi_{n=1}^{\infty}(\Omega_n, \mathscr{F}_n, \mu_n)$ is the product probability space. Then $\mathscr{H}$ is the countable tensor product of the sequence $\{\mathscr{H}_n\}$ with respect to the stabilising sequence $\{\phi_n\}$ where ϕ_n is the constant function 1 in Ω_n for each n. This may be used to construct examples where different stabilising sequences may lead to different countable tensor products. (Hint: If λ and ν are two distinct probability measures such that $\lambda \equiv \nu$ then $\lambda \times \lambda \times \cdots$ and $\nu \times \nu \times \cdots$ are singular with respect to each other.)

Exerecise 15.12: Let $\mathscr{H} = L^2([0,1], h)$ be the Hilbert space defined in Section 2 where μ is the Lebesgue measure in $[0, 1]$. Then any continuous map $f : [0,1] \to \mathscr{H}$ is an element of $\mathscr{H}$. For any such continuous map f define the element $e_n(f)$ in $(\mathbb{C} \oplus h)^{\otimes n}$ by

$$e_n(f) = \otimes_{j=1}^{n}\{1 \oplus n^{-\frac{1}{2}} f(\frac{j}{n})\}.$$

Then

$$\lim_{n \to \infty} \langle e_n(f), e_n(g) \rangle = \exp\langle f, g \rangle.$$

Exercise 15.13: Let $\mathscr{H}_1, \mathscr{H}_2$ be Hilbert spaces and let $\{f_1, f_2, \ldots\}$ be an orthonormal basis in $\mathscr{H}_2$. Then there exists a unitary isomorphism $U : \mathscr{H}_1 \otimes \mathscr{H}_2 \to \mathscr{H}_1 \oplus \mathscr{H}_1 \oplus \cdots$ satisfying $Uu \otimes v = \oplus_i \langle f_i, v \rangle u$ for all $u \in \mathscr{H}_1$, $v \in \mathscr{H}_2$.

Exercise 15.14: Let $L^2(\mu, h)$ be as defined in Section 2. Then there exists a unique unitary isomorphism $U : L^2(\mu) \otimes h \to L^2(\mu, h)$ satisfying $(Uf \otimes u)(\omega) = f(\omega)u$.

Exercise 15.15: Let $\mathscr{H}$ be a Hilbert space and let $\mathcal{I}_2(\mathscr{H})$ be the Hilbert space of all Hilbert-Schmidt operators in $\mathscr{H}$ with scalar product $\langle T_1, T_2 \rangle = \operatorname{tr} T_1^* T_2$. For any fixed conjugation J there exists a unique unitary isomorphism $U : \mathscr{H} \otimes \mathscr{H} \to \mathcal{I}_2(\mathscr{H})$ satisfying $Uu \otimes v = |u\rangle\langle Jv|$ for all u, v in $\mathscr{H}$. (See Section 9.)

Example 15.16: [20] Let H be any selfadjoint operator in a Hilbert space $\mathscr{H}$ with pure point spectrum $\Sigma(H) = S$. Then S is a finite or countable subset of $\mathbb{R}$. Denote by G the countable additive group generated by S and endowed with the discrete topology. Let $\tilde{G}$ be its compact character group with the normalised Haar measure. For any bounded operator X on $\mathscr{H}$ and $\lambda \in G$ define the bounded operator

$$X_\lambda = \int_{\tilde{G}} \overline{\chi}(\lambda)\chi(H) X \overline{\chi}(H) d\chi.$$

Let $\mu, \nu \in S$, $u, v \in \mathscr{H}$, $Hu = \lambda u$, $Hv = \nu v$. Then

$$\langle u, X_\lambda v \rangle = \{\int_{\tilde{G}} \chi(\mu - \nu - \lambda) d\chi\} \langle u, Xv \rangle.$$

Hence

$$\langle u, X_\lambda v \rangle = \begin{cases} \langle u, Xv \rangle & \text{if } \lambda = \mu - \nu, \\ 0 & \text{if } \lambda \neq \mu - \nu. \end{cases}$$

Thus, for any non-zero bounded operator X there exists a $\lambda \in S - S = \{\mu - \nu | \mu, \nu \in S\}$ such that $X_\lambda \neq 0$ and on the linear manifold D generated by all the eigenvectors of H

$$[H, [H, X_\lambda]] = \lambda^2 X_\lambda,$$
$$Xu = X_0 u + \sum_{\substack{\lambda > 0 \\ \lambda \in S - S}} (X_{-\lambda} + X_\lambda) u, \quad u \in D.$$

If X is selfadjoint $(X_\lambda)^* = X_{-\lambda}$ and X is a "superposition" of bounded "harmonic" observables $X_0, \{X_{-\lambda} + X_\lambda | \lambda \in S - S, \lambda > 0\}$ with respect to H. (See Example 6.2.) Whenever X, Y are Hilbert-Schmidt operators

$$\mathrm{tr}\, X^* Y = \mathrm{tr}\, X_0^* Y_0 + \sum_{\substack{\lambda \in S - S \\ \lambda > 0}} \mathrm{tr}\, X_\lambda^* Y_\lambda$$

and

$$X = X_0 + \sum_{\substack{\lambda \in S - S \\ \lambda > 0}} (X_\lambda + X_{-\lambda})$$

converges in Hilbert-Schmidt norm (i.e., the norm in $\mathcal{I}_2(\mathcal{H})$). See Example 6.2.

Notes

The presentation here is based on the notes of Parthasarathy and Schmidt [99]. Proposition 15.1 is known as Schur's Lemma. Proposition 15.4 and Exercise 15.6 constitute what may be called a probability theorist's translation of the famous Gelfand-Neumark-Segal or G.N.S. Theorem. Its origin may be traced back to the theory of second order stationary stochastic processes developed by A.N. Kolmogorov, N. Wiener, A.I. Khinchine and K. Karhunen [31]. Proposition 7.2, 15.4 and 19.4 more or less constitute the basic principles around which the fabric of our exposition in this volume is woven.

It is an interesting idea of Journé and Meyer [88] that $(\mathbb{C} \oplus h)^{\otimes^n}$ in Exercise 15.12 may be looked upon as a toy Fock space where $e_n(f)$ may be imagined as a toy exponential or coherent vector which in the limit becomes the boson Fock space $\Gamma(L^2[0,1] \otimes h)$ where $e(f)$ is the true exponential or coherent vector. See Exercise 29.12, 29.13, Parthasarathy [107], Lindsay and Parthasarathy [78].

16 Operators in tensor products of Hilbert spaces

We shall now define tensor products of operators. To begin with let $\mathcal{H}_i$ be a finite dimensional Hilbert space of dimension m_i, $i = 1, 2, \ldots, n$ and $\mathcal{H} = \bigotimes_{i=1}^n \mathcal{H}_i$. Suppose T_i is a selfadjoint operator in $\mathcal{H}_i$ with eigenvalues $\{\lambda_{ij}, 1 \leq j \leq m_i\}$ and corresponding orthnormal set of eigenvectors $\{e_{ij}, 1 \leq j \leq m_i\}$ so that

$$T_i e_{ij} = \lambda_{ij} e_{ij}, 1 \leq j \leq m_i, i = 1, 2, \ldots, n.$$

Using Exercise 15.8 define a selfadjoint operator T on $\mathcal{H}$ by putting

$$T \bigotimes_{i=1}^n e_{ij_i} = \Pi_{i=1}^n \lambda_{ij_i} \bigotimes_{i=1}^n e_{ij_i}, \quad 1 \leq j_i \leq m_i$$

and extending linearly on $\mathscr{H}$. The operator T has eigenvalues $\Pi_{i=1}^n \lambda_{ij_i}$ and satisfies

$$T \bigotimes_{i=1}^n u_i = \bigotimes_{i=1}^n T_i u_i \quad \text{for all} \quad u_i \in \mathscr{H}_i, 1 \leq i \leq n.$$

Furthermore

$$\begin{aligned} \|T\| &= \max\left(|\Pi_{i=1}^n \lambda_{ij_i}|, 1 \leq j_i \leq m_i\right) \\ &= \Pi_{i=1}^n \max(|\lambda_{ij}|, 1 \leq j \leq m_i) = \Pi_{i=1}^n \|T_i\|. \end{aligned}$$

In particular, we have the identity

$$\begin{aligned} \Big|\sum_{1 \leq j,k \leq N} \overline{a}_j \alpha_k \Pi_{i=1}^n \langle u_{ij}, T_i u_{ik} \rangle\Big| &= \Big|\Big\langle \sum_{j=1}^N \alpha_j \bigotimes_{i=1}^n u_{ij}, T \sum_{j=1}^N \alpha_j \bigotimes_{i=1}^n u_{ij} \Big\rangle\Big| \\ &\leq \Pi_{i=1}^n \|T_i\| \; \Big\|\sum_{j=1}^N \alpha_j \bigotimes_{i=1}^n u_{ij}\Big\|^2 \end{aligned} \tag{16.1}$$

for all scalars α_j, $u_{ij} \in \mathscr{H}_i$, $1 \leq j \leq N$, $1 \leq i \leq n$, $N = 1, 2, \ldots$. We write $T = \bigotimes_{i=1}^n T_i = T_1 \otimes \cdots \otimes T_n$ and call it the *tensor product* of operators T_i, $1 \leq i \leq n$. The next proposition extends this elementary notion to all bounded operators on Hilbert spaces.

Proposition 16.1: Let $\mathscr{H}_i$, $1 \leq i \leq n$ be Hilbert spaces and let T_i be a bounded operator in $\mathscr{H}_i$ for each i. Then there exists a unique bounded operator T in $\mathscr{H} = \bigotimes_{i=1}^n \mathscr{H}_i$ satisfying

$$T \bigotimes_{i=1}^n u_i = \bigotimes_{i=1}^n T_i u_i \quad \text{for all} \quad u_i \in \mathscr{H}_i, \quad 1 \leq i \leq n. \tag{16.2}$$

Furthermore $\|T\| = \Pi_i \|T_i\|$.

Proof: Let $S = \{\bigotimes_{i=1}^n u_i | u_i \in \mathscr{H}_i, 1 \leq i \leq n\}$. For all scalars α_j, $1 \leq j \leq N$ and product vectors $\bigotimes_{i=1}^n u_{ij} \in S$, $1 \leq j \leq N$ we have

$$\begin{aligned} \Big\|\sum_{j=1}^N \alpha_j \bigotimes_{i=1}^n T_i u_{ij}\Big\|^2 &= \sum_{1 \leq j,k \leq N} \overline{\alpha}_j \alpha_k \Pi_{i=1}^n \langle u_{ij}, T_i^* T_i u_{ik} \rangle \\ &= \sum_{1 \leq j,k \leq N} \overline{\alpha}_j \alpha_k \Pi_{i=1}^n \langle u_{ij}, P_i T_i^* T_i P_i u_{ik} \rangle \end{aligned} \tag{16.3}$$

where P_i is the projection on the finite dimensional subspace M_i spanned by $\{u_{ij}, 1 \leq j \leq N\}$ in $\mathscr{H}_i$ for each i. Since $P_i T_i^* T_i P_i$ is a positive operator in M_i, it follows from (16.1) and (16.3) that

$$\begin{aligned} \Big\|\sum_{j=1}^N \alpha_j \bigotimes_{i=1}^n T_i u_{ij}\Big\|^2 &\leq \Pi_{i=1}^n \|P_i T_i^* T_i P_i\| \; \Big\|\sum_{j=1}^N \alpha_j \bigotimes_{i=1}^n u_{ij}\Big\|^2 \\ &\leq \Pi_{i=1}^n \|T_i^* T_i\| \; \Big\|\sum_{j=1}^N \alpha_j \bigotimes_{i=1}^n u_{ij}\Big\|^2. \end{aligned} \tag{16.4}$$

Define T on the set S by (16.2). Then (16.4) shows that T extends uniquely as a linear map on the linear manifold generated by S. Since S is total in $\mathcal{H}$ this linear extension can be closed to define a bounded operator T on $\mathcal{H}$ satisfying

$$\|T\| \leq \Pi_{i=1}^{n} \|T_i^* T_i\|^{1/2} = \Pi_{i=1}^{n} \|T_i\|.$$

Let now $0 < \varepsilon < 1$ be arbitrary. Choose unit vectors $u_i \in \mathcal{H}_i$ such that $\|T_i u_i\| \geq (1-\varepsilon)\|T_i\|$ for each i, assuming $T_i \neq 0$. Then $\bigotimes_{i=1}^{n} u_i$ is a unit vector in $\mathcal{H}$ and

$$\|T \bigotimes_{i=1}^{n} u_i\| = \|\bigotimes_{i=1}^{n} T_i u_i\| = \Pi_{i=1}^{n} \|T_i u_i\| \geq (1-\varepsilon)^n \Pi_{i=1}^{n} \|T_i\|.$$

Thus $\|T\| \geq \Pi_{i=1}^{n} \|T_i\|$. ■

The operator T determined by Proposition 16.1 is called the *tensor product* of the operators T_i, $1 \leq i \leq n$. We write $T = \bigotimes_{i=1}^{n} T_i = T_1 \otimes \cdots \otimes T_n$. If $\mathcal{H}_i = h$, $T_i = S$ for all $i = 1, 2, \ldots, n$ we write $T = S^{\otimes^n}$ and call it the *n-th tensor power* of the operator S.

Proposition 16.2: Let $\mathcal{H}_i$, $1 \leq i \leq n$ be Hilbert spaces and let S_i, T_i be bounded operators in $\mathcal{H}_i$ for each i. Let $S = \bigotimes_{i=1}^{n} S_i$, $T = \bigotimes_{i=1}^{n} T_i$. Then the following relations hold:

(i) The mapping $(T_1, T_2, \ldots, T_n) \to T$ is multilinear from $\mathcal{B}(\mathcal{H}_1) \times \cdots \times \mathcal{B}(\mathcal{H}_n)$ into $\mathcal{B}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$;

(ii) $ST = \bigotimes_{i=1}^{n} S_i T_i$, $T^* = \bigotimes_{i=1}^{n} T_i^*$;

(iii) If each T_i has a bounded inverse then T has a bounded inverse and $T^{-1} = \otimes_i T_i^{-1}$;

(iv) T is a selfadjoint, unitary, normal or projection operator according to whether each T_i is a selfadjoint, unitary, normal or projection operator;

(v) T is positive if each T_i is positive;

(vi) If $T_i = |u_i\rangle\langle v_i|$ where $u_i, v_i \in \mathcal{H}_i$ for each i then

$$T = |u_1 \otimes u_2 \otimes \cdots \otimes u_n\rangle\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n|.$$

Proof: This is straightforward from definitions and we omit the proof. ■

Proposition 16.3: Let T_i be a compact operator in $\mathcal{H}_i$, $i = 1, 2, \ldots, n$, $\mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_i$, $T = \bigotimes_{i=1}^{n} T_i$. If T_i has the canonical decomposition in the sense of Proposition 9.6

$$T_i = \sum_j s_j(T_i) |v_{ij}\rangle\langle u_{ij}|, \quad i = 1, 2, \ldots, n \tag{16.5}$$

then T is a compact operator with canonical decomposition

$$T = \sum_{j_1, j_2, \ldots, j_n} s_{j_1}(T_1) \cdots s_{j_n}(T_n) |v_{1j_1} \otimes \cdots \otimes v_{nj_n}\rangle\langle u_{1j_1} \otimes \cdots \otimes u_{nj_n}|. \tag{16.6}$$

If $T_i \in \mathcal{I}_1(\mathcal{H}_i)$ for each i then $T \in \mathcal{I}_1(\mathcal{H})$ and

$$\|T\|_1 = \Pi_{i=1}^n \|T_i\|_1, \quad \operatorname{tr} T = \Pi_{i=1}^n \operatorname{tr} T_i.$$

In particular, if each T_i is a state so is T.

Proof: Since $\{u_{ij}, j = 1, 2, \ldots\}$, $\{v_{ij}, j = 1, 2, \ldots\}$ are orthonormal sets in $\mathcal{H}_i$ for each i it follows that $\{u_{1j_1} \otimes \cdots \otimes u_{nj_n}\}$, $\{v_{1j_1} \otimes \cdots \otimes v_{nj_n}\}$ are orthonormal sets in $\mathcal{H}$. Equation (16.5), the second part of Proposition 16.1 and (vi) in Proposition 16.2 yield (16.6) as an operator norm convergent sum over the indices $(j_1, j_2, \ldots, j_n)$. If $\|T_i\|_1 = \sum_j |s_j(T_i)| < \infty$ for each i then we have

$$\|T\|_1 = \sum_{j_1, j_2, \ldots, j_n} |s_{j_1}(T_1) \cdots s_{j_n}(T_n)| = \Pi_{i=1}^n \|T_i\|_1.$$

If $\{e_{ij}, j = 1, 2, \ldots\}$ is an orthonormal basis in $\mathcal{H}_i$ for each i it follows that

$$\{e_{1j_1} \otimes \cdots \otimes e_{nj_n}\}$$

is an orthonormal basis in $\mathcal{H}$ and

$$\begin{aligned}
\operatorname{tr} T &= \sum_{j_1, \ldots, j_n} \langle e_{1j_1} \otimes \cdots \otimes e_{nj_n}, T_1 \otimes \cdots \otimes T_n e_{1j_1} \otimes \cdots \otimes e_{nj_n} \rangle \\
&= \sum_{j_1, \ldots, j_n} \Pi_{i=1}^n \langle e_{ij_i}, T_i e_{ij_i} \rangle \\
&= \Pi_{i=1}^n \sum_j \langle e_{ij}, T_i e_{ij} \rangle \\
&= \Pi_{i=1}^n \operatorname{tr} T_i,
\end{aligned}$$

due to the absolute convergence of all the sums involved. Finally, if each T_i is positive so is T. If $\operatorname{tr} T_i = 1$ for each i then $\operatorname{tr} T = 1$. ∎

If $(\mathcal{H}_i, \mathcal{P}(\mathcal{H}_i), \rho_i)$ is a quantum probability space for each $i = 1, 2, \ldots, n$ then putting $\mathcal{H} = \bigotimes_{i=1}^n \mathcal{H}_i$, $\rho = \bigotimes_{i=1}^n \rho_i$ we obtain a new quantum probability space $(\mathcal{H}, \mathcal{P}(\mathcal{H}), \rho)$ called the *product* of the quantum probability spaces

$$(\mathcal{H}_i, \mathcal{P}(\mathcal{H}_i), \rho_i), \quad i = 1, 2, \ldots, n.$$

If X_i is a bounded selfadjoint operator in $\mathcal{H}_i$ or, equivalently, a bounded real valued observable in $\mathcal{H}_i$ then $X = X_1 \otimes \cdots \otimes X_n$ is a bounded real valued observable in $\mathcal{H}$ and the expectation of X in the *product state* ρ is equal to $\Pi_{i=1}^n \operatorname{tr} \rho_i X_i$, the product of the expectations of X_i in the state ρ_i, $i = 1, 2, \ldots, n$.

Now the stage is set for achieving our goal of combining the description of events concerning several statistical experiments into those of a single experiment. Suppose that the events of the i-th experiment are described by the elements of $\mathcal{P}(\mathcal{H}_i)$ where $\mathcal{H}_i$ is a Hilbert space, $i = 1, 2, \ldots, n$. Let $\mathcal{H} = \otimes_i \mathcal{H}_i$. If $P_i \in \mathcal{P}(\mathcal{H}_i)$ then we view the event P_i as the element

$$\hat{P}_i = 1 \otimes \cdots \otimes 1 \otimes P_i \otimes \cdots \otimes 1,$$

(called the *ampliation* of P_i in $\mathscr{H}$) in $\mathscr{P}(\mathscr{H})$ and interpret $P = \bigotimes_{i=1}^{n} P_i = \Pi_{i=1}^{n} \hat{P}_i \in \mathscr{P}(\mathscr{H})$ (see Proposition 16.2) as the event signifying the simultaneous occurrence of P_i in the i-th experiment for $i = 1, 2, \ldots, n$. These are the quantum probabilistic equivalents of measurable rectangles in products of sample spaces. The next proposition indicates how $\mathscr{P}(\mathscr{H})$ is "generated" by $\mathcal{P}(\mathscr{H}_i)$, $i = 1, 2, \ldots, n$.

Proposition 16.4: Every element P in $\mathscr{P}(\mathscr{H})$ can be obtained as a strong limit of linear combinations of projections of the form $\bigotimes_{i=1}^{n} P_i$, $P_i \in \mathcal{P}(\mathscr{H}_i)$.

Proof: Choose an increasing sequence $\{Q_{ik}, k = 1, 2, \ldots\}$ of finite dimensional projections so that $\text{s.lim}_{k \to \infty} Q_{ik} = 1$ in $\mathscr{H}_i$ for each i. Then $Q_k = \bigotimes_{i=1}^{n} Q_{ik}$ increases to the identity strongly in $\mathscr{H}$ and for any bounded operator T in $\mathscr{H}$, $\text{s.lim}_{K \to \infty} Q_k T Q_k = T$. Thus it suffices to show that for fixed k the operator $Q_k T Q_k$ can be expressed as a linear combination of product projections of the form $\bigotimes_{i=1}^{n} P_i$, $P_i \in \mathcal{P}(\mathscr{H}_i)$. Replacing $\mathscr{H}_i$ by the range of Q_{ik} if necessary we may assume that each $\mathscr{H}_i$ has dimension $m_i < \infty$. Let $\{e_{ir}, 1 \leq r \leq m_i\}$ be an orthonormal basis in $\mathscr{H}_i$ and let $E^i_{rs} = |e_{is}\rangle\langle e_{ir}|$. Then every operator T on $\mathscr{H}$ can be expressed as a linear combination of operators of the form $\bigotimes_{i=1}^{n} E^j_{r_j s_j}$. Each E^j_{rs} can be expressed as a linear combination of selfadjoint operators:

$$E^j_{rs} = \frac{1}{2}(E^j_{rs} + E^{j*}_{rs}) + i\{\frac{1}{2i}(E^j_{rs} - E^{j^*}_{rs})\}.$$

Since every selfadjoint operator in $\mathscr{H}_i$ admits a spectral decomposition in terms of eigen projections the proof is complete. ∎

Proposition 16.5: Let $(\Omega_i, \mathscr{F}_i)$ be measurable spaces and let $\xi_i : \mathscr{F}_i \to \mathcal{P}(\mathcal{H}_i)$ be an Ω_i-valued observable in $\mathscr{H}_i$ for $i = 1, 2, \ldots, n$. Let $(\Omega, \mathscr{F}) = \Pi_{i=1}^{n}(\Omega_i, \mathscr{F}_i)$ be the product measurable space and $\mathscr{H} = \bigotimes_{i=1}^{n} \mathscr{H}_i$. Then there exists a unique Ω-valued observable $\xi : \mathscr{F} \to \mathscr{P}(\mathscr{H})$ such that

$$\xi(F_1 \times \cdots \times F_n) = \bigotimes_{i=1}^{n} \xi_i(F_i) \text{ for all } F_i \in \mathscr{F}_i, \quad i = 1, 2, \ldots, n. \tag{16.7}$$

Proof: By Proposition 7.4 there exists a finite or countable family of σ-finite measures $\{\mu_{ij}, j = 1, 2, \ldots, \}$ and a unitary isomorphism $U_i : \mathscr{H}_i \to \bigoplus_j L^2(\mu_{ij})$ such that

$$U_i \xi_i(E) U_i^{-1} = \bigoplus_j \xi^{\mu_{ij}}(E) \text{ for all } E \in \mathscr{F}_i, \quad 1 \leq i \leq n.$$

We have

$$\bigotimes_{i=1}^{n} \{\oplus_j L^2(\mu_{ij})\} = \bigoplus_{j_1, \ldots, j_n} L^2(\mu_{1j_1} \times \mu_{2j_2} \times \cdots \times \mu_{nj_n}).$$

Define $U = U_1 \otimes \cdots \otimes U_n$ and

$$\xi(F) = U^{-1}\{\bigoplus_{j_1, \ldots, j_n} \xi^{\mu_{1j_1} \times \cdots \times \mu_{nj_n}}(F)\}U, F \in \mathscr{F}.$$

Then (16.7) is obtained. ∎

The unique observable ξ satisfying (16.7) in Proposition 16.5 is called the *tensor product* of the observables ξ_i, $i = 1, 2, \ldots, n$. We write $\xi = \xi_1 \otimes \cdots \otimes \xi_n$.

Proposition 16.6: Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and let T be a trace class operator in $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Then there exists a unique trace class operator T_1 in $\mathcal{H}_1$ such that

$$\operatorname{tr} T_1 X = \operatorname{tr} T(X \otimes 1) \text{ for all } X \text{ in } \mathcal{B}(\mathcal{H}_1). \tag{16.8}$$

If T is a state in $\mathcal{H}$ then T_1 is a state in $\mathcal{H}_1$.

Proof: For any compact operator X in $\mathcal{H}_1$ define $\lambda(X) = \operatorname{tr} T(X \otimes 1)$. By Proposition 9.12, (iii) we have $|\lambda(X)| \leq \|T\|_1 \|X\|$. In particular, λ is a continuous linear functional on $\mathcal{I}_\infty(\mathcal{H}_1)$. By Schatten's Theorem (Theorem 9.17) there exists a T_1 in $\mathcal{I}_1(\mathcal{H}_1)$ such that $\operatorname{tr} T_1 X = \operatorname{tr} T(X \otimes 1)$ for all X in $\mathcal{I}_\infty(\mathcal{H}_1)$. Since the maps $X \to \operatorname{tr} T_1 X$ and $X \to \operatorname{tr} T(X \otimes 1)$ are strongly continuous in $\mathcal{B}(\mathcal{H}_1)$ we obtain (16.8). If T is positive we have

$$\langle u, T_1 u \rangle = \operatorname{tr} T_1 |u\rangle\langle u| = \operatorname{tr} T(|u\rangle\langle u| \otimes 1) \geq 0$$

for all u in $\mathcal{H}_1$. Thus $T_1 \geq 0$. Putting $X = 1$ in (16.8) we get $\operatorname{tr} T_1 = \operatorname{tr} T$. This proves that T_1 is a state if T is a state. ■

The operator T_1 in Proposition 16.6 is called the *relative trace* of T in $\mathcal{H}_1$. If T is a state then the relative trace of T_1 is the analogue of marginal distribution in classical probability.

Exercise 16.7: Suppose $\mathcal{H}_1, \mathcal{H}_2$ are two real finite dimensional Hilbert spaces of dimensions m_1, m_2 respectively and $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Let $\mathcal{O}(\mathcal{H}_1), \mathcal{O}(\mathcal{H}_2)$ and $\mathcal{O}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ be the real linear spaces of all selfadjoint operators in $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_1 \otimes \mathcal{H}_2$ respectively. Then we have $\dim \mathcal{O}(\mathcal{H}_i) = \frac{1}{2} m_i(m_i + 1)$, $i = 1, 2$ and $\dim \mathcal{O}(\mathcal{H}_1 \otimes \mathcal{H}_2) = \frac{1}{2} m_1 m_2 (m_1 m_2 + 1)$. In particular,

$$\dim \mathcal{O}(\mathcal{H}_1 \otimes \mathcal{H}_2) > \dim \mathcal{O}(\mathcal{H}_1) \cdot \dim \mathcal{O}(\mathcal{H}_2) \text{ if } m_i > 1,\ i = 1, 2, .$$

On the other hand, if $\mathcal{H}_1, \mathcal{H}_2$ are complex Hilbert spaces $\dim \mathcal{O}(\mathcal{H}_i) = m_i^2$ and

$$\dim \mathcal{O}(\mathcal{H}_1 \otimes \mathcal{H}_2) = \dim \mathcal{O}(\mathcal{H}_1) \dim \mathcal{O}(\mathcal{H}_2).$$

(In the light of Proposition 16.4 this indicates the advantage of working with complex Hilbert spaces in dealing with observables concerning several quantum statistical experiments).

Exercise 16.8: (i) Let T_j be (a not necessarily bounded) selfadjoint operator in $\mathcal{H}_j$ with spectral representation

$$T_j = \int_{\mathbb{R}} x \xi_j(dx), \quad j = 1, 2, \ldots, n,$$

ξ_j being a real valued observable in $\mathcal{H}_j$ for each j. Define the selfadjoint operator

$$T_1 \otimes \cdots \otimes T_n = \int_{\mathbb{R}^n} x_1 x_2 \cdots x_n \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n (dx_1 dx_2 \cdots dx_n)$$

by Proposition 12.1 and Theorem 12.2. If $D_j = D(T_j)$ is a core for T_j for each j then the linear manifold $\mathcal{M}$ generated by $\{u_1 \otimes \cdots \otimes u_n | u_j \in D_j \text{ for each } j\}$ is a core for $T_1 \otimes \cdots \otimes T_n$ and

$$T_1 \otimes \cdots \otimes T_n \;\; u_1 \otimes \cdots \otimes u_n = T_1 u_1 \otimes \cdots \otimes T_n u_n$$

for all $u_j \in D(T_j)$, $1 \leq j \leq n$.

(ii) Let $\hat{T}_i = 1 \otimes \cdots \otimes 1 \otimes T_i \otimes 1 \cdots 1 \otimes 1$ (be the i-th *ampliation of* T_i) where T_i is in the i-th position. Then $\hat{T}_1 + \cdots + \hat{T}_n$ is essentially selfadjoint on $\mathcal{M}$ with its closure being the selfadjoint operator $\int_{\mathbb{R}^n} (x_1 + \cdots + x_n) \xi_1 \otimes \cdots \otimes \xi_n (dx_1 \cdots dx_n)$.

(iii) If ρ_j is a state in $\mathcal{H}_j$ and T_j has finite expectation in the state ρ_j for each j then $T_1 \otimes \cdots \otimes T_n$ has finite expectation in the product state $\rho = \rho_1 \otimes \cdots \otimes \rho_n$ and

$$\langle T_1 \otimes \cdots \otimes T_n \rangle_{\rho_1 \otimes \cdots \otimes \rho_n} = \Pi_{i=1}^n \langle T_i \rangle_{\rho_i}$$

in the notation of Proposition 13.6.

Exercise 16.9: Let $T \in \mathcal{I}_1(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and let T_1 be its relative trace in $\mathcal{H}_1$. If $\{f_j\}$ is an orthonormal basis in $\mathcal{H}_2$ then

$$\langle u, T_1 v \rangle = \sum_j \langle u \otimes f_j, Tv \otimes f_j \rangle \text{ for all } u, v \text{ in } \mathcal{H}_1$$

where the right hand side converges absolutely.

Exercise 16.10: Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Then the *-algebra generated by $\{X_1 \otimes X_2 | X_i \in \mathcal{B}(\mathcal{H}_i), i = 1, 2\}$ is strongly dense in $\mathcal{B}(\mathcal{H})$.

(ii) For any trace class operator ρ in $\mathcal{H}_2$ there exists a unique linear map $\mathbb{E}_\rho : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}_1)$ satisfying

$$\langle u, \mathbb{E}_\rho(X) v \rangle = \operatorname{tr} X(|v\rangle\langle u| \otimes \rho) \text{ for all } u, v \in \mathcal{H}_1, X \in \mathcal{B}(\mathcal{H})$$

where

$$\|\mathbb{E}_\rho(X)\| \leq \|\rho\|_1 \|X\|.$$

(iii) If ρ is a state then $\mathbb{E}_\rho$ is called the ρ-*conditional* expectation map from $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H}_1)$. The ρ conditional expectation map satisfies the following properties:

(1) $\mathbb{E}_\rho 1 = 1$, $\mathbb{E}_\rho X^* = (\mathbb{E}_\rho X)^*$, $\|\mathbb{E}_\rho X\| \leq \|X\|$;

(2) $\mathbb{E}_\rho (A \otimes 1) X (B \otimes 1) = A(\mathbb{E}_\rho X) B$, for all $A, B \in \mathcal{B}(\mathcal{H}_1)$, $X \in \mathcal{B}(\mathcal{H})$;

(3) $\sum_{1 \leq i,j \leq k} Y_i^* (\mathbb{E}_\rho X_i^* X_j) Y_j \geq 0$ for all $X_i \in \mathcal{B}(\mathcal{H})$, $Y_i \in \mathcal{B}(\mathcal{H}_1)$. In particular, $\mathbb{E}_\rho X \geq 0$ whenever $X \geq 0$.

Exercise 16.11: Let $\{\mathcal{H}_n, n \geq 0\}$ be a sequence of Hilbert spaces and $\{\phi_n, n \geq 1\}$ be unit vectors, $\phi_n \in \mathcal{H}_n$. Define $\mathcal{H}_{[n+1} = \mathcal{H}_{n+1} \otimes \mathcal{H}_{n+2} \otimes \cdots$ with respect to the stabilising sequence $\phi_{n+1}, \phi_{n+2}, \ldots$ and $\mathcal{H}_{n]} = \mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. In the Hilbert space

$$\tilde{\mathcal{H}} = \mathcal{H}_0 \otimes \mathcal{H}_{[1} = \mathcal{H}_{n]} \otimes \mathcal{H}_{[n+1}, \quad n = 1, 2, \ldots$$

consider the increasing sequence of *-algebras

$$\mathcal{B}_{n]} = \{X \otimes 1_{[n+1} | X \in \mathcal{B}(\mathcal{H}_{n]})\}, \quad n = 0, 1, 2 \ldots$$

Define $\mathcal{B}_n = \mathcal{B}(\mathcal{H}_{n]})$ and $1_{n]}$ to be the identity in $\mathcal{B}_n$. There exists a unique linear map $\mathbb{E}_{n]} : \mathcal{B}_\infty \to \mathcal{B}_{n]}$ satisfying

$$\langle u, \mathbb{E}_{n]}(X) v\rangle = \langle u \otimes \phi_{[n+1}, Xv \otimes \phi_{[n+1}\rangle \text{ for all } u, v \in \mathcal{H}_{n]}, \ X \in \mathcal{B}_\infty$$

where $\phi_{[n+1} = \phi_{n+1} \otimes \phi_{n+2} \otimes \cdots$ and $\mathcal{B}_\infty = \mathcal{B}(\tilde{\mathcal{H}})$. Indeed,

$$\mathbb{E}_{n]}(X) = \mathbb{E}_{|\phi_{[n+1}\rangle\langle\phi_{[n+1}|}(X) \otimes 1_{[n+1}.$$

The maps $\{\mathbb{E}_{n]}\}_{n \geq 0}$ satisfy the following properties:

(i) $\mathbb{E}_{n]} 1 = 1$, $\mathbb{E}_{n]} X^* = (\mathbb{E}_{n]} X)^*$, $\|\mathbb{E}_{n]} X\| \leq \|X\|$;

(ii) $\mathbb{E}_{n]} AXB = A \mathbb{E}_{n]}(X) B$ whenever $A, B \in \mathcal{B}_{n]}$;

(iii) $\mathbb{E}_{m]} \mathbb{E}_{n]} = \mathbb{E}_{n]} \mathbb{E}_{m]} = \mathbb{E}_{m]}$ whenever $m \leq n$;

(iv) $\sum_{1 \leq i,j \leq k} Y_i^* \mathbb{E}_{n]}(X_i^* X_j) Y_j \geq 0$ for all $Y_i \in \mathcal{B}_{n]}$, $X_i \in \mathcal{B}_\infty$. In particular $\mathbb{E}_{n]} X \geq 0$ whenever $X \geq 0$;

(v) $\text{s.lim}_{n \to \infty} \mathbb{E}_{n]} X = X$ for all X in $\mathcal{B}_\infty$.

Exercise 16.12: (i) In the notations of Exercise 16.11 a sequence $\{X_n\}$ in $\mathcal{B}_\infty$ is said to be *adapted* if $X_n \in \mathcal{B}_{n]}$ for every n. It is called a *martingale* if

$$\mathbb{E}_{n-1]} X_n = X_{n-1} \text{ for all } n \geq 1$$

Suppose $\underline{A} = \{A_n\}_{n \geq 1}$ is any sequence of operators where

$$A_n \in \mathcal{B}(\mathcal{H}_n), \langle \phi_n, A_n \phi_n\rangle = 0, \quad n = 1, 2, \ldots$$

Define

$$\tilde{A}_n = 1_{n-1]} \otimes A_n \otimes 1_{[n+1},$$

$$M_n(\underline{A}) = \begin{cases} 0 & \text{if } n = 0, \\ \tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_n & \text{if } n \geq 1. \end{cases}$$

Then $\{M_n(\underline{A})\}_{n \geq 0}$ is a martingale. For any two sequences $\underline{A}, \underline{B}$ of operators where $A_n, B_n \in \mathcal{B}(\mathcal{H}_n), \langle \phi_n, A_n \phi_n\rangle = \langle \phi_n, B_n \phi_n\rangle = 0$ for each n

$$\mathbb{E}_{m]}\{M_n(\underline{A}) - M_m(\underline{A})\}^*\{M_n(\underline{B}) - M_m(B)\}$$

$$= \sum_{j=m+1}^{n} \langle A_j \phi_j, B_j \phi_j\rangle \quad \text{for all } n > m \geq 0.$$

(ii) For any sequence $\underline{E} = (E_1, E_2, \ldots)$ of operators such that

$$E_n \in \mathcal{B}_{n-1]}, \quad n = 1, 2, \ldots$$

define

$$I_n(\underline{A}, \underline{E}) = \begin{cases} 0 & \text{if } n = 0, \\ \sum_1^n E_n\{M_n(\underline{A}) - M_{n-1}(\underline{A})\} & \text{if } n \geq 1. \end{cases}$$

Then $\{I_n(\underline{A}, \underline{E})\}_{n \geq 0}$ is a martingale. Furthermore

$$\begin{aligned} \mathbb{E}_{n-1]} I_n(\underline{A}, \underline{E})^* I_n(\underline{B}, \underline{F}) &= I_{n-1}(\underline{A}, \underline{E})^* I_{n-1}(\underline{B}, \underline{F}) \\ &\quad + \langle A_n \phi_n, B_n \phi_n \rangle E_n^* F_n \end{aligned}$$

for all $n \geq 1$.

Exercise 16.13: For any selfadjoint operator X in the Hilbert space $\mathcal{H}$ define the operator $S(X)$ in $\mathbb{C}^2 \otimes \mathcal{H}$ by $S(X) = (\sigma_1 \otimes 1) \exp[i\sigma_2 \otimes X]$ where σ_j, $1 \leq j \leq 3$ are the Pauli spin matrices. Then $S(X) = S(X)^{-1} = S(X)^*$ and $\frac{1}{2}[S(X) + S(-X)] = \sigma_1 \otimes \cos\ X$. Thus $S(X)$ and $S(-X)$ are spin observables with two-point spectrum $\{-1, 1\}$ but their average can have arbitrary spectrum in the interval $[-1, 1]$. (See also Exercise 4.4, 13.11.)

Notes

The role of tensor products of Hilbert spaces and operators in the construction of observables concerning multiple quantum systems is explained in Mackey [84]. For a discussion of conditional expectation in non-commutative probability theory, see Accardi and Cecchini [4]. Exercise 16.13 arose from discussions with B.V. R. Bhat.

17 Symmetric and antisymmetric tensor products

There is a special feature of quantum mechanics which necessitates the introduction of symmetric and antisymmetric tensor products of Hilbert spaces. Suppose that a physical system consists of n identical particles which are indistinguishable from one another. A transition may occur in the system resulting in merely the interchange of particles regarding some physical characteristic (like position for example) and it may not be possible to detect such a change by any observable means. Suppose the statistical features of the dynamics of each particle in isolation are described by states in some Hilbert space $\mathcal{H}$. According to the procedure outlined in Section 15, 16 the events concerning all the n particles are described by the elements of $\mathcal{P}(\mathcal{H}^{\otimes^n})$. If $P_i \in \mathcal{P}(\mathcal{H})$, $1 \leq i \leq n$ then $\bigotimes_{i=1}^n P_i$ signifies the event that P_i occurs for each i. If the particles i and j $(i < j)$ are interchanged and a change cannot be detected then we should not distinguish between the events $P_1 \otimes \cdots \otimes P_n$ and $P_1 \otimes \cdots \otimes P_{i-1} \otimes P_j \otimes P_{i+1} \otimes \cdots \otimes P_{j-1} \otimes P_i \otimes P_{j+1} \otimes \cdots \otimes P_n$, where in the second product the positions of P_i and P_j are interchanged. This suggests that the Hilbert space $\mathcal{H}^{\otimes^n}$ is too large and therefore admits too many projections and does not qualify for the description of events concerning n identical particles. In order to take into account this curtailment in the degrees of freedom, the desired reduction of the Hilbert space may be achieved by restriction to a suitable subspace of $\mathcal{H}^{\otimes^n}$ which is "invariant under permutations". We shall make this statement more precise in the sequel.

Let S_n denote the group of all permutations of the set $\{1, 2, \ldots, n\}$. Thus any $\sigma \in S_n$ is a one-to-one map of $\{1, 2, \ldots, n\}$ onto itself. For each $\sigma \in S_n$ let U_σ be defined on the product vectors in $\mathcal{H}^{\otimes^n}$ by

$$U_\sigma u_1 \otimes \cdots \otimes u_n = u_{\sigma^{-1}(1)} \otimes \cdots \otimes u_{\sigma^{-1}(n)} \tag{17.1}$$

where σ^{-1} is the inverse of σ. Then U_σ is a scalar product preserving the map of the total set of product vectors in $\mathcal{H}^{\otimes^n}$ onto itself. Hence by Proposition 7.2, U_σ extends uniquely to a unitary operator on $\mathcal{H}^{\otimes^n}$, which we shall denote by U_σ itself. Clearly

$$U_\sigma U_{\sigma'} = U_{\sigma\sigma'} \text{ for all } \sigma, \sigma' \in S_n. \tag{17.2}$$

Thus $\sigma \to U_\sigma$ is a homomorphism from the finite group S_n into $\mathcal{U}(\mathcal{H}^{\otimes^n})$. The closed subspaces

$$\mathcal{H}^{\textcircled{s}^n} = \{u \in \mathcal{H}^{\otimes^n} | U_\sigma u = u \text{ for all } \sigma \in S_n\}, \tag{17.3}$$

$$\mathcal{H}^{\textcircled{a}^n} = \{u \in \mathcal{H}^{\otimes^n} | U_\sigma u = \varepsilon(\sigma) u \text{ for all } \sigma \in S_n\}, \tag{17.4}$$

where $\varepsilon(\sigma) = \pm 1$ according to whether the permutation σ is even or odd are called respectively the n-fold *symmetric* and *antisymmetric tensor products* of $\mathcal{H}$. They are left invariant by the unitary operators $U_\sigma, \sigma \in S_n$. There do exist other such permutation invariant subspaces of $\mathcal{H}^{\otimes^n}$ but it seems that they do not feature frequently in a significant form in the physical description of n identical particles. If the statistical features of the dynamics of a single particle are described by states on $\mathcal{P}(\mathcal{H})$ and the dynamics of n such identical particles is described by states in $\mathcal{P}(\mathcal{H}^{\textcircled{s}^n})$ for $n = 2, 3, \ldots$ then such a particle is called *boson*. Instead, if it is described by states on $\mathcal{P}(\mathcal{H}^{\textcircled{a}^n})$ for every n then such a particle is called *fermion*. (This nomenclature is in honour of the physicists S.N. Bose and E. Fermi who pioneered the investigation of statistics of such particles).

We shall now present some of the basic properties of $\mathcal{H}^{\textcircled{s}^n}$ and $\mathcal{H}^{\textcircled{a}^n}$ in the next few propositions.

Proposition 17.1: Let E and F be operators in $\mathcal{H}^{\otimes^n}$ defined by

$$E = \frac{1}{n!} \sum_{\sigma \in S_n} U_\sigma, \tag{17.5}$$

$$F = \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma) U_\sigma, \tag{17.6}$$

where U_σ is the unitary operator satisfying (17.1) and $\varepsilon(\sigma)$ is the signature of the permutation σ. Then E and F are projections onto the subspaces $\mathcal{H}^{\textcircled{s}^n}$ and $\mathcal{H}^{\textcircled{a}^n}$ respectively. If $K \subset S_n$ is any subgroup

$$E = E\frac{1}{\#K}\sum_{\sigma\in K} U_\sigma, \tag{17.7}$$

$$F = F\frac{1}{\#K}\sum_{\sigma\in K} \varepsilon(\sigma)U_\sigma \tag{17.8}$$

where $\#K$ is the cardinality of K. Furthermore, for any $u_i, v_i \in \mathcal{H}$, $1 \le i \le n$

$$\begin{aligned}\langle Eu_1 \otimes \cdots \otimes u_n, Ev_1 \otimes \cdots \otimes v_n\rangle &= \langle u_1 \otimes \cdots \otimes u_n, Ev_1 \otimes \cdots \otimes v_n\rangle \\ &= \frac{1}{n!}\sum_{\sigma\in S_n} \Pi_{i=1}^n \langle u_i, v_{\sigma(i)}\rangle, \qquad (17.9)\\ \langle Fu_1 \otimes \cdots \otimes u_n, Fv_1 \otimes \cdots \otimes v_n\rangle &= \langle u_1 \otimes \cdots \otimes u_n, Fv_1 \otimes \cdots \otimes v_n\rangle \\ &= \frac{1}{n!}\sum_{\sigma\in S_n} \varepsilon(\sigma)\Pi_{i=1}^n \langle u_i, v_{\sigma(i)}\rangle \\ &= \frac{1}{n!}\det((\langle u_i, v_j\rangle)). \qquad (17.10)\end{aligned}$$

Proof: From (17.2) we have

$$\begin{aligned} E^* &= \frac{1}{n!}\sum_{\sigma\in S_n} U_\sigma^* = \frac{1}{n!}\sum_{\sigma\in S_n} U_{\sigma^{-1}} = E, \\ EU_\sigma &= U_\sigma E = E^2, \qquad (17.11)\\ F^* &= \frac{1}{n!}\sum_{\sigma\in S_n} \varepsilon(\sigma)U_\sigma^* = \frac{1}{n!}\sum_{\sigma\in S_n} \varepsilon(\sigma^{-1})U_{\sigma^{-1}} = F, \\ FU_\sigma &= U_\sigma F = \varepsilon(\sigma)F, F^2 = F. \end{aligned}$$

Thus E and F are projections. Furthermore (17.3), (17.4) and (17.11) imply that $Eu = u$ if and only if $u \in \mathcal{H}^{\textcircled{s}^n}$ and $Fv = v$ if and only if $v \in \mathcal{H}^{\textcircled{a}^n}$. This proves the first part. Summing up over $\sigma \in K$ and dividing by $\#K$ in the second equation of (17.11) we obtain (17.7). Multiplying by $\varepsilon(\sigma)$, summing over $\sigma \in K$ and dividing by $\#K$ in the last equation of (17.11) we obtain (17.8).

The first part of (17.9) is a consequence of the fact that E is a projection. By definition

$$\begin{aligned}\langle u_1 \otimes \cdots \otimes u_n, Ev_1 \otimes \cdots \otimes v_n\rangle &= \frac{1}{n!}\sum_{\sigma\in S_n} \langle u_1 \otimes \cdots \otimes u_n, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}\rangle \\ &= \frac{1}{n!}\sum_{\sigma\in S_n} \Pi_i \langle u_i, v_{\sigma(i)}\rangle.\end{aligned}$$

This proves (17.9). (17.10) follows exactly along the same lines. ∎

It may be mentioned here that the quantity $\sum_{\sigma \in S_n} \Pi_{i=1}^n \langle u_i, v_{\sigma(i)} \rangle$ occurring on the right hand side of (17.9) is the *permanent* of the matrix $((\langle u_i, v_j \rangle))$ in contrast to the determinant occurring in (17.10).

Corollary 17.2: Let E_n and F_n denote respectively the projections E and F defined on $\mathcal{H}^{\otimes^n}$ by (17.5) and (17.6) for each $n = 1, 2, \ldots$. Then for $u \in \mathcal{H}^{\otimes^\ell}$, $v \in \mathcal{H}^{\otimes^m}$, $w \in \mathcal{H}^{\otimes^n}$

$$\begin{aligned} E_{\ell+m+n}\{E_{\ell+m}(u \otimes v) \otimes w\} &= E_{\ell+m+n}\{u \otimes E_{m+n}(v \otimes w)\} \\ &= E_{\ell+m+n}(u \otimes v \otimes w), \\ F_{\ell+m+n}\{F_{\ell+m}(u \otimes v) \otimes w\} &= F_{\ell+m+n}\{u \otimes F_{m+n}(v \otimes w)\} \\ &= F_{\ell+m+n}(u \otimes v \otimes w). \end{aligned}$$

Proof: This is immediate from (17.7) and (17.8) if we consider the permutation groups $S_{\ell+m}$ and S_{m+n} as subgroups of $S_{\ell+m+n}$ so that $\sigma \in S_{\ell+m}$ leaves the last n elements of $\{1, 2, \ldots, \ell+m+n\}$ fixed whereas $\sigma \in S_{m+n}$ leaves the first ℓ elements of $\{1, 2, \ldots, \ell+m+n\}$ fixed. ■

Proposition 17.3: Let $\{e_i, i = 1, 2, \ldots\}$ be an orthonormal basis for $\mathcal{H}$. For any $u_i \in \mathcal{H}$, $i = 1, 2, \ldots, k$ and positive integers $r_1, r_2, \ldots, r_k$ satisfying $r_1 + \cdots + r_k = n$ denote by $\bigotimes_{i=1}^k u_i^{\otimes^{r_i}}$ the element $u_1 \otimes \cdots \otimes u_1 \otimes u_2 \otimes \cdots \otimes u_2 \otimes \cdots \otimes u_k \otimes \cdots \otimes u_k$ where u_i is repeated r_i times for each i. Let E and F be the projections defined respectively by (17.5) and (17.6) in $\mathcal{H}^{\otimes^n}$. Then the sets

$$\begin{aligned} &\{(\frac{n!}{r_1! \cdots r_k!})^{1/2}\, E \bigotimes_{j=1}^k e_{i_j}^{\otimes^{r_j}} | i_1 < i_2 < \cdots < i_k, \\ &\quad r_j \geq 1 \text{ for each } 1 \leq j \leq k, r_1 + r_2 + \cdots + r_k = n, k = 1, 2, \ldots, n\}, \\ &\quad \{(n!)^{1/2} F\, e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} | i_1 < i_2 < \cdots < i_n\} \end{aligned}$$

are orthonormal bases in $\mathcal{H}^{\circledS^n}$ and $\mathcal{H}^{\circledA^n}$ respectively. In particular, if $\dim \mathcal{H} = N < \infty$ then

$$\dim \mathcal{H}^{\circledS^n} = \binom{N+n-1}{n},$$

$$\dim \mathcal{H}^{\circledA^n} = \begin{cases} \binom{N}{n} & \text{if } n \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: The first part follows from (17.9) and (17.10). The formula for $\dim \mathcal{H}^{\circledS^n}$ is immediate if we identify it as the number of ways in which n indistinguishable balls can be thrown in N cells. Similarly the dimension of $\mathcal{H}^{\circledA^n}$ can be identified with the number of ways in which n indistinguishable balls can be thrown in N cells so that no cell has more than one ball. ■

Using Proposition 17.3 it is possible to compare the different probability distributions that arise in the "statistics of occupancy". More precisely, let ρ be a state in $\mathcal{H}$ with $\dim \mathcal{H} = N < \infty$. Let $\{e_j, j = 1, 2, \ldots, N\}$ be an orthonormal basis of $\mathcal{H}$ such that $\rho = \sum_j p_j |e_j\rangle\langle e_j|$. Let $|e_j\rangle\langle e_j|$ signify the event "particle occupies cell number j". Now consider n distinguishable particles whose statistics are described by the quantum probability space $(\mathcal{H}^{\otimes^n}, \mathcal{P}(\mathcal{H}^{\otimes^n}), \rho^{\otimes^n})$. The projection $|e_{j_1} \otimes \cdots \otimes e_{j_n}\rangle\langle e_{j_1} \otimes \cdots \otimes e_{j_n}|$ signifies the event "particle i occupies cell j_i for each $i = 1, 2, \ldots, n$". Let $\underline{r} = (r_1, \ldots, r_N)$ where r_j is the number of particles in cell j so that $r_1 + \cdots + r_N = n$. Define

$$E^0_{\underline{r}} = \sum |e_{j_1} \otimes \cdots \otimes e_{j_n}\rangle\langle e_{j_1} \otimes \cdots \otimes e_{j_n}| \tag{17.12}$$

where the summation on the right hand side is over all $(j_1, \ldots, j_n)$ such that the cardinality of $\{i | j_i = j\}$ is r_j for $j = 1, 2, \ldots, N$. $E^0_{\underline{r}}$ is a projection whose range has dimension $\frac{n!}{r_1! \cdots r_N!}$ and it signifies the event that cell j has r_j particles for each j. Then

$$\operatorname{tr} \rho^{\otimes^n} E^0_{\underline{r}} = \frac{n!}{r_1! \cdots r_N!} p_1^{r_2} p_1^{r_2} \cdots p_N^{r_N}. \tag{17.13}$$

In this case we say that the particles obey the *Maxwell-Boltzmann statistics* and the probability that there are r_j particles in cell j for each j in the state $\rho^{\otimes^n}$ is given by (17.13).

Suppose that the n particles under consideration are n identical bosons. Then the Hilbert space $\mathcal{H}^{\otimes^n}$ is replaced by $\mathcal{H}^{\circledS^n}$ and correspondingly $\rho^{\otimes^n}$ by its restricton to $\mathcal{H}^{\circledS^n}$. To make this restriction a state we put

$$\rho^{\circledS^n} = \rho^{\otimes^n}|_{\mathcal{H}^{\circledS^n}}.$$

$$c = \operatorname{tr} \rho^{\circledS^n} = \operatorname{tr} \rho^{\otimes^n} E = \sum_{s_1 + \cdots + s_N = n} p_1^{s_1} p_2^{s_2} \cdots p_N^{s_N}$$

and observe that $c^{-1}\rho^{\circledS^n}$ is a state, E being defined by (17.5). The quantum probability space $(\mathcal{H}^{\circledS^n}, \mathcal{P}(\mathcal{H}^{\circledS^n}), c^{-1}\rho^{\circledS^n})$ describes the statistics of n identical bosons. Denote by $E^{(b)}_{\underline{r}}$ the projection on the one dimensional subspace generated by the vector

$$e_{\underline{r}} = E \bigotimes_{j=1}^{N} e^{\otimes^{r_j}}, \ \underline{r} = (r_1, \ldots, r_N), \ r_1 + \cdots + r_N = n.$$

Then the probability of finding r_j particles in cell j for each $j = 1, 2, \ldots, N$ is

$$\operatorname{tr} c^{-1} \rho^{\circledS^n} E^{(b)}_{\underline{r}} = \frac{p_1^{r_1} \cdots p_N^{r_N}}{\sum_{s_1 + \cdots s_N = n} p_1^{s_1} \cdots p_N^{s_N}} \tag{17.14}$$

In this case we say that the particles obey the *Bose-Einstein statistics*.

When the particles are n identical fermions $\mathcal{H}^{\otimes^n}$ is replaced by $\mathcal{H}^{\textcircled{a}^n}$ which is non-trivial if and only if $n \leq N$, i.e., the number of particles does not exceed the number of cells. The state $\rho^{\otimes^n}$ is replaced by its restriction to $\mathcal{H}^{@^n}$. Once again to make this restriction a state we have to divide it by

$$c' = \operatorname{tr} \rho^{\otimes^n}|_{\mathcal{H}^{@^n}} = \sum_{1 \le i_1 < i_2 < \cdots < i_n \le N} p_{i_1} p_{i_2} \cdots p_{i_n}.$$

Then the quantum probability space describing the statistics of n identical fermions is $(\mathcal{H}^{@^n}, \mathcal{P}(\mathcal{H}^{@^n}), c'^{-1}\rho^{@^n})$ where

$$\rho^{@^n} = \rho^{\otimes^n}|_{\mathcal{H}^{@^n}}$$

If $E_{\underline{r}}^{(f)}$ denotes the one dimensional projection on the subspace generated by the vector

$$f_{\underline{r}} = F e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}$$

where $\underline{r} = (r_1, \ldots, r_N)$, $r_{i_j} = 1$, $j = 1, 2, \ldots, n$ and the remaining r_i are 0 and F is defined by (17.6) then the probability that cell i_j is occupied by one particle for each $j = 1, 2, \ldots, n$ and the remaining cells are unoccupied is given by

$$\operatorname{tr} c'^{-1} \rho^{@^n} E_{\underline{r}}^{(f)} = \frac{p_{i_1} p_{i_2} \cdots p_{i_n}}{\sum_{1 \le j_1 < j_2 < \cdots < j_n \le N} p_{j_1} p_{j_2} \cdots p_{j_n}}. \tag{17.15}$$

In this case we say that the particles obey the *Fermi-Dirac statistics*.

For a comparison of the three distributions (17.13)–(17.15) consider the case of two cells and n particles. Let the state ρ of a single particle be given by

$$\rho e_1 = \frac{1}{2} e_1, \quad \rho e_2 = \frac{1}{2} e_2$$

where $\{e_1, e_2\}$ is an orthonormal basis in $\mathcal{H}$. According to Maxwell-Boltzmann statistics the number of particles in cell 1 has a binomial distribution given by

$$\Pr(\text{cell 1 has } k \text{ particles}) = \binom{n}{k} 2^{-n}, \quad 0 \le k \le n$$

whereas Bose-Einstein statistics yield

$$\Pr(\text{cell 1 has } k \text{ particles}) = \frac{1}{n+1}, \quad 0 \le k \le n.$$

According to the first distribution the probability that all the particles occupy a particular cell is 2^{-n} whereas the second distribution assigns the enhanced probability $\frac{1}{n+1}$ to the same event.

In the case of fermions it is impossible to have more than two of them when there are only two cells available and if a cell is occupied by one particle the second one has to be occupied by the other. Bosons tend to crowd more than particles obeying Maxwell-Boltzmann statistics and fermions tend to avoid each other.

Exercise 17.4: Let π be an irreducible unitary representation of S_n and let $\chi(\sigma) = \operatorname{tr} \pi(\sigma)$ be its character. Suppose $\chi(1) = d_\chi$ denotes the dimension of the representation π. Define

$$E_\chi = \frac{d_\chi}{n!} \sum_{\sigma \in S_n} \chi(\sigma) U_\sigma$$

where U_σ is the unitary operator satisfying (17.1). Then E_χ is a projection for each χ, $E_\chi U_\sigma = U_\sigma E_\chi$ for all $\sigma \in S_n, E_{\chi_1} E_{\chi_2} = 0$ if $\chi_1 \neq \chi_2$ and $\sum_\chi E_\chi = 1$. (Hint: Use Schur orthogonality relations).

Example 17.5: [74] (i) The volume of the region

$$\Delta = \{(x_1, x_2, \dots, x_{N-1}) | x_j \geq 0 \text{ for all } j, 0 \leq x_1 + \cdots + x_{N-1} \leq 1\}$$

in $\mathbb{R}^{N-1}$ is $[(N-1)!]^{-1}$.

(ii) Let $r_1, \dots, r_N$ be non-negative integers such that $r_1 + \cdots + r_N = n$. Then

$$\int_\Delta \frac{n!}{r_1! \cdots r_N!} p_1^{r_1} \cdots p_N^{r_N} (N-1)! dp_1 dp_2 \cdots dp_{N-1} = \binom{N+n-1}{n}^{-1}$$

where $p_N = (1 - p_1 - p_2 - \cdots - p_{N-1})$.

This identity has the following interpretation. Suppose all the probability distributions $(p_1, p_2, \dots, p_N)$ for the occupancy of the cells $(1, 2, \dots, N)$ by a particle are equally likely and for any chosen prior distribution $(p_1, p_2, \dots, p_N)$ the particle obeys Maxwell-Boltzmann statistics. Then one obtains the Bose-Einstein distribution (17.4) with $p_j = \frac{1}{N}$, $j = 1, 2, \dots, N$.

Notes

Regarding the role of symmetric and antisymmetric tensor products of Hilbert spaces in the statistics of indistinguishable particles, see Dirac [29]. For an interesting historical account of indistinguishable particles and Bose-Einstein statistics, see Bach [11]. Example 17.5 linking Bose-Einstein and Maxwell-Boltzmann statistics in the context of Bayesian inference is from Kunte [74].

18 Examples of discrete time quantum stochastic flows

Using the notion of a countable tensor product of a sequence of Hilbert spaces with respect to a stabilising sequence of unit vectors and properties of conditional expectation (see Exercise 16.10, 16.11) we shall now outline an elementary procedure of constructing a "quantum stochastic flow" in discrete time which is an analogue of a classical Markov chain induced by a transition probability matrix.

For a Hilbert space $\mathcal{H}$ any subalgebra $\mathcal{B} \subset \mathcal{B}(\mathcal{H})$ which is closed under the involution * and weak topology is called a W^* *algebra* or a *von Neumann algebra*. If $\mathcal{H}_i$, $i = 1, 2$ are Hilbert spaces, $\mathcal{B}_i \subset \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$ are von Neumann algebras denote by $\mathcal{B}_1 \otimes \mathcal{B}_2$ the smallest von Neumann algebra containing $\{X_1 \otimes X_2 | X_i \in$

$\mathcal{B}_i, i = 1,2\}$ in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. If ρ is a trace class operator in $\mathcal{H}_2$, following Exercise 16.10, define the operator $\mathbb{E}_\rho(Z), Z \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by the relation

$$\langle u, \mathbb{E}_\rho(Z)v\rangle = \operatorname{tr} Z\,|v\rangle\langle u| \otimes \rho, \quad u, v \in \mathcal{H}_1 \tag{18.1}$$

in $\mathcal{H}_1$.

Proposition 18.1: $\mathbb{E}_\rho(Z) \in \mathcal{B}_1$ if $Z \in \mathcal{B}_1 \otimes \mathcal{B}_2$.

Proof: If $Z = X_1 \otimes X_2$ then (18.1) implies that $\mathbb{E}_\rho(Z) = (\operatorname{tr} \rho X_2)X_1$. Thus the proposition holds for any finite linear combination of product operators in $\mathcal{B}_1 \otimes \mathcal{B}_2$. Suppose that $\rho = \Sigma p_j |e_j\rangle\langle e_j|$ is a state where $p_j > 0, \Sigma p_j = 1$ and $\{e_j\}$ is an orthonormal set and $\operatorname{w.lim}_{n\to\infty} Z_n = Z$ in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Then by (18.1)

$$\begin{aligned}\lim_{n\to\infty} \langle u, \mathbb{E}_\rho(Z_n)v\rangle &= \lim_{n\to\infty} \sum_j p_j \langle u \otimes e_j, Z_n v \otimes e_j\rangle \\ &= \Sigma p_j \langle u \otimes e_j, Zv \otimes e_j\rangle \\ &= \langle u, \mathbb{E}_\rho(Z)v\rangle \text{ for all } u, v \in \mathcal{H}_1.\end{aligned}$$

In other words $\mathbb{E}_\rho$ is weakly continuous if ρ is a state. Since $\mathbb{E}_\rho$ is linear in ρ the same property follows for any trace class operator. Now the required result is immediate from the definition of $\mathcal{B}_1 \otimes \mathcal{B}_2$. ■

Let $\mathcal{H}_0, \mathcal{H}$ be Hilbert spaces where $\dim \mathcal{H} = d < \infty$. Let $\{e_0, e_1, \ldots, e_{d-1}\}$ be a fixed orthonormal basis in $\mathcal{H}$ and let $\mathcal{B}_0 \subset \mathcal{B}(\mathcal{H}_0)$ be a von Neumann algebra with identity. Putting $\mathcal{H}_n = \mathcal{H}$, $\phi_n = e_0$ for all $n \geq 1$ in Exercise 16.11 construct the Hilbert spaces $\mathcal{H}_{n]}, \mathcal{H}_{[n+1}$ for each $n \geq 0$. Define the von Neumann algebras

$$\begin{aligned}\mathcal{B}_{n]} &= \{X \otimes 1_{[n+1} | X \in \mathcal{B}_0 \otimes \mathcal{B}(\mathcal{H}^{\otimes^n})\}, \quad n \geq 0, \\ \mathcal{B} &= \mathcal{B}_0 \otimes \mathcal{B}(\mathcal{H}_{[1}).\end{aligned}$$

Property (v) in Exercise 16.11 implies that $\mathcal{B}$ is the smallest von Neumann algebra containing all the $\mathcal{B}_n, n \geq 0$. $\{\mathcal{B}_{n]}\}$ is increasing in n. By Proposition 18.1 the $\phi_{[n+1}$-conditional expectation $\mathbb{E}_{n]}$ of Exercise 16.11 maps $\mathcal{B}$ onto $\mathcal{B}_{n]}$.

Any algebra with identity and an involution * is called a **-unital* algebra. If $\mathcal{B}_1, \mathcal{B}_2$ are *-unital algebras and $\theta : \mathcal{B}_1 \to \mathcal{B}_2$ is a mapping preserving * and identity then θ is called a **-unital* map.

Proposition 18.2: Let $\theta : \mathcal{B}_0 \to \mathcal{B}_0 \otimes \mathcal{B}(\mathcal{H})$ be a *-unital homomorphism. Define the linear maps $\theta^i_j : \mathcal{B}_0 \to \mathcal{B}_0$ by

$$\theta^i_j(X) = \mathbb{E}_{|e_j\rangle\langle e_i|}(\theta(X)), \quad 0 \leq i,\ j \leq d-1, \quad X \in \mathcal{B}_0 \tag{18.2}$$

Then the following holds:

(i) $\theta^i_j(1) = \delta^i_j$, $\theta^i_j(X^*) = \theta^j_i(X)^*$;

(ii) $\theta^i_j(XY) = \sum_{k=0}^{d-1} \theta^i_k(X)\theta^k_j(Y)$ for all $X, Y \in \mathcal{B}_0$.

Proof: From (18.1) and (18.2) we have

$$\langle u, \theta^i_j(X)v\rangle = \langle u \otimes e_i, \theta(X)v \otimes e_j\rangle.$$

If $X = 1$ the right hand side of this equation is $\langle u, v\rangle \delta^i_j$. Furthermore

$$\begin{aligned}\langle u, \theta^i_j(X^*)v\rangle &= \langle u \otimes e_i, \theta(X^*)v \otimes e_j\rangle = \overline{\langle v \otimes e_j, \theta(X)u \otimes e_i\rangle}\\ &= \overline{\langle v, \theta^j_i(X)u\rangle} = \langle u, \theta^j_i(X)^* v\rangle.\end{aligned}$$

This proves (i). To prove (ii) choose an orthonormal basis $\{u_n\}$ in $\mathscr{H}_o$ and observe that

$$\begin{aligned}\langle u, \theta^i_j(XY)v\rangle &= \langle u \otimes e_i, \theta(X)\theta(Y)v \otimes e_j\rangle\\ &= \langle \theta(X^*)u \otimes e_i, \theta(Y)v \otimes e_j\rangle\\ &= \sum_{r,k} \langle u \otimes e_i, \theta(X)u_r \otimes e_k\rangle\langle u_r \otimes e_k, \theta(Y)v \otimes e_j\rangle\\ &= \sum_{k,r} \langle u, \theta^i_k(X)u_r\rangle\langle u_r, \theta^k_j(Y)v\rangle\\ &= \sum_k \langle \theta^i_k(X)^* u, \theta^k_j(Y)v\rangle\\ &= \langle u, \sum_k \theta^i_k(X)\theta^k_j(Y)v\rangle.\end{aligned}$$

■

Proposition 18.3: Let $\theta, \mathscr{H}_0, \mathscr{H}_1, \mathscr{B}_0$ be as in Proposition 18.2. Define the maps $j_n : \mathscr{B}_0 \to \mathscr{B}_{n]}$, $n = 0, 1, 2, \ldots$ inductively by

$$\begin{aligned}&j_0(X) = X \otimes 1_{[1}, j_1(X) = \theta(X) \otimes 1_{[2},\\ &j_n(X) = \sum_{0 \le i,j \le d-1} j_{n-1}(\theta^i_k(X))1_{n-1]} \otimes |e_i\rangle\langle e_k| \otimes 1_{[n+1}.\end{aligned} \tag{18.3}$$

Then j_n is a *-unital homomorphism for every n. Furthermore

$$\mathbb{E}_{n-1]} j_n(X) = j_{n-1}(\theta^0_0(X)) \text{ for all } n \ge 1,\ X \in \mathscr{B}_0,$$

where $\mathbb{E}_{n-1]}$ is the $\phi_{[n}$-conditional expectation map of Exercise 16.11.

Proof: We prove by induction. For $n = 0, 1$ it is immediate. Let $n \ge 2$. Then by (i) in Proposition 18.2 and the induction hypothesis we have

$$\begin{aligned}j_n(1) &= \sum_{i,j} j_{n-1}(\delta^i_j)1_{n-1]} \otimes |e_i\rangle\langle e_j| \otimes 1_{[n+1}\\ &= 1_{n-1]} \otimes \sum_i |e_i\rangle\langle e_i| \otimes 1_{[n+1} = 1\end{aligned}$$

and

$$\begin{aligned} j_n(X)^* &= \sum_{i,j} j_{n-1}(\theta_i^j(X^*))1_{n-1]} \otimes |e_j\rangle\langle e_i| \otimes 1_{[n+1} \\ &= j_n(X^*). \end{aligned}$$

By (ii) in Proposition 18.2 and induction hypothesis we have

$$\begin{aligned} j_n(X)j_n(Y) &= \sum_{i,j,k,\ell} j_{n-1}(\theta_j^i(X))j_{n-1}(\theta_\ell^k(Y))1_{n-1]} \otimes \delta_j^k |e_i\rangle\langle e_\ell| \otimes 1_{[n+1} \\ &= \sum_{i,\ell} j_{n-1} \sum_k \theta_k^i(X)\theta_\ell^k(Y))1_{n-1]} \otimes |e_i\rangle\langle e_\ell| \otimes 1_{[n+1} \\ &= \sum_{i,\ell} j_{n-1}(\theta_\ell^i(XY))1_{n-1]} \otimes |e_i\rangle\langle e_\ell| \otimes 1_{[n+1}. \end{aligned}$$

This proves the first part. By the definition of $\mathbb{E}_{n]}$ in Exercise 16.11 and the fact that $\langle e_0, |e_i\rangle\langle e_j|e_0\rangle = \delta_0^i\delta_j^0$, the second part follows from (18.3). ■

Corollary 18.4: Let $\{j_n, n \geq 0\}$ be the *-unital homomorphisms of Proposition 18.3. Write $T = \theta_0^0$. Then for $0 \leq n_0 < n_1 < \cdots < n_k < \infty$, $X_i \in \mathcal{B}_0$, $1 \leq i \leq k$

$$\begin{aligned} &\mathbb{E}_{n_0]} j_{n_1}(X_1) j_{n_2}(X_2) \cdots j_{n_k}(X_k) = \\ &\quad j_{n_0}(T^{n_1-n_0}(X_1 T^{n_2-n_1}(X_2 \cdots (X_{k-1}T^{n_k-n_{k-1}}(X)) \cdots) \end{aligned} \tag{18.4}$$

Proof: By Exercise 16.11 $\mathbb{E}_{n_0]} = \mathbb{E}_{n_0]}\mathbb{E}_{n_{k-1}]}$. Since $j_{n_1}(X_1)\cdots j_{n_{k-1}}(X_{k-1})$ is an element of $\mathcal{B}_{n_{k-1}]}$ it follows from the same exercise that

$$\begin{aligned} &\mathbb{E}_{n_0]} j_{n_1}(X_1) \cdots j_{n_k}(X_k) \\ &\quad = \mathbb{E}_{n_0]} j_{n_1}(X_1) \cdots j_{n_{k-1}}(X_{k-1}) \mathbb{E}_{n_{k-1}]}(j_{n_k}(X_k)). \end{aligned} \tag{18.5}$$

Since $\mathbb{E}_{n_{k-1}]} = \mathbb{E}_{n_{k-1}]}\mathbb{E}_{n_{k-1}+1]} \cdots \mathbb{E}_{n_k-1]}$ it follows from Proposition 18.3 that

$$\mathbb{E}_{n_{k-1}]} j_{n_k}(X_k) = j_{n_{k-1}}(T^{n_k-n_{k-1}}(X)).$$

Substituting this in (18.5), using the fact that $j_{n_{k-1}}$ is a homomorphism and repeating this argument successively we arrive at (18.4). ■

Proposition 18.5: The map $T = \theta_0^0$ from $\mathcal{B}_0$ into itself satisfies the following: (i) T is a *-unital linear map on $\mathcal{B}(\mathcal{H}_0)$; (ii) for any $X_i, Y_i \in \mathcal{B}_0$, $1 \leq i \leq k$, $\sum_{1 \leq i,j \leq k} Y_i^* T(X_i^* X_j) Y_j \geq 0$ for every k. In particular, $T(X) \geq 0$ whenever $X \geq 0$.

Proof: Since θ is a *-unital homomorphism from $\mathscr{B}_0$ into $\mathscr{B}_0 \otimes \mathscr{B}(\mathscr{H})$ and $T(X) = \mathbb{E}_{|e_0\rangle\langle e_0|}(\theta(X))$, (i) is immediate from Exercise 16.10. Using the same exercise once again we have

$$\begin{aligned}\sum_{i,j} Y_i^* T(X_i^* X_j))Y_j &= \sum_{i,j} Y_i^* \mathbb{E}_{|e_0\rangle\langle e_0|}(\theta(X_i)^*\theta(X_j))Y_j \\ &= \mathbb{E}_{|e_0\rangle\langle e_0|}(\{\sum_i \theta(X_i)Y_i \otimes 1\}^*\{\sum_j \theta(X_j)Y_j \otimes 1\}) \geq 0.\end{aligned}$$

Putting $k = 1$, $Y_1 = 1$, $X_1 = X$ in this relation we get the last part. ■

We may now compare the situation in Proposition 18.3, Corollary 18.4 and Proposition 18.5 with the one that is obtained in the theory of classical Markov chains. Consider a Markov chain with state space $S = \{1, 2, \ldots, N\}$ and transition probability matrix $P = ((p_{ij}))$, $1 \leq i, j \leq N$. Denote by B_∞ the *-unital commutative algebra of all bounded complex valued measurable functions on the space $S^\infty = S_0 \times S_1 \times \cdots \times S_n \times \cdots$ where $S_n = S$ for every n. Let $B_{n]} \subset B_\infty$ be the *-subalgebra of all functions which depend only on the first $n+1$ coordinates. Denote by $\mathbb{E}_{n]}$ the conditional expectation map determined by

$$(\mathbb{E}_{n]}g)(i_0, i_1, \ldots, i_n) = \mathbb{E}(g|X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n)$$

where $X_0, X_1, \ldots$ is the Markov chain starting in the state $X_0 = i_0$ with stationary transition probability matrix P. For any function f on S define

$$j_n(f)(\underline{i}) = f(i_n) \text{ where } \underline{i} = (i_0, i_1, \ldots, i_n, \ldots) \in S^\infty.$$

Then j_n is a *-unital homomorphism from B_0 into $B_{n]}$ and the Markov property implies that

$$\begin{aligned}&\mathbb{E}_{n_0]} j_{n_1}(f_1) j_{n_2}(f_2) \cdots j_{n_k}(f_k) = \\ &\quad j_{n_0}(T^{n_1-n_0}(f_1(T^{n_2-n_1} f_2(\cdots(f_{k-1}T^{n_k-n_{k-1}}(f_k))\cdots)\end{aligned} \tag{18.6}$$

where

$$(Tf)(i) = \sum_{j=1}^{N} p_{ij} f(j), \quad f \in B_0 = B_{0]},$$

$n_0 < n_1 < \cdots < n_k$ and $f_1, \ldots, f_k \in B_0$. T is a *-unital positive linear map on B_0. Then (18.4) is the non-commutative or quantum probabilistic analogue of the classical Markov property (18.6) expressed in the language of *-unital commutative algebras. For this reason we call the family $\{j_n, n \geq 0\}$ of homomorphisms in Proposition 18.3 a *quantum stochastic flow induced by* the *-unital homomorphism $\theta : \mathscr{B}_0 \to \mathscr{B}_0 \otimes \mathscr{B}(\mathscr{H})$.

Proposition 18.6: Suppose that the von Neumann algebra $\mathscr{B}_0$ in Proposition 18.3 is abelian. Then for any $X, Y \in \mathscr{B}_0$, $m, n \geq 0$

$$[j_m(X), j_n(Y)] = 0. \tag{18.7}$$

Proof: Since j_n is a homomorphism we have $[j_n(X), j_n(Y)] = j_n([X,Y]) = 0$. Thus (18.7) is trivial if $m = n$. Suppose $m < n$. By induction on (18.3) we have

$$j_n(X) = \sum_{\substack{0 \le i_1, i_2, \dots, \\ k_1, k_2, \dots, \le d-1}} j_m([\theta_{k_1}^{i_1} \cdots \theta_{k_{n-m}}^{i_{n-m}}(X))1_{m]} \otimes |e_{i_1}\rangle\langle e_{k_1}| \otimes \cdots \otimes |e_{i_{n-m}}\rangle\langle e_{k_{n-m}}| \otimes 1_{[n+1}$$

from which it follows that

$$[j_n(X), j_m(Y)] = \sum_{\substack{i_1, i_2 \dots \\ k_1, k_2, \dots}} j_m([\theta_{k_1}^{i_1} \cdots \theta_{k_{n-m}}^{i_{n-m}}(X), Y])1_{m]} \otimes |e_{i_1}\rangle\langle e_{k_1}| \otimes \cdots \otimes |e_{i_{n-m}}\rangle\langle e_{k_{n-m}}| \otimes 1_{[n+1}$$

$= 0$ for all $X, Y \in \mathscr{B}_0$.

If $\mathscr{B}_0$ is abelian and $X_1, X_2, \dots, X_k$ is any finite set of selfadjoint elements in $\mathscr{B}_0$ then $j_{n_1}(X_1), j_{n_2}(X_2), \dots, j_{n_k}(X_k)$ is a commuting family of observables and hence possesses a joint distribution in $\mathbb{R}^k$ in any state ρ on the countable tensor product $\mathscr{H}_0 \otimes \{\mathscr{H} \otimes \mathscr{H} \otimes \cdots\}$ where the Hilbert space within the braces $\{\ \}$ is with respect to the constant stabilising sequence of unit vectors e_0. For any state ρ_0 in $\mathscr{H}_0$, the family $\{j_n(X) | X \in \mathscr{B}_0, n \ge 0\}$ can be interpreted as a classical Markov flow in the state $\rho_0 \otimes |e_0 \otimes e_0 \otimes \cdots\rangle\langle e_0 \otimes e_0 \otimes \cdots|$.

Example 18.7: Let $(S, \mathscr{F}, \mu)$ be any measure space and let $\phi_j : S \to S$ be measurable maps satisfying $\mu\phi_j^{-1} \ll \mu$, $0 \le j \le d-1$. Suppose $p_j : S \to [0,1], 0 \le j \le d-1$ are measurable functions satisfying $\sum_j p_j(x) \equiv 1$. Let $\mathscr{B}_0 = L^\infty(\mu) \subset \mathscr{B}(L^2(\mu))$ when bounded measurable functions are considered as bounded multiplication operators. We write $\mathscr{H}_0 = L^2(\mu), \mathscr{H} = \mathbb{C}^d$ and choose $\{e_0, e_1, \dots, e_{d-1}\}$ to be the canonical orthonormal basis in $\mathscr{H}$. Define the map $T : \mathscr{B}_0 \to \mathscr{B}_0$ by

$$Tf = \sum_{j=0}^{d-1} p_j f o \phi_j \tag{18.8}$$

where o denotes composition. The map T can be interpreted as the transition operator of a Markov chain with state space S for which the state changes in one step from x to one of the states $\phi_0(x), \phi_1(x), \dots, \phi_{d-1}(x)$ with respective probabilities $p_0(x), p_1(x), \dots, p_{d-1}(x)$. However, it is possible that $\phi_i(x) = \phi_j(x)$ for some $i \neq j$. If S is a finite set of cardinality k it can be shown that every Markov transition operator is of the form (18.8) with μ being counting measure. In many practically interesting models of classical probability d is small.

Consider a unitary $d \times d$ matrix valued measurable function U on S for which $U = ((u_{ij})), 0 \le i,j \le d-1, u_{0j} = \sqrt{p_j}$ for all j. For example one may choose the orthogonal matrix

$$U = \left(\begin{array}{c|c} p_0^{1/2} & p_1^{1/2} \quad \cdots\cdots \quad p_{d-1}^{1/2} \\ \hline -p_1^{1/2} & \\ \vdots & 1 - Q \\ -p_{d-1}^{1/2} & \end{array} \right)$$

where $Q = ((q_{ij})), q_{ij} = (p_i p_j)^{1/2}(1 + p_0^{1/2})^{-1}, i, j \geq 1$. Define the *-unital homomorphism

$$\theta : f \to ((\theta_j^i(f))) = U \begin{pmatrix} fo\phi_0 & & & 0 \\ & fo\phi_1 & & \\ 0 & & \ddots & fo\phi_{d-1} \end{pmatrix} U^* \tag{18.9}$$

from $\mathcal{B}_0$ into $\mathcal{B}_0 \otimes \mathcal{B}(\mathcal{H})$ so that

$$\theta_j^i(f) = \sum_{r=0}^{d-1} u_{ir} \overline{u}_{jr} fo\phi_r,$$

$$\theta_0^0(f) = Tf.$$

where T is defined by (18.8). Note that in (18.9) the right hand side is to be interpreted as a matrix multiplication operator in the Hilbert space

$$L^2(\mu) \otimes \mathbb{C}^d = \underbrace{L^2(\mu) \oplus \cdots \oplus L^2(\mu)}_{d-\text{fold}}$$

and any element in the right hand side version of the Hilbert space is expressed as a column vector of elements in $L^2(\mu)$. By Proposition 18.3 and 18.6 there exists a quantum stochastic flow $\{j_n, n \geq 0\}$ of *-unital homomorphisms from $\mathcal{B}_0$ into $\mathcal{B}$ induced by the *-unital homomorphism θ of (18.9) satisfying

$$[j_m(f), j_n(g)] = 0,$$

$$\mathbb{E}_{n-1]} j_n(f) = j_{n-1}(Tf)$$

for all $f, g \in \mathcal{B}_0 = L^\infty(\mu)$. If S is a countable or finite set and μ is the counting measure then for any $x \in S$, in the quantum theoretical state $\delta_x \otimes e_0 \otimes e_0 \otimes \cdots$ in $L^2(\mu) \otimes \mathcal{H} \otimes \mathcal{H} \otimes \cdots$ the sequence of observables $j_0(f) = f, j_1(f), \ldots, j_n(f), \ldots$ has the same probability distribution as the sequence of random variables $f(\xi_0(x, \omega)), f(\xi_1(x, \omega)), f(\xi_n(x, \omega)), \ldots$ where $\{\xi_n(x, \omega)\}$ is a discrete time classical Markov chain with state space $S, \xi_0(x, \omega) = x$ and transition operator T, f is any element of $\mathcal{B}_0$. In other words $\{j_n\}$ can be identified with a classical Markovian stochastic flow with transition operator T.

It is interesting to note that in the quantized construction of the classical Markov chain we get the following bonus. By confining ourselves to $\{j_n, 0 \leq n \leq N\}$ for any finite time period in the Hilbert space $\mathcal{H}_0 \otimes \mathcal{H}^{\otimes^n}$ and defining the conditional expectations $\mathbb{E}^{\psi}_{n]}$ with respect to an arbitrary unit vector ψ, in $\mathcal{H}$ replacing e_0, we get a Markov flow with the property

$$\mathbb{E}^{\psi}_{n_0]} j_{n_1}(f_1)\cdots j_{n_k}(f_k) = j_{n_0}(T_{\psi}^{n_1-n_0}(f_1 T_{\psi}^{n_2-n_1}(f_2\cdots(f_{k-1}T_{\psi}^{n_k-n_{k-1}}(f_k))\cdots)$$

for all $0 \leq n_0 < n_1 < \cdots < n_k \leq N$ where T_ψ is the Markov transition operator

$$T_\psi f = \sum_{j=0}^{d-1} |\langle e_j, U(\cdot)\psi\rangle|^2 f o \phi_j.$$

T_ψ describes the chain where the state changes from x to one of the states $\phi_0(x), \phi_1(x), \ldots, \phi_{d-1}(x)$ with respective probabilities $|\langle e_j, U(x)\psi\rangle|^2, j = 0, 1, \ldots, d-1$. Thus a quantum probabilistic description enables us to describe a whole class of Markov chains with transition operators $T_\psi, \psi \in \mathcal{H}, \|\psi\| = 1$ within the framework of a single Hilbert space $\mathcal{H}_0 \otimes \mathcal{H}^{\otimes^N}$ if we confine ourselves to the finite time period $[0, N]$.

Example 18.8: We examine Example 18.7 when $d = 2$. Denote the maps ϕ_0 and ϕ_1 on S by ϕ and ψ respectively. Write $p_0 = p$, $p_1 = q$ so that $p + q = 1$. Then the *-unital homomorphism θ in (18.9) assumes the simple form

$$\theta(f) = ((\theta^i_j(f))) = \begin{pmatrix} pfo\phi + qfo\psi & \sqrt{pq}(fo\psi - fo\phi) \\ \sqrt{pq}(fo\psi - fo\phi) & qfo\phi + pfo\psi \end{pmatrix} \tag{18.10}$$

when

$$U = \begin{pmatrix} \sqrt{p} & \sqrt{q} \\ -\sqrt{q} & \sqrt{p} \end{pmatrix}.$$

Define

$$\begin{aligned} a_n &= 1_{n-1]} \otimes |e_0\rangle\langle e_1| \otimes 1_{[n+1}, \\ a_n^\dagger &= 1_{n-1]} \otimes |e_1\rangle\langle e_0| \otimes 1_{[n+1}, \quad n = 1, 2, \ldots, \end{aligned}$$

so that

$$\begin{aligned} a_n a_n^\dagger &= 1_{n-1]} \otimes |e_0\rangle\langle e_0| \otimes 1_{[n+1}, \\ a_n^\dagger a_n &= 1_{n-1]} \otimes |e_1\rangle\langle e_1| \otimes 1_{[n+1}, \end{aligned}$$

and $a_n a_n^\dagger + a_n^\dagger a_n = 1$ for all $n \geq 1$. Then the flow $\{j_n, n \geq 0\}$ induced by θ in (18.3) assumes the form

$$j_n(f) = j_{n-1}(Tf) + j_{n-1}(Lf)(a_n + a_n^\dagger) + j_{n-1}(Kf)a_n^\dagger a_n \tag{18.11}$$

for all $f \in L^\infty(\mu), n \geq 1$ where

$$\begin{aligned} &Tf = pfo\phi + qfo\psi, \\ &Lf = \sqrt{pq}(fo\psi - fo\phi),\ Kf = (p-q)\{fo\psi - fo\phi\} \end{aligned} \tag{18.12}$$

If $A_n = 0$ for $n = 0$ and $a_1 + \cdots + a_n$ for $n \geq 1$ and $\Lambda_n = 0$ for $n = 0$ and $a_1^\dagger a_1 + \cdots + a_n^\dagger a_n$ for $n \geq 1$ we may express (18.11) in the form

$$j_n(f) = j_{n-1}(Tf) + j_{n-1}(Lf)(A_n - A_{n-1} + A_n^\dagger - A_{n-1}^\dagger) + j_{n-1}(Kf)(\Lambda_n - \Lambda_{n-1}) \tag{18.13}$$

where A_n, $A_n^\dagger$ and Λ_n, $n \geq 0$ are three martingales with respect to the conditional expectations $\mathbb{E}_{n]}$ (see Exercise 16.12). In Chapter III we shall treat continuous time analogues of flows satisfying (18.13) where difference equations will become differential equations.

Example 18.9: (Hypergeometric model) Consider an urn with a white balls and b black balls. Draw a ball at random successively without replacement. The state of the Markov chain at any time is denoted (x, y) where x is the number of white balls and y is the number of black balls. Then

$$S = \{(x, y) | 0 \leq x \leq a, 0 \leq y \leq b\}$$

is the state space. Define maps ϕ, ψ on S by

$$\phi(x, y) = \begin{cases} (x-1, y) & \text{if } x > 0 \\ (0, y-1) & \text{if } x = 0,\ y > 0, \\ (0, 0) & \text{if } x = 0,\ y = 0, \end{cases}$$

$$\psi(x, y) = \begin{cases} (x, y-1) & \text{if } y > 0, \\ (x-1, 0) & \text{if } y = 0,\ x > 0, \\ (0, 0) & \text{if } x = y = 0. \end{cases}$$

Define

$$p(x, y) = \begin{cases} \frac{x}{x+y} & \text{if } x > 0, \\ \frac{1}{2} & \text{if } x = 0. \end{cases}$$

Then

$$(Tf)(x, y) = \begin{cases} \dfrac{x}{x+y} f(x-1, y) + \dfrac{y}{x+y} f(x, y-1) & \text{if } x > 0,\ y > 0, \\ f(x-1, 0) & \text{if } x > 0,\ y = 0, \\ f(0, y-1) & \text{if } x = 0,\ y > 0, \\ f(0, 0) & \text{if } x = 0,\ y = 0. \end{cases}$$

We may call $\{j_n, n \geq 0\}$ defined by the *-unital homomorphism θ in (18.10)–(18.12) the *hypergeometric flow*.

Example 18.10: (Ehrenfest's model) There are two urns with a and b balls so that $a + b = c$. One of the c balls is chosen at random and shifted from its urn to the other. The state of the system is the number of balls in the first urn. Then

$$S = \{0, 1, 2, \ldots, c\}.$$

Define the maps ϕ, ψ on S by

$$\phi(x) = \begin{cases} x - 1 & \text{if } x > 0, \\ 1 & \text{if } x = 0, \end{cases}$$

$$\psi(x) = \begin{cases} x + 1 & \text{if } x < c, \\ c - 1 & \text{if } x = c. \end{cases}$$

Define

$$p(x) = \begin{cases} c^{-1}x & \text{if } 0 < x < c, \\ \frac{1}{2} & \text{if } x = 0 \text{ or } c. \end{cases}$$

We may call the corresponding $\{j_n, n \geq 0\}$ the *Ehrenfest flow*.

Example 18.11: (Polya's urn scheme) Consider an urn with a white balls and b black balls where $a > 0, b > 0$. Draw a ball at random, replace it and add c balls of its colour. The state of the system can be described by (x, y) where x and y are respectively the number of white and black balls. Thus $S = \mathbb{N} \times \mathbb{N}$ where $\mathbb{N} = \{1, 2, \ldots\}$. Define

$$\phi(x, y) = (x + c, y), \psi(x, y) = (x, y + c)$$

$$p(x, y) = x(x + y)^{-1}, q(x, y) = y(x + y)^{-1}$$

Then

$$(Tf)(x, y) = (x + y)^{-1}\{xf(x + c, y) + yf(x, y + c)\}.$$

We may call the corresponding $\{j_n, n \geq 0\}$ the *Polya flow*.

Exercise 18.12: (i) Let $\mathcal{H}_0 = L^2(\mu)$ where μ is the counting measure in $S = \{1, 2, \ldots, N\}$ and let $\mathcal{H}$ be any Hilbert space. For any function f on S denote by the same letter the operator of multiplication by f. Let $\mathcal{B}_0$ be the *-unital abelian algebra of all complex valued functions viewed as an abelian von Neumann subalgebra of $\mathcal{B}(\mathcal{H}_0)$. A map $\theta : \mathcal{B}_0 \to \mathcal{B}_0 \otimes \mathcal{B}(\mathcal{H})$ is a *-unital homomorphism if and only if there exists a matrix $P_\theta = ((P_{ij}))_{1 \leq i,j \leq N}$ of projections in $\mathcal{H}$ satisfying the following:

$$P_{i1} + \cdots + P_{iN} = 1, \tag{1}$$

$$\theta(I_{\{i\}}) = \sum_j I_{\{j\}} \otimes P_{ji} \text{ for every } \ i. \tag{2}$$

where $I_{\{i\}}$ is the indicator function of the singleton $\{i\}$ in S.

(ii) If $\mathcal{H}_i$, $i = 1, 2$ are Hilbert spaces, $\theta_i : \mathcal{B}_0 \to \mathcal{B}_0 \otimes \mathcal{B}(\mathcal{H}_0)$ are *-unital homomorphisms and $P_{\theta_i} = ((P_{k\ell}^{(i)}))$ are the corresponding matrices of projections in (i) then

$$P_{\theta_1} * P_{\theta_2} = ((P_{k\ell})), P_{k\ell} = \sum_r P_{kr}^{(1)} \otimes P_{r\ell}^{(2)}$$

is a matrix of projections in $\mathcal{H}_1 \otimes \mathcal{H}_2$ satisfying property (1) of part (i). Thus $P_{\theta_1} * P_{\theta_2}$ determines a *-unital homomorphism $\theta : \mathcal{B}_0 \to \mathcal{B}_0 \otimes \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$ where

$$\theta(I_{\{i\}}) = \sum_{j,r} I_{\{j\}} \otimes P_{jr}^{(1)} \otimes P_{ri}^{(2)} \ \text{ for every } \ i.$$

(iii) Let θ, P_θ be as in (i). Then the quantum stochastic flow $\{j_n, n \geq 0\}$ induced by θ is given by

$$j_n(f) = \Sigma I_{\{j_0\}} \otimes P_{j_0 j_1} \otimes P_{j_1 j_2} \otimes \cdots \otimes P_{j_{n-1} j_n} \otimes f(j_n) 1_{[n+1}.$$

If Ω is the unit vector in $\mathcal{H}$ with respect to which the conditional expectation maps $\mathbb{E}_{n]}$, $n \geq 0$ are defined then

$$\mathbb{E}_{n-1]} j_n(f) = j_{n-1}(Tf) \text{ where } (Tf)(i) = \sum_j p_{ij} f(j), p_{ij} = \langle \Omega, P_{ij} \Omega \rangle.$$

Example 18.13: [115] Let G be a compact second countable group and let $g \to L_g$ denote its left regular representation in the complex Hilbert space $L^2(G)$ of all absolutely square integrable functions on G with respect to its normalised Haar measure so that $(L_g f)(x) = f(g^{-1}x)$, $f \in L^2(G)$. Denote by $\mathcal{B}_0$ the von Neumann algebra generated by $\{L_g | g \in G\}$ and its centre by Z_0. Let $\Gamma(G)$ denote the countable set of all characters of irreducible unitary representations of G. For any $\chi \in \Gamma(G)$ let U^χ be an irreducible unitary representation of G with character χ and dimension $d(\chi)$. If $\chi_1, \chi_2 \in \Gamma(G)$ the map $g \to U_g^{\chi_1} \otimes U_g^{\chi_2}, g \in G$ defines a unitary representation which decomposes into a direct sum of irreducible representations. Denote by $m(\chi_1, \chi_2\,; \chi)$ the multiplicity with which the type U^χ appears in such a decomposition of $U^{\chi_1} \otimes U^{\chi_2}$. Define

$$p^{\chi}_{\chi_1, \chi_2} = \frac{m(\chi, \chi_2\,; \chi_1) d(\chi_2)}{d(\chi) d(\chi_1)}.$$

Then

$$\sum_{\chi_2 \in \Gamma(G)} p^{\chi}_{\chi_1, \chi_2} = 1 \text{ for each } \chi, \chi_1 \in \Gamma(G).$$

In other words, for every fixed $\chi \in \Gamma(G)$, the matrix $P^\chi = ((p^{\chi}_{\chi_1, \chi_2}))$ is a stochastic matrix over the state space $\Gamma(G)$. In each row of P^χ all but a finite number of entries are 0 and each entry is rational. Thanks to the Peter-Weyl Theorem $L^2(G)$ admits the Plancherel decomposition

$$L^2(G) = \bigoplus_{\chi \in \Gamma(G)} \mathcal{H}_\chi$$

where $\dim \mathcal{H}_\chi = d(\chi)^2, L_g$ leaves each $\mathcal{H}_\chi$ invariant and $L_g|_{\mathcal{H}_\chi}, g \in G$ is a direct sum of $d(\chi)$ copies of the representation U^χ. If π_χ denotes the orthogonal projection onto the component $\mathcal{H}_\chi$ then

$$\pi_\chi = d(\chi) \int_G \chi(g) L_g dg.$$

Fix $\chi_0 \in \Gamma(G)$. Let U^{χ_0} act in the Hilbert space $\mathcal{H}$. Denote by ρ the state $d(\chi_0)^{-1} I$ in $\mathcal{H}$. Fix a positive integer N and consider in $\mathcal{H}^{\otimes^N}$ the increasing sequence of von Neumann subalgebras

$$\mathcal{B}_{n]} = \{X \otimes 1_{[n+1,N]} | X \in \mathcal{B}(\mathcal{H}^{\otimes^n})\}, n = 1, 2, \ldots, N$$

where $\mathcal{B}_{N]} = \mathcal{B}(\mathcal{H}^{\otimes^N})$ and $1_{[a,b]}$ denotes the identity in $\mathcal{H}^{\otimes^{b-a+1}}$, the tensor product of the a-th, $a+1$-th,..., b-th copies of $\mathcal{H}$ in $\mathcal{H}^{\otimes^N}$. Consider the conditional expectation maps $\mathbb{E}_{n]} : \mathcal{B}_{N]} \to \mathcal{B}_{n]}$ defined by

$$\mathbb{E}_{n]} X = (\mathbb{E}_{\rho^{\otimes N-n}} X) \otimes 1_{[n+1,N]}, \quad 1 \leq n \leq N-1$$

(see Exercise 16.10) so that

$$\mathbb{E}_{n]} X_1 \otimes \cdots \otimes X_N = (\Pi_{i=n+1}^{N} \operatorname{tr} \rho X_i)(X_1 \otimes \cdots \otimes X_n) \otimes 1_{[n+1,N]}$$

for all $X_i \in \mathcal{B}(\mathcal{H})$. By the Peter-Weyl Theorem there exists a unique *-unital homomorphism $j_n^{\chi_0} : \mathcal{B}_0 \to \mathcal{B}_{n]}$ satisfying

(i) $j_n^{\chi_0}(L_g) = (U_g^{\chi_0})^{\otimes^n} \otimes 1_{[n+1,N]}$

(ii) $\mathbb{E}_{n-1]} j_n^{\chi_0}(X) = j_{n-1}(T(X))$ for all $X \in \mathcal{B}_0$ where

$$T(L_g) = d(\chi_0)^{-1}\chi_0(g)L_g, \quad T(\pi_\chi) = \sum_{\chi' \in \Gamma(G)} p_{\chi',\chi}^{\chi_0} \pi_{\chi'}.$$

(iii) The centre Z_0 of $\mathcal{B}_0$ is generated by $\{\pi_\chi | \chi \in \Gamma(G)\}$ and $[j_m^{\chi_0}(Z), j_r^{\chi_0}(X)] = 0$ for all $Z \in Z_0, X \in \mathcal{B}_0, m \le n$. In particular, the family $\{j_n^{\chi_0}(Z) | 1 \le n \le N, z \in Z_0\}$ is commutative and hence $\{j_n^{\chi_0}|_{Z_0}, 1 \le n \le N\}$ induces a classical Markov chain with state space $\Gamma(G)$ and transition probability matrix P^{χ_0} for every $\chi_0 \in \Gamma(G)$.

(iv) When $G = SU(2)$ and χ_n denotes the character of the unique equivalence class of an irreducible unitary representation of dimension n the Clebsch-Gordan formula implies that

$$p_{\chi_i,\chi_j}^{\chi_2} = \begin{cases} \dfrac{i-1}{2i} & \text{if } j = i-1, \\ \dfrac{i+1}{2i} & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 18.14: Let $(S, \mathcal{F}, \mu)$ be as in Example 18.7. Suppose that

$$T_1 f = \sum_{i=0}^{d-1} p_i f o \phi_i, \quad T_2 f = \sum_{i=0}^{d-1} q_i f o \psi_i$$

are two Markov transition operators as described in (18.8) and

$$\theta_1(f) = U \begin{pmatrix} fo\phi_0 & & 0 \\ & fo\phi_1 & \\ & \ddots & \\ & 0 & fo\phi_{d-1} \end{pmatrix} U^{-1},$$

$$\theta_2(f) = V \begin{pmatrix} fo\psi_0 & & 0 \\ & fo\psi_1 & \\ & \ddots & \\ & 0 & fo\psi_{d-1} \end{pmatrix} V^{-1},$$

are the *-unital homomorphisms corresponding to (18.9). Define

$$W = \begin{pmatrix} U\cos\alpha & V\sin\alpha \\ -V^*\sin\alpha & V^*U^*V\cos\alpha \end{pmatrix},$$

$$\theta(f) = W \begin{pmatrix} fo\phi_0 & & & & & 0 \\ & \ddots & & & & \\ & & fo\phi_{d-1} & & & \\ & & & fo\psi_0 & & \\ & & & & \ddots & \\ 0 & & & & & fo\psi_{d-1} \end{pmatrix} W^{-1}$$

where α is any fixed angle. Then W is a unitary matrix valued function and θ is a *-unital homomorphism for which $\theta_0^0 = T_1\cos^2\alpha + T_2\sin^2\alpha$. Thus the Markov transition operator of the quantum stochastic flow induced by θ is a superposition (or convex combination) of the two transition operators T_i, $i = 1, 2$.

Notes

The idea of describing a general quantum stochastic process in terms of a time indexed family $\{j_n\}$ or $\{j_t\}$ of *-unital homomorphisms from a *-unital initial or system algebra $\mathscr{B}_0$ into a larger *-unital algebra $\tilde{\mathscr{B}}$ made up of $\mathscr{B}_0$ and noise or heat bath elements has its origin in Accardi, Frigerio and Lewis [6] modelled on the description of classical processes by Nelson (J. Funct. Anal., 12 (1973) 97–112, 211–277) and Guerra, Rosen and Simon (Ann. Math., 101 (1975) 111–259).

For a detailed account of Markov chains (or discrete time flows) in *-algebras of the form $\mathscr{B}_0 \otimes \mathscr{B} \otimes \mathscr{B} \otimes \cdots$ where $\mathscr{B}_0$ is the initial or system algebra and $\mathscr{B}$ is the noise algebra see Kummerer [71,73]. The account given here is in anticipation of Evans-Hudson flows which are discussed in Section 28. It is based on Parthasarathy [112] and inspired by Meyer [91]. Example 18.9–18.11 are based on classical probability theory as described in Feller [40]. Example 18.12 was suggested to me by B.V.R. Bhat. Example 18.13 is based on the work of Biane [22], von Waldenfels [138] and Parthasarathy [115].

19 The Fock Spaces

In Section 16–18 we saw how the notion of tensor products of Hilbert spaces enables us to combine several quantum probability spaces into one. In this context there is yet another basic construction leading to the combination of an "indefinite" number of such systems. This idea is illustrated by first raising the following question: if the events concerning the dynamics of a single particle are described by the elements of $\mathscr{P}(\mathscr{H})$ where $\mathscr{H}$ is a separable Hilbert space, how does one construct the Hilbert space for an indefinite number of such particles in a system where the indefiniteness is due to the fact that "births" and "deaths" of particles take place or, equivalently, particles are being "created" and "annihilated" subject to certain laws of chance? We shall try to achieve this by pooling all the finite order tensor products into a single direct sum. To this end we collect some of the elementary properties of direct sums of operators in the form of a proposition without proof.

Proposition 19.1: Let $A_n \in \mathcal{B}(\mathcal{H}_n), n = 1, 2, \ldots$ where $\{\mathcal{H}_n\}$ is a sequence of Hilbert spaces. Suppose $\sup_n \|A_n\| < \infty$. Then there exists a unique operator $A = \oplus_n A_n$ on $\mathcal{H} = \oplus_n \mathcal{H}_n$ satisfying

(i) $A \oplus_n u_n = \oplus_n A_n u$; (ii) $\|A\| = \sup_n \|A_n\|$.

If $\{A_n\}, \{B_n\}$ are two sequences of operators such that $A_n, B_n \in \mathcal{B}(\mathcal{H}_n)$ for each n and $\sup_n(\|A_n\| + \|B_n\|) < \infty$ then their direct sums $A = \oplus_n A_n$ and $B = \oplus_n B_n$ satisfy the following:

(a) $A + B = \oplus_n (A_n + B_n)$, $AB = \oplus_n A_n B_n$, $A^* = \oplus_n A_n^*$;

(b) If each A_n has a bounded inverse and $\sup_n \|A_n^{-1}\| < \infty$ then A has a bounded inverse and $A^{-1} = \bigoplus_n A_n^{-1}$;

(c) A is a selfadjoint, normal, unitary, positive or projection operator according to whether each A_n has the same property;

(d) If $A \in \mathcal{I}_\infty(\mathcal{H})$ and therefore has finite trace then each $A_n \in \mathcal{I}_\infty(\mathcal{H}_n)$, $n = 1, 2, \ldots$, and $\|A\|_1 = \sum_n \|A_n\|_1$, $\operatorname{tr} A = \sum_n \operatorname{tr} A_n$;

(e) If $A = \rho$ is a state in $\mathcal{H}$ then there exist states ρ_n and scalars $p_n \geq 0$, $n = 1, 2, \ldots$ such that $\sum_n p_n = 1$ and $\rho = \oplus_n p_n \rho_n$.

Proof: Omitted. ■

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{H}^{\otimes^n}, \mathcal{H}^{\circledS^n}$ and $\mathcal{H}^{@^n}$ be the n-fold tensor product, symmetric tensor product and antisymmetric tensor product of $\mathcal{H}$ respectively, where the 0-fold product is the one dimensional complex plane and the 1-fold product is $\mathcal{H}$ itself in all the three cases. The Hilbert spaces

$$\Gamma_{fr}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes^n}, \Gamma_s(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\circledS^n}, \Gamma_a(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{@^n}$$

are respectively called the *free* (or Maxwell-Boltzman), the *symmetric* (or boson) and the *antisymmetric* (or fermion) *Fock space* over $\mathcal{H}$. The n-th direct summand in each case is called the *n-particle subspace*. When $n = 0$ it is called the *vacuum subspace*. Any element of the n-particle subspace is called an *n-particle vector*. The vector $1 \oplus 0 \oplus 0 \oplus \cdots$ is called the *vacuum vector* which we shall denote by Φ. We denote by $\Gamma^0_{fr}(\mathcal{H}), \Gamma^0_s(\mathcal{H})$ and $\Gamma^0_a(\mathcal{H})$ the dense linear manifold generated by all n-particle vectors, $n = 0, 1, 2, \ldots$ in the corresponding Fock space and call any element in it a *finite particle vector*. For any $u \in \mathcal{H}$ the element

$$e(u) = \oplus_n (n!)^{-1/2} u^{\otimes^n} \tag{19.1}$$

(where $0! = 1$, $u^{\otimes^0} = 1$) belongs to $\Gamma_s(\mathcal{H}) \subset \Gamma_{fr}(\mathcal{H})$ and is called the *exponential* (or *coherent*) vector associated with u. For any $u, v \in \mathcal{H}$

$$\langle e(u), e(v) \rangle = \exp\langle u, v \rangle. \tag{19.2}$$

The projections E and F from $\Gamma_{fr}(\mathcal{H})$ onto the subspaces $\Gamma_s(\mathcal{H})$ and $\Gamma_a(\mathcal{H})$ can be expressed as

$$E = \oplus_n E_n, F = \oplus_n F_n \tag{19.3}$$

where E_n and F_n are the projections from $\mathcal{H}^{\otimes^n}$ onto $\mathcal{H}^{\circledS^n}$ and $\mathcal{H}^{@^n}$ respectively described by Proposition 17.1. If $\dim \mathcal{H} = N < \infty$ then the direct sum in $\Gamma_a(\mathcal{H})$ terminates at the N-th stage and by the second part of Proposition 17.3

$$\dim \Gamma_a(\mathcal{H}) = \sum_{n=0}^{N} \binom{N}{n} = 2^N.$$

Let $U_{rs} : \mathcal{H}^{\otimes^r} \otimes \mathcal{H}^{\otimes^s} \to \mathcal{H}^{\otimes^{r+s}}$ be the unique unitary isomorphism satisfying the relations

$$U_{rs}(u_1 \otimes \cdots \otimes u_r) \otimes (v_1 \otimes \cdots \otimes v_s) = u_1 \otimes \cdots \otimes u_r \otimes v_1 \otimes \cdots \otimes v_s$$

for all $u_i, v_j \in \mathcal{H}$, $1 \le i \le r$, $1 \le j \le s$. Such an isomorphism is well-defined in view of Proposition 7.2 and 15.5. We use this isomorphism to identify $\mathcal{H}^{\otimes^r} \otimes \mathcal{H}^{\otimes^s}$ with $\mathcal{H}^{\otimes^{r+s}}$. In each of the Fock spaces let P_n denote the projection on the n-particle subspace for every n. If $\mathcal{P}(\mathcal{H})$ is interpreted as the collection of events concerning the dynamics of a single particle then $\mathcal{P}(\Gamma_{fr}(\mathcal{H}))$ can be interpreted as the collection of events concerning an indefinite number of identical but distinguishable particles obeying Maxwell-Boltzmann statistics (see (17.13)). Similarly $\mathcal{P}(\Gamma_s(\mathcal{H}))$ and $\mathcal{P}(\Gamma_a(\mathcal{H}))$ may be considered as the collection of events concerning an indefinite number of identical bosons and fermions respectively. (See (17.14), (17.15) and the succeeding remarks.) In such a case the projection P_n signifies the event that the number of particles in the system is n.

Proposition 19.2: Define multiplications $(u, v) \to u \otimes v, uv, u \wedge v$ respectively in the finite particle Fock spaces $\Gamma^0_{fr}(\mathcal{H}), \Gamma^0_s(\mathcal{H}), \Gamma^0_a(\mathcal{H})$ by

$$u \otimes v = \oplus_n \sum_{r+s=n} P_r u \otimes P_s v,\ uv = Eu \otimes v,\ u \wedge v = Fu \otimes v$$

where $1 \otimes u = u \otimes 1 = u$ and E, F are defined by (19.3). Then the following properties hold: (i) $\Gamma^0_{fr}(\mathcal{H})$ is an associative algebra; (ii) $\Gamma^0_s(\mathcal{H})$ is a commutative and associative algebra; (iii) $\Gamma^0_a(\mathcal{H})$ is an associative algebra in which

$$u \wedge v = (-1)^{mn} v \wedge u \text{ for all } u \in \mathcal{H}^{@^m}, v \in \mathcal{H}^{@^n}.$$

Proof: The associativity of the three multiplications is immediate from Exercise 15.7 and Corollary 17.2. (ii) is immediate from definitions. The last part follows from the fact that the permutation from $(1, 2, \ldots, m+n)$ to $(m+1, m+2, \ldots, m+n, 1, 2, \ldots, m)$ can be achieved by mn successive elementary permutations which are transpositions. ■

If we say that the *degree* or *rank* of any n-particle vector is equal to n then the product of two vectors of degrees m and n leads to a vector of degree $m+n$ in all the three algebras $\Gamma^0_{fr}(\mathcal{H}), \Gamma^0_s(\mathcal{H})$ and $\Gamma^0_a(\mathcal{H})$. These are examples of "graded algebras". $\Gamma^0_{fr}(\mathcal{H})$ is the *tensor algebra* over $\mathcal{H}$ whereas $\Gamma^0_s(\mathcal{H})$ and $\Gamma^0_a(\mathcal{H})$ are the *symmetric* and *exterior* algebras over $\mathcal{H}$. When $\dim \mathcal{H} = \infty$ the products of m-particle and n-particle vectors generate only a dense linear manifold in the subspace of $m+n$-particle vectors. In this sense the definitions here are different from the corresponding notions in the algebraic appoach to tensor products over a vector space.

If $\Gamma^+_{fr}(\mathcal{H})$ and $\Gamma^-_{fr}(\mathcal{H})$ denote the subspaces spanned by vectors of even and odd degrees respectively then any element of $\Gamma^+_{fr}(\mathcal{H})$ is called *even* and any element of $\Gamma^-_{fr}(\mathcal{H})$ is called *odd*. If $u, v \in \Gamma^0_a(\mathcal{H})$ then $u \wedge v = v \wedge u$ if either u or v is even and $u \wedge v = -v \wedge u$ if both u, v are odd. $u \wedge v$ is odd if one of them is odd and the other is even. $u \wedge v$ is even if both are odd or both are even. In other words $\Gamma^0_a(\mathcal{H})$ is a Z_2-graded algebra under the multiplication $\wedge$.

Proposition 19.3: Suppose $\{e_j, j = 1, 2, \dots,\}$ is an orthonormal basis in $\mathcal{H}$. Then the three sets

$$\{\Phi, e_{i_1} \otimes \cdots \otimes e_{i_n} | i_j = 1, 2, \dots; j = 1, 2, \dots, n;\ n = 1, 2, \dots,\},$$

$$\{\Phi, \left(\frac{n!}{r_1! \cdots r_k!}\right)^{1/2} e_{i_1}^{r_1} e_{i_2}^{r_2} \cdots e_{i_k}^{r_k} | r_1 + \cdots + r_k = n,$$
$$1 \leq i_1 < i_2 < \cdots < i_k,\ k, n = 1, 2, \dots\},$$

$$\{\Phi, (n!)^{1/2} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} | 1 \leq i_1 < i_2 < \cdots < i_n,\ n = 1, 2, \dots\}$$

are respectively orthonormal bases in the Fock spaces $\Gamma_{fr}(\mathcal{H}), \Gamma_s(\mathcal{H})$ and $\Gamma_a(\mathcal{H})$, the multiplications being defined according to Proposition 19.2.

Proof: This is immediate from Exercise 15.8 and Proposition 17.3. ■

Proposition 19.4: The set $\{e(u) | u \in \mathcal{H}\}$ of all exponential vectors is linearly independent and total in $\Gamma_s(\mathcal{H})$.

Proof: Let $\{u_j | 1 \leq j \leq n\}$ be a finite subset of $\mathcal{H}$. Since $\{u | \langle u, u_j \rangle \neq \langle u, u_k \rangle\}$, $j \neq k$ are open and dense in $\mathcal{H}$ there exists a v in $\mathcal{H}$ such that the scalars $\theta_j = \langle v, u_j \rangle$, $1 \leq j \leq n$ are distinct. Suppose α_j, $1 \leq j \leq n$ are scalars such that $\sum_j \alpha_j e(u_j) = 0$. Then for any scalar z

$$0 = \langle e(\bar{z}v), \sum_j \alpha_j e(u_j) \rangle = \sum_j \alpha_j e^{z\theta_j}.$$

Hence $\alpha_j = 0$ for all j. This proves linear independence. To prove the second part consider an orthonormal basis $\{e_i\}$ in $\mathcal{H}$. If $u = z_1 e_{i_1} + \cdots + z_k e_{i_k}$, z_j being scalars then the coefficient of $z_1^{r_1} z_2^{r_2} \cdots z_k^{r_k}, r_1 + \cdots + r_k = n$, in $u^{\otimes^n}$ belongs to $\mathcal{H}^{\circledS^n}$. Suppose that S is the closed subspace spanned by all the exponential vectors. Then the vacuum vector $e(0) \in S$. By (19.1)

$$u^{\otimes^{n+1}} = (n+1!)^{1/2} \lim_{\varepsilon \to 0} \varepsilon^{-(n+1)} \{e(\varepsilon u) - \bigoplus_{r=0}^{n} (r!)^{-\frac{1}{2}} \varepsilon^r u^{\otimes^r}\}.$$

Hence it follows by induction that $u^{\otimes^n} \in S$ for all n and u in $\mathcal{H}$. Thus $S = \Gamma_s(\mathcal{H})$. ■

Corollary 19.5: Let S be a dense set in $\mathcal{H}$. Then the linear manifold $\mathcal{E}(S)$ generated by $\tilde{S} = \{e(u) | u \in S\}$ is dense in $\Gamma_s(\mathcal{H})$. If $T : \tilde{S} \to \Gamma_s(\mathcal{H})$ is any map there exists a unique linear operator $\hat{T}$ on $\Gamma_s(\mathcal{H})$ with domain $\mathcal{E}(S)$ satisfying

$$\hat{T} e(u) = T e(u) \text{ for all } u \in S.$$

Proof: For any $u, v \in \mathcal{H}$

$$\|e(u) - e(v)\|^2 = e^{\|u\|^2} + e^{\|v\|^2} - 2 \operatorname{Re} e^{\langle u,v \rangle}.$$

Since the scalar product is continuous in its arguments this shows that the map $u \to e(u)$ from $\mathcal{H}$ into $\Gamma_s(\mathcal{H})$ is continuous and, in particular, $\overline{\{e(u)|u \in S\}} = \{e(u)|u \in \mathcal{H}\}$. The second part follows from the linear independence of the set of all exponential vectors. ■

The next two propositions indicate how the correspondences $\mathcal{H} \to \Gamma_s(\mathcal{H})$ and $\mathcal{H} \to \Gamma_a(\mathcal{H})$ share a functorial property in the "category of Hilbert spaces".

Proposition 19.6: Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Then there exists a unique unitary isomorphism $U : \Gamma_s(\mathcal{H}) \to \Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2)$ satisfying the relation

$$U e(u \oplus v) = e(u) \otimes e(v) \text{ for all } u \in \mathcal{H}_1, v \in \mathcal{H}_2. \tag{19.4}$$

Proof: We may assume without loss of generality that $\mathcal{H}_1, \mathcal{H}_2$ are mutually orthogonal subspaces of $\mathcal{H}$. By Proposition 19.4 the sets $\{e(u)|u \in \mathcal{H}\}, \{e(u)|u \in \mathcal{H}_i\}$, $i = 1, 2$ are respectively total in $\Gamma_s(\mathcal{H}), \Gamma_s(\mathcal{H}_i)$, $i = 1, 2$. By Proposition 15.6 the set $\{e(u) \otimes e(v)|u \in \mathcal{H}_1, v \in \mathcal{H}_2\}$ is total in $\Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2)$. By (19.2), for u_i, v_i in $\mathcal{H}_i$

$$\begin{aligned} \langle e(u_1 + u_2), e(v_1 + v_2) \rangle &= \exp \langle u_1 + u_2, v_1 + v_2 \rangle \\ &= e^{\langle u_1, v_1 \rangle} e^{\langle u_2, v_2 \rangle} \\ &= \langle e(u_1), e(v_1) \rangle \langle e(u_2), e(v_2) \rangle. \end{aligned}$$

In other words the map U defined by (19.4) on the set $\{e(u+v)|u \in \mathcal{H}_1, v \in \mathcal{H}_2\}$ is scalar product preserving. Hence by Proposition 7.2 U extends uniquely to the required unitary isomorphism. ■

Proposition 19.7: Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Then there exists a unique unitary isomorphism $U : \Gamma_a(\mathcal{H}_1 \oplus \mathcal{H}_2) \to \Gamma_a(\mathcal{H}_1) \otimes \Gamma_a(\mathcal{H}_2)$ satisfying the relations:

$$\begin{aligned} &U[(m+n)!]^{1/2} u_1 \wedge \cdots \wedge \ u_m \wedge v_1 \wedge \cdots \wedge \ v_n \\ &\quad = \{(m!)^{1/2} u_1 \wedge \cdots \wedge u_m\} \otimes \{(n!)^{1/2} v_1 \wedge \cdots \wedge \ v_n\} \end{aligned} \tag{19.5}$$

for all $u_i \in \mathcal{H}_1$, $v_j \in \mathcal{H}_2$, $1 \leq i \leq m$, $1 \leq j \leq n$, $m = 1, 2, \ldots$, $n = 1, 2, \ldots$ and $U\Phi = \Phi_1 \otimes \Phi_2, \Phi, \Phi_1, \Phi_2$ being the vacuum vectors in $\Gamma_a(\mathcal{H}), \Gamma_a(\mathcal{H}_1), \Gamma_a(\mathcal{H}_2)$ respectively.

Proof: Once again, as in the proof of Proposition 19.6, we assume without loss of generality that $\mathcal{H}_1, \mathcal{H}_2$ are orthogonal subspaces of $\mathcal{H}$. Let

$$\begin{aligned} S &= \{\Phi, [(m+n)!]^{1/2} u_1 \wedge \cdots \wedge u_m \wedge v_1 \wedge \cdots \wedge v_n | u_i \in \mathcal{H}_1, v_j \in \mathcal{H}_2, \} \\ &\qquad 1 \leq i \leq m, \ 1 \leq j \leq n, m = 1, 2, \ldots \ ; \ n = 1, 2, \ldots\}, \\ S_1 &= \{\Phi_1, (m!)^{1/2} u_1 \wedge \cdots \wedge u_m, u_i \in \mathcal{H}_1, \ 1 \leq i \leq m, \ m = 1, 2, \ldots\}, \\ S_2 &= \{\Phi_2, (n!)^{1/2} v_1 \wedge \cdots \wedge v_n, v_i \in \mathcal{H}_1, \ 1 \leq i \leq n, \ n = 1, 2, \ldots\}. \end{aligned}$$

By Proposition 19.3, S, S_1, S_2 are total in $\Gamma_a(\mathcal{H}), \Gamma_a(\mathcal{H}_1), \Gamma_a(\mathcal{H}_2)$ respectively. Furthermore, the set $\{u \otimes v | u \in S_1, v \in S_2\}$ is total in $\Gamma_a(\mathcal{H}_1) \otimes \Gamma_a(\mathcal{H}_2)$. Thus it suffices to show that the map U defined by (19.5) is scalar product preserving. Let $u_i, u'_k \in \mathcal{H}_1$, $1 \leq i \leq m$, $1 \leq k \leq m'$, $v_j, v'_\ell \in \mathcal{H}_2$, $1 \leq j \leq n$, $1 \leq \ell \leq n'$. We now consider three different cases:

Case 1: $m + n \neq m' + n'$.

Define $u_1 \wedge \cdots \wedge u_m = \Phi_1$, $v_1 \wedge \cdots \wedge v_n = \Phi_2$, $u_1 \wedge \cdots \wedge u_m \wedge v_1 \wedge \cdots \wedge v_n = \Phi$ when $m = n = 0$. Then we see that since $u_1 \wedge \cdots \wedge u_m \wedge v_1 \wedge \cdots \wedge v_n$ and $u'_1 \wedge \cdots \wedge u'_{m'} \wedge v'_1 \wedge \cdots \wedge v'_1 \wedge \cdots \wedge v'_{n'}$ are $m + n$ and $m' + n'$-particle vectors, they are orthogonal. Furthermore, either $m \neq m'$ or $n \neq n'$ and hence

$$\langle u_1 \wedge \cdots \wedge u_m, u'_1 \wedge \cdots \wedge u'_{m'} \rangle \langle v_1 \wedge \cdots \wedge v_n, v'_1 \wedge \cdots \wedge v'_{n'} \rangle = 0. \tag{19.6}$$

Case 2: $m + n = m' + n'$, $m \neq m'$.

Without loss of generality let $m < m'$, $n > n'$. Then (19.6) holds. On the other hand, writing

$$\begin{aligned} (u_1, \ldots, u_m, v_1, \ldots, v_n) &= (w_1, \ldots, w_{m+n}), \\ (u'_1, \ldots, u'_{m'}, v'_1, \ldots, v'_{n'}) &= (w'_1, \ldots, w'_{m+n}) \end{aligned}$$

we observe that the matrix $(((\langle w_i, w'_j \rangle)))$, $1 \leq i, j \leq m + n$ has the partitioned form

$$\begin{pmatrix} A_{m \times m} & B_{m \times (m'-m)} & 0 \\ 0 & 0 & C_{(m'-m) \times n'} \\ 0 & 0 & D_{n' \times n'} \end{pmatrix}$$

and hence has rank $\leq m+n' < m+n$. Thus its determinant vanishes. By (17.10) and the definition of $\wedge$ we have

$$\langle u_1 \wedge \cdots \wedge u_m \wedge v_1 \wedge \cdots \wedge v_n, u_1' \wedge \cdots \wedge u_{m'}' \wedge v_1' \wedge \cdots \wedge v_{n'}' \rangle = 0.$$

Case 3: $m = m'$, $n = n'$.

Define $(w_1, \ldots, w_{m+n})$ and $(w_1', \ldots, w_{m+n}')$ as in case 2 and observe that the matrix $((\langle w_i, w_j' \rangle))$ has the partitioned form

$$\begin{pmatrix} A_{m\times m} & 0 \\ 0 & B_{n\times n} \end{pmatrix}$$

where $A = ((\langle u_i, u_k' \rangle)), B = ((\langle v_j, v_\ell' \rangle))$. By (17.10)

$$\begin{aligned}
&\langle u_1 \wedge \cdots \wedge u_m \wedge v_1 \wedge \cdots \wedge v_n, u_1' \wedge \cdots \wedge u_m' \wedge v_1' \wedge \cdots \wedge v_n' \rangle \\
&= \frac{1}{m+n!} \det((\langle w_i, w_j' \rangle)) = \frac{1}{m+n!} \det A \det B \\
&= \frac{m!n!}{m+n!} \langle u_1 \wedge \cdots \wedge u_m \otimes v_1 \wedge \cdots \wedge v_n, u_1' \wedge \cdots \wedge u_m' \otimes v_1' \wedge \cdots \wedge v_n' \rangle.
\end{aligned}$$

■

It is interesting to note that the correspondence

$$U u \otimes v = \oplus_n \sum_{r+s=n} P_r u \otimes P_s v, \ u \in \Gamma_{fr}(\mathcal{H}_1), \ v \in \Gamma_{fr}(\mathcal{H}_2)$$

can be extended by linearity and closure to an isometry from $\Gamma_{fr}(\mathcal{H}_1) \otimes \Gamma_{fr}(\mathcal{H}_2)$ into $\Gamma_{fr}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ but not a unitary isomorphism. Thus the functorial property established for the boson and fermion Fock spaces fails for the free Fock space.

The next few examples illustrate the connections between Fock spaces and probability theory.

Example 19.8: Let μ be the standard normal (or Gaussian) distribution on the real line. Consider the Hilbert spaces $L^2(\mu)$ and the boson Fock space $\Gamma_s(\mathbb{C})(= \Gamma_{fr}(\mathbb{C}))$. Since $\mathbb{C}^{\otimes^n} = \mathbb{C}$ for all n we have

$$\Gamma_s(\mathbb{C}) = \mathbb{C} \oplus \mathbb{C} \oplus \cdots = \ell^2.$$

For any $z \in \mathbb{C}$ the associated exponential vector $e(z)$ is the sequence

$$e(z) = (1, z, (2!)^{-\frac{1}{2}} z^2, \ldots, (n!)^{-\frac{1}{2}} z^n, \ldots).$$

In $L^2(\mu)$ consider the generating function of the Hermite polynomials

$$e^{zx - \frac{1}{2}z^2} = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x), \tag{19.7}$$

H_n being the n-th degree Hermite polynomial. There exists a unique unitary isomorphism $U : \Gamma_s(\mathbb{C}) \to L^2(\mu)$ satisfying

$$[U e(z)](x) = e^{zx - \frac{1}{2}z^2} \text{ for all } z \in \mathbb{C}. \tag{19.8}$$

Indeed, the set of functions $\{\exp zx | z \in \mathbb{C}\}$ is total in $L^2(\mu)$ and

$$\int_{\mathbb{R}} e^{\overline{z}_1 x - \frac{1}{2}\overline{z}_1^2} \cdot e^{z_2 x - \frac{1}{2}z_2^2} d\mu(x) = e^{\overline{z}_1 z_2} = \langle e(z_1), e(z_2)\rangle.$$

By Proposition 19.4 and 7.2 the correspondence (19.8) extends uniquely to a unitary isomorphism. Under this unitary isomorphism we have

$$U(1,0,0,\ldots) = 1,$$

$$U(0,0,\ldots,0,1,0,\ldots) = (n!)^{-\frac{1}{2}} H_n(x)$$

where, on the left hand side, 1 appears in the n-th position and n varies from 0 to ∞. In particular the sequence $\{(n!)^{-\frac{1}{2}} H_n(x), n = 0, 1, 2, \ldots\}$ is an orthonormal basis in $L^2(\mu)$.

In this example we may replace $\mathbb{C}$ by $\mathbb{C}^k$ and the distribution μ by its k-fold cartesian product $\mu^{\otimes^k}$ and replace (19.8) by

$$[Ue(\underline{z})](\underline{x}) = \exp(\underline{z} \cdot \underline{x} - \frac{1}{2}\sum_{j=1}^{k} z_j^2) \tag{19.9}$$

where $\underline{z} \cdot \underline{x} = \sum_j z_j x_j$. Then U is a unitary isomorphism between $L^2(\mu^{\otimes^k})$ and $\Gamma_s(\mathbb{C}^k)$. When k is the countably infinite cardinal and correspondingly $\mu^{\otimes^k}$ is the countable cartesian product of standard normal distributions we obtain a unitary correspondence between the L^2-space of an independent and identically distributed sequence of standard Gaussian random variables and the boson Fock space $\Gamma_s(\ell^2), \ell^2$ denoting the Hilbert space of absolutely square summable sequences. This leads us to the connection between Brownian motion and the boson Fock space.

Example 19.9: Let $\{w(t) | t \geq 0\}$ be the standard Brownian motion stochastic process described by the Wiener probability measure P on the space of continuous functions in $[0, \infty)$. Then $w(0) = 0$ and for any $0 < t_1 < t_2 < \cdots < t_k < \infty, w(t_1), w(t_2) - w(t_1), \ldots, w(t_k) - w(t_{k-1})$ are independent Gaussian random variables with mean 0 and $\mathbb{E}(w(t) - w(s))^2 = t - s$. For any complex valued function f in $L^2(\mathbb{R}_+)$ where $\mathbb{R}_+ = [0, \infty)$ is equipped with Lebesgue measure, let $\int_0^\infty f dw$ denote the stochastic integral (in the sense of Wiener) of f with respect to the path w of the Brownian motion. Then the argument outlined in Example 19.1 shows that there exists a unique unitary isomorphism $U : \Gamma_s(L^2(\mathbb{R}_+)) \rightarrow L^2(P)$ satisfying

$$[Ue(f)](w) = \exp(\int_0^\infty f dw - \frac{1}{2}\int_0^\infty f(t)^2 dt). \tag{19.10}$$

For any $t \geq 0$, let $f_{t]} = f I_{[0,t]}$ which agrees with f in the interval $[0, t]$ and vanishes in the interval (t, ∞). Then

$$[Ue(f_{t]})](w) = \exp(\int_0^t f dw - \frac{1}{2}\int_0^t f(s)^2 ds), \quad t \geq 0 \tag{19.11}$$

where the right hand side as a stochastic process is the well-known exponential martingale of classical stochastic calculus.

A multidimensional analogue of this example can be constructed as follows. Suppose $\{\underline{w}(t) = (w_1(t), w_2(t), \ldots, w_n(t)) | t \geq 0\}$ is the n-dimensional standard Brownian motion process whose probability measure in the space of continuous sample paths with values in $\mathbb{R}^n$ is $P^{\otimes^n}$. Let $\{e_j | 1 \leq j \leq n\}$ be the canonical basis of column vectors in $\mathbb{C}^n$. Then there exists a unique unitary isomorphism $U : \Gamma_s(L^2(\mathbb{R}_+) \otimes \mathbb{C}^n) \to L^2(P^{\otimes^n})$ satisfying

$$[Ue(\sum_{j=1}^{n} f_{t]}^{(j)} \otimes e_j)](\underline{w}) = \exp \sum_{j=1}^{n} \{ \int_0^t f^{(j)}(s) dw_j(s) - \frac{1}{2} \int_0^t f^{(j)}(s)^2 ds \}$$

for all $f^{(j)}$ in $L^2(\mathbb{R}_+)$, $j = 1, 2, \ldots, n$.

Example 19.10: Let μ be the Poisson distribution with mean value λ in the space Z_+ of non-negative integers. In $L^2(\mu)$ consider the generating function of the Charlier-Poisson polynomials defined by

$$e^{-\sqrt{\lambda}\, z}(1 + \frac{z}{\sqrt{\lambda}})^x = \sum_{n=0}^{\infty} \frac{z^n}{n!} \pi_n(\lambda, x), x \in Z_+. \qquad (19.12)$$

Then there exists a unique unitary isomorphism $U : \Gamma_s(\mathbb{C}) \to L^2(\mu)$ satisfying

$$[Ue(z)](x) = e^{-\sqrt{\lambda}\, z}(1 + \frac{z}{\sqrt{\lambda}})^x \text{ for all } z \in \mathbb{C}. \qquad (19.13)$$

Under this isomorphism

$$U(1, 0, 0, \ldots) = 1,$$

$$U(0, 0, \ldots, 0, 1, 0, \ldots) = (n!)^{-\frac{1}{2}} \pi_n(\lambda, x), \quad n = 1, 2, \ldots$$

where 1 appears in the n-th position and $n = 0, 1, 2, \ldots$, and in particular, $\{(n!)^{-\frac{1}{2}} \pi_n(\lambda, x) | n = 0, 1, 2, \ldots\}$ is an orthonormal basis in $L^2(\mu)$.

Example 19.11: Let $\{N(t) | t \geq 0\}$ be the Poisson process with stationary independent increments, right continuous trajectories and intensity parameter λ and let its distribution be described by the probability measure P. Then

$$P\{N(t) - N(s) = j\} = e^{-(t-s)} \frac{[\lambda(t-s)]^j}{j!}, \quad j = 0, 1, 2, \ldots$$

and the conditional distribution of the jump points $0 < \xi_1 < \xi_2 < \cdots < \xi_n < t$, given the fact that the process has n jumps in $[0, t]$, is uniform in the simplex $\{\underline{s} | 0 < s_1 < \cdots < s_n < t\}$. This enables us to construct a natural unitary isomorphism U between $\Gamma_s(L^2(\mathbb{R}_+))$ and $L^2(P)$ by putting for every $t \geq 0$

$$[Ue(f_{t]})](N) = e^{-\sqrt{\lambda} \int_0^t f(s) ds} \Pi_{0 \leq s \leq t} \{1 + \frac{[N(s+) - N(s-)]}{\sqrt{\lambda}} f(s)\} \quad (19.14)$$

(where $N(0-) = 0$) for all f in $L^2(\mathbb{R}_+)$ and $f_{t]}$ is defined as in (19.11). Indeed,

$$\begin{aligned}
&\langle Ue(f_{t]}), Ue(g_{t]})\rangle = \\
&\quad e^{-\sqrt{\lambda}\int_0^t [\overline{f}(s)+g(s)]ds} \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!} \times \\
&\quad n!t^{-n}\int_{0<s_1<\cdots<s_n<t} \Pi_{j=1}^n (1+\frac{\overline{f(s_j)}}{\sqrt{\lambda}})(1+\frac{g(s_j)}{\sqrt{\lambda}})ds_1\cdots ds_n \\
&= \exp\{-\sqrt{\lambda}\int_0^t [\overline{f}(s)+g(s)]ds - \lambda t + \lambda\int_0^t (1+\frac{\overline{f(s)}}{\sqrt{\lambda}})(1+\frac{g(s)}{\sqrt{\lambda}})ds\} \\
&= \exp\int_0^t \overline{f}(s)g(s)ds = \langle e(f_{t]}), e(g_{t]})\rangle.
\end{aligned}$$

In $L^2(P)$ the totality of the set of random variables occurring on the right hand side of (19.14) as f varies in $L^2(\mathbb{R}_+)$ and t in $\mathbb{R}_+$ is an exercise for the reader.

Example 19.12: [51] Let $(S, \mathscr{F}, \mu)$ be a non-atomic, σ-finite and separable measure space. For any finite set $\sigma \subset S$ denote by $\#\sigma$ its cardinality. Let $\Gamma(S)$ be the space of all finite subsets of S and $\Gamma_n(S) = \{\sigma | \sigma \subset S, \#\sigma = n\}$, $\Gamma_0(S) = \{\emptyset\}$ where $\emptyset$ is the empty subset of S. (Thus the empty subset of S is a point in $\Gamma(S)$!). For any symmetric measurable function f on S^n define the function f_n on $\Gamma(S)$ by

$$f_n(\sigma) = \begin{cases} f(s_1, \ldots, s_n) & \text{if } \#\sigma = n, \\ & \sigma = \{s_1, \ldots, s_n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathscr{F}_\Gamma$ be the smallest σ-algebra which makes all such functions f_n measurable for $n = 1, 2, \ldots$. Define the measure μ_Γ on $\mathscr{F}_\Gamma$ as follows. Let $\Delta_n \subset S^n$ denote the subset $\{\underline{s} = (s_1, \ldots, s_n) | s_i \neq s_j \text{ for } i \neq j\}$. The non-atomicity of μ implies that $\mu^{\otimes^n}(S^n \backslash \Delta_n) = 0$. For any $E \in \mathscr{F}_\Gamma$ put

$$\mu_\Gamma(E) = I_E(\emptyset) + \sum_{n=1}^{\infty}\frac{1}{n!}\int_{\Delta_n} I_E|_{\Gamma_n(S)}(\{s_1, \ldots, s_n\})\mu(ds_1)\cdots\mu(ds_n). \quad (19.15)$$

Then μ_Γ is a σ-finite measure whose only atom is $\emptyset$ with mass at $\emptyset$ being unity. $(\Gamma(S), \mathscr{F}_\Gamma, \mu_\Gamma)$ is called the *symmetric measure space* over $(S, \mathscr{F}, \mu)$ in the sense of A. Guichardet [51]. We write $d\sigma = d\mu_\Gamma(\sigma)$ when integration is with respect to μ_Γ.

For any function f on S let

$$\pi_f(\sigma) = \begin{cases} 1 & \text{if } \sigma = \emptyset, \\ \Pi_{s\in\sigma} f(s) & \text{otherwise.} \end{cases} \quad (19.16)$$

Then $\pi_f \pi_g = \pi_{fg}$ and $\overline{\pi}_f = \pi_{\overline{f}}$. If f is integrable with respect to μ then (19.15) implies

$$\begin{aligned} \int_{\Gamma(S)} \pi_f(\sigma) d\sigma &= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{S^n} \Pi_{j=1}^n f(s_j) \mu(ds_1) \cdots \mu(ds_n) \\ &= \exp \int_S f d\mu. \end{aligned} \tag{19.17}$$

Now consider the Hilbert space $L^2(\mu_\Gamma)$. It is an exercise for the reader to show that $\{\pi_f | f \in L^2(\mu)\}$ is total in $L^2(\mu)$. Equation (19.17) implies that

$$\langle \pi_f, \pi_g \rangle = \int \overline{\pi}_f \pi_g(\sigma) d\sigma = \int \pi_{\overline{f}g}(\sigma) d\sigma = \exp \int_S \overline{f} g \, d\mu.$$

This at once enables us to see the unitary isomorphism $U : \Gamma_s(L^2(\mu)) \to L^2(\mu_\Gamma)$ satisfying the relations

$$[Ue(f)](\sigma) = \pi_f(\sigma) \text{ for all } f \text{ in } L^2(\mu).$$

Exercise 19.13: Let $\mathscr{H} = \mathbb{C}^n$ or ℓ^2 according to whether n is the finite or countably infinite cardinal. Suppose $\{\xi_j | j = 1, 2, \ldots\}$ is an n-length sequence of independent and identically distributed Bernoulli random variables assuming the values ± 1 with equal probability. Choose and fix an orthonormal basis $\{e_1, e_2, \ldots\}$ in $\mathscr{H}$. Then there exists a unique unitary isomorphism $U : \Gamma_a(\mathscr{H}) \to L^2(P)$ satisfying

$$U\Phi = 1,$$

$$U(k!)^{1/2} e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_k} = \xi_{j_1} \xi_{j_2} \cdots \xi_{j_k}, \; j_1 < j_2 < \cdots < j_k,$$

$k < 1 + \dim \mathscr{H}$ where Φ is the vacuum vector and P is the probability measure of the sequence $\{\xi_j\}$. (Hint: Use Proposition 19.3.)

Exercise 19.14: Let μ be any (not necessarily non-atomic!) σ-finite measure in $\mathbb{R}_+ = [0, \infty)$. Define the measurable space $(\Gamma(\mathbb{R}_+), \mathscr{F}_\Gamma)$ and the symmetric measure μ_Γ on $\mathscr{F}_\Gamma$ as in Example 19.12. Then there exists a unique unitary isomorphism $V : \Gamma_a(L^2(\mu)) \to L^2(\mu_\Gamma)$ satisfying

$$V\Phi = I_{\{\emptyset\}},$$

$$(Vf_1 \wedge f_2 \wedge \cdots \wedge f_n)(\sigma) = \begin{cases} 0 & \text{if } \#\sigma \neq n, \\ (n!)^{-\frac{1}{2}} \det((f_i(t_j))) & \text{if } \sigma = \{t_1, \ldots, t_n\} \\ \text{and } t_1 < t_2 < \cdots < t_n & \end{cases}$$

for all $n \geq 1$ and $f_1, f_2, \ldots, f_n$ in $L^2(\mu)$. (Hint: See Proposition 19.2 and (17.10).)

Exercise 19.15: [116] In Exercise 19.14 let μ be Lebesgue measure. Then

$$\Gamma_{fr}(L^2(\mathbb{R}_+)) = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2(\mathbb{R}_+^n).$$

There exists a unique unitary isomorphism $V : \Gamma_{fr}(L^2(\mathbb{R}_+)) \to L^2(\mu_\Gamma)$ satisfying

$$V\Phi = I_{\{\emptyset\}},$$

$$(Vf)(\sigma) = \begin{cases} 0 & \text{if } \#\sigma \neq n \\ f(t_1, t_2 - t_1, \ldots, t_n - t_{n-1}) & \text{if } \sigma = \{t_1, t_2, \ldots, t_n\} \\ \text{and } t_1 < t_2 < \cdots < t_n & \end{cases}$$

for all $n \geq 1$ and $f \in L^2(\mathbb{R}_+^n)$.

Notes

Fock spaces were introduced by Fock [43] in his investigations of quantum electrodynamics. For a mathematical account of this topic, see Cook [26]. The connection between Gaussian processes and Fock space appears in Segal [122]. Example 19.12 is from Guichardet [51] and Maassen [83]. This identification of Fock space is the key idea on which Maassen's kernel approach to quantum stochastic calculus [83, 76] and Meyer's investigations of chaos [88] are based. For further applications of this idea of kernels, see Parthasarathy [110], Lindsay and Parthasarathy [79].

20 The Weyl Representation

In Section 13 we had already remarked that the route to construct observables lies in looking at unitary representations of Lie groups and evaluating the Stone generators of their restrictions to one parameter subgroups. Any Hilbert space $\mathcal{H}$, being a vector space, is an additive group. Thanks to the existence of a scalar product in $\mathcal{H}$, we have the group $\mathcal{U}(\mathcal{H})$ of all unitary operators in $\mathcal{H}$. A pair (u, U), $u \in \mathcal{H}$, $U \in \mathcal{U}(\mathcal{H})$ acts on any element v as follows:

$$(u, U)v = Uv + u,$$

first through a "rotation" by U and then a "translation" by u. The map $v \to (u, U)v$ is a homeomorphism of $\mathcal{H}$ with inverse being given by the action of the pair $(-U^{-1}u, U^{-1})$. Successive applications by the pairs (u_2, U_2) and (u_1, U_1) on v leads to

$$\begin{aligned} (u_1, U_1)\{(u_2, U_2)v\} &= (u_1, U_1)\{U_2 v + u_2\} \\ &= U_1 U_2 v + U_1 u_2 + u_1 \\ &= ((u_1 + U_1 u_2), U_1 U_2)v. \end{aligned}$$

This suggests the following composition for the pairs (u_j, U_j), $j = 1, 2$.

$$(u_1, U_1)(u_2, U_2) = (u_1 + U_1 u_2, U_1 U_2). \tag{20.1}$$

The cartesian product $\mathcal{H} \times \mathcal{U}(\mathcal{H})$ as a set becomes a group with multiplication defined by (20.1), identity element $(0, 1)$ and inverse $(u, U)^{-1} = (-U^{-1}u, U^{-1})$. $\mathcal{H}$ inherits a topology from its norm and $\mathcal{U}(\mathcal{H})$ the corresponding strong topology. With the product topology and group operation (20.1), $\mathcal{H} \times \mathcal{U}(\mathcal{H})$ becomes a topological group which we denote by $E(\mathcal{H})$ and call the *Euclidean group* over

$\mathcal{H}$. The separability of $\mathcal{H}$ implies that $E(\mathcal{H})$ is, indeed, a complete and separable metric group. If $d = \dim \mathcal{H} < \infty$ then $E(\mathcal{H})$ is a connected Lie group of dimension $n(n+2)$. In any case $E(\mathcal{H})$ has a rich supply of one parameter subgroups. Since $(u,U)(v,1)(u,U)^{-1} = (Uv,1), \mathcal{H}$ is a normal subgroup of $E(\mathcal{H})$ if we identify any element $u \in \mathcal{H}$ as $(u,1)$ in $E(\mathcal{H})$. The quotient group $E(\mathcal{H})/\mathcal{H}$ is isomorphic to $\mathcal{U}(\mathcal{H})$. Any element U in $\mathcal{U}(\mathcal{H})$ can be identified with $(0,U)$ in $E(\mathcal{H})$.

We shall now construct a certain canonical (projective) unitary representation of $E(\mathcal{H})$ in the boson Fock space $\Gamma_s(\mathcal{H})$ over $\mathcal{H}$ and reap a rich harvest of observables which constitute the building blocks of quantum stochastic calculus.

Let $S = \{\alpha e(v) | \alpha \in \mathbb{C}, v \in \mathcal{H}\}$. By Proposition 19.4 S is total in $\Gamma_s(\mathcal{H})$. Consider, for any fixed $(u,U) \in E(\mathcal{H})$, the action induced on S by the map $e(v) \to e(Uv+u)$. This is not inner product preserving. Indeed, for any v_1, v_2 in $\mathcal{H}$

$$\langle e(Uv_1+u), e(Uv_2+u)\rangle = \langle e(v_1), e(v_2)\rangle \exp\{\|u\|^2 + \langle Uv_1, u\rangle + \langle u, Uv_2\rangle\}.$$

This shows that the correspondence

$$e(v) \to \{\exp(-\frac{1}{2}\|u\|^2 - \langle u, Uv\rangle)\} e(Uv+u)$$

yields an isometry of S onto itself. Hence by Proposition 7.2 there exists a unique unitary opeator $W(u,U)$ in $\Gamma_s(\mathcal{H})$ satisfying

$$W(u,U)e(v) = \{\exp(-\frac{1}{2}\|u\|^2 - \langle u, Uv\rangle)\} e(Uv+u) \text{ for all } v \text{ in } \mathcal{H}. \quad (20.2)$$

$W(u,U)$ is called the *Weyl operator* associated with the pair (u,U) in $E(\mathcal{H})$.

Proposition 20.1: The correspondence $(u,U) \to W(u,U)$ from $E(\mathcal{H})$ into $\mathcal{U}(\Gamma_s(\mathcal{H}))$ is strongly continuous. Furthermore

$$W(u_1,U_1)W(u_2,U_2) = e^{-i \operatorname{Im}\langle u_1, U_1 u_2\rangle} W((u_1,U_1)(u_2,U_2)) \quad \text{for all } (u_j,U_j) \text{ in } E(\mathcal{H}), \quad j = 1,2. \quad (20.3)$$

Proof: Since the scalar product in $\mathcal{H}$ is continuous in its arguments and the map $u \to e(u)$ is also continuous (see the proof of Corollary 19.5) it follows from (20.2) that $(u,U) \to W(u,U)e(v)$ is a continuous map from $E(\mathcal{H})$ into $\Gamma_s(\mathcal{H})$ for every fixed v. The totality of exponential vectors and the unitarity of Weyl operators imply the first part. The second part of the proposition is immediate from (20.2) on successive applications of $W(u_2,U_2)$ and $W(u_1,U_1)$ on an exponential vector. ■

Proposition 20.1 shows that the correspondence $(u,U) \to W(u,U)$ is a homomorphism from $E(\mathcal{H})$ into $\mathcal{U}(\Gamma_s(\mathcal{H}))$ modulo a phase-factor of modulus unity. Such a correspondence is called a *projective unitary representation* [135]. As a special case of (20.3) we obtain the following relations by putting

$$W(u) = W(u,1), \quad \Gamma(U) = W(0,U) \quad (20.4)$$

for u in $\mathcal{H}$ and U in $\mathcal{U}(\mathcal{H})$:

$$W(u)W(v) = e^{-i\,\mathrm{Im}\langle u,v\rangle}W(u+v), \tag{20.5}$$

$$W(u)W(v) = W(v)W(u)\{\exp -2i\,\mathrm{Im}\langle u,v\rangle\}, \tag{20.6}$$

$$\Gamma(U)\Gamma(V) = \Gamma(UV), \tag{20.7}$$

$$\Gamma(U)W(u)\Gamma(U)^{-1} = W(Uu), \tag{20.8}$$

$$W(su)W(tu) = W(\overline{s+t}u), \quad s,t \in \mathbb{R}. \tag{20.9}$$

It is fruitful to compare (20.6) with the Weyl commutation relations in (13.7) of Example 13.3. Equation (20.9) shows that every element u in $\mathcal{H}$ yields a one parameter unitary group $\{W(tu)|t \in \mathbb{R}\}$ and hence an observable $p(u)$ through its Stone generator so that

$$W(tu) = e^{-it\,p(u)}, t \in \mathbb{R}, u \in \mathcal{H}. \tag{20.10}$$

If S is any completely real subspace of $\mathcal{H}$ then $\langle u,v\rangle$ is real for all u,v in S and the family $\{p(u)|u \in S\}$ is commutative (or non-interfering) in the sense that all their spectral projections commute with each other.

The operator $\Gamma(U)$ defined by (20.4) is called the *second quantization* of U. Equation (20.7) shows that for every one parameter unitary group $U_t = e^{-itH}$ in $\mathcal{H}$ there corresponds a one parameter unitary group $\{\Gamma(U_t)|t \in \mathbb{R}\}$ in $\Gamma_s(\mathcal{H})$. We denote its Stone generator by $\lambda(H)$ so that

$$\Gamma(e^{-itH}) = e^{-it\lambda(H)}, \quad t \in \mathbb{R}. \tag{20.11}$$

The observable $\lambda(H)$ is called the *differential second quantization* of H.

Through (20.10) and (20.11) we thus obtain the families $\{p(u)|u \in \mathcal{H}\}$, $\{\lambda(H)|H$ an observable in $\mathcal{H}\}$ of observables in $\Gamma_s(\mathcal{H})$. The quantum stochastic calculus that we develop in the sequel depends very much on the basic properties of these observables. We shall now investigate the commutation relations obeyed by them.

Proposition 20.2: For an arbitrary finite set $\{v_1, v_2, \dots, v_n\} \subset \mathcal{H}$ the map $\underline{z} \to e(z_1v_1 + \cdots + z_nv_n)$ from $\mathbb{C}^n$ into $\Gamma_s(\mathcal{H})$ is analytic.

Proof: Since $\sum_{n=0}^{\infty}|z|^n \frac{\|v\|^n}{\sqrt{n!}} < \infty$ it follows that the map $z \to e(zv) = \sum_{n=0}^{\infty} z^n \frac{v^{\otimes n}}{\sqrt{n!}}$ is analytic so that the proposition holds for $n = 1$. When $n > 1$ choose an orthonormal basis $\{u_1, u_2, \dots, u_m\}$ for the subspace spanned by $\{v_1, v_2, \dots, v_n\}$ and note that $\Sigma_i z_i v_i = \Sigma_j L_j(\underline{z})u_j$ where $L_j(\underline{z}) = \Sigma_i z_i\langle u_j, v_i\rangle$ is linear in $\underline{z}$ for each j. Denote by $\mathcal{H}_j$ the one dimensional subspace $\mathbb{C}u_j$ for $1 \le j \le m$ and put $\mathcal{H}_{m+1} = \{u_1, \dots, u_m\}^{\perp}$. Since $\mathcal{H} = \bigoplus_{j=1}^{m+1}\mathcal{H}_j$ we can, by Proposition 19.6, identify $\Gamma_s(\mathcal{H})$ with the tensor product $\bigotimes_{j=1}^{m+1}\Gamma_s(\mathcal{H}_j)$ so that

$$e(\sum_i z_i v_i) = \{\bigotimes_{j=1}^{m} e(L_j(\underline{z})u_j)\} \otimes e(0)$$

and the required analyticity follows from the case $n = 1$. ■

Proposition 20.3: For any $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n, v$ in $\mathcal{H}$ the map $(\underline{s}, \underline{t}) \to W(s_1 u_1) \cdots W(s_m u_m) e(t_1 v_1 + t_2 v_2 + \cdots + t_n v_n + v)$ from $\mathbb{R}^{m+n}$ into $\Gamma_s(\mathcal{H})$ is analytic.

Proof: By (20.2) and (20.5) we have

$$W(s_1 u_1) \cdots W(s_m u_m) e(\sum_{j=1}^{n} t_j v_j + v) = \phi(\underline{s}, \underline{t}) e(\sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} t_j v_j + v)$$

where $\phi(\underline{s}, \underline{t})$ is the exponential of a second degree polynomial in the variables s_i, t_j, $1 \leq i \leq m$, $1 \leq j \leq n$. The required result is immediate from Proposition 20.2. ∎

For any set $S \subset \mathcal{H}$ recall (from Corollary 19.5) that $\mathcal{E}(S)$ denotes the linear manifold generated by $\{e(u) | u \in S\}$. When $S = \mathcal{H}$ and there is no confusion we write $\mathcal{E} = \mathcal{E}(\mathcal{H})$.

Proposition 20.4: For each u in $\mathcal{H}$ let $p(u)$ be the observable defined by (20.10). Then the following holds:

(i) $\mathcal{E} \subset D(p(u_1) p(u_2) \cdots p(u_n))$ for all n and $u_1, u_2, \ldots, u_n \in \mathcal{H}$;

(ii) $\mathcal{E}$ is a core for $p(u)$ for any u in $\mathcal{H}$;

(iii) $[p(u), p(v)] e(w) = \{2i \operatorname{Im}\langle u, v \rangle\} e(w)$. (20.12)

Proof: (i) is immediate from Proposition 20.3 by putting $t_1 = t_2 = \cdots = 0$, applying Stone's Theorem (Theorem 13.1) and differentiating successively with respect to $s_m, s_{m-1}, \ldots, s_1$ at the origin. To prove (ii) first observe that for any real $s \neq 0$, $p(su) = sp(u)$ and hence we may assume without loss of generality that $\|u\| = 1$. Let $\mathcal{H}_0 = \mathbb{C}u$, $\mathcal{H}_1 = \mathcal{H}_0^{\perp}$. By Proposition 19.6 $\Gamma_s(\mathcal{H}) = \Gamma_s(\mathcal{H}_0) \otimes \Gamma_s(\mathcal{H}_1)$ and for any $v \in \mathcal{H}$

$$\begin{aligned} e(v) &= e(\langle u, v \rangle u) \otimes e(v - \langle u, v \rangle u), \\ W(tu) e(v) &= e^{-\frac{1}{2}t^2 - t\langle u, v \rangle} e(v + tu) \\ &= \{W_0(tu) e(\langle u, v \rangle u)\} \otimes e(v - \langle u, v \rangle u) \end{aligned}$$

where W_0 indicates Weyl operator in $\Gamma_s(\mathcal{H}_0)$. The totality of exponential vectors implies that

$$W(tu) = W_0(tu) \otimes 1,$$

1 denoting the identity operator in $\Gamma_s(\mathcal{H}_1)$. Thus

$$p(u) = p_0(u) \otimes 1$$

where $p_0(u)$ is the Stone generator of $\{W_0(tu) | t \in \mathbb{R}\}$ in $\Gamma_s(\mathcal{H}_0)$. In other words it suffices to prove (ii) when $\dim \mathcal{H} = 1$ or $\mathcal{H} = \mathbb{C}$. Let U denote multipliction by $e^{i\theta}$ in $\mathbb{C}$. Then $\Gamma(U) W_0(u) \Gamma(U)^{-1} = W_0(e^{i\theta} u)$ and $\Gamma(U) e(v) = e(e^{i\theta} v)$. Thus

$\Gamma(U)p_0(u)\Gamma(U)^{-1} = p_0(e^{i\theta}u)$. Hence it is enough to prove (ii) when $\mathcal{H} = \mathbb{C}$ and $u = 1$. Write $p_0 = p_0(1)$. We have

$$\Gamma_s(\mathbb{C}) = \mathbb{C} \oplus \mathbb{C} \oplus \cdots$$

where any element ψ can be expressed as

$$\psi = (z_0, z_1, \ldots, z_n, \ldots),\ z_j \in \mathbb{C},\ \sum_j |z_j|^2 < \infty. \tag{20.13}$$

Let $\psi \in D(p_0)$ be such that $(\psi, p_0\psi)$ is orthogonal to every vector $(e(x), p_0 e(x))$, $x \in \mathbb{R}$ in $\Gamma_s(\mathbb{C}) \oplus \Gamma_s(\mathbb{C})$. Then

$$\langle \psi, e(x)\rangle + \langle p_0\psi, p_0 e(x)\rangle = 0.$$

By (i), $e(x) \in D(p_0^2)$ and hence

$$\langle \psi, (1 + p_0^2)e(x)\rangle = 0 \text{ for all } x \in \mathbb{R}, \tag{20.14}$$

where

$$e(x) = (1, x, \frac{x^2}{\sqrt{2!}}, \ldots, \frac{x^n}{\sqrt{n!}}, \ldots).$$

Since by definition

$$\begin{aligned} p_0^2 e(x) &= -\frac{d^2}{dt^2} W(t)e(x)|_{t=0} \\ &= -\frac{d^2}{dt^2} e^{-\frac{1}{2}t^2 - tx} e(x+t)_{t=0}, \end{aligned}$$

(20.13) and (20.14) yield

$$(2 - x^2)\sum_{n=0}^{\infty} \frac{\overline{z}_n}{\sqrt{n!}} x^n + 2\sum_{n=1}^{\infty} \frac{n\overline{z}_n}{\sqrt{n!}} x^n - \sum_{n=2}^{\infty} \frac{n(n-1)}{\sqrt{n!}} \overline{z}_n x^{n-2} = 0. \tag{20.15}$$

Putting

$$f(x) = \sum_{n=0}^{\infty} \frac{\overline{z}_n}{\sqrt{n!}} x^n$$

(20.15) becomes

$$(2 - x^2)f(x) + 2xf'(x) - f''(x) = 0$$

or

$$f(x) = (\alpha e^x + \beta e^{-x})e^{\frac{1}{2}x^2}$$

where α, β are scalars. This shows that

$$\overline{z}_n = \frac{\alpha + (-1)^n\beta}{\sqrt{n!}} \mathbb{E}(\xi + 1)^n$$

where ξ is a standard normal random variable. Now (20.13) implies

$$\sum_{n=0}^{\infty} \frac{|\alpha + (-1)^n\beta|^2}{n!} \{\mathbb{E}(\xi + 1)^n\}^2 < \infty. \tag{20.16}$$

Let $n = 2k$. Then

$$\mathbb{E}(\xi + 1)^{2k} \geq \mathbb{E}\xi^{2k} = 1.3.5 \cdots 2k - 1.$$

By Stirling's formula

$$\frac{(\mathbb{E}\xi^{2k})^2}{2k!} \cong \frac{c}{\sqrt{k}} \text{ as } k \to \infty$$

where $c > 0$ is a constant and $\cong$ denotes asymptotic equality. Thus

$$\sum_{k=0}^{\infty} \frac{\{\mathbb{E}(\xi + 1)^{2k}\}^2}{2k!} = \infty$$

and (20.16) is possible only if $\alpha + \beta = 0$. Now consider $n = 2k + 1$. Then

$$\mathbb{E}(\xi + 1)^{2k+1} \geq (2k + 1)\mathbb{E}\xi^{2k} = 1.3.5 \cdots (2k + 1)$$

and

$$\frac{\{\mathbb{E}(\xi + 1)^{2k+1}\}^2}{2k + 1!} \geq (2k + 1)\frac{(1.3.5 \cdots 2k - 1)^2}{2k!} \cong c\sqrt{k} \text{ as } k \to \infty$$

for some $c > 0$. Thus (20.16) is possible only if $\alpha - \beta = 0$. In other words $\alpha = \beta = 0$ and hence $f(x) = 0$. Thus $\psi = 0$ or $G(p_0) \cap \{(e(x), p_0 e(x)) | x \in \mathbb{R}\}^{\perp} = 0, G(p_0)$ denoting the graph of p_0. This proves (ii).

To prove (iii) we observe that for any u, v, w_1, w_2 in $\mathcal{H}$

$$\begin{aligned}
&\langle p(u)e(w_1), p(v)e(w_2)\rangle = \\
&\qquad \frac{\partial^2}{\partial s \partial t}\langle W(su)e(w_1), W(tv)e(w_2)\rangle|_{s=t=0} \\
&\quad = \frac{\partial^2}{\partial s \partial t} \exp\{-\frac{1}{2}s^2\|u\|^2 - \frac{1}{2}t^2\|v\|^2 - s\langle\overline{u, w_1}\rangle \\
&\qquad -t\langle v_1, w_2\rangle + \langle w_1 + su, w_2 + tv\rangle\}|_{s=t=0} \\
&\quad = \{\langle u, v\rangle + (\langle u, w_2\rangle - \langle w_1, u\rangle)(\langle w_1, v\rangle - \langle v, w_2\rangle)\}e^{\langle w_1, w_2\rangle}.
\end{aligned} \tag{20.17}$$

Interchanging u and v in this equation, using (i) of the proposition and the totality of exponential vectors we obtain

$$\{p(u)p(v) - p(v)p(u)\}e(w_2) = (\langle u, v\rangle - \langle v, u\rangle)e(w_2).$$

■

Corollary 20.5: Let $S \subset \mathcal{H}$ be any dense set. Then $p(u)$ is essentially selfadjoint in the domain $\mathcal{E}(S)$.

Proof: For any v, w in $\mathcal{H}$ define

$$\begin{aligned} B(v,w) &= \|e(v)-e(w)\|^2 + \|p(u)(e(v)-e(w))\|^2 \\ &= \|e(v)-e(w)\|^2 \\ &\quad + \frac{\partial^2}{\partial s \partial t}\langle W(su)(e(v)-e(w)), W(tu)(e(v)-e(w))\rangle|_{s=t=0} \end{aligned}$$

Then $B(v, w)$ is a continuous function of v and w. Thus any element of the form $(e(w), p(u)e(w))$ in $G(p(u))$ can be approximated by a sequence of the form $\{(e(v_n), p(u)e(v_n))\}$ when $v_n \in S$. The rest is immediate from (ii) in Proposition 20.4. ∎

Corollary 20.6: The linear manifold of all finite particle vectors in $\Gamma_s(\mathcal{H})$ is a core for every observable $p(u)$, $u \in \mathcal{H}$.

Proof: This follows easily from Proposition 20.3, 20.4. ∎

Proposition 20.7: For any observable H in $\mathcal{H}$ let $\lambda(H)$ denote its differential second quantization in $\Gamma_s(\mathcal{H})$. Then the following holds;

(i) $\mathcal{E}(D(H)) \subset D(\lambda(H))$;

(ii) $\mathcal{E}(D(H^2))$ is a core for $\lambda(H)$;

(iii) For any two bounded observables H_1, H_2 in $\mathcal{H}$ and any v in $\mathcal{H}$

$$i[\lambda(H_1), \lambda(H_2)]e(v) = \lambda(i[H_1, H_2])e(v).$$

Proof: Let $u \in D(H)$. Stone's Theorem implies that the map $t \to e^{-itH}u$ is differentiable. By Proposition 20.2 the map $t \to e^{-it\lambda(H)}e(u) = e(e^{-itH}u)$ is differentiable. Hence $e(u) \in D(\lambda(H))$. This proves (i).

If $v \in D(H^2)$ then $t \to e^{-itH}v$ is twice differentiable and Proposition 20.2 implies that $t \to e^{-it\lambda(H)}e(v)$ is a twice differentiable map and hence $e(v) \in D(\lambda(H)^2)$. Now we proceed as in the proof of (ii) in Proposition 20.4. Let $\psi \in D(\lambda(H))$ be such that $(\psi, \lambda(H)\psi)$ is orthogonal to $(e(v), \lambda(H)e(v))$ for all $v \in D(H^2)$. Then

$$\langle \psi, e(v)\rangle + \langle \lambda(H)\psi, \lambda(H)e(v)\rangle = 0.$$

Since $e(v) \in D(\lambda(H)^2)$ we have

$$\langle \psi, \{1 + \lambda(H)^2\}e(v)\rangle = 0 \text{ for all } v \in D(H^2). \tag{20.18}$$

Let $\tilde{\mathcal{H}}_n$ denote the n-particle subspace in $\Gamma_s(\mathcal{H})$ and let $\tilde{E}_n$ be the projection on $\tilde{\mathcal{H}}_n$. Then $\lambda(H) = \bigoplus_{n=0}^{\infty}\lambda_n(H)$ where $\lambda_n(H)$ is the generator of the n-fold tensor product $e^{-itH} \otimes \cdots \otimes e^{-itH}$ restricted to $\tilde{\mathcal{H}}_n$. Since $e(tv) = \bigoplus_{n=0}^{\infty} t^n \frac{v^{\otimes^n}}{\sqrt{n!}}$ we obtain by changing v to tv in (20.18)

$$\langle \tilde{E}_n\psi, (1 + \lambda_n(H)^2)v^{\otimes^n}\rangle = 0, \quad n = 1, 2, \ldots, \; v \in D(H^2).$$

A polarisation argument yields

$$\begin{aligned} &\langle \tilde{E}_n\psi, \{1+\lambda_n(H)^2\}E_n v_1 \otimes \cdots \otimes v_n\rangle = 0 \\ &\quad \text{for all } v_j \in D(H^2), \qquad 1 \le j \le n. \end{aligned} \tag{20.19}$$

where E_n denotes the symmetrization projection. Since the linear manifold generated by $\{E_n v_1 \otimes \cdots \otimes v_n | v_j \in D(H^2), 1 \le j \le n\}$ is a core for $1+\lambda_n(H)^2$, (20.19) implies

$$\langle \tilde{E}_n\psi, \{1+\lambda_n(H)^2\}\xi\rangle = 0 \text{ for all } \xi \in D(\lambda_n(H)^2).$$

Since $1+\lambda_n(H)^2$ has a bounded inverse (thanks to the spectral theorem) its range is $\tilde{\mathscr{H}}_n$. Thus $\tilde{E}_n\psi = 0$ for all n. In other words

$$G(\lambda(H)) \cap \{(e(v), \lambda(H)e(v)) | v \in D(H^2)\}^{\perp} = 0.$$

This proves (ii).

To prove (iii) we first observe by using (i)

$$\begin{aligned} \langle e(u), \lambda(H)e(v)\rangle &= i\frac{d}{dt}\langle e(u), e(e^{-itH}v)\rangle|_{t=0} \\ &= i\frac{d}{dt}\exp\langle u, e^{-itH}v\rangle|_{t=0} \\ &= \langle u, Hv\rangle e^{\langle u,v\rangle} \end{aligned} \tag{20.20}$$

for any observable H in $\mathscr{H}$ and $v \in D(H)$. If H_1, H_2 are bounded observables the map $(s,t) \to e^{-isH_1}e^{-itH_2}v$ is analytic for every $v \in \mathscr{H}$ and, in particular, by Proposition 20.2 the map $(s,t) \to e(e^{-isH_1}e^{-itH_2}v)$ is differentiable and

$$\lambda(H_1)\lambda(H_2)e(v) = -\frac{\partial^2}{\partial s \partial t}e(e^{-isH_1}e^{-itH_2}v)|_{s=t=0}.$$

Thus for any $u, v \in \mathscr{H}$,

$$\begin{aligned} &\langle e(u), \lambda(H_1)\lambda(H_2)e(v)\rangle = \frac{-\partial^2}{\partial s \partial t}\exp\langle e^{isH_1}u, e^{-itH_2}v\rangle|_{s=t=0} \\ &= \{\langle u, H_1H_2v\rangle + \langle u, H_1v\rangle\langle u, H_2v\rangle\}\exp\langle u, v\rangle. \end{aligned}$$

Now the totality of exponential vectors and (20.20) imply (iii). ■

Proposition 20.8: Let H be an observable in $\mathscr{H}$ and $u, v \in D(H^2)$. Then

$$i[p(u), \lambda(H)]e(v) = -p(iHu)e(v).$$

Proof: Let $u, v, w \in D(H^2)$, $U_t = e^{-itH}$. Then the $\mathscr{H}$-valued functions

$$\Gamma(U_t)W(su)e(w) = e^{-\frac{1}{2}s^2\|u\|^2 - s\langle u,w\rangle} e(U_t(su+w)), \quad (20.21)$$

$$W(su)\Gamma(U_t^{-1})e(w) = e^{-\frac{1}{2}s^2\|u\|^2 - s\langle U_t u,w\rangle} e(U_t^{-1}w+su), \quad (20.22)$$

$$\begin{aligned}\Gamma(U_t)W(su)\Gamma(U_t^{-1})e(w) &= W(sU_t u)e(w) \\ &= e^{-\frac{1}{2}s^2\|u\|^2 - s\langle U_t u,w\rangle} e(w+sU_t w) \end{aligned} \quad (20.23)$$

are twice differentiable in (s,t). Hence by Stone's Theorem and (20.23) we have

$$\begin{aligned}&\frac{\partial^2}{\partial t \partial s}\langle e(v), \Gamma(U_t)W(su)\Gamma(U_t^{-1})e(w)\rangle|_{s=t=0} \\ &= \{\langle v, -iHu\rangle - \langle -iHu, w\rangle\}e^{\langle v,w\rangle}.\end{aligned} \quad (20.24)$$

On the other hand the left hand side of the above equation can be written as

$$\begin{aligned}&\frac{\partial^2}{\partial s \partial t}\langle \Gamma(U_t^{-1})e(v), W(su)\Gamma(U_t^{-1})e(w)\rangle|_{s=t=0} \\ &= \frac{\partial}{\partial s}\{\langle i\lambda(H)e(v), W(su)e(w)\rangle + \langle e(v), W(su)i\lambda(H)e(w)\rangle\}|_{s=0} \\ &= \langle e(v), \{p(u)\lambda(H) - \lambda(H)p(u)\}e(w)\rangle.\end{aligned} \quad (20.25)$$

Furthermore, for any u, v, w in $\mathscr{H}$

$$\begin{aligned}\langle e(v), p(u)e(w)\rangle &= i\frac{d}{dt}\langle e(v), W(tu)e(w)\rangle|_{t=0} \\ &= i\frac{d}{dt}\exp\{-\frac{1}{2}t^2\|u\|^2 - t\langle u,w\rangle + \langle v, w+tu\rangle\}|_{t=0} \\ &= i(\langle v,u\rangle - \langle u,w\rangle)\exp\langle v,w\rangle.\end{aligned} \quad (20.26)$$

Equating the right hand side expressions in (20.24) and (20.25) and using (20.26) we obtain

$$i[p(u), \lambda(H)]e(w) = -p(iHu)e(w).$$ ■

Proposition 20.9: Let T be any bounded operator in $\Gamma_s(\mathscr{H})$ such that $TW(u) = W(u)T$ for all u in $\mathscr{H}$. Then T is a scalar multiple of the identity.

Proof: Without loss of generality we assume that $\mathscr{H} = \ell^2$. The same proof will go through when $\dim \mathscr{H} < \infty$. Consider the unitary isomorphism $U : \Gamma_s(\mathscr{H}) \to L^2(P)$ discussed in Example 19.8 where P is the probability measure of an independent and identically distributed sequence of standard Guassian random variables $\underline{\xi} = (\xi_1, \xi_2, \ldots)$. Then

$$[Ue(\underline{z})](\underline{\xi}) = \exp\sum_j (z_j\xi_j - \frac{1}{2}z_j^2).$$

Elementary computation shows that the Weyl operators obey the following relations thanks to the totality of exponential vectors in $\Gamma_s(\mathcal{H})$ and the set

$$\{\exp \sum_j a_j \xi_j | \underline{a} \in \ell^2\} \text{ in } L^2(P):$$

for any square summable real sequence $\underline{a}$

$$\{UW(-i\underline{a})U^{-1}f\}(\underline{\xi}) = e^{-i\sum_j a_j\xi_j} f(\underline{\xi}) \tag{20.27}$$

$$\{UW(-\frac{1}{2}\underline{a})U^{-1}f\}(\underline{\xi}) = e^{-\frac{1}{4}\|\underline{a}\|^2 - \frac{1}{2}\sum_j a_j\xi_j} f(\underline{\xi}+\underline{a}) \tag{20.28}$$

for all $f \in L^2(P)$. Let $UTU^{-1} = S$. Then S commutes with $UW(-i\underline{a})U^{-1}$ and $UW(-\frac{1}{2}\underline{a})U^{-1}$ for all real square summable sequences $\underline{a}$. Equation (20.27) implies that S commutes with the operator of multiplication of $\phi(\underline{\xi})$ for every bounded random variable $\phi(\underline{\xi})$. In particular, for any indicator random variable I_E we have

$$SI_E = SI_E 1 = I_E S1.$$

If $S1 = \psi$ then we conclude that $(Sf)(\underline{\xi}) = \psi(\underline{\xi})f(\underline{\xi})$ for every f in $L^2(P)$. Now using the commutativity of S with the operators defined by (20.28) we conclude that

$$\psi(\underline{\xi}+\underline{a})f(\underline{\xi}+\underline{a}) = \psi(\underline{\xi})f(\underline{\xi}+\underline{a}) \text{ a.e. } \underline{\xi}$$

for each real square summable sequence $\underline{a}$ and f in $L^2(P)$. Thus

$$\psi(\underline{\xi}+\underline{a}) = \psi(\underline{\xi}) \text{ a.e. } \underline{\xi}$$

for each $\underline{a}$ of the form $(a_1, a_2, \ldots, a_n, 0, 0, \ldots), a_j \in \mathbb{R}$. This means that $\psi(\underline{\xi})$ is independent of $\xi_1, \xi_2, \ldots, \xi_n$ for each n. An application of Kolmogorov's 0-1 law shows that ψ is a constant. ∎

Suppose $\{T_\alpha | \alpha \in J\}$ is a family of bounded operators in a Hilbert space $\mathcal{H}$, which is closed under the adjoint operation $*$. Such a family is said to be *irreducible* if for any bounded operator S the identity $ST_\alpha = T_\alpha S$ for all $\alpha \in J$ implies that S is a scalar multiple of the identity. For such an irreducible family the only closed subspaces invariant under all the T_α are either $\{0\}$ or $\mathcal{H}$. Indeed, if P is an orthogonal projection on such an invariant subspace then $PT_\alpha = T_\alpha P$ for all α. In this sense the family of Weyl operators $\{W(u)|u \in \mathcal{H}\}$ is irreducible in $\Gamma_s(\mathcal{H})$ according to Proposition 20.9. We now summarise our conclusions in the form of a theorem.

Theorem 20.10: Let $\mathcal{H}$ be any complex separable Hilbert space and let $\Gamma_s(\mathcal{H})$ be the boson Fock space over $\mathcal{H}$. Let $W(u, U), (u, U) \in E(\mathcal{H})$ be the Weyl operator defined by (20.2). The mapping $(u, U) \to W(u, U)$ is a strongly continuous, irreducible and unitary projective representation of the group $E(\mathcal{H})$. For any u in $\mathcal{H}$ and any observable H on $\mathcal{H}$ there exist observables $p(u)$ and $\lambda(H)$ satisfying

$$W(tu) = W(tu, 1) = e^{-itp(u)}$$

$$\Gamma(e^{-itH}) = W(0, e^{-itH}) = e^{-it\lambda(H)}$$

for all t in $\mathbb{R}$. The observables $p(u)$ and $\lambda(H)$ obey the following commutation relations:

(i) $[p(u), p(v)]e(w) = 2i\,\mathrm{Im}\langle u, v\rangle e(w)$ for all u, v, w in $\mathcal{H}$;

(ii) $i[p(u), \lambda(H)]e(v) = -p(iHu)e(v)$ for all $u, v \in D(H^2)$;

(iii) for any two bounded observables H_1, H_2 in $\mathcal{H}$ and $v \in \mathcal{H}$

$$i[\lambda(H_1), \lambda(H_2)]e(v) = \lambda(i[H_1, H_2])e(v).$$

Proof: This is contained in Proposition 20.4, 20.7–20.9. ■

We shall now introduce a family of operators in terms of which computations involving the Weyl operators or, equivalently, the operators $p(u)$ and $\lambda(H)$ become considerably simplified. We write

$$q(u) = -p(iu), a(u) = \frac{1}{2}(q(u) + ip(u)), a^\dagger(u) = \frac{1}{2}(q(u) - ip(u)) \qquad (20.29)$$

for any u in $\mathcal{H}$. For any bounded operator H in $\mathcal{H}$ we write

$$\lambda(H) = \lambda(\frac{1}{2}(H + H^*)) + i\lambda(\frac{1}{2i}(H - H^*)), \lambda^\dagger(H) = \lambda(H^*). \qquad (20.30)$$

Proposition 20.11: Let T be any operator of the form $T = T_1T_2\cdots T_n$ where $T_i = p(u_i)$ or $\lambda(H_i)$ for some $u_i \in \mathcal{H}$ or some bounded observable H_i in $\mathcal{H}$. Then the linear manifold $\mathcal{E}$ generated by all the exponential vectors in $\Gamma_s(\mathcal{H})$ satisfies the relation $\mathcal{E} \subset D(T)$.

Proof: The proof is analogous to those of (i) and (iii) in Proposition 20.4, 20.7 and we leave it to the reader. ■

Proposition 20.12: Let $a(u), a^\dagger(u), \lambda(H), \lambda^\dagger(H)$ be defined as in (20.29), (20.30) for $u \in \mathcal{H}$, $H \in \mathcal{B}(\mathcal{H})$. For any operator of the form $T = T_1T_2\cdots T_n$ where each T_j is one of the operators $a(u_j), a^\dagger(u_j), \lambda(H_j), u_j \in \mathcal{H}$, $H_j \in \mathcal{B}(\mathcal{H})$, $1 \le j \le n$, $n = 1, 2, \ldots$, the relation $\mathcal{E} \subset D(T)$ holds. Furthermore, for any $\psi, \psi_1, \psi_2 \in \mathcal{E}$ the following relations hold:

(i) $a(u)e(v) = \langle u, v\rangle e(v)$;

(ii) $\langle a^\dagger(u)\psi_1, \psi_2\rangle = \langle \psi_1, a(u)\psi_2\rangle$;

(iii) $\langle \lambda^\dagger(H)\psi_1, \psi_2\rangle = \langle \psi_1, \lambda(H)\psi_2\rangle$;

(iv) The restrictions of $a(u)$ and $a^\dagger(u)$ to $\mathcal{E}$ are respectively antilinear and linear in the variable u. The restriction of $\lambda(H)$ to $\mathcal{E}$ is linear in the variable H;

$$[a(u), a(v)]\psi = [a^\dagger(u), a^\dagger(v)]\psi = 0,$$

$$[a(u), a^\dagger(v)]\psi = \langle u, v\rangle\psi,$$

$$[\lambda(H_1), \lambda(H_2)]\psi = \lambda([H_1, H_2])\psi,$$

$$[a(u), \lambda(H)]\psi = a(H^*u)\psi,$$

$$[a^\dagger(u), \lambda(H)]\psi = -a^\dagger(Hu)\psi.$$

Proof: From (20.29) and (20.26) we have

$$\langle e(w), a(u)e(v)\rangle = \langle u, v\rangle\langle e(w), e(v)\rangle \text{ for all } w \text{ in } \mathcal{H}.$$

Since exponential vectors are total in $\Gamma_s(\mathcal{H})$ we obtain (i). For the remaining parts we may assume without loss of generality that ψ, ψ_1, ψ_2 are exponential vectors. Since $\mathcal{E} \subset D(p(u))$ for any u in $\mathcal{H}$ and $p(u)$ is selfadjoint we have

$$\begin{aligned}\langle a^\dagger(u)e(w), e(v)\rangle &= \langle \frac{1}{2}(-p(iu) - ip(u))e(w), e(v)\rangle \\ &= \langle e(w), \frac{1}{2}(-p(iu) + ip(u))e(v)\rangle \\ &= \langle e(w), a(u)e(v)\rangle.\end{aligned}$$

This proves (ii). When $H = H^*$, $\lambda^\dagger(H) = \lambda(H)$ and (iii) is trivial. Now (iii) is immediate from (20.30). The antilinearity of $a(u)$ in the variable u follows from (i). The linearity of $a^\dagger(u)$ in u follows from (i) and (ii). When H is selfadjoint

$$\begin{aligned}\langle e(u), \lambda(H)e(v)\rangle &= i\langle e(u), \frac{d}{dt}e(e^{-itH}v)|_{t=0}\rangle \\ &= i\frac{d}{dt}\exp\langle u, e^{-itH}v\rangle \\ &= \langle u, Hv\rangle \exp\langle u, v\rangle.\end{aligned}$$

Thus the linearity of $\lambda(H)$ in H follows from (20.30). This proves (iv). (v) follows from Theorem 20.10 and definitions (20.29), (20.30). ■

Proposition 20.13: The operators $a^\dagger(u), u \in \mathcal{H}$ and $\lambda(H), H \in \mathcal{B}(\mathcal{H})$ obey the following relations:

(i) $\langle e(v), \lambda(H)e(w)\rangle = \langle v, Hw\rangle e^{\langle v,w\rangle}$;

(ii) $\langle a^\dagger(u_1)e(v), a^\dagger(u_2)e(w)\rangle = \{\langle u_1, w\rangle\langle v, u_2\rangle + \langle u_1, u_2\rangle\}e^{\langle v,w\rangle}$;

(iii) $\langle a^\dagger(u)e(v), \lambda(H)e(w)\rangle = \{\langle u, w\rangle\langle v, Hw\rangle + \langle u, Hw\rangle\}e^{\langle v,w\rangle}$;

(iv) $\langle \lambda(H_1)e(v), \lambda(H_2)e(w)\rangle = \{\langle H_1v, w\rangle\langle v, H_2w\rangle + \langle H_1v, H_2w\rangle\}e^{\langle v,w\rangle}$.

Proof: (i) already occurs in the proof of Proposition 20.12. (ii) follows from (i), (ii) and the second commutation relation in (v) of Proposition 20.12. (iii) follows from the fourth commutation relation of (v) and (i) in Proposition 20.12 and property (i) of the present proposition. (iv) results from

$$\langle \lambda(H_1)e(v), \lambda(H_2)e(w)\rangle = \frac{\partial^2}{\partial s\partial t}\exp\langle e^{-isH_1}v, e^{-itH_2}w\rangle|_{s=t=0}$$

when H_1, H_2 are selfadjoint and the general case from (20.30). ■

We now make a notational remark in the context of Dirac's bra and ket symbols. Write

$$a(u) = a(\langle u|),\ a^\dagger(u) = a^\dagger(|u\rangle).$$

Then we have the suggestive relations:

$$a(\langle u|)e(v) = \langle u, v\rangle e(v),$$

$$[a(\langle u|), a^\dagger(|v\rangle)] = \langle u, v\rangle,$$

$$[a(\langle u|), \lambda(H)] = a(\langle u|H) = a(\langle H^* u|),$$

$$[\lambda(H), a^\dagger(|u\rangle)] = a^\dagger(H|u\rangle) = a^\dagger(|Hu\rangle)$$

in the domain $\mathcal{E}$. The remaining relations in Proposition 20.12 and 20.13 also acquire a natural significance especially in the context of the quantum stochastic calculus that will be developed later.

Proposition 20.14: The operators $a(u), a^\dagger(u), u \in \mathcal{H}$ and $\lambda(H), H \in \mathcal{B}(\mathcal{H})$ satisfy the following relations:

(i) $a^\dagger(u)e(v) = \frac{d}{dt}e(v+tu)|_{t=0}$;

(ii) the linear manifold of all finite particle vectors is contained in the domains of $a(u), a^\dagger(u)$ and $\lambda(H)$. Furthermore

$$a(u)e(0) = 0, a(u)v^{\otimes^n} = \sqrt{n}\langle u, v\rangle v^{\otimes^{n-1}} \text{ if } n \geq 1 \tag{20.31}$$

$$a^\dagger(u)v^{\otimes^n} = (n+1)^{-\frac{1}{2}}\sum_{r=0}^{n} v^{\otimes^r} \otimes u \otimes v^{\otimes^{n-r}}, \tag{20.32}$$

$$\lambda(H)v^{\otimes^n} = \sum_{r=0}^{n-1} v^r \otimes Hv \otimes v^{n-r-1}. \tag{20.33}$$

Proof: (i) follows from the two relations

$$\langle a^\dagger(u)e(v), e(w)\rangle = \langle e(v), a(u)e(w)\rangle = \langle u, w\rangle e^{\langle v,w\rangle},$$

$$\langle \frac{d}{dt}e(v+tu)|_{t=0}, e(w)\rangle = \frac{d}{dt}e^{\langle v+tu,w\rangle}|_{t=0} = \langle u, w\rangle e^{\langle v,w\rangle}$$

and the totality of exponential vectors. The first part of (ii) is an easy consequence of Stone's Theorem and Proposition 20.2. By (i) in Proposition 20.12 $a(u)e(tv) = t\langle u, v\rangle e(tv)$ for all $t \in \mathbb{C}$. Identifying coefficients of t^n on both sides of this equation we obtain (20.31). Expanding $e(sv+tu)$ as $\bigoplus_{n=0}^\infty \frac{(sv+tu)^{\otimes^n}}{\sqrt{n!}}$, differentiating at $t = 0$ and using (i) we obtain (20.32). When H is selfadjoint $e(e^{-itH}v) = e^{-it\lambda(H)}e(v) = \bigoplus_{n=0}^\infty \frac{(e^{-itH}v)^{\otimes^n}}{\sqrt{n!}}$. Change v to sv and identify the coefficients of $s^n t^0$. Then we get (20.33). When H is not selfadjoint (20.33) follows from (20.30). ■

Proposition 20.15: In the domain $\mathcal{E}$ the Weyl operator $W(u, U)$ admits the factorisation:

$$W(u, U) = e^{-\frac{1}{2}\|u\|^2} e^{a^\dagger(u)}\Gamma(U)e^{-a(U^{-1}u)} \tag{20.34}$$

for all $u \in \mathcal{H}$, $U \in \mathcal{U}(\mathcal{H})$.

Proof: By (i) in Proposition 20.12 we have

$$e^{-a(U^{-1}u)}e(v) = e^{-\langle U^{-1}u,v\rangle}e(v) = e^{-\langle u,Uv\rangle}e(v).$$

By (20.2) and (20.4)

$$\Gamma(U)e^{-a(U^{-1}u)}e(v) = e^{-\langle u,Uv\rangle}e(Uv).$$

By (i) in Proposition 20.14 and (20.2) we get

$$\begin{aligned} e^{a^\dagger(u)}\Gamma(U)e^{-a(U^{-1}u)}e(v) &= e^{-\langle u,Uv\rangle}e(Uv+u) \\ &= e^{\frac{1}{2}\|u\|^2}W(u,U)e(v). \end{aligned}$$ ■

Proposition 20.14 shows that $a((u)e(0) = 0$ and $a(u)$ transforms an n-particle vector into an $(n-1)$-particle vector whereas $a^\dagger(u)$ sends an n-particle vector into an $(n+1)$-particle vector. $\lambda(H)$ leaves the n-particle subspace invariant. In view of these properties we call $a(u)$ the *annihilation operator* associated with u and $a^\dagger(u)$ the *creation operator* associated with u. $\lambda(H)$ is called the *conservation operator* associated with H. The quantum stochastic calculus that we shall develop in the sequel will depend heavily on the properties of these operators described in Proposition 20.12–20.15.

Exercise 20.16: (i) Let G be a connected Lie group with Lie algebra $\mathscr{G}$. Suppose $g \to U_g$ is a unitary representation of G in $\mathscr{H}$. For any $X \in \mathscr{G}$ let $\pi(X)$ denote the Stone generator of $\{U_{\exp tX} | t \in \mathbb{R}\}$. Let

$$D = \{v \in \mathscr{H} | \text{the map } g \to U_g v \text{ from } G \text{ into } \mathscr{H} \text{ is infinitely differentiable}\}.$$

Then

$$i[\lambda(\pi(X)), \lambda(\pi(Y))]e(v) = -\lambda(\pi([X,Y]))e(v)$$

for all $v \in D, X, Y \in \mathscr{G}$.

(ii) Let $\alpha : \mathscr{G} \to \mathscr{H}$ be a linear map satisfying

$$\pi(X)\alpha(Y) - \pi(Y)\alpha(X) = i\alpha([X,Y]) \text{ for all } X,Y \in \mathscr{G}. \tag{20.35}$$

Define

$$\Phi(X) = a^\dagger(\alpha(X)) - a(\alpha(X)) - i\lambda(\pi(X)).$$

Then

$$[\Phi(X), \Phi(Y)] = \Phi([X,Y]) - 2i\,\mathrm{Im}\langle\alpha(X), \alpha(Y)\rangle$$

in the domain $\mathscr{E}(D)$.

(iii) If $u \in D$ then $\alpha(X) = \pi(X)u$ satisfies (20.35). If $\Psi(X) = \Phi(X) - i\langle u, \pi(X)u\rangle$ then

$$[\Psi(X), \Psi(Y)] = \Psi([X,Y]) \text{ for all } X,Y \in \mathscr{G}.$$

Exercise 20.17: Let $h = \bigoplus_{k=1}^{\infty} h_k$ be a direct sum of Hilbert spaces $\{h_k\}$ and let

$$D = \{u | u \in h, u = \oplus_k u_k, u_k = 0 \text{ for all but a finite number of } k's\}.$$

Suppose T is an operator on h with domain D such that $T|_{h_k}$ is bounded for every k and $T(h_k) \subseteq \bigoplus_{j \geq n_k} h_j$ where $n_k \to \infty$ as $k \to \infty$. Then T is closable. In particular, $a(u)$ and $a^\dagger(u)$ restricted to the domain of finite particle vectors are closable and $a^\dagger(u) \subset a(u)^*$.

Exercise 20.18: (a) Let $\{e_j | j = 1, 2, \ldots ,\}$ be an orthonormal basis in $\mathcal{H}$. Define on the domain $\Gamma_s^0(\mathcal{H})$ of all finite particle vectors the operators a_j and $a_j^\dagger$ as the restrictions of $a(e_j)$ and $a^\dagger(e_j)$ respectively for each $j = 1, 2, \ldots$. Let

$$q_j = 2^{-\frac{1}{2}}(a_j + a_j^+), p_j = -i2^{-\frac{1}{2}}(a_j - a_j^\dagger).$$

Then q_j, p_j are essentially selfadjoint in $\Gamma_s^0(\mathcal{H})$. Furthermore

$$[a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0, \ [a_j, a_k^\dagger] = \delta_{jk},$$

$$[q_j, q_k] = [p_j, p_k] = 0, \ [q_j, p_k] = i\delta_{jk}.$$

(b) When $\mathcal{H} = \mathbb{C}, e_1^{\otimes^0} = \Phi$, $e_1^{\otimes^j} = f_j$, $j = 1, 2, \ldots$ then $\{\Phi, f_1, f_2, \ldots\}$ is an orthonormal basis for $\Gamma_s(\mathbb{C})$. If $a_1 = a$, $a_1^\dagger = a^\dagger$ then the matrices of a and $a^\dagger$ are respectively

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \ldots & \ldots & \ldots \\ 0 & 0 & \sqrt{2} & 0 & 0 & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \sqrt{3} & 0 & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots & \sqrt{n} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix},$$

$$a^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots & \ldots & \ldots & \ldots \\ 1 & 0 & 0 & 0 & \ldots & \ldots & \ldots & \ldots \\ 0 & \sqrt{2} & 0 & 0 & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \sqrt{3} & 0 & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \ldots & \sqrt{n} & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

where f_j is considered as a column vector with 1 in the j-th position and 0 elsewhere and $f_0 = \Phi$. This may be expressed in Dirac's notation as

$$a = \sum_{j=1}^{\infty} \sqrt{j} |f_{j-1}\rangle\langle f_j|, \ a^\dagger = \sum_{j=1}^{\infty} \sqrt{j} |f_j\rangle\langle f_{j-1}|.$$

The operator $a^\dagger a = N$ is called the *number operator* because the j-th particle vector with 1 in the j-th position and 0 elsewhere is an eigenvector for the eigenvalue j, f_0 denoting Φ. If L denotes the coisometry $\sum_{j=1}^{\infty} |f_{j-1}\rangle\langle f_j|$ we can express

$$a = \sqrt{N+1}L, \; a^\dagger = L^* \sqrt{N+1}$$

on the domain of finite particle vectors. In the pure state Φ the observable $\frac{1}{2}(L+L^*)$ has standard Wigner distribution whereas the closure of $a+a^*$ has standard normal distribution. (See Exercise 4.5, 6.3.)

Exercise 20.19: Let H be a selfadjoint operator in $\mathcal{H}$ with a complete orthonormal eigenbasis $\{e_j | j = 1, 2, \ldots\}$ satisfying $He_j = \lambda_j e_j$ for each j. Then its differential second quantization $\lambda(H)$ in $\Gamma_s(\mathcal{H})$ has the orthonormal eigenbasis

$$\left\{ \left[\frac{(r_1 + \cdots + r_k)!}{r_1! \cdots r_k!} \right]^{1/2} e_1^{r_1} e_2^{r_2} \cdots e_k^{r_k} | r_j \geq 0, \; j = 1, 2, \ldots, k, \; k = 1, 2, \ldots \right\} \tag{20.36}$$

(expressed in the notation of Proposition 19.3) with corresponding eigenvalues $\{r_1\lambda_1 + \cdots + r_k\lambda_k\}$. In the linear manifold generated by the set (20.36) $\lambda(H)$ coincides with the (formal) sum $\sum_j \lambda_j a_j^\dagger a_j$ where $a_j, a_j^\dagger$ are as in Exercise 20.18.

Exercise 20.20: (a) There exists a unique unitary isomorphism $U : \Gamma_s(\mathbb{C}) \rightarrow L^2(\mathbb{R})$ satisfying

$$\{Ue(z)\}(x) = (2\pi)^{-\frac{1}{4}} \exp -(\frac{x^2}{4} - zx + \frac{z^2}{2}) \text{ for all } z \in \mathbb{C};$$

(b) Let $D_\theta, \theta > 0$ denote the unitary dilation operator $L^2(\mathbb{R})$ defined by

$$(D_\theta f)(x) = \theta^{1/2} f(\theta x), \quad f \in L^2(\mathbb{R})$$

and let $V_\theta = D_\theta U$. Then the image under V_θ of the domain of all finite particle vectors in $\Gamma_s(\mathbb{C})$ is the linear manifold

$$\mathcal{M}_\theta = \{P(x)e^{-\frac{1}{4}\theta^2 x^2} | P \text{ a polynomial in the real variable } x\}.$$

If $q = 2^{-\frac{1}{2}}(a + a^\dagger), p = -2^{-\frac{1}{2}}i(a - a^\dagger)$ where $a, a^\dagger$ are as in Exercise 20.18 (b) then for any f in $\mathcal{M}_\theta$ the following holds:

(i) $(V_\theta q V_\theta^{-1} f)(x) = 2^{-\frac{1}{2}}\theta x f(x)$;

(ii) $(V_\theta p V_\theta^{-1} f)(x) = -2^{-\frac{1}{2}} i \theta^{-1} f'(x)$;

(iii) for $\theta = \sqrt{2c}$

$$\{cV_{\sqrt{2c}}(a^\dagger a + \frac{1}{2})V_{\sqrt{2c}}^{-1} f\}(x) = -\frac{1}{2} f''(x) + \frac{1}{2}c^2 x^2 f(x).$$

(c) The functions

$$\psi_j(x) = (2\pi)^{-\frac{1}{4}}(j!)^{-\frac{1}{2}} H_j(\sqrt{2c}x)e^{-\frac{1}{2}x^2}, \quad j = 0, 1, 2, \ldots,$$

where $\{H_j\}$ is the sequence of Hermitian polynomials as defined by (19.7) satisfy

$$-\frac{1}{2}\psi_j''(x) + \frac{1}{2}c^2 x^2 \psi_j(x) = c(j + \frac{1}{2})\psi_j(x), \quad j = 0, 1, 2, \ldots$$

(**Hint:** Use the properties of the number operator $a^\dagger a$ defined in Exercise 20.18 (b).)

(d) $U\Gamma(-i)U^{-1} = D_{1/2}\mathbb{F}$ where $\Gamma(-i)$ is the second quantization of multiplication by $-i$ in $\Gamma_s(\mathbb{C})$ and $\mathbb{F}$ is the unitary Fourier transform defined by (12.8);

(e) $\mathbb{F}D_\theta = D_{\theta^{-1}}\mathbb{F}, D_{\theta_1}D_{\theta_2} = D_{\theta_1\theta_2}$ and the Stone generator of the unitary group $\{D_{e^t}|t \in \mathbb{R}\}$ is the closure of $\frac{1}{2}U(qp+pq)U^{-1}, q, p$ being as in (b) (i) and (ii).

Exercise 20.21: Let the Hilbert space $\mathcal{H}$ be a direct sum: $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. In the factorisation $\Gamma_s(\mathcal{H}) = \Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2)$ determined by Proposition 19.6 the Weyl operators $W(u,U)$ and the observables $p(u), \lambda(H)$ defined by (20.2), (20.10) and (20.11) satisfy the following:

(i) $W(u_1 \oplus u_2, U_1 \oplus U_2) = W(u_1, U_1) \otimes W(u_2, U_2)$ for all $u_j \in \mathcal{H}_j$, $U_j \in \mathcal{U}(\mathcal{H}_j)$, $j = 1, 2$;

(ii) $p(u_1 \oplus u_2)e(v_1 \oplus v_2) = \{p(u_1)e(v_1)\} \otimes e(v_2) + e(v_1) \otimes \{p(u_2)e(v_2)\}$ for all $u_j, v_j \in \mathcal{H}_j$, $j = 1, 2$;

(iii) for any two observables H_j in $\mathcal{H}_j$, $j = 1, 2$

$$\lambda(H_1 \oplus H_2)e(v_1 \oplus v_2) = \{\lambda(H_1)e(v_1)\} \otimes e(v_2) + e(v_1) \otimes \{\lambda(H_2)e(v_2)\}$$

for all $v_j \in \mathcal{H}_j$, $j = 1, 2, \ldots$.

Exercise 20.22: (i) Let $\mathcal{C}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ be the set of all contraction operators. Then $\mathcal{C}(\mathcal{H})$ is a *-weakly closed convex set. In particular, $\mathcal{C}(\mathcal{H})$ is strongly closed. Under the strong topology $\mathcal{C}(\mathcal{H})$ is a multiplicative topological semigroup. If $T \in \mathcal{C}(\mathcal{H})$ and T leaves a subspace $\mathcal{H}_0$ invariant then $T|_{\mathcal{H}_0}$ is a contraction. Direct sums and tensor products of contractions are also contractions.

(ii) For any $T \in \mathcal{C}(\mathcal{H})$ the operator $\Gamma_{fr}(T)$ in the free Fock space $\Gamma_{fr}(\mathcal{H})$ is defined by

$$\Gamma_{fr}(T) = 1 \oplus T \oplus T^{\otimes^2} \oplus \cdots \oplus T^{\otimes^n} \oplus \cdots$$

Write

$$\Gamma_s(T) = \Gamma_{fr}(T)|_{\Gamma_s(\mathcal{H})}, \Gamma_a(T) = \Gamma_{fr}(T)|_{\Gamma_a(\mathcal{H})}$$

where $\Gamma_s(\mathcal{H})$ and $\Gamma_a(\mathcal{H})$ are respectively the boson and fermion Fock spaces over $\mathcal{H}$. Then the maps $T \to \Gamma_{fr}(T), \Gamma_s(T), \Gamma_a(T)$ are strongly continuous *-unital homomorphisms from the topological semigroup $\mathcal{C}(\mathcal{H})$ into $\mathcal{C}(\Gamma_{fr}(\mathcal{H}))$, $\mathcal{C}(\Gamma_s(\mathcal{H}))$, $\mathcal{C}(\Gamma_a(\mathcal{H}))$ respectively. These are called *second quantization homomorphisms*. Second quantization homomorphisms are positivity-preserving.

(iii) If T is a trace class operator in $\mathcal{H}$ such that $\|T\|_1 < 1$ then $\Gamma_{fr}(T)$ is also a trace class operator and $\|\Gamma_{fr}(T)\|_1 = (1 - \|T\|_1)^{-1}$, $\operatorname{tr}\Gamma_{fr}(T) = (1 - \operatorname{tr} T)^{-1}$.

(iv) Let T be a positive operator of finite trace with eigenvalues $\{\lambda_j | j = 1, 2, \ldots\}$ inclusive of multiplicity and $\sup_j |\lambda_j| < 1$. Then

$$\operatorname{tr}\Gamma_s(T) = \Pi_j(1 - \lambda_j)^{-1}, \ \operatorname{tr}\Gamma_a(T) = \Pi_j(1 + \lambda_j).$$

In particular, $\{\operatorname{tr}\Gamma_s(T)\}^{-1}\Gamma_s(T)$ and $\{\operatorname{tr}\Gamma_a(T)\}^{-1}\Gamma_a(T)$ are states in $\Gamma_s(\mathcal{H})$ and $\Gamma_a(\mathcal{H})$ respectively.

(v) Let $\dim \mathscr{H} < \infty$ and let ρ be any positive operator in $\mathscr{H}$ such that $\|\rho\| < 1$. Then for any observable X in $\mathscr{H}, \det(1 - \rho e^{itX})^{-1}(1 - \rho)$ and $\det(1 + \rho e^{itX})(1 + \rho)^{-1}$ are characteristic functions of probability distributions.

(Hint: Use Proposition 19.3 for computing traces.)

Exercise 20.23: [83] Let $(S, \mathscr{F}, \mu)$ and μ_Γ be as in Example 19.12. Write $d\sigma$ for $\mu_\Gamma(d\sigma)$. Then for any $f \in L^2(\mu_\Gamma \times \mu_\Gamma \times \cdots \times \mu_\Gamma)$, where the product is n-fold, the following holds:

$$\int_{\Gamma(S)^n} f(\sigma_1, \sigma_2, \ldots, \sigma_n) d\sigma_1 \cdots d\sigma_n = \int_{\Gamma(S)} \sum_{\sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_n = \sigma} f(\sigma_1, \ldots, \sigma_n) d\sigma$$

where $\cup$ signifies disjoint union. (This is known as the *sum-integral formula*.) Under the unitary isomorphism $U : \Gamma_s(L^2(\mu)) \to L^2(\mu_\Gamma)$ for any $f \in U(\mathscr{E})$, $u \in L^2(\mu)$

$$(Ua^\dagger(u)U^{-1}f)(\sigma) = \sum_{s \in \sigma} u(s) f(\sigma \backslash s)$$

$$(Ua(u)U^{-1}f)(\sigma) = \int_S \overline{u}(s) f(\sigma \cup \{s\}) d\mu(s)$$

and for any real valued $\mathscr{F}$ measurable function ϕ on S

$$(U\Gamma(e^{i\phi})U^{-1}f)(\sigma) = f(\sigma) \exp i \sum_{s \in \sigma} \phi(s) \text{ for all } f \in L^2(\mu_\Gamma)$$

where $e^{i\phi}$ denotes the unitary operator of multiplication by $e^{i\phi}$ in $L^2(\mu)$.

Exercise 20.24: [136] For any $u \in \mathscr{H}$ there exists a unique bounded operator $\ell(u)$ in the free Fock space $\Gamma_{fr}(\mathscr{H})$ satisfying the relations:

$$\ell(u)\Phi = 0, \ell(u)v_1 \otimes \cdots \otimes v_n = \langle u, v_1 \rangle v_2 \otimes \cdots \otimes v_n \text{ for all } n \geq 1, \ v_j \in \mathscr{H}$$

(where $v_2 \otimes \cdots \otimes v_n = \Phi$ when $n = 1$). The adjoint $\ell^*(u)$ of $\ell(u)$ satisfies the relations:

$$\ell^*(u)\Phi = u, \ell^*(u)v_1 \otimes \cdots \otimes v_n = u \otimes v_1 \otimes \cdots \otimes v_n \text{ for all } n \geq 1, \ v_j \in \mathscr{H}.$$

If T is a bounded operator in $\mathscr{H}$ then there exists a unique bounded operator $\lambda_0(T)$ in $\Gamma_{fr}(\mathscr{H})$ satisfying

$$\lambda_0(T)\Phi = 0, \lambda_0(T)v_1 \otimes \cdots \otimes v_n = (Tv_1) \otimes v_2 \otimes \cdots \otimes v_n \text{ for all } n \geq 1, \ v_j \in \mathscr{H}.$$

The operators $\ell(u), \ell^*(u)$ and $\lambda_0(T)$ satisfy the following:

(i) $\ell(u)\ell^*(v) = \langle u, v \rangle$;

(ii) $\|\ell(u)\| = \|\ell^*(u)\| = \|u\|$;

(iii) $\ell(u)$ is antilinear in u whereas $\ell^*(u)$ is linear in u;

(iv) $\ell(u)\lambda_0(T) = \ell(T^*u), \lambda_0(T)\ell^*(u) = \ell^*(Tu)$;

(v) $\lambda_0(T)^* = \lambda_0(T^*)$, $\|\lambda_0(T)\| = \|T\|$;

(vi) λ_0 is a *-(non-unital) homomorphism from $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\Gamma_{fr}(\mathcal{H}))$;

(vii) If $\ell(u)\psi = 0$ for all $u \in \mathcal{H}$ then ψ is a scalar multiple of the vacuum vector Φ. If $B \in \mathcal{B}(\mathcal{H})$ and B commutes with $\ell(u)$ for all u then $B\Phi = b\Phi$ for some scalar b. If, in addition, B commutes with $\ell^*(u)$ for all u then $B = b$. In other words the family $\{\ell(u), \ell^*(u) | u \in \mathcal{H}\}$ is irreducible.

(It is instructive to compare the properties of $\ell(u), \ell^*(u)$ and $\lambda_0(T)$ in $\Gamma_{fr}(\mathcal{H})$ with the properties of $a(u), a^\dagger(u)$ and $\lambda(T)$ in $\Gamma_s(\mathcal{H})$. The operators $\ell(u)$ and $\ell^*(u)$ are respectively called the *free annihilation* and *creation operators associated with* u. $\lambda_0(T)$ is called the *free conservation operator associated with* T. It is appropriate to call $\lambda(T)$ the *boson conservation operator* associated with T).

Exercise 20.25: [116] Consider the unitary isomorphism $V : \Gamma_{fr}(L^2(\mathbb{R}_+)) \to L^2(\mu_\Gamma)$ where μ is Lebesgue measure in $\mathbb{R}_+$ and μ_Γ and V are as in Exercise 19.15. For any $\sigma \in \Gamma(\mathbb{R}_+)$, $s \in \mathbb{R}_+$ define $\sigma + s = \{t + s | t \in \sigma\}$ if $\sigma \neq \emptyset$ and $= \emptyset$ if $\sigma = \emptyset$; $\sigma - s = \{t - s | t \in \sigma\}$ if $\sigma \neq \emptyset$ and if every element in σ exceeds s. Denote by $\min \sigma$ the smallest element in σ. Then

(i) $(V\ell(u)V^{-1}f)(\sigma) = \int_0^\infty \overline{u}(s) f((\sigma + s) \cup \{s\}) ds$;

(ii) $(V\ell^*(u)V^{-1}f)(\sigma) = u(\min \sigma)\ f(\sigma \backslash \{\min \sigma\} - \min \sigma)$ if $\sigma \neq \emptyset$ and $= 0$ if $\sigma = \emptyset$;

(iii) For any bounded measurable function ϕ on $\mathbb{R}_+$
$$\begin{aligned}(V\lambda_0(\phi)V^{-1}f(\sigma) &= \phi(\min \sigma) f(\sigma) \text{ if } \sigma \neq \emptyset, \\ &= 0 \quad \text{if} \quad \sigma = \emptyset,\end{aligned}$$
where, on the left hand side, ϕ denotes the operator of multiplication by ϕ. (Compare (i), (ii) and (iii) with the properties of $a(u), a^\dagger(u)$ and $\lambda(\phi)$ in Exercise 20.23 after noting that $\Gamma(e^{i\phi}) = e^{i\lambda(\phi)}$).

Notes

For an extensive discussion of second quantization, CCR and CAR, see Cook [26], Garding and Wightman [45], [46], Segal [120], [121], Berezin [19], Bratteli and Robinson [23]. Exercise 20.23 is from Maassen [83]. Exercise 20.24 [136] is the starting point of a free Fock space stochastic calculus developed by Speicher [126,127]. Exercise 20.25 is from Parthasarathy and Sinha [116].

21 Weyl Representation and infinitely divisible distributions

Using the Weyl operators defined by (20.2) we constructed the observables $\{p(u) |\ u \in \mathcal{H}\}$, $\{\lambda(H) | H$ an observable in $\mathcal{H}\}$ in $\Gamma_s(\mathcal{H})$ through (20.10) and (20.11). We shall now analyse their probability distributions in every pure state of the form
$$\psi(v) = e^{-\frac{1}{2}\|v\|^2} e(v), \quad v \in \mathcal{H}. \tag{21.1}$$
Any state of the form (21.1) is called a *coherent state* (associated with v). Such an analysis together with the factorisability property indicated in Exercise 20.21

shows how one can realise every infinitely divisible probability distribution as the distribution of an observable in $\Gamma_s(\mathcal{H})$ in the vacuum state. It may be recalled that a probability distribution μ on the real line (or $\mathbb{R}^n$) is called *infinitely divisible* if for every positive integer k there exists a probability distribution μ_k such that $\mu = \mu_k^{*k}$, the k-fold convolution of μ_k. The investigation of such distributions in the present context leads us to the construction of stochastic processes with independent increments as linear combinations of creation, conservation and annihilation operators in $\Gamma_s(\mathcal{H})$.

Proposition 21.1: Let $S \subset \mathcal{H}$ be a completely real subspace such that $\mathcal{H} = S + iS = \{u + iv | u \in S, v \in S\}$. Then the following holds:

(i) $\{p(u)|u \in S\}$ and $\{q(u) = -p(iu)|u \in S\}$ are two commuting families of observables with $\mathcal{E}$ as a common core in $\Gamma_s(\mathcal{H})$;

(ii) $[q(u), p(v)]\psi = 2i\langle u, v\rangle\psi$ for all $u, v \in S$, $\psi \in \mathcal{E}$;

(iii) For any $v \in \mathcal{H}$, $u_1, \dots, u_n \in S$ the joint distribution of $p(u_1), \dots, p(u_n)$ in the coherent state $\psi(v)$ is Gaussian with mean vector $-2(\mathrm{Im}\langle v, u_1\rangle, \mathrm{Im}\langle v, u_2\rangle, \dots, \mathrm{Im}\langle v, u_n\rangle)$ and covariance matrix $(((\langle u_i, u_j\rangle)))$, $1 \le i, j \le n$. In the same state the joint distribution of $q(u_1), \dots, q(u_n)$ is Gaussian with mean vector $2(\mathrm{Re}\langle u, u_1\rangle, \dots, \mathrm{Re}\langle v, u_n\rangle)$ and the same covariance matrix.

Proof: For any $u, v \in S$, $\mathrm{Im}\langle u, v\rangle = 0$ and (20.3) and (20.4) imply

$$W(u)W(v) = W(u+v) = W(v)W(u),$$
$$W(iu)W(iv) = W(i(u+v)) = W(iv)W(iu).$$

Now (20.10) implies that $\{p(u)|u \in S\}$ and $\{q(u)|u \in S\}$ are commuting families of observables. That $\mathcal{E}$ is a core for each $p(u)$ and $q(u)$ is just (ii) in Proposition 20.4. This proves (i). Now (iii) in Proposition 20.4 implies that for $u, v \in S$

$$\begin{aligned}[q(u), p(v)]\psi &= -[p(iu), p(v)]\psi \\ &= -2i\,\mathrm{Im}\langle iu, v\rangle = 2i\langle u, v\rangle\psi, \ \psi \in \mathcal{E}.\end{aligned}$$

This proves (ii).

For any $u, v \in \mathcal{H}$ we have from (21.1), (20.2) and (20.4)

$$\begin{aligned}\langle\psi(v), W(u)\psi(v)\rangle &= \exp(-\|v\|^2 - \frac{1}{2}\|u\|^2 - \langle u, v\rangle + \langle v, u+v\rangle) \\ &= \exp(2i\,\mathrm{Im}\langle v, u\rangle - \frac{1}{2}\|u\|^2).\end{aligned}$$

Thus for any $u_j \in S$, $t_j \in \mathbb{R}$, $1 \le j \le n$ and $v \in \mathcal{H}$ we have

$$\begin{aligned}\langle\psi(v), e^{-it_1p(u_1)}\cdots e^{-it_np(u_n)}\psi(v)\rangle &= \langle\psi(v), W(\sum_j t_ju_j)\psi(v)\rangle \\ = \exp\{2i\sum_j t_j\,\mathrm{Im}\langle v, u_j\rangle - \frac{1}{2}\sum_{i,j} t_it_j\langle u_i, u_j\rangle\}\end{aligned} \tag{21.2}$$

and

$$\langle \psi(v), e^{-it_1 q(u_1)} \cdots e^{-it_n q(u_n)} \psi(v)\rangle = \langle \psi(v), W(-\sum_j it_j u_j)\psi(v)\rangle$$
$$= \exp\{-2i\sum_j t_j \operatorname{Re}\langle v, u_j\rangle - \frac{1}{2}\sum_{i,j} t_i t_j \langle u_i, u_j\rangle\}. \tag{21.3}$$

Equations (21.2) and (21.3) express the characteristic functions of the joint distributions of $p(u_1), \ldots, p(u_n)$ and $q(u_1), \ldots, q(u_n)$ respectively in the pure state $\psi(v)$. ■

In Proposition 21.1, S is a real Hilbert space and we have $p(\alpha u + \beta v)\psi = \alpha p(u)\psi + \beta p(v)\psi$, $q(\alpha u + \beta v)\psi = \alpha q(u)\psi + \beta q(v)\psi$ for $\psi \in \mathscr{E}$, $\alpha, \beta \in \mathbb{R}$, $u, v \in S$. In view of Property (iii) in Proposition 21.1, $p(\cdot)$ and $q(\cdot)$ may be looked upon as classical Gaussian random fields over the real Hilbert space S in each coherent state.

Example 21.2: Let $S \subset \mathscr{H}$ be a completely real subspace and let $\{U_t | t \in \mathbb{R}\}$ be a one parameter unitary group in $\mathscr{H}$ leaving S invariant. Consider $X_t = p(U_t u)$ where $u \in S$ is fixed. Then $\{X_t | t \in \mathbb{R}\}$ is a commuting family of observables whose distribution in the vacuum state yields a stationary Gaussian stochastic process with mean 0 and spectral distribution $\langle u, \xi(E)u\rangle = \mu(E)$, $E \subseteq \mathbb{R}$ being a Borel set where $U_t = \int e^{-itx}\xi(dx)$, $t \in \mathbb{R}$. As a special case we may consider $\mathscr{H} = L^2(\mathbb{R})$, $S =$ the subspace of all real valued functions in $\mathbb{R}, (U_t f)(x) = f(x-t)$, $f \in \mathscr{H}$. In this case the spectral density function is $|\hat{u}|^2$ where $\hat{u}$ is the Fourier transform of u.

Example 21.3: Let $S \subset \mathscr{H}$ be a completely real subspace and let ξ be a spectral measure on the real line for which S is invariant. For any fixed S let $X_t = p(\xi((-\infty, t]u), F(t) = \langle u, \xi((-\infty, t])u\rangle$. Then $\{X_t | t \in \mathbb{R}\}$ is a commuting family of observables (with common core $\mathscr{E}$) whose distribution in the vacuum state is a Gaussian process with independent increments, mean 0 and $\operatorname{cov}(X_t, X_s) = F(t \wedge s), t \wedge s$ being the minimum of t and s.

One may interpret ξ as a time observable in $\mathscr{H}$. This example is meaningful when ξ is a spectral measure in any interval of the real line. When ξ is a spectral measure in the unit interval and $\langle u, \xi([0,t])u\rangle = t$ for all $0 \le t \le 1$ then the distribution of $\{X_t | 0 \le t \le 1\}$ is the same as that of standard Brownian motion.

Proposition 21.4: Let H be an observable in $\mathscr{H}$ with spectral resolution $H = \int_{\mathbb{R}} x\xi(dx)$ and let $\mu_v(E) = \langle v, \xi(E)v\rangle$, $E \subseteq \mathbb{R}$ being a Borel set. The characteristic function of the distribution of $\lambda(H)$ in the coherent state $\psi(v)$ is equal to $\exp\langle v, (e^{itH} - 1)v\rangle$ and the corresponding distribution ν is given by

$$\nu = e^{-\|v\|^2}\{\delta_0 + \mu_v + \frac{1}{2!}\mu_v^{*^2} + \cdots + \frac{1}{n!}\mu_v^{*^n} + \cdots\} \tag{21.4}$$

where δ_0 is the Dirac measure at the origin and $*^n$ denotes n-fold convolution.

Proof: From (21.1), (20.4) and (20.11) we have

$$\begin{aligned}\langle\psi(v), e^{-it\lambda(H)}\psi(v)\rangle &= \langle\psi(v), \Gamma(e^{-itH})\psi(v)\rangle \\ &= \exp\{-\|v\|^2 + \langle v, e^{-itH}v\rangle\}.\end{aligned}$$

This proves the first part. On the other hand the last expression is equal to $\exp\int(e^{-itx}-1)\mu_v(dx)$ which implies (21.4). ■

Example 21.5: Let $(\Omega, \mathscr{F})$ be any measurable space and let η be an Ω-valued observable in $\mathscr{H}$. For any $E \in \mathscr{F}, \eta(E)$ is a projection in $\mathscr{H}$ and $X(E) = \lambda(\eta(E))$ is an observable in $\Gamma_s(\mathscr{H})$. $\{X(E)|E \in \mathscr{F}\}$ is a commuting family of observables and by Proposition 20.7 has a common core $\mathscr{E}$. For any sequence $\{E_j\}$ of disjoint sets from $\mathscr{F}$

$$X(\cup_i E_j)\psi = \sum_j X(E_j)\psi, \quad \psi \in \mathscr{E}.$$

In any coherent state $\psi(v)$, for any $E \in \mathscr{F}, X(E)$ has Poisson distribution with mean $\langle v, \eta(E)v\rangle$ and for any finite sequence E_j, $1 \leq j \leq n$ of disjoint sets from $\mathscr{F}$, the joint distribution of $X(E_j)$, $1 \leq j \leq n$ is the product of its marginal distributions. In short, we have realised a Poisson point process over $(\Omega, \mathscr{F})$ with intensity measure $\langle v, \eta(\cdot)v\rangle$ in terms of observables in the quantum probability space $(\Gamma_s(\mathscr{H}), \mathscr{P}(\Gamma_s(\mathscr{H})), \psi(v))$.

Proposition 21.6: For any $u \in \mathscr{H}$ and any observable H in $\mathscr{H}$ let

$$\lambda(H, u) = W(-u)\lambda(H)W(u). \tag{21.5}$$

The distribution of the observable $\lambda(H, u)$ in the vacuum state is also the distribution of $\lambda(H)$ in the coherent state $\psi(u)$. $\lambda(H, u)$ is essentially selfadjoint on the linear manifold generated by $\{e(v+u)|v \in D(H^2)\}$. When H is a bounded observable, $\mathscr{E}$ is a core for $\lambda(H, u)$ for every u and

$$\lambda(H, u)|_{\mathscr{E}} = \lambda(H) + a(Hu) + a^\dagger(Hu) + \langle u, Hu\rangle|_{\mathscr{E}}.$$

Proof: The first part is immediate from the identity

$$\langle\Phi, W(-u)e^{it\lambda(H)}W(u)\Phi\rangle = \langle\psi(u), e^{it\lambda(H)}\psi(u)\rangle.$$

The second part follows from Proposition 20.7 and also implies that $\mathscr{E}$ is a core for $\lambda(H, u)$ when H is bounded. For any ψ in $\mathscr{E}$ we have from Theorem 20.10 and (20.29)

$$\begin{aligned}W(-u)\lambda(H)W(u)\psi &= e^{ip(u)}\lambda(H)e^{-ip(u)}\psi \\ &= \lambda(H)\psi - p(iHu)\psi + \langle u, Hu\rangle\psi \\ &= \{\lambda(H) + a(Hu) + a^\dagger(Hu) + \langle u, Hu\rangle\}\psi.\end{aligned}$$

In order to combine observables of the form $\lambda(H, u)$ for varying H, u the next proposition is very useful.

Proposition 21.7: Let X_j be an observable in the Hilbert space $\mathscr{H}_j, j = 1, 2, \ldots,$ and let $\phi_j \in \mathscr{H}_j, \|\phi_j\| = 1$ for each j. Suppose $\mathscr{H} = \bigotimes_{j=1}^{\infty} \mathscr{H}_j$ is the countable tensor product with respect to the stabilising sequence $\{\phi_j\}$. Let

$$\rho_j(t) = \langle \phi_j, e^{-itX_j} \phi_j \rangle, \quad j = 1, 2, \ldots .$$

In order that there may exist a one parameter unitary group $\{U_t | t \in \mathbb{R}\}$ in $\mathscr{H}$ satisfying the relation

$$U_t \bigotimes_{j=1}^{\infty} u_j = \lim_{n \to \infty} \bigotimes_{j=1}^{n} e^{-itX_j} u_j \otimes \bigotimes_{j=n+1}^{\infty} \phi_j \tag{21.6}$$

for every sequence $\{u_j\}$ where $u_j \in \mathscr{H}_j$ for each j and $u_j = \phi_j$ for all but a finite number of j's, it is necessary and sufficient that the infinite product $\rho(t) = \Pi_{j=1}^{\infty} \rho_j(t)$ is defined as a continuous function of t in some open interval containing 0.

Proof: First we prove sufficiency. Let $P \subset \mathscr{H}$ denote the set of all product vectors of the form $\bigotimes_{j=1}^{\infty} u_j$, $u_j \in \mathscr{H}_j$, $u_j = \phi_j$ for all but a finite number of j's. Choose any $\psi = \bigotimes_{j=1}^{\infty} u_j$ in P and let $u_j = \phi_j$ for all $j > j_0$. For $m, n > j_0$ we have

$$\begin{aligned} &\| \bigotimes_{j=1}^{n} e^{-itX_j} u_j \otimes \bigotimes_{j=n+1}^{\infty} \phi_j - \bigotimes_{j=1}^{m} e^{-itX_j} u_j \otimes \bigotimes_{j=m+1}^{\infty} \phi_j \|^2 \\ &= 2\Pi_{j=1}^{j_0} \|u_j\|^2 \{1 - \operatorname{Re} \Pi_{j=m+1}^{n} \rho_j(t)\} \end{aligned} \tag{21.7}$$

provided $n > m$. Each $\rho_j(t)$ is the characteristic function of a probability distribution μ_j on the real line and by hypothesis $\rho(t)$ is the characteristic function of the weakly convergent product $\mu = \mu_1 * \mu_2 *, \ldots, *$ denoting convolution. If $\{\zeta_j\}$ is a sequence of independent random variables where ζ_j has distribution μ_j then $\sum_j \zeta_j$ converges in distribution and hence converges almost surely (see Theorem 5.3.4 [24]). Thus $\lim_{m,n\to\infty} \sum_{j=m+1}^{n} \zeta_j = 0$ almost surely and

$$\lim_{m,n \to \infty} \Pi_{j=m+1}^{n} \rho_j(t) = 1 \text{ for all } t.$$

Thus the right hand side of (21.7) converges to 0 as $m, n \to \infty$. In other words the limit on the right hand side of (21.6) exists and U_t defined on P by (21.6) is an isometry. By Proposition 7.2, U_t extends uniquely to an isometry on $\mathscr{H}$. Denote $U_t \bigotimes_{j=1}^{\infty} u_j = \bigotimes_{j=1}^{\infty} e^{-itX_j} u_j$ where $\bigotimes_{j=1}^{\infty} u_j \in P$. It is clear that for any fixed t the range of U_t includes all vectors of the form $\psi \otimes \bigotimes_{j=n+1}^{\infty} e^{-itX_j} u_j$, $\psi \in \mathscr{H}_1 \otimes \cdots \otimes \mathscr{H}_n$. Choose

$$\psi_n = u_1 \otimes \cdots \otimes u_k \otimes \phi_{k+1} \otimes \cdots \otimes \phi_n, n > k$$

where $u_j \in \mathscr{H}_j$. Then

$$\lim_{n \to \infty} \psi_n \otimes \bigotimes_{j=n+1}^{\infty} e^{-itX_j} \phi_j = u_1 \otimes \cdots \otimes u_k \otimes \bigotimes_{j=k+1}^{\infty} \phi_j.$$

This shows that $R(U_t) = \mathcal{H}$ and U_t is unitary for each t. By definition $\{U_t | t \in \mathbb{R}\}$ is a unitary group. The strong continuity of the map $t \to U_t$ is easily established. This proves sufficiency.

Necessity is immediate from the relation

$$\langle \bigotimes_{j=1}^{\infty} \phi_j, U_t \bigotimes_{j=1}^{\infty} \phi_j \rangle = \Pi_{j=1}^{\infty} \rho_j(t). \qquad \blacksquare$$

We shall denote the unitary group $\{U_t\}$ defined by Proposition 21.7 by

$$U_t = \bigotimes_{j=1}^{\infty} e^{-itX_j}.$$

It may be noted that $\bigotimes_{j=1}^{\infty} \phi_j$ need not be in the domain of the Stone generator X of U_t since ϕ_j need not be in the domain of X_j. It is far from clear from this analysis how a core for X can be constructed from the domains of the individual operators X_j.

Proposition 21.8: Let H_j be an observable in $\mathcal{H}_j$ with spectral resolution

$$H_j = \int_{\mathbb{R}} x \xi_j(dx), \quad j = 1, 2, \dots ,$$

and let $u_j \in \mathcal{H}_j$ for each j be such that

$$\mu(E) = \sum_{j=1}^{\infty} \langle u_j, \xi_j(E) u_j \rangle, \quad E \in \mathcal{F}_{\mathbb{R}}$$

is a σ-finite measure satisfying the condition

$$\int_{\mathbb{R}} \frac{x^2}{1+x^2} d\mu(x) < \infty. \tag{21.8}$$

Let X_j be the observable in $\Gamma_s(\mathcal{H}_j)$ defined by

$$X_j = \lambda(H_j, u_j) - \langle u_j, H_j(1 + H_j^2)^{-1} u_j \rangle \tag{21.9}$$

where $\lambda(H_j, u_j)$ is given by (21.5). Denote by Φ_j the vacuum vector in $\Gamma_s(\mathcal{H}_j)$ for each j. Then there exists an observable X in the Hilbert space

$$\Gamma_s(\bigoplus_{j=1}^{\infty} \mathcal{H}_j) = \bigotimes_{j=1}^{\infty} \Gamma_s(\mathcal{H}_j)$$

defined with respect to the stabilising sequence $\{\Phi_j\}$ such that the following conditions are fulfilled:

(i) $e^{-itX} = \bigotimes_{j=1}^{\infty} e^{-itX_j}$;

(ii) If $\Phi = \bigotimes_{j=1}^{\infty} \Phi_j$ denotes the vacuum vector in $\Gamma_s(\bigoplus_{j=1}^{\infty} \mathcal{H}_j)$ then

$$\langle \Phi, e^{-itX} \Phi \rangle = \exp \int_{\mathbb{R}} (e^{-itx} - 1 + \frac{itx}{1+x^2}) d\mu(x);$$

(iii) the distribution of the observable X in the vacuum state Φ is infinitely divisible and has characteristic function

$$\exp \int_{\mathbb{R}} (e^{itx} - 1 - \frac{itx}{1+x^2}) d\mu(x).$$

Proof: By Proposition 21.4, 21.6 and (21.9) we have

$$\langle \Phi_j, e^{-itX_j} \Phi_j \rangle = \exp \int (e^{-itx} - 1 + \frac{itx}{1+x^2}) d\mu_j(x)$$

where $\mu_j(E) = \langle u_j, \xi_j(E) u_j \rangle$. Since $\mu(E) = \sum_j \mu_j(E)$ the integrability condition on μ implies that

$$\Pi_{j=1}^{\infty} \langle \Phi_j, e^{-itX_j} \Phi_j \rangle = \exp \int (e^{-itx} - 1 + \frac{itx}{1+x^2}) d\mu(x)$$

is a continuous function of t. By Proposition 21.7 the existence of the observable X satisfying (i) and (ii) follows. (iii) is immediate from (ii). ■

Corollary 21.9: Let X_0 be the observable in $\Gamma_s(\mathscr{H}_0)$ defined by

$$X_0 = m + p(u_0)$$

for some vector $u_0 \in \mathscr{H}_0$ and real scalar m. If X is the observable defined by Proposition 21.8 and Y is the Stone generator of $e^{-itX_0} \otimes e^{-itX}$ then the distribution of Y in the vacuum state of $\Gamma_s(\bigoplus_{j=0}^{\infty} \mathscr{H}_j)$ has characteristic function $\exp\{imt - \frac{1}{2} t^2 \|u_0\|^2 + \int_{\mathbb{R}} (e^{itx} - 1 - \frac{itx}{1+x^2}) d\mu(x)\}$.

Proof: Immediate. ■

Corollary 21.10: Let $\mathscr{H}_j, H_j, u_j$, $j = 1, 2, \ldots, \mu$ be as in Proposition 21.8 and let $\mathscr{H}_0, u_0, m$ be as in Corollary 21.9. Define

$$\mathscr{H}'_j = \mathscr{H}_j \otimes L^2(\mathbb{R}_+), \mathscr{H} = \bigoplus_{j=0}^{\infty} \mathscr{H}_j$$

$$u'_j(t) = u_j \otimes I_{[0,t]}, \ \ j = 0, 1, 2, \ldots$$

$$H'_j(t) = H_j \otimes \xi^L([0,t]), \ \ j = 1, 2, \ldots,$$

where ξ^L denotes the canonical spectral measure in $L^2(\mathbb{R}_+)$ with respect to the Lebesgue measure. Let

$$X'_j(t) = \lambda(H'_j(t), u'_j(t)) - \langle u'_j(t), H'_j(t)[1 + H'_j(t)^2]^{-1} u'_j(t) \rangle$$

for $j = 1, 2, \ldots$. Then there exists a commuting family of observables $\{Y(t), t \geq 0\}$ in $\Gamma_s(\mathscr{H} \otimes L^2(\mathbb{R}_+)) = \bigotimes_{j=0}^{\infty} \Gamma_s(\mathscr{H}'_j)$ satisfying the following:

(i) $e^{-iaY(t)} = e^{-iamt} e^{-iap(u'_0(t))} \otimes \bigotimes_{j=1}^{\infty} e^{-iaX'_j(t)}$ for all $a \in \mathbb{R}$, $t \geq 0$;

(ii) If Φ' denotes the vacuum vector in $\Gamma_s(\mathcal{H} \otimes L^2(\mathbb{R}_+))$ then

$$\langle \Phi', \Pi_{j=1}^n e^{-ia_j Y(t_j)} e^{ia_j Y(t_{j-1})} \Phi' \rangle =$$

$$\exp\{\sum_{j=1}^{n} (t_j - t_{j-1})[-ia_j m - \frac{1}{2}\|u_0\|^2 a_j^2$$

$$+ \int (e^{-ia_j x} - 1 + \frac{ia_j x}{1+x^2} d\mu(x)]\}$$

for all $0 = t_0 < t_1 < \cdots < t_n$, $a_1, \ldots, a_n \in \mathbb{R}$, $n = 1, 2, \ldots$

(iii) In the vacuum state Φ' the observables $\{Y(t) | t \geq 0\}$ form a classical stochastic process with stationary independent increments where $Y(t)$ has characteristic function $\exp t\{iam - \frac{1}{2}\|u_0\|^2 a^2 + \int_{\mathbb{R}} (e^{iax} - 1 - \frac{iax}{1+x^2}) d\mu(x)\}$ as a function of a for each $t \geq 0$.

Proof: The existence of $Y(t)$ satisfying (i) is immediate from the fact that $\{H_j'(t)\}$ and $\{u_j'(t)\}$ satisfy the conditions of Proposition 21.8. (ii) is proved by straightforward calculations and (iii) is immediate from (ii). That $Y(s)$ and $Y(t)$ commute for all s and t follows from (20.3). ∎

Exercise 21.11: In Corollary 21.9 put

$$\mathcal{H} = \bigoplus_{j=0}^{\infty} \mathcal{H}_j, \; H = 1 \oplus \bigoplus_{j=1}^{\infty} H_j, \; U_t = e^{-itH},$$

$$v(t) = tu_0 \oplus \bigoplus_{j=1}^{\infty} (e^{-itH_j} u_j - u_j),$$

$$V_t = W(v(t), e^{-itH}) \exp[-i \int (\sin tx - \frac{tx}{1+x^2}) d\mu(x)].$$

Then $v(t) \in \mathcal{H}$ for each t and

$$v(t+s) = v(t) + U_t v(s) \text{ for all } s, t \in \mathbb{R}.$$

The map $t \to V_t$ is strongly continuous and $\{V_t | t \in \mathbb{R}\}$ is a one parameter unitary group in $\Gamma_s(\mathcal{H})$ and

$$\langle \Phi, V_t \Phi \rangle = \exp[-\frac{1}{2} t^2 \|u_0\|^2 - \int_{\mathbb{R}} (e^{-itx} - 1 + \frac{itx}{1+x^2}) d\mu(x)].$$

Exercise 21.12: Let μ be a σ-finite measure in $\mathbb{R}^n$ satisfying

$$\int_{\mathbb{R}^n} \frac{|\underline{x}|^2}{1+|\underline{x}|^2} d\mu(\underline{x}) < \infty.$$

Put $\mathcal{H} = L^2(\mu)$,

$$(U_{\underline{t}} f)(\underline{x}) = e^{-i\underline{t}\cdot\underline{x}} f(\underline{x}), \quad v(\underline{t})(\underline{x}) = \begin{cases} C\underline{t} & \text{if } \underline{x} = \underline{0}, \\ e^{-i\underline{t}\cdot\underline{x}} - 1 & \text{if } \underline{x} \neq \underline{0} \end{cases}$$

for each $\underline{t} \in \mathbb{R}^n$, where C is any $n \times n$ matrix with complex entries. Then $\{U_{\underline{t}} | \underline{t} \in \mathbb{R}^n\}$ is an n-parameter unitary group, $v(\underline{t}) \in \mathcal{H}$ and

$$v(\underline{t} + \underline{s}) = v(\underline{t}) + U_{\underline{t}} v(\underline{s}) \text{ for all } \underline{s}, \underline{t} \in \mathbb{R}^n.$$

If

$$V_{\underline{t}} = W(v(\underline{t}), U_{\underline{t}}) \exp[-i \int_{\mathbb{R}^n} (\sin \underline{t} \cdot \underline{x} - \frac{\underline{t} \cdot \underline{x}}{1 + |\underline{x}|^2}) d\mu(\underline{x})]$$

then $\{V_{\underline{t}} | \underline{t} \in \mathbb{R}^n\}$ is an n-parameter unitary group in $\Gamma_s(\mathcal{H})$ and

$$\langle \Phi, V_{\underline{t}} \Phi \rangle = \exp \left[-\frac{1}{2} \mu(\{0\}) \|C\underline{t}\|^2 + \int_{\mathbb{R}^n} (e^{-i\underline{t} \cdot \underline{x}} - 1 + \frac{i\underline{t} \cdot \underline{x}}{1 + |\underline{x}|^2}) d\mu(\underline{x}) \right]. \tag{21.10}$$

Thus $\{V_{\underline{t}}\}$ determines an $\mathbb{R}^n$-valued observable in $\Gamma_s(\mathcal{H})$ whose distribution in the vacuum state Φ is the infinitely divisible distribution with characteristic function given by (21.10).

Exercise 21.13: Let G be any group and let $g \to U_g$ be any homomorphism from G into the unitary group $\mathcal{U}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$. Suppose $g \to v(g)$ is a map from G into $\mathcal{H}$ satisfying

$$v(gh) = v(g) + U_g v(h) \text{ for } g, h \in G. \tag{21.11}$$

Let

$$V_g = W(v(g), U_g), \quad g \in G.$$

Then

$$V_g V_h = e^{-i \operatorname{Im}\langle v(g), U_g v(h) \rangle} V_{gh} \text{ for all } g, h \in G.$$

In other words $g \to V_g$ is a projective unitary representation of G with *multiplier* $\sigma(g, h) = \exp[-i \operatorname{Im}\langle v(g), U_g v(h) \rangle] = \exp i \operatorname{Im}\langle v(g^{-1}), v(h) \rangle$. (See [135].) The function $\alpha(g, h) = \langle v(g^{-1}), v(h) \rangle$ satisfies the identity

$$\alpha(g_1, g_2) + \alpha(g_1 g_2, g_3) = \alpha(g_1, g_2 g_3) + \alpha(g_2, g_3) \tag{21.12}$$

for all $g_1, g_2, g_3 \in G$. Equations (21.11) and (21.12) are known as *first* and *second order cocycle identities* respectively. If there exists a real valued function $\beta(g)$, $g \in G$ such that

$$\operatorname{Im}\langle v(g^{-1}), v(h) \rangle = \beta(gh) - \beta(g) - \beta(h) \text{ for all } g, h$$

then the correspondence $g \to e^{i\beta(g)} V_g$ is a homomorphism from G into $\mathcal{U}(\Gamma_s(\mathcal{H}))$. In such a case

$$\langle \Phi, e^{i\beta(g)} V_g \Phi \rangle = \exp \left[i\beta(g) - \frac{1}{2} \|v(g)\|^2 \right]$$

is an infinitely divisible positive definite function on the group G in the sense that $\exp(it\ \beta(g) - \frac{1}{2} t \|v(g)\|^2)$ is positive definite in G for each $t \geq 0$. Conversely, every infinitely divisible positive definite function has this form. The example in Exercise 21.12 is a special case when $G = \mathbb{R}^n$.

Exercise 21.14: [100] Let $\mathcal{G}$ be any real Lie algebra and let $X \to \pi(X)$ be a representation of $\mathcal{G}$ in $\mathcal{B}(\mathcal{H})$ so that $\pi : \mathcal{G} \to \mathcal{B}(\mathcal{H})$ is a linear map satisfying $[\pi(X), \pi(Y)] = \pi([X, Y])$ for all X, Y in $\mathcal{G}$. Suppose $\alpha, \beta : \mathcal{G} \to \mathcal{H}$ are linear maps satisfying

$$\pi(X)\alpha(Y) - \pi(Y)\alpha(X) = \alpha([X, Y]),$$
$$\pi(X)^*\beta(Y) - \pi(Y)^*\beta(X) = \beta([Y, X])$$

for all X, Y. In $\Gamma_s(\mathcal{H})$ define the operators

$$\Phi(X) = a^\dagger(\alpha(X)) + \lambda(\pi(X)) + a(\beta(X))$$

with domain $\mathcal{E}$. Then

$$[\Phi(X), \Phi(Y)] = \Phi([X, Y]) + \langle \beta(X), \alpha(Y) \rangle - \langle \beta(Y), \alpha(X) \rangle$$

for all X, Y in $\mathcal{G}$.

If $\pi(X)^* \equiv -\pi(X)$, $\alpha(X) = \pi(X)u$, $\beta(X) = -\alpha(X)$ and $\Psi(X) = \Phi(X) + \langle u, \pi(X)u \rangle$ then $i\Psi(X)$ is essentially selfadjoint on $\mathcal{E}$ and $[\Psi(X), \Psi(Y)] = \psi([X, Y])$. Furthermore

$$\langle e(0), \exp \tilde{\Phi}(X) e(0) \rangle = \exp\langle u, (e^{\pi(X)} - 1)u \rangle,$$

$\sim$ denoting closure. (See also Exercise 20.16.)

Exercise 21.15: Let $\xi : \mathcal{F}_{\mathbb{R}_+} \to \mathcal{P}(\mathcal{H})$ be an $\mathbb{R}_+$-valued observable in $\mathcal{H}$ satisfying the condition $\xi(\{t\}) = 0$ for every $t \geq 0$. Define

$$X_t = \Gamma(-\xi([0, t]) + \xi((t, \infty))), \quad t \geq 0.$$

Then $\{X_t | t \geq 0\}$ is a commuting family of selfadjoint operators satisfying the following:

(i) $X_t^2 = 1$ for every $t, X_0 = 1$;

(ii) For any u in $\mathcal{H}$ and $0 < t_1 < t_2 < \cdots < t_{2n+1} < \infty$

$$\langle \psi(u), X_{t_1} X_{t_2} \cdots X_{t_{2n}} \psi(u) \rangle = \Pi_{j=0}^{n-1} \alpha(t_{2j+1}) \alpha(t_{2j+2})^{-1}$$

where $\alpha(t) = \exp 2\langle u, \xi([0, t])u \rangle$;

(iii) Kolmogorov's Consistency Theorem implies that, in the coherent state $\psi(u)$, the distribution of the observables $\{X_t | t \geq 0\}$ is the same as that of a Markov chain $\{x((t) | t \geq 0\}$ with initial value $x(0) = 1$, state space $\{1, -1\}$ and transition probability matrix

$$P(s, t) = \begin{pmatrix} \frac{1}{2}(1 + \alpha(s)\alpha(t)^{-1}) & \frac{1}{2}(1 - \alpha(s)\alpha(t)^{-1}) \\ \frac{1}{2}(1 - \alpha(s)\alpha(t)^{-1}) & \frac{1}{2}(1 - \alpha(s)\alpha(t)^{-1}) \end{pmatrix}.$$

Notes

The decisive role of the Weyl representation in the geometric approach to infinitely divisble distributions can be traced to Streater [132], Araki [9], Parthasarathy and Schmidt [98, 99]. For recent developments on infinitely divisible completely positive maps on groups, see Holevo [55], [56], Fannes and Quaegebeur [38], [39]. Exercise 21.14 is adapted from Parthasarathy and Schmidt [100]. For an introduction to the theory of infinitely divisible distributions in classical probability, see Gnedenko and Kolmogorov [49].

22 The symplectic group of $\mathcal{H}$ and Shale's Theorem

Let $\mathcal{H}$ be a complex separable Hilbert space and let $S : \mathcal{H} \to \mathcal{H}$ be any real linear bijective map satisfying the following conditions: (i) S and S^{-1} are continuous; (ii) $\mathrm{Im}\langle Su, Sv\rangle = \mathrm{Im}\langle u, v\rangle$ for all $u, v \in \mathcal{H}$. Then S is called a *symplectic automorphism* of $\mathcal{H}$. All such symplectic automorphisms constitute a group $\mathcal{S}(\mathcal{H})$ under multiplication. Every unitary operator in $\mathcal{H}$ is also a symplectic automorphism of $\mathcal{H}$. Thus $\mathcal{U}(\mathcal{H})$ is a subgroup of $\mathcal{S}(\mathcal{H})$. For any Weyl operator $W(u) = W(u, 1)$ define

$$W_S(u) = W(Su). \tag{22.1}$$

By (20.5) we have

$$W_S(u)W_S(v) = e^{-i\,\mathrm{Im}\langle u,v\rangle} W_S(u+v). \tag{22.2}$$

In other words the correspondence $u \to W_S(u)$ is another projective unitary representation of the additive group $\mathcal{H}$ with multiplier $\sigma(u, v) = \exp(-i\,\mathrm{Im}\langle u, v\rangle)$. Proposition 20.9 implies that this representation is even irreducible. The question arises whether W_S is a new representation or there exists a unitary operator $\Gamma(S)$ in $\Gamma_s(\mathcal{H})$ such that

$$\Gamma(S)W(u)\Gamma(S)^{-1} = W_S(u) \text{ for all } u \in \mathcal{H}. \tag{22.3}$$

When $S \in \mathcal{U}(\mathcal{H})$ its second quantization satisfies (22.3). We shall now investigate how far we can deviate from a unitary operator within the symplectic group $\mathcal{S}(\mathcal{H})$ but preserving (22.3).

Let $\mathcal{H}_0 \subset \mathcal{H}$ be a completely real subspace such that $\mathcal{H} = \{u + iv | u, v \in \mathcal{H}_0\} = \mathcal{H}_0 + i\mathcal{H}_0$. For any $S \in \mathcal{S}(\mathcal{H})$ define operators S_{ij} in the real Hilbert space $\mathcal{H}_0$ for $1 \le i, j \le 2$ by putting

$$S(u + iv) = S_{11}u + iS_{21}u + S_{12}v + iS_{22}v \tag{22.4}$$

Express any vector in $\mathcal{H}_0 \oplus \mathcal{H}_0$ as a column $\binom{u}{v}$, $u, v \in \mathcal{H}_0$ and define

$$S_0\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{22.5}$$

We have

$$\begin{aligned}\mathrm{Im}\langle u_1 + iv_1, u_2 + iv_2\rangle &= \langle u_1, v_2\rangle - \langle v_1, u_2\rangle \\ &= \langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}\rangle\end{aligned}$$

for all $u_j, v_j \in \mathcal{H}_0$, $1 \le i, j \le 2$. In $\mathcal{H}_0 \oplus \mathcal{H}_0$ introduce the operator J by

$$J\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -u \end{pmatrix}.$$

The fact that S preserves the imaginary part of scalar product in $\mathcal{H}$ can be expressed in terms of S_0 in $\mathcal{H}_0 \oplus \mathcal{H}_0$ by

$$\langle S_0\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, JS_0\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}\rangle = \langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, J\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}\rangle$$

for all $u_j, v_j \in \mathscr{H}_0$, $j = 1, 2$, i.e.,

$$S_0^* J S_0 = J. \tag{22.6}$$

Conversely, if S_0 is any bounded operator with a bounded inverse in $\mathscr{H}_0 \oplus \mathscr{H}_0$ defined by operators S_{ij}, $1 \leq i, j \leq 2$ in $\mathscr{H}_0$ through (22.5) and if (22.6) holds then the map S defined by (22.4) in $\mathscr{H}$ is a symplectic automorphism of $\mathscr{H}$.

Proposition 22.1: Let $\mathscr{H}_0 \subset \mathscr{H}$ be any completely real subspace of $\mathscr{H}$ such that $\mathscr{H} = \mathscr{H}_0 + i\mathscr{H}_0$. Then every symplectic automorphism S of $\mathscr{H}$ admits a factorisation $S = U_1 T U_2$ where U_1, U_2 are unitary operators in $\mathscr{H}$ and T is a symplectic automorphism of the form

$$T(u + iv) = Au + iA^{-1}v, \quad u, v \in \mathscr{H}_0,$$

A being a positive bounded operator with a bounded inverse in the real Hilbert space $\mathscr{H}_0$.

Proof: Define the operator S_0 in $\mathscr{H}_0 \oplus \mathscr{H}_0$ through (22.4) and (22.5) so that (22.6) holds. Let $S_0 = U_0 H_0$ be the polar decomposition of S_0 so that U_0 is unitary and H_0 is a bounded positive operator with a bounded inverse in $\mathscr{H}_0 \oplus \mathscr{H}_0$. Then (22.6) can be expressed as

$$JU_0 H_0 = U_0 H_0^{-1} J = U_0 J J^{-1} H_0^{-1} J$$

where JU_0 and $U_0 J$ are unitary and H_0 and $J^{-1} H_0^{-1} J$ are positive. Thus the uniqueness of polar decomposition implies

$$JU_0 = U_0 J, \; H_0 = J^{-1} H_0^{-1} J. \tag{22.7}$$

In particular, there exist symplectic automorphisms U and H of $\mathscr{H}$ such that U_0 is unitary and H_0 is positive in $\mathscr{H}_0 \oplus \mathscr{H}_0$. Furthermore U is a unitary operator in $\mathscr{H}$. The second relation in (22.7) implies that H_0 and H_0^{-1} are unitarily equivalent through J. Let ξ denote the spectral measure of H_0 in $(0, \infty)$ so that $H_0 = \int x \xi(dx)$. Let R_1, R_2, R_3 be the ranges of the spectral projections $\xi((0,1)), \xi(\{1\})$ and $\xi((1, \infty))$ respectively. Then $R_1 \oplus R_2 \oplus R_3 = \mathscr{H}_0 \oplus \mathscr{H}_0$. The second relation in (22.7) implies that J maps R_1 onto R_3, R_3 onto R_1 and R_2 onto itself. Since

$$\langle \begin{pmatrix} u \\ v \end{pmatrix}, \; J \begin{pmatrix} u \\ v \end{pmatrix} \rangle = 0 \text{ for all } u, v \in \mathscr{H}_0$$

we can and do select an orthonormal basis of the form $\{\binom{u_j}{v_j}, \; J\binom{u_j}{v_j} | j = 1, 2, \ldots, \}$ for the subspace R_2. Let R_{21} and R_{22} be the closed subspaces spanned by $\{\binom{u_j}{v_j} | j = 1, 2, \ldots\}$ and $\{J\binom{u_j}{v_j} | j = 1, 2, \ldots\}$ respectively so that $R_2 = R_{21} \oplus R_{22}$. Write $M_1 = R_1 \oplus R_{21}$, $M_2 = R_{22} \oplus R_3$. Then $\mathscr{H}_0 \oplus \mathscr{H}_0 = M_1 \oplus M_2, J$ maps M_1 onto M_2 and vice versa. Furthermore H_0 leaves M_1 and M_2 invariant. Choose orthonormal bases $\{w_1, w_2, \ldots\}$ and $\{e_1, \, e_2, \, \ldots\}$ in M_1 and $\mathscr{H}_0$ respectively. Define a unitary operator V_0 in $\mathscr{H}_0 \oplus \mathscr{H}_0$ by putting

$$V_0 w_j = \begin{pmatrix} -e_j \\ 0 \end{pmatrix}, V_0 J w_j = \begin{pmatrix} 0 \\ e_j \end{pmatrix}, \quad j = 1, 2, \ldots$$

Then V_0 commutes with J and $V_0 H_0 V_0^{-1}$ leaves the subspaces $\mathcal{H}_0 \oplus \{0\}$ and $\{0\} \oplus \mathcal{H}_0$ invariant. Thus $V_0 H_0 V_0^{-1}$ has the form $\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$ where A is a bounded positive operator with bounded inverse in $\mathcal{H}_0$. Hence

$$S_0 = U_0 V_0^{-1} \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} V_0.$$

If U and V are the unitary operators in $\mathcal{H}$ which determine the unitary operators U_0, V_0 in $\mathcal{H}_0 \oplus \mathcal{H}_0$ respectively and T is the symplectic automorphism for which $T_0 = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$ then $S = U_1 T U_2$ where $U_1 = UV^{-1}$, $U_2 = V$. ∎

Proposition 22.2: For any u, v, w in $\mathcal{H}$

$$\langle e(v), e^{-\frac{1}{2}p(u)^2} e(w)\rangle = (1 + \|u\|^2)^{-\frac{1}{2}} \exp\{\langle v, w\rangle + \frac{(\langle v, u\rangle - \langle u, w\rangle)^2}{2(1 + \|u\|^2)}\}. \quad (22.8)$$

Proof: The left hand side of (22.8) can be expressed as

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \langle e(v), e^{-ixp(u)} e(w)\rangle e^{-\frac{1}{2}x^2} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \langle e(v), W(xu) e(w)\rangle e^{-\frac{1}{2}x^2} dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\{-\frac{1}{2}(1 + \|u\|^2)x^2 + x(\langle v, u\rangle - \langle u, w\rangle) + \langle u, w\rangle\} dx.$$

Now (22.8) follows by a straightforward evaluation of Gaussian integrals. ∎

Corollary 22.3: Let $u_1, u_2, \ldots, u_n$ be mutually orthogonal vectors in $\mathcal{H}$. Then

$$\langle e(v), \Pi_{j=1}^n e^{-\frac{1}{2}p(u_j)^2} e(w)\rangle =$$
$$\left\{\Pi_{j=1}^n (1 + \|u_j\|^2)^{-\frac{1}{2}}\right\} \exp\left\{\langle v, w\rangle + \sum_{j=1}^n \frac{(\langle v, u_j\rangle - \langle u_j, w\rangle)^2}{2(1 + \|u_j\|^2)}\right\}. \quad (22.9)$$

Proof: Denote by P the projection on the subspace spanned by $\{u_1, \ldots, u_n\}$. By Proposition 19.6 $\Gamma_s(\mathcal{H})$ can be identified with the tensor product

$$\{\bigotimes_{j=1}^n \Gamma_s(\mathbb{C}u_j)\} \bigotimes \Gamma_s(\{u_1, \ldots, u_n\}^\perp)$$

so that

$$e(v) = \{\bigotimes_{j=1}^n e(\frac{\langle u_j, v\rangle}{\|u_j\|^2} u_j)\} \otimes e((1 - P)v) \text{ for all } v \in \mathcal{H}.$$

By Exercise 20.21

$$\Pi_{j=1}^n e^{-\frac{1}{2}p(u_j)^2} = \{\bigotimes_{j=1}^n e^{-\frac{1}{2}p(u_j)^2}\} \otimes 1.$$

Then the left hand side of (22.9) is equal to

$$e^{\langle v,(1-P)w\rangle}\Pi_{j=1}^{n}\langle e(\frac{\langle u_j,v\rangle}{\|u_j\|^2}u_j), e^{-\frac{1}{2}p(u_j)^2}e(\frac{\langle u_j,w\rangle}{\|u_j\|^2}u_j)\rangle.$$

Now the corollary is immediate from Proposition 22.2. ■

Proposition 22.4: Let $\{u_n\}$ be a sequence of mutually orthogonal vectors in $\mathscr{H}$ such that $\lim_{n\to\infty}\|u_n\|^2 = \lambda$. Then

$$\underset{n\to\infty}{\text{w.lim}}\ \Pi_{j=1}^{n}e^{-\frac{t}{2n}p(u_j)^2} = e^{-\lambda t/2} \text{ for all } \ t \geq 0. \tag{22.10}$$

Proof: For any $v, w \in \mathscr{H}$ we have from (22.9)

$$\begin{aligned}&\langle e(v), \Pi_{j=1}^{n}e^{-\frac{t}{2n}p(u_j)^2}e(w)\rangle = \langle e(v), \Pi_{j=1}^{n}e^{-\frac{1}{2}p(\sqrt{\frac{t}{n}}u_j^2)}e(w)\rangle \\ &= \{\Pi_{j=1}^{n}(1+\frac{t}{n}\|u_j\|^2)^{-\frac{1}{2}}\}\exp\{\langle v,w\rangle + \frac{t}{n}\sum_{j=1}^{n}\frac{(\langle v,u_j\rangle - \langle u_j,w\rangle)^2}{2(1+\frac{t}{n}\|u_j\|^2)}\}.\end{aligned} \tag{22.11}$$

By hypothesis

$$\lim_{n\to\infty}\Pi_{j=1}^{n}(1+\frac{t}{n}\|u_j\|^2) = e^{\lambda t}.$$

Since $\{u_j\}$ are mutually orthogonal $\lim_{n\to\infty}\langle v, u_n\rangle = 0$ for every v. Hence the right hand side of (22.11) converges to $\exp\{-\frac{1}{2}\lambda t + \langle v,w\rangle\}$ as $n \to \infty$. Since exponential vectors are total in $\Gamma_s(\mathscr{H})$ and $\exp -\frac{1}{2}p(u)^2$ is a contraction operator for every u in $\mathscr{H}$ we obtain (22.10). ■

Proposition 22.5: Let $\mathscr{H}_0 \subset \mathscr{H}$ be a completely real subspace such that $\mathscr{H} = \mathscr{H}_0 + i\mathscr{H}_0$ and let A be a bounded positive operator in $\mathscr{H}_0$ with a bounded inverse. Suppose there exists a unitary operator Γ in $\Gamma_s(\mathscr{H}_0)$ satisfying the relations

$$\Gamma W(u)\Gamma^{-1} = W(Au) \text{ for all } u \in \mathscr{H}_0. \tag{22.12}$$

Then every point $\lambda \neq 1$ in the spectrum of A is an isolated eigenvalue of finite multiplicity.

Proof: By hypothesis the spectrum of A is a closed bounded set contained in the open interval $(0,\infty)$. If $\lambda \neq 1$ is an isolated eigenvalue of infinite multiplicity then there exists an orthonormal sequence $\{u_n\}$ such that $Au_n = \lambda u_n$ and $\|Au_n\|^2 = \lambda^2$ for $n = 1, 2, \ldots$. If $\lambda \neq 1$ is a limit point in the spectrum of A then there exist sequences $\{\lambda_n\}, \{\varepsilon_n\}$ of positive numbers such that λ_n is in the spectrum of A, λ_n converges monotonically to $\lambda, \varepsilon_n \to 0$ as $n \to \infty$ and the intervals $(\lambda_j - \varepsilon_j, \lambda_j + \varepsilon_j)$, $j = 1, 2, \ldots$ are disjoint. If $A = \int x\xi(dx)$ is the spectral resolution of A we can choose a unit vector u_j in the range of $\xi((\lambda_j - \varepsilon_j, \lambda_j + \varepsilon_j))$ for each j. This shows that if the proposition is not true we can construct an orthonormal sequence $\{u_n\}$ such that $\{Au_n\}$ is an orthogonal sequence of vectors and

$$\lim_{n\to\infty}\|Au_n\|^2 = \lambda^2. \tag{22.13}$$

From (22.12) we get

$$\Gamma\, \Pi_{j=1}^{n} e^{-\frac{t}{2n}p(u_j)^2}\Gamma^{-1} = \Pi_{j=1}^{n} e^{-\frac{t}{2n}p(Au_j)^2}. \tag{22.14}$$

By Proposition 22.4 and (22.13) the left hand side of (22.14) converges weakly to $e^{-\frac{1}{2}t}$ whereas the right hand side converges weakly to $e^{-\frac{1}{2}\lambda^2 t}$ for each $t > 0$ and $\lambda \neq 1$. This is a contradiction. ■

Proposition 22.6: Let $\dim \mathscr{H} < \infty, \mathscr{H} = \mathscr{H}_0 + i\mathscr{H}_0$ where $\mathscr{H}_0$ is a completely real subspace and let G denote the general linear group of all linear invertible transformations of $\mathscr{H}_0$. For any $C \in G$ let S_C denote the symplectic automorphism of $\mathscr{H}$ defined by $S_C(u + iv) = Cu + iC^{*^{-1}}v$ for all u, v in $\mathscr{H}_0$. Then there exists a unitary representation $C \to V_C$ of G in $\Gamma_s(\mathscr{H})$ satisfying

(i) $$V_C W(u) V_C^{-1} = W(S_C u) \text{ for all } C \in G, u \in \mathscr{H}; \tag{22.15}$$

(ii) $$\langle \Phi, V_C \Phi \rangle = |\det \frac{1}{2}(C + C^{*^{-1}})|^{-1/2}, \tag{22.16}$$

where Φ is the vacuum state.

Proof: Let $\dim \mathscr{H} = n$. Choose and fix an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ in $\mathscr{H}_0$. Consider the unitary isomorphism $U : \Gamma_s(\mathscr{H}) \to L^2(\mathbb{R}^n)$ satisfying

$$[Ue(u)](\underline{x}) = (2\pi)^{-\frac{n}{4}} \exp \sum_{j=1}^{n} \{-\frac{1}{4}x_j^2 + \langle e_j, u\rangle x_j - \frac{1}{2}\langle e_j, u\rangle^2\} \tag{22.17}$$

for all u in $\mathscr{H}$. Then (20.27) and (20.28) in the proof of Proposition 20.9 imply

$$[UW(u)U^{-1}f](\underline{x}) = f(\underline{x} - 2\underline{u}), \tag{22.18}$$

$$[UW(iu)U^{-1}f](\underline{x}) = e^{i\underline{u}\cdot\underline{x}} f(\underline{x}) \tag{22.19}$$

for any $u \in \mathscr{H}_0, f \in L^2(\mathbb{R}^n)$ and $\underline{u} = (\langle e_1, u\rangle, \ldots, \langle e_n, u\rangle)$. Define the operators L_C in $L^2(\mathbb{R}^n)$ by

$$(L_C f)(\underline{x}) = |\det C|^{-\frac{1}{2}} f(C^{-1}\underline{x}), C \in G, f \in L^2(\mathbb{R}^n). \tag{22.20}$$

Put $V_C = U^{-1}L_C U$. Then $C \to V_C$ is a unitary representation of G and (22.17)–(22.20) imply that for any u in $\mathscr{H}_0$

$$V_C W(u) V_C^{-1} = W(Cu), V_C W(iu) V_C^{-1} = W(C^{*^{-1}}u).$$

Thus for any u in $\mathscr{H}$ we obtain (22.15). From (22.17) and (22.20) we have

$$\begin{aligned} \langle \Phi, V_C \Phi \rangle &= \langle Ue(0), L_C Ue(0)\rangle \\ &= |\det C|^{-\frac{1}{2}} (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \{\exp -\frac{1}{4}\langle \underline{x}, (C^{*^{-1}}C^{-1} + 1)\underline{x}\rangle\} d\underline{x} \\ &= |\det \frac{1}{2}(C + C^{*^{-1}})|^{-\frac{1}{2}}. \end{aligned}$$

■

We shall now try to extend Proposition 22.6 to the infinite dimensional case. Choose $\mathcal{H} = \ell^2 = \mathbb{C} \oplus \mathbb{C} \oplus \cdots$ and denote by $\mathcal{H}_k$ the subspace of all sequences whose coordinates vanish from the $k+1$-th stage. We have $\Gamma_s(\mathcal{H}) = \Gamma_s(\mathcal{H}_k) \otimes \Gamma_s(\mathcal{H}_k^{\perp}) = \Gamma_s(\mathbb{C}) \otimes \Gamma_s(\mathbb{C}) \otimes \cdots$ where the countable tensor product is with respect to the stabilising sequence of vacuum vectors. For any $\lambda > 0$ let V_λ be the unitary operator in $\Gamma_s(\mathbb{C})$ defined by Proposition 22.6 so that

$$V_\lambda W(z) V_\lambda^{-1} = W(\lambda x + i\lambda^{-1} y), z = x + iy, \tag{22.21}$$

$$\langle \Phi, V_\lambda \Phi \rangle = (\frac{\lambda + \lambda^{-1}}{2})^{\frac{1}{2}}. \tag{22.22}$$

With these notations we have the following proposition.

Proposition 22.7: Let $\{\lambda_n\}$ be a sequence of scalars such that $\lambda_n > 0$ for each n and $\lim_{n\to\infty} \lambda_n = 1$. Let

$$\Gamma_n = V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_n} \otimes 1_{[n+1} \tag{22.23}$$

where $1_{[n+1}$ is the identity operator in $\Gamma_s(\mathcal{H}_n^{\perp})$. Then there exists a sequence $\{c_n\}$ of scalars such that $c_n \Gamma_n$ converges strongly to a unitary operator Γ as $n \to \infty$ if and only if $c_n \to c$ as $n \to \infty$, $|c| = 1$ and $\sum_j (\lambda_j - 1)^2 < \infty$.

Proof: First we prove necessity. In the course of the proof we shall view $\psi(u)$ for any u in $\mathcal{H}_k$ as an element of $\Gamma_s(\mathcal{H}_k)$ as well as $\Gamma_s(\mathcal{H})$. Then

$$\Gamma_n \psi(u) = [V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_k} \psi(u)] \otimes [\bigotimes_{j=k+1}^{n} V_{\lambda_j} \Phi] \otimes \Phi \otimes \Phi \otimes \cdots$$

for any u in $\mathcal{H}_k, n > k$ where Φ denotes the vacuum in $\Gamma_s(\mathbb{C})$. In particular, $\|\Gamma_n \psi(u)\| = \|\psi(u)\| = 1$. Since $c_n \Gamma_n$ converges to a unitary operator it follows that $|c_n| \to 1$ as $n \to \infty$. From (22.22) and (22.23) we have for $u \in \mathcal{H}_k$

$$\begin{aligned} \langle \Phi \otimes \Phi \otimes \cdots, \Gamma \psi(u) \rangle &= \lim_{n\to\infty} c_n \langle \Phi \otimes \Phi \otimes \cdots, \Gamma_n \psi(u) \rangle \\ &= \langle \Phi^{\otimes^k}, V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_k} \psi(u) \rangle \lim_{n\to\infty} c_n \Pi_{j=k+1}^n \left(\frac{\lambda_j + \lambda_j^{-1}}{2} \right)^{-\frac{1}{2}}. \end{aligned} \tag{22.24}$$

The totality of all vectors of the form $\{\psi(u) | u \in \mathcal{H}_k, k = 1, 2, \ldots\}$ in $\Gamma_s(\mathcal{H})$ implies that the left hand side of (22.24) is not equal to 0 for some u in $\mathcal{H}_k$ for some k. This can happen only if $c_n \to c$ as $n \to \infty$, $|c| = 1$ and

$$\sum_j \{1 - (\frac{2}{\lambda_j + \lambda_j^{-1}})^{1/2}\} < \infty \tag{22.25}$$

which is equivalent to $\sum_j (\lambda_j - 1)^2 < \infty$ since $\lambda_n \to 1$ as $n \to \infty$. This proves necessity.

Conversely, let (22.25) hold. Then for any $n > m > k$, we have

$$\begin{aligned}&\|[V_{\lambda_{k+1}}\Phi \otimes \cdots \otimes V_{\lambda_n}\Phi] \otimes \Phi \otimes \Phi \otimes \cdots \\ &\quad -[V_{\lambda_{k+1}}\Phi \otimes \cdots \otimes V_{\lambda_m}\Phi] \otimes \Phi \otimes \Phi \otimes \cdots \|^2 \\ &= 2\{1 - \Pi_{j=m+1}^{n}(\frac{\lambda_j + \lambda_j^{-1}}{2})^{-\frac{1}{2}}\}\end{aligned}$$

which converges to 0 as $m, n \to \infty$. Thus for any fixed k and any u in $\mathcal{H}_k$

$$\lim_{n\to\infty} \Gamma_n \psi(u) = [V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_k}\psi(u)] \otimes \bigotimes_{j=k+1}^{\infty} V_{\lambda_j}\Phi$$

exists. Furthermore $\|\Gamma\psi(u)\| = \|\psi(u)\| = 1$. Thus Γ extends to an isometry on $\Gamma_s(\mathcal{H})$. By the same argument, for any $u \in \mathcal{H}_k$

$$\lim_{n\to\infty} [V_{\lambda_1^{-1}} \otimes \cdots \otimes V_{\lambda_k^{-1}}\psi(u)] \otimes \bigotimes_{j=k+1}^{n} V_{\lambda_j^{-1}}\Phi \otimes \Phi \otimes \Phi \otimes \cdots = \Gamma'\psi(u)$$

exists and Γ' extends to an isometry on $\Gamma_s(\mathcal{H})$. Clearly $\Gamma' = \Gamma^*$ and Γ is a unitary operator. ■

Proposition 22.8: Let $\mathcal{H} = h \otimes k$ be the tensor product of two Hilbert spaces h, k. Suppose A is a bounded operator in $\mathcal{H}$ satisfying the relation

$$A(B \otimes 1) = (B \otimes 1)A \text{ for all } B \in \mathcal{B}(h),$$

1 being the identity in k. Then there exists a $C \in \mathcal{B}(k)$ such that $A = 1 \otimes C$ where 1 denotes the identity in h.

Proof: Choose and fix a unit vector u_0 in h. For any v in k we have

$$\{(1 - |u_0\rangle\langle u_0|) \otimes 1\}Au_0 \otimes v = A(1 - |u_0\rangle\langle u_0|)u_0 \otimes v = 0.$$

Thus $Au_0 \otimes v \in R(|u_0\rangle\langle u_0| \otimes 1)$. In other words there exists a v' in k such that $Au_0 \otimes v = u_0 \otimes v'$. The correspondence $v \to v'$ yields a linear operator $C \in \mathcal{B}(k)$ such that $Au_0 \otimes v = u_0 \otimes Cv$. Let now u be any element in h. Choose an operator $B \in \mathcal{B}(h)$ such that $Bu_0 = u$. Then

$$Au \otimes v = A(B \otimes 1)u_0 \otimes v = (B \otimes 1)u_0 \otimes Cv = u \otimes Cv.$$

i.e., $A = 1 \otimes C$. ■

Proposition 22.9: (von Neumann's Double Commutant Theorem [133]): For any von Neumann algebra $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ let $\mathcal{N}' = \{A | A \in \mathcal{B}(\mathcal{H}), AB = BA$ for every $B \in \mathcal{N}\}$. Then $\mathcal{N}'$ is also a von Neumann algebra and $\mathcal{N}'' = \mathcal{N}$.

Proof: We omit the proof and refer to [133]. ■

Proposition 22.10: Let $\mathcal{H}$ be any Hilbert space. Then the smallest weakly closed algebra containing $\{W(u) | u \in \mathcal{H}\}$ is $\mathcal{B}(\Gamma_s(\mathcal{H}))$.

Proof: Since $W(u)^* = W(-u)$ it follows that the smallest weakly closed algebra $\mathcal{N}$ containing $\{W(u) | u \in \mathcal{H}\}$ is closed under * and hence a von Neumann algebra. By Proposition 20.9 $\mathcal{N}' = \mathbb{C}$ and $\mathcal{N}'' = \mathcal{B}(\Gamma_s(\mathcal{H}))$. An application of Proposition 22.9 completes the proof. ■

Theorem 22.11: (Shale's Theorem [123]): Let $\mathcal{H} = \mathcal{H}_0 + i\mathcal{H}_0$ where $\mathcal{H}_0$ is a completely real subspace. For any symplectic automorphism S of $\mathcal{H}$ let S_0 be the operator in $\mathcal{H}_0 \oplus \mathcal{H}_0$ defined by (22.4) and (22.5). Then there exists a unitary operator $\Gamma(S)$ in $\Gamma_s(\mathcal{H})$ such that

$$\Gamma(S)W(u)\Gamma(S)^{-1} = W(Su) \text{ for all } u \in \mathcal{H} \tag{22.26}$$

if and only if $S_0^* S_0 - 1$ is a Hilbert-Schmidt operator in $\mathcal{H}_0 \oplus \mathcal{H}_0$. In such a case $\Gamma(S)$ is determined uniquely up to a scalar multiple of modulus unity.

Proof: Let S be any symplectic automorphism of $\mathcal{H}$. By Proposition 22.1 we can express

$$S = U_1 T U_2, T(u + iv) = Au + iA^{-1}v, \quad u, v \in \mathcal{H}_0 \tag{22.27}$$

where A is a bounded positive operator with positive inverse in $\mathcal{H}_0$ and $U_1, U_2 \in \mathcal{U}(\mathcal{H})$. If $\Gamma(U_j)$ is the second quantization of U_j we have

$$\Gamma(U_j)W(u)\Gamma(U_j)^{-1} = W(U_j u), u \in \mathcal{H}, \quad j = 1, 2.$$

Thus a unitary operator $\Gamma(S)$ satisfying (22.16) exists if and only if a unitary operator $\Gamma(T)$ satisfying

$$\Gamma(T)W(u)\Gamma(T)^{-1} = W(Tu), u \in \mathcal{H} \tag{22.28}$$

exists when T is of the form described in (22.27). Suppose $\dim \mathcal{H} < \infty$. Then the condition that $S_0^* S_0 - 1$ is Hilbert-Schmidt is superfluous. Put $\Gamma(T) = V_A$ where V_A is determined by Proposition 22.6. Then (22.28) obtains and the proof is complete in this case.

Let now $\dim \mathcal{H} = \infty$. First we prove sufficiency. The condition that $S_0^* S_0 - 1$ is Hilbert-Schmidt implies that $A^2 - 1$ and $A^{-2} - 1$ are Hilbert-Schmidt in $\mathcal{H}_0$. Hence there exists an orthonormal basis $\{e_j\}$ in $\mathcal{H}_0$ such that $Ae_j = \lambda_j e_j$, $\lambda_j > 0$ for each j and $\sum_j (\lambda_j - 1)^2 < \infty$. Now identify $\mathcal{H}_0$ with real ℓ^2 via the basis $\{e_j\}$, use Proposition 22.7 and define

$$\Gamma(T) = \operatorname*{s.lim}_{n \to \infty} \Gamma_n$$

where Γ_n is defined by (22.23). Then $\Gamma(T)$ satisfies (22.28) and the proof of sufficiency is complete.

To prove necessity suppose that a unitary $\Gamma(T)$ satisfying (22.28) exists. Then $\Gamma(T)W(u)\Gamma(T)^{-1} = W(Au)$ for all u in $\mathcal{H}_0$. By Proposition 22.5 there exists an orthonormal basis $\{e_j\}$ in $\mathcal{H}_0$ such that $Ae_j = \lambda_j e_j$, $\lambda_j > 0$ for each j and $\lim_{j \to \infty} \lambda_j = 1$. Once again identify $\mathcal{H}$ with ℓ^2 via the basis $\{e_j\}$ and define the unitary operators Γ_n by (22.23). Then

$$\Gamma_n W(u) \Gamma_n^{-1} = W(Tu) \text{ for all } u \in \mathcal{H}_n, \quad n = 1, 2, \ldots$$

This together with (22.28) implies that

$$\Gamma_n^{-1}\Gamma(T)W(u) = W(u)\Gamma_n^{-1}\Gamma(T) \text{ for all } u \in \mathcal{H}_n.$$

By Proposition 22.10, the von Neumann algebra generated by $\{W(u)|u \in \mathcal{H}_n\}$ consists of all operators of the form $B \otimes 1$ where $B \in \mathcal{B}(\Gamma_s(\mathcal{H}_n))$, 1 denotes the identity operator in $\Gamma_s(\mathcal{H}_n^\perp)$ and $\Gamma_s(\mathcal{H}) = \Gamma_s(\mathcal{H}_n) \otimes \Gamma_s(\mathcal{H}_n^\perp)$. By Proposition 22.8 there exists a unitary operator Γ_n' in $\Gamma_s(\mathcal{H}_n^\perp)$ such that

$$\Gamma(T) = V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_n} \otimes \Gamma_n'$$

for every n. Let Φ_n' denote the vacuum vector in $\Gamma_s(\mathcal{H}_n^\perp)$ and let

$$P_n = 1 \otimes |\Phi_n'\rangle\langle\Phi_n'|$$

where 1 is the identity in $\Gamma_s(\mathcal{H}_n)$. Then for any $u \in \mathcal{H}_k$ and $n > k$

$$P_n\Gamma(T)e(u) = c_n[V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_k} e(u)] \otimes \bigotimes_{j=k+1}^{n} V_{\lambda_j}\Phi \otimes \Phi \otimes \Phi \otimes \cdots$$

where

$$c_n = \langle \Phi_n', \Gamma_n'\Phi_n'\rangle.$$

Since $\lim_{n\to\infty} P_n\Gamma(T)e(u) = \Gamma(T)e(u)$ for every $u \in \mathcal{H}_k$, $k = 1, 2, \ldots, c_n\Gamma_n$ converges strongly to a unitary operator where Γ_n is defined by (22.23). By Proposition 22.7, $c_n \to c$ as $n \to \infty$, $|c| = 1$ and $\sum_j(\lambda_j - 1)^2 < \infty$. This implies that $S_0^*S_0 - 1$ is Hilbert-Schmidt.

We now turn to the uniqueness of $\Gamma(S)$ in (22.26). Suppose there is another unitary operator $\Gamma'(S)$ satisfying (22.26) when $\Gamma(S)$ is replaced by $\Gamma'(S)$. Then $\Gamma'(S)\Gamma(S)^{-1}$ commutes with all the operators $W(u)$, $u \in \mathcal{H}$. By Proposition 20.9 there exists a scalar λ of modulus unity such that $\Gamma'(S) = \lambda\Gamma(S)$. ■

Let $\mathcal{H}_0 \subset \mathcal{H}$ be a completely real subspace such that $\mathcal{H} = \mathcal{H}_0 + i\mathcal{H}_0$. Define

$$\mathcal{S}_0(\mathcal{H}) = \{S|S \in \mathcal{S}(\mathcal{H}), S_0^*S_0 - 1 \text{ is Hilbert-Schmidt in } \mathcal{H}_0 \oplus \mathcal{H}_0\}$$

where $\mathcal{S}(\mathcal{H})$ is the group of all symplectic automorphisms of $\mathcal{H}$ and S_0 is defined by (22.4) and (22.5). For each $S \in \mathcal{S}_0(\mathcal{H})$ select a unitary operator $\Gamma(S)$ according to Shale's Theorem so that

$$\Gamma(S)W(u)\Gamma(S)^{-1} = W(Su) \text{ for all } u \in \mathcal{H}. \tag{22.29}$$

Then

$$\Gamma(S_1)\Gamma(S_2) = \sigma(S_1, S_2)\Gamma(S_1S_2) \text{ for all } S_1, S_2 \in \mathcal{S}_0(\mathcal{H}), \tag{22.30}$$

where $\sigma(S_1, S_2)$ is a scalar of modulus unity depending on S_1, S_2. It is not clear whether one can choose $\Gamma(S)$ in such a way that $\sigma(S_1, S_2) = 1$. If dim $\mathcal{H} < \infty$ it is clear from Proposition 22.6 that the answer is in the affirmative.

Define

$$\mathcal{I}\mathcal{S}_0(\mathcal{H}) = \{(u, S)|u \in \mathcal{H}, S \in \mathcal{S}_0(\mathcal{H})\}.$$

Under the multiplication

$$(u_1, S_1)(u_2, S_2) = (u_1 + S_1 u_2, S_1 S_2)$$

$\mathcal{IS}_0(\mathcal{H})$ is a group. If $W(u, S) = W(u)\Gamma(S)$ where $\Gamma(\cdot)$ satisfies (22.29) and (22.30) then $(u, S) \to W(u, S)$ is a projective unitary representation of $\mathcal{IS}_0(\mathcal{H})$. It is natural to raise the question of an appropriate topology for $\mathcal{IS}_0(\mathcal{H})$ and explore the continuity of the map $(u, S) \to W(u, S)$.

Exercise 22.12: Let $\mathcal{H} = \mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$, $G = GL(n, \mathbb{R})$ the general linear group of all real non-singular matrices of order n and let $e_1, e_2, \ldots, e_n$ be the canonical basis of $\mathbb{R}^n$. Write

$$p_j = p(\frac{1}{2}e_j), q_j = p(-ie_j), \quad 1 \le j \le n$$

where $p(u)$ is defined by (20.10). Suppose $C \to V_C$ is the unitary representation of G in $\Gamma_s(\mathcal{H})$ described in the proof of Proposition 22.6. For any real $n \times n$ matrix $L = ((\ell_{ij}))$ define the observable X_L as the Stone generator of the one parameter unitary group $\{V_{\exp tL} | t \in \mathbb{R}\}$. Then the following holds:

(i) the family $\{q_j, p_j | j = 1, 2, \ldots\}$ obeys the canonical commutation relations in the domain of finite particle vectors in $\Gamma_s(\mathcal{H})$;

(ii) X_L is the closure of the essentially selfadjoint operator $\frac{1}{2}\sum_j \ell_{jj}(q_j p_j + p_j q_j) + \sum_{i \neq j} \ell_{ij} p_i q_j$ on the domain of all finite particle vectors;

(iii) In the coherent state $\psi(\underline{\alpha} + i\underline{\beta}), \underline{\alpha}, \underline{\beta} \in \mathbb{R}^n$ the distribution of X_L has the characteristic function

$$\phi^L_{\underline{\alpha},\underline{\beta}}(t) =$$

$$| \det \frac{1}{2}(e^{tL} + e^{-tL'})|^{-\frac{1}{2}} \exp\{\underline{\alpha}'[(1 + e^{tL})(1 + e^{tL'}e^{tL})^{-1}(1 + e^{tL'}) - 2]\underline{\alpha}$$

$$-\underline{\beta}'(1 - e^{tL})(1 + e^{tL'}e^{tL})^{-1}(1 - e^{tL'})\underline{\beta}$$

$$-2i\underline{\alpha}'(1 + e^{tL})(1 + e^{tL'}e^{tL})^{-1}(1 - e^{tL'})\underline{\beta}\}$$

where the prime $'$ denotes transpose. In particular, the characteristic function of X_L in the vacuum state is

$$\phi^L_{\underline{0},\underline{0}}(t) = | \det \frac{1}{2}(e^{tL} + e^{-tL'})|^{-\frac{1}{2}};$$

(iv) When L is symmetric

$$\phi^L_{\underline{\alpha},\underline{0}}(t) = | \det \operatorname{sech} \, tL|^{1/2} \exp \underline{\alpha}'(\operatorname{sech} \, tL - 1)\underline{\alpha}$$

is the characteristic function of an absolutely continuous, symmetric and infinitely divisible distribution;

(v) When L is skew symmetric

$$\phi^L_{\underline{\alpha},\underline{\beta}}(t) = \exp\{\underline{\alpha}'(\frac{e^{tL} + e^{-tL}}{2} - 1)\underline{\alpha} + \underline{\beta}'(\frac{e^{tL} + e^{-tL}}{2} - 1)\underline{\beta}$$

$$-i \, \underline{\alpha}'(e^{tL} - e^{-tL})\underline{\beta}\}$$

is the characteristic function of a discrete infinitely divisible distribution. For suitable values of $\underline{\alpha}, \underline{\beta}, L, X_L$ has Poisson distribution.

(vi) When $n = 2$, $L = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$\phi^L_{\underline{0},\underline{0}}(t) = (1 + \frac{t^2}{4})^{-\frac{1}{2}}$$

is an infinitely divisible characteristic function. (It is not clear, in general, when $\phi^L_{\underline{\alpha},\underline{\beta}}$ is an infinitely divisible characteristic function.)

Exercise 22.13: Let S_1, S_2 be bounded operators in a Hilbert space $\mathcal{H}$ and let J be an antiunitary operator in $\mathcal{H}$. Define the unitary operators $\{\tilde{W}(u) | u \in \mathcal{H}\}$ in $\Gamma_s(\mathcal{H} \oplus \mathcal{H}) = \Gamma_s(\mathcal{H}) \otimes \Gamma_s(\mathcal{H})$ by

$$\tilde{W}(u) = W(S_1 u \oplus J S_2 u).$$

If $S_1^* S_1 - S_2^* S_2 = 1$ then

$$\tilde{W}(u)\tilde{W}(v) = e^{-i \operatorname{Im}\langle u,v \rangle} \tilde{W}(u+v) \text{ for all } u, v \in \mathcal{H}.$$

and

$$\langle \Phi \otimes \Phi, \tilde{W}(u) \Phi \otimes \Phi \rangle = \exp[-\frac{1}{2} \langle u, (S_1^* S_1 + S_2^* S_2) u \rangle]$$

where Φ is the vacuum vector in $\Gamma_s(\mathcal{H})$.

Notes

The proof of Shale's Theorem is adapted from Shale [123].

23 Creation, conservation and annihilation operators in $\Gamma_a(\mathcal{H})$

There does not seem to exist a natural analogue of the Weyl representation in the fermion Fock space $\Gamma_a(\mathcal{H})$ even though the second quantization homomorphism $U \to \Gamma_a(U)$ from $\mathcal{U}(\mathcal{H})$ into $\mathcal{U}(\Gamma_a(\mathcal{H}))$ is well-defined by Exercise 20.22. We shall drop the suffix a from $\Gamma_a(U)$ when there is no confusion. The projective unitary representation $u \to W(u)$ of the additive group $\mathcal{H}$ in $\Gamma_s(\mathcal{H})$ is described equivalently through the family $\{p(u) | u \in \mathcal{H}\}$ of its Stone generators obeying the commutation relations $[p(u), p(v)] = 2i \operatorname{Im}\langle u, v \rangle$. These can as well be described by the creation and annihilation operators $a^\dagger(u), a(u), u \in \mathcal{H}$ which obey the commutation relations of Proposition 20.14, $a^\dagger(u)$ maps $\mathcal{H}^{\circledS^n}$ into $\mathcal{H}^{\circledS^{n+1}}$, whereas $a(u)$ maps $\mathcal{H}^{\circledS^n}$ into $\mathcal{H}^{\circledS^{n-1}}$ for each n. We shall now introduce analogues of $a^\dagger(u)$ and $a(u)$ by exploiting the multiplication $\wedge$ (defined by Proposition 19.2) for increasing the rank of antisymmetric tensors.

Proposition 23.1: For any u in $\mathcal{H}$ let $M_n(u) : \mathcal{H}^{@^n} \to \mathcal{H}^{@^{n+1}}$ denote the linear map defined by

$$M_n(u)v_1 \wedge v_2 \wedge \cdots \wedge v_n = u \wedge v_1 \wedge \cdots \wedge v_n$$

for all v_j in $\mathcal{H}$, $1 \leq j \leq n$. Then the adjoint $M_n(u)^*$ of $M_n(u)$ is defined by

$$M_n^*(u)v_1 \wedge \cdots \wedge v_{n+1} = \frac{1}{n+1}\sum_{j=1}^{n+1}(-1)^{j-1}\langle u, v_j\rangle v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_{n+1}$$

where the hat ˆ over v_j indicates its omission.

Proof: For any u_j, $1 \leq j \leq n+1$ we have from (17.10) and Proposition 19.2

$$\langle u_1 \wedge \cdots \wedge u_{n+1}, u \wedge v_1 \wedge \cdots \wedge v_n\rangle =$$

$$\frac{1}{n+1!}\begin{vmatrix} \langle u_1, u\rangle & \langle u_1, v_1\rangle & \cdots & \langle u_1, v_n\rangle \\ \langle u_2, u\rangle & \langle u_2, v_1\rangle & \cdots & \langle u_2, v_2\rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle u_{n+1}, u\rangle & \langle u_{n+1}, v_1\rangle & \cdots & \langle u_{n+1}, v_n\rangle \end{vmatrix}$$

$$= \frac{1}{(n+1)!}\sum_{j=1}^{n+1}(-1)^{j-1}\langle u_j, u\rangle \begin{vmatrix} \langle u_1, v_1\rangle & \cdots & \langle u_1, v_n\rangle \\ \cdots & \cdots & \cdots \\ \langle u_{j-1}, v_1\rangle & \cdots & \langle u_{j-1}, v_n\rangle \\ \langle u_{j+1}, v_1\rangle & \cdots & \langle u_{j+1}, v_n\rangle \\ \cdots & \cdots & \cdots \\ \langle u_{n+1}, v_1\rangle & \cdots & \langle u_{n+1}, v_n\rangle \end{vmatrix}$$

$$= \frac{1}{n+1}\sum_{j=1}^{n+1}(-1)^{j-1}\langle u_j, u\rangle\langle u_1 \wedge \cdots \wedge \hat{u}_j \wedge \cdots \wedge u_{n+1}, v_1 \wedge \cdots \wedge v_n\rangle.$$

■

Proposition 23.2: For every u in $\mathcal{H}$ define the linear operators $a(u)$ and $a^\dagger(u)$ on the domain of all finite particle vectors in $\Gamma_a(\mathcal{H})$ by putting

$$a(u)\Phi = 0, a(u)|_{\mathcal{H}^{@^n}} = \sqrt{n}M_{n-1}^*(u),$$
$$a^\dagger(u)\Phi = u, a^\dagger(u)|_{\mathcal{H}^{@^n}} = \sqrt{n+1}M_n(u), \quad n = 1, 2, \ldots$$

where $M_n(u)$ and $M_n(u)^*$ are as in Proposition 23.1. Then

$$\{a(u)a^\dagger(v) + a^\dagger(v)a(u)\}\psi = \langle u, v\rangle\psi$$

for every finite particle vector ψ in $\Gamma_a(\mathcal{H})$.

Proof: By definitions and Proposition 23.1

$$
\begin{aligned}
a^{\dagger}(v) w_1 \wedge \cdots \wedge w_n &= \sqrt{n+1} v \wedge w_1 \wedge \cdots \wedge w_n, \\
a(u) a^{\dagger}(v) w_1 \wedge \cdots \wedge w_n &= \langle u, v\rangle w_1 \wedge \cdots \wedge w_n \\
&+ \sum_{j=1}^{n} (-1)^j \langle u, w_j\rangle v \wedge w_1 \wedge \cdots \wedge \hat{w}_j \wedge \cdots \wedge w_n,
\end{aligned} \tag{23.1}
$$

$$
a(u) w_1 \wedge \cdots \wedge w_n = n^{-\frac{1}{2}} \sum_{j=1}^{n} (-1)^{j-1} \langle u, w_j\rangle w_1 \wedge \cdots \wedge \hat{w}_j \wedge \cdots \wedge w_n,
$$

$$
a^{\dagger}(v) a(u) w_1 \wedge \cdots \wedge w_n = \sum_{j=1}^{n} (-1)^{j-1} \langle u, w_j\rangle v \wedge w_1 \wedge \cdots \wedge \hat{w}_j \wedge \cdots \wedge w_n. \tag{23.2}
$$

Adding (23.1) and (23.2) we obtain the required result. ■

Proposition 23.3: In the fermion Fock space $\Gamma_a(\mathcal{H})$ there exists a unique family $\{a(u), a^{\dagger}(u) | u \in \mathcal{H}\}$ of bounded operators satisfying the following conditions:

(i) $a(u)\Phi = 0, a^{\dagger}(u)\Phi = u, \Phi$ being the vacuum and u a 1-particle vector;

(ii) $a(u) v_1 \wedge \cdots \wedge v_n = n^{-\frac{1}{2}} \sum_{j=1}^{n} (-1)^{j-1} \langle u, v_j\rangle v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_n, a^{\dagger}(u) v_1 \wedge \cdots \wedge v_n = (n+1)^{1/2} u \wedge v_1 \wedge \cdots \wedge v_n$ for all $v_j \in \mathcal{H}$, $1 \leq j \leq n$, $n = 1, 2, \ldots$;

(iii) $a^{\dagger}(u) = a(u)^*$ for all u in $\mathcal{H}$.

Proof: First define $a(u)$ and $a^{\dagger}(u)$ on the dense linear manifold $\Gamma_a^0(\mathcal{H})$ of all finite particle vectors in $\Gamma_a(\mathcal{H})$ through Proposition 23.2. By Proposition 23.1, $a(u)$ and $a^{\dagger}(u)$ are adjoint to each other on $\Gamma_a^0(\mathcal{H})$. From Proposition 23.2 we have for any $\psi \in \Gamma_a^0(\mathcal{H})$

$$
\begin{aligned}
\|a(u)\psi\|^2 + \|a^{\dagger}(u)\psi\|^2 &= \langle \psi, a^{\dagger}(u) a(u) + a(u) a^{\dagger}(u) \psi\rangle \\
&= \|u\|^2 \|\psi\|^2.
\end{aligned}
$$

In particular,

$$
\|a(u)\psi\| \leq \|u\| \, \|\psi\|, \|a^{\dagger}(u)\psi\| \leq \|u\| \, \|\psi\|, \psi \in \Gamma_a^0(\mathcal{H}).
$$

Hence we can and do extend $a(u)$ and $a^{\dagger}(u)$ uniquely to bounded operators on $\Gamma_a(\mathcal{H})$ and denote them by the same symbols so that $\|a(u)\| \leq \|u\|, \|a^{\dagger}(u)\| \leq \|u\|$. The rest is immediate. ■

Proposition 23.4: The operators $a(u)$ and $a^{\dagger}(u)$ defined by Proposition 25.3 satisfy the following:

(i) $a(u)a(v) + a(v)a(u) = 0$, $a^{\dagger}(u)a^{\dagger}(v) + a^{\dagger}(v)a^{\dagger}(u) = 0$, $a(u)a^{\dagger}(v) + a^{\dagger}(v)a(u) = \langle u, v\rangle$ for all $u, v \in \mathcal{H}$;

(ii) For any $U \in \mathcal{U}(\mathcal{H})$ its second quantization $\Gamma(U)$ in $\Gamma_a(\mathcal{H})$ satisfies $\Gamma(U) a(u) \Gamma(U)^{-1} = a(Uu)$ for all $u \in \mathcal{H}$;

(iii) The set $\{\Phi, a^\dagger(u_1)\cdots a^\dagger(u_n)\Phi | u_j \in \mathcal{H}, 1 \leq j \leq n, n = 1, 2, \ldots\}$ is total in $\Gamma_a(\mathcal{H})$;

(iv) If $T \in \mathcal{B}(\Gamma_a(\mathcal{H}))$, $Ta(u) = a(u)T$, $Ta^\dagger(u) = a^\dagger(u)T$ for all u in $\mathcal{H}$ then T is a scalar multiple of the identity.

Proof: By (i) and (ii) in Proposition 23.3 the vectors $\{a^\dagger(u)a^\dagger(v) + a^\dagger(v)a^\dagger(u)\}\Phi$ and $\{a^\dagger(u)a^\dagger(v) + a^\dagger(v)a^\dagger(u)\}w_1 \wedge \cdots \wedge w_n$ are respectively scalar multiples of $u \wedge v + v \wedge u$ and $(u \wedge v + v \wedge u) \wedge w_1 \wedge \cdots \wedge w_n$ which vanish. The density of finite particle vectors and the boundedness of the operators $a^\dagger(u)$ imply the second relation in (i). The first relation in (i) follows from the second by taking adjoints. The third relation in (i) follows from Proposition 23.2. Since $\Gamma(U)\Phi = \Phi$ and $\Gamma(U)v_1 \wedge \cdots \wedge v_n = Uv_1 \wedge \cdots \wedge Uv_n$ for U in $\mathcal{U}(\mathcal{H})$ we obtain (ii) from Proposition 23.3. (iii) is just a restatement of the fact that finite particle vectors are dense in $\Gamma_a(\mathcal{H})$. We now prove (iv). For any $u_1, \ldots, u_n$ in $\mathcal{H}$ we have

$$\begin{aligned}\langle a^\dagger(u_1)\cdots a^\dagger(u_n)\Phi, T\Phi\rangle &= \langle a^\dagger(u_2)\cdots a^\dagger(u_n)\Phi, a(u_1)T\Phi\rangle \\ &= \langle a^\dagger(u_2)\cdots a^\dagger(u_n)\Phi, Ta(u_1)\Phi\rangle = 0.\end{aligned}$$

In other words $T\Phi$ is orthogonal to every n-particle vector for $n \geq 1$. Hence there exists a scalar λ such that $T\Phi = \lambda\Phi$. Now

$$\begin{aligned}Ta^\dagger(u_1)\cdots a^\dagger(u_n)\Phi &= a^\dagger(u_1)\cdots a^\dagger(u_n)T\Phi \\ &= \lambda a^\dagger(u_1)\cdots a^\dagger(u_n)\Phi\end{aligned}$$

and (iv) follows from (iii). ∎

The operators $a^\dagger(u), a(u)$ defined by Proposition 23.3 are respectively called the *fermion creation* and *annihilation* operators associated with u. The relations (i) in Proposition 23.4 are called the *canonical anticommutation relations* or CAR. It is instructive to compare CCR, CAR and the relations satisfied by the free creation and annihilation operators described in Exercise 20.24 and explore the connections between them. (See Example 25.18, Exercise 25.35.)

Exercise 23.5: Suppose $b(u), b^\dagger(u), u \in \mathcal{H}$ are bounded operators in a Hilbert space $\tilde{\mathcal{H}}$ such that $b^\dagger(u) = b(u)^*$ and the map $u \to b^\dagger(u)$ is linear. Let the following conditions be fulfilled.

(i) $b(u)b(v) + b(v)b(u) = 0$,

(ii) There exists a unit vector $\tilde{\Phi} \in \tilde{\mathcal{H}}$ such that $b(u)\tilde{\Phi} = 0$ for all u and the set $\{\tilde{\Phi}, b^\dagger(u_1)\cdots b^\dagger(u_n)\tilde{\Phi} | u_j \in \mathcal{H}, 1 \leq j \leq n, n = 1, 2, \ldots\}$ is total in $\tilde{\mathcal{H}}$.

Then there exists a unitary isomorphism $V : \tilde{\mathcal{H}} \to \Gamma_a(\mathcal{H})$ such that $V\tilde{\Phi} = \Phi$, $Vb^\dagger(u_1)\cdots b^\dagger(u_n)\tilde{\Phi} = a^\dagger(u_1)\cdots a^\dagger(u_n)\Phi$ for all $u_j \in \mathcal{H}$, $1 \leq j \leq n$, $n = 1, 2, \ldots$ and $Vb(u)V^{-1} = a(u)$ for all $u \in \mathcal{H}$. (Hint: Use Proposition 7.2.)

Exercise 23.6: Let $p(u) = i\{a^\dagger(u) - a(u)\}, q(u) = a(u) + a^\dagger(u)$ in $\Gamma_a(\mathcal{H})$. Then $p(u)$ and $q(u)$ are selfadjoint operators satisfying the relations:

$$p(u)p(v) + p(v)p(u) = 2\,\mathrm{Re}\langle u, v\rangle,$$

$$q(u)q(v) + q(v)q(u) = 2\,\mathrm{Re}\langle u, v\rangle,$$

$$p(u)q(v) + q(v)p(u) = 2\,\mathrm{Im}\langle u, v\rangle,$$

$$\Gamma(i)q(u)\Gamma(i)^{-1} = p(u) \text{ for all } \ u, v \in \mathcal{H}.$$

In particular, $p(u)^2 = q(u)^2 = \|u\|^2$. In the vacuum state as well as any pure n-particle state of the form $\|u_1 \wedge \cdots \wedge u_n\|^{-1}(u_1 \wedge \cdots \wedge u_n)$ the observables $q(u)$ and $p(u)$ assume the values $\pm\|u\|$ with equal probability.

Exercise 23.7: Let $U_t \in \mathcal{U}(\mathcal{H})$, $0 \neq u_t \in \mathcal{H}$ for each $t \in \mathbb{R}$ be such that

$$U_t^2 = 1, \quad U_s U_t = U_t U_s \text{ for all } \ s, t$$

and

$$U_s u_t = \begin{cases} u_t - 2u_s & \text{if } s \leq t, \\ -u_t & \text{if } s > t. \end{cases}$$

Then $\langle u_s, u_t - u_s\rangle = 0$ and $\|u_t\|$ is a non-decreasing function of t. Define

$$X_t = i\Gamma(U_t)p(u_t)$$

where $p(u)$ is as in Exercise 23.6. Then $\{X_t|\ t \in \mathbb{R}\}$ is a commuting family of observables satisfying the relations:

$$X_t^2 = \|u_t\|^2,$$

$$\langle \Phi, X_{t_1} X_{t_2} \cdots X_{t_{2n}} \Phi\rangle = \|u_{t_1}\|^2 \|u_{t_3}\|^2 \cdots \|u_{t_{2n-1}}\|^2,$$

$$\langle \Phi, X_{t_1} X_{t_2} \cdots X_{t_{2n+1}} \Phi\rangle = 0$$

for all $t_1 < t_2 < \cdots < t_{2n+1}$, $n = 0, 1, 2, \ldots$

The distribution of the family $\{\|u_t\|^{-1} X_t | t \in \mathbb{R}\}$ in the vacuum state Φ is the same as that of a Markov process $\{x_t | t \in \mathbb{R}\}$ with state space $\{1, -1\}, P(x_t = 1) = P(x_t = -1) = \frac{1}{2}$ for all t and the transition probability matrix

$$P(s,t) = \frac{1}{2}\begin{pmatrix} 1 + \|u_s\|\,\|u_t\|^{-1} & 1 - \|u_s\|\,\|u_t\|^{-1} \\ 1 - \|u_s\|\,\|u_t\|^{-1} & 1 - \|u_s\|\,\|u_t\|^{-1} \end{pmatrix}, \quad s < t.$$

We can realise the process $\{x_t|\ t \in \mathbb{R}\}$ as

$$x_t = \begin{cases} 1 & \text{if } y_t \text{ is even}, \\ -1 & \text{if } y_t \text{ is odd} \end{cases}$$

where $\{y_t | t \in \mathbb{R}\}$ is a Poisson process with $y_0 = 0$ or 1 with probability $\frac{1}{2}$ each and $y_t - y_s$ has Poisson distribution with mean $\log\|u_t\| - \log\|u_s\|$ for all $s < t$. (Compare with Exercise 21.15.)

As a special case one has the following example: Choose $\xi : \mathcal{F}_{\mathbb{R}_+} \to \mathcal{P}(\mathcal{H})$, an $\mathbb{R}_+$ valued observable satisfying the condition $\xi(\{t\}) = 0$ for every t. Let $u \in \mathcal{H}$ be such that $u_t = \xi([0,t]u) \neq 0$ for each t. Define $U_t = -\xi([0,t]) + \xi((t,\infty))$. Then the required conditions of the exercise are fulfilled.

Example 23.8: Let X be any set and let $K(x, y)$, $x, y \in X$ be a positive definite kernel with $K(x, x) = 1$ for all x. Using Proposition 15.4 choose a map $v : X \to \mathcal{H}$ when $\mathcal{H}$ is a Hilbert space and $K(x, y) = \langle v(x), v(y) \rangle$ for all x, y. Choose $S_x = p(v(x))$ where $p(v)$ is as in Exercise 23.6. Since $v(x)$ is a unit vector $S_x^2 = 1$ and hence S_x is a spin observable, i.e., S_x assumes only the values ± 1. In the vacuum state Φ of $\Gamma_a(\mathcal{H})$, S_x assumes the values ± 1 with equal probability and the covariance between S_x and S_y is $K(x, y)$. Thus an arbitrary correlation kernel $K(x, y)$ in any set X can be realised as the correlation kernel of a family of spin observables. See Section 5, Proposition 5.3, 5.5.

Exercise 23.9: Suppose $m(u), m^\dagger(u)$, $u \in \mathcal{H}$ are bounded operators in a Hilbert space $\tilde{\mathcal{H}}$ such that $m^\dagger(u) = m(u)^*$ and the following conditions are fulfilled:

(i) The map $u \to m^\dagger(u)$ is linear;

(ii) $m(u)m^\dagger(v) = \langle u, v \rangle$ for all $u, v \in \mathcal{H}$;

(iii) There exists a unit vector $\tilde{\Phi} \in \tilde{\mathcal{H}}$ such that $m(u)\tilde{\Phi} = 0$ for all u and the set $\{\tilde{\Phi}, m^\dagger(u_1) \cdots m^\dagger(u_n)\tilde{\Phi} | u_j \in \mathcal{H}, 1 \leq j \leq n, n = 1, 2, \ldots\}$ is total in $\tilde{\mathcal{H}}$.

Then there exists a unitary isomorphism $V : \tilde{\mathcal{H}} \to \Gamma_{fr}(\mathcal{H})$ such that $V\tilde{\Phi} = \Phi, V m^\dagger(u_1) \cdots m^\dagger(u_n)\tilde{\Phi} = \ell^*(u_1) \cdots \ell^*(u_n)\Phi$ for all $u_j \in \mathcal{H}$, $1 \leq j \leq n$, $n = 1, 2, \ldots$ where $\Gamma_{fr}(\mathcal{H})$ and $\ell^*(\cdot)$ are as in Exercise 20.24.

Notes

This is a small part of the theory of second quantization and the literature cited at the end of Section 20 covers it.

Springer
http://www.springer.com/978-3-0348-0565-0

An Introduction to Quantum Stochastic Calculus
Parthasarathy, K.R.
1992, XI, 290 p., Softcover
ISBN: 978-3-0348-0565-0
A product of Birkhäuser Basel