

MATHEMATICS FROM THE GROUND UP

The social activities of everyday life gave rise to arithmetic and geometry, the two major forms of mathematical work, in all the ancient civilizations. The development of arithmetic was associated with problems in accounting, taxation, and trade, as well as with certain problems in religious, magical, artistic, and astronomical activities. Problems in land surveying, construction and engineering in general gave rise to geometry. Arithmetic and geometry appear in association with the rise of literacy. The emergence and development of these mathematical systems in particular civilizations is the product of both diffusion and independent invention.

It is now possible to *argue* that even fundamental geometrical shapes such as the circle and the square originated in social activities. According to A. Seidenberg's (1981) admittedly controversial conjecture, participants in early human societies identified with and imitated the motions of the stars. The ritual scene thus took on a circular shape; and this, Seidenberg argues, is the origin of the circle. The circle was eventually bisected and split into quadrants representing four basic organizational sectors within a society. Representatives of the four sectors "placed themselves about the center of the circle in positions corresponding to the positions of the four sections [of the circle], thereby giving rise to the square. The square was valued as a figure dual to the circle" (Seidenberg, 1981:324).

The *discipline* of mathematics emerged when and to the extent that sets of arithmetic and geometrical problems were assembled for purposes of codification and teaching, and to facilitate mathematical studies. Assembling problems was an important step toward unifying mathematics and stimulating abstraction. An even more important step was the effort to state general rules for solving all problems of a given type. A further step could be taken once problems were arranged so that they could be treated in more general and abstract terms. Problems that had arisen in practical settings could now be transformed into purely hypothetical puzzles, and problems could be invented without explicit reference to practical issues. The three famous puzzles proposed by Greek geometers of the 5th and 4th centuries BCE are among the earliest examples of such

puzzles: to double the volume of a cube (duplication of the cube), to construct a square with the same area as a given circle (quadrature of the circle), and to divide a given angle into three equal parts (trisection of the angle). Such problems may have been related to the non-mathematical riddles religious oracles commonly posed for one another. One account of the origin of the problem of duplicating the cube, for example, is that the oracle at Delos, in reply to an appeal from the Athenians concerning the plague of 430 BCE, recommended doubling the size of the altar of Apollo. The altar was a cube. The early Hindu literature already refers to problems about the size and shape of altars, and these may have been transmitted to Greece by the Pythagoreans, a secret religious and political society. The problem is also a translation into spatial geometric algebra of the Babylonian cubic equation $x^3 = v$.

The duplication, quadrature, and trisection problems were popular with the Sophists, who made a specialty out of debates of all kinds. A generation or two later, Plato introduced the constraint that the only valid solutions to these problems were those in which only an unmarked straightedge and a compass were used. This meant that special mechanical devices for drawing geometrical forms could not be used in mathematical competitions. The goal was apparently to control the competitive process and make it more rigorous by stressing intellectual means and “purely gentlemanly” norms. This development was related to social factors in the Platonist era. Plato’s Academy was organized to help an elite group of intellectuals gain political power; and it represented the opposition of an aristocracy to democratization and commercialization. It is not surprising that this elite group of intellectuals developed an ideology of extreme intellectual purity, glorifying the extreme separation of hand and brain in the slave economy of classical Greece.

The three famous Greek puzzles and other problems became the basis of a mathematical game of challenge-and-response. Various forms of this game are important throughout most of the subsequent history of Western mathematics. Prior to the nineteenth and twentieth centuries, however, such competitions were often initiated, endorsed, or rewarded by patrons, scientific academies, and governments. Prizes were sometimes offered for solutions to practical problems. Economic concerns as well as governmental prestige were often mixed in with the struggles for intellectual preeminence.

At about the same time that they initiated mathematical contests, the Greek mathematicians took two further steps that led to new mathemati-

cal forms. They stipulated that a formal, logical mode of argument must be used in solving problems. This represented an extension of earlier methods of proof. And by extending the proof idea, the mathematicians created *systems* of interrelated proofs. This culminated in the *Elements* of Euclid shortly after 300 BCE. In addition to a collection of problems, Euclid presented an explicit body of abstractions in the form of definitions, postulates, and axioms. Euclid, like Aristotle, did not use the term “axiom” but something closer to “common notion”. They both self-consciously worked at codifying past human experiences. The process of “systematization-and-abstraction” is one of the two major paths to new mathematical forms. The other major path is an “empirical” one.

The empirical path to new mathematical forms involves applying existing mathematical concepts and methods to new areas of experience. Most of the early Greek geometrical puzzles, for example, concerned flat figures. But the methods of plane geometry could be easily extended to solid geometry, and then to the properties of spheres or of conic sections; the work on conic sections eventually led to work on curves of various shapes. The intermittent periods of creativity in Alexandrian mathematics (especially from 300 to 200 BCE, and 150–200) were largely devoted to these extensions. No new level of abstraction (with the exception of trigonometry, considered below) was reached, but a number of new specialties appeared.

In arithmetical work the effort to find general rules for solving numerical problems led gradually to what we call algebra. Here again we see mathematicians developing the practice of posing problems primarily to challenge other mathematicians. For example, there is the famous problem, attributed to Archimedes (287 to 212 BCE): find the number of bulls and cows of various colors in a herd, if the number of white cows is one third plus one quarter of the total number of black cattle; the number of black bulls is one quarter plus one fifth the number of the spotted bulls in excess of the number of brown bulls, etc. Such problems, involving unknown quantities, led over a very long period to the introduction of various kinds of notations and symbolisms. These took quite different directions in ancient and medieval China and India, the Arab world, and later in medieval and Renaissance Europe. The creation of a highly abstract symbolism which could be mechanically manipulated to find solutions did not appear until the late 1500s and 1600s in Europe. To different degrees in different parts of the world, algebra developed through empirical extension. Problems were deliberately created to

increase the number of unknowns, and to raise them to successively higher powers. Equations of the form $ax + b = c$ gave way to equations such as $ax^4 + by^3 + cz^2 = 9$. The complexity of these equations, of course, could be extended indefinitely (in the sixteenth century, for example, Vieta was challenged to solve an equation involving x^{45}) but the extensions also gave rise to efforts to find general rules for solving higher order equations. (That is, empirical extensions tended to promote abstract extensions). Of course, concrete problems in areas such as astronomy could lead to “complicated” equations. But the point here is that the social network itself could foster such developments more or less directly.

At the same time, arithmetic was developing in other directions. Elementary arithmetic (solving numerical problems in, for example, addition, subtraction, multiplication, and division) continued to stimulate efforts to find general rules for solving particular problems. There was tremendous variation from one system of numerical symbols and calculating rules to another in terms of the ease or difficulty with which they could be applied to solving practical problems. Most of the ancient forms of notation made working with large numbers, fractions, or complex operations like division or the extraction of roots difficult; the exposition of problems was usually rhetorical, that is, problems were expressed in words. A great deal of mathematical creativity went into the development of notational systems that could be readily manipulated. Among the most important of these innovations were the invention of decimal place notation and the zero sign in India; the standardization of positional methods for multiplication and division (in early seventeenth century Europe); and the invention of logarithms by Napier in 1614 for use in astronomy, navigation, and commerce.

A different development in arithmetic led to what we now call “number theory”. This focused on the properties of numbers themselves. As early as Eratosthenes (ca. 230BCE), efforts were made to find prime numbers and to produce a general formula for doing so. There were also various propositions about how numbers are composed of other numbers. The work by the Pythagoreans on “triangular” and “square” numbers, for example, anticipated work that led to Fermat’s famous theorem that every prime number of the form $4n + 1$ is a sum of two squares. Number theory was particularly popular in the Alexandrian period in an occultist, cabalistic form. In its more standard puzzle-solving form, it has remained popular among mathematicians from the Renaissance through the twentieth century.

One more branch of mathematics, based on a combination of arithmetic and geometry, developed in the Alexandrian period. Measuring angles and lines, and calculating their ratios, led to the creation of trigonometry (notably in the works of Hipparchus, ca. 140 BCE, and Menelaus ca. 100 BCE). Trigonometry spread to medieval India and the Arab world, and in Renaissance Europe provided the basis of Napier's development of logarithms.

The creation of new fields continued in modern Europe. They grew out of the processes of abstraction, the extension of results to new empirical areas, and the combination of existing mathematical fields into hybrid fields. The combination of algebra with a new coordinate representation in geometry by Descartes and Fermat produced analytic geometry. Consideration of the problems of motion and the study of curves gave rise to the calculus in the 1600s. Calculus was then applied to successively more complex functions (empirical extension); and eventually (in the 1800s) it was generalized into an abstract theory concerning such things as the rules for solving equations, and the general properties of all functions (abstract extension). The drive towards creating new fields by abstraction and extension seems to be characteristic of highly competitive periods.

Geometry itself experienced a rapid series of branchings around 1800 and thereafter, the best known being the non-Euclidean geometries. Other developments included descriptive geometry (Monge), projective geometry (Poncelet), higher analytical geometry (Plucker), modern synthetic geometry (Steiner and von Staudt), and topology (Möbius, Klein, and Poincaré). In the late nineteenth and early twentieth centuries, systems unifying these different geometries were developed by Klein, Hilbert, and E. Cartan.

In algebra, there was a parallel set of developments after 1800. The effort to find a general solution for the quintic and other higher order equations led to the creation of the theory of groups by Abel, Galois, Cauchy, and others. This theory focused on an abstract pattern among the coefficients of equations, and opened up a new area of inquiry in abstract mathematics.

Abstract algebras were created by Boole, Cayley, Sylvester, Hamilton, and Grassman. All of these new tools were applied to other branches of mathematics. Dedekind applied set theory to the calculus, Cantor applied it to the concept of infinity, and others applied it to topology, number theory, and geometry. These developments led to the creation of yet



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