

CHAPTER II

\mathbb{Z}^d -actions on compact abelian groups

5. The dual module

According to Theorem 4.2, \mathbb{Z}^d is of Markov type for every $d \geq 1$, and \mathbb{Z}^d -actions by automorphisms of compact groups enjoy the properties described in (4.10), Propositions 4.9–4.10, Remark 4.15, and Theorem 4.11. Just as compact, abelian groups like $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ have automorphisms with very intricate dynamical properties, there is an abundance of examples of interesting \mathbb{Z}^d -actions by automorphisms of compact abelian groups. In this section we introduce a general formalism for the investigation of such actions which will also give us a systematic approach to constructing actions with specified properties.

Let $d \geq 1$, and let $\alpha: \mathbf{n} \mapsto \alpha_{\mathbf{n}}$ be an action of \mathbb{Z}^d by automorphisms of X . For every $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ we denote by $\hat{\alpha}_{\mathbf{n}}$ the automorphism of \hat{X} dual to $\alpha_{\mathbf{n}}$ and write $\hat{\alpha}: \mathbb{Z}^d \mapsto \text{Aut}(\hat{X})$ for the resulting \mathbb{Z}^d -action dual to α . Under the action $\hat{\alpha}$ the group \hat{X} becomes a \mathbb{Z}^d -module, and hence a module over the group ring $\mathbb{Z}[\mathbb{Z}^d]$. In order to make this explicit we denote by

$$\mathfrak{R}_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}] \quad (5.1)$$

the ring of Laurent polynomials in the (commuting) variables u_1, \dots, u_d with coefficients in \mathbb{Z} . A typical element $f \in \mathfrak{R}_d$ will be written as

$$f = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) u^{\mathbf{n}}, \quad (5.2)$$

where $c_f(\mathbf{n}) \in \mathbb{Z}$ and $u^{\mathbf{n}} = u_1^{n_1} \cdots u_d^{n_d}$ for all $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$, and where $c_f(\mathbf{n}) \neq 0$ for only finitely many $\mathbf{n} \in \mathbb{Z}^d$. Then $\mathfrak{R}_d \cong \mathbb{Z}[\mathbb{Z}^d]$, \mathfrak{R}_d acts on \hat{X} by

$$(f, a) \mapsto f \cdot a = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) \hat{\alpha}_{\mathbf{n}}(a) \quad (5.3)$$

for every $f \in \mathfrak{R}_d$, $a \in \hat{X}$, and \hat{X} is an \mathfrak{R}_d -module. Note that

$$\hat{\alpha}_{\mathbf{n}}(a) = \hat{\alpha}_{\mathbf{n}}(a) = u^{\mathbf{n}} \cdot a \quad (5.4)$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $a \in \hat{X}$. Conversely, if \mathfrak{M} is an \mathfrak{R}_d -module (always assumed to be countable), then \mathbb{Z}^d has an obvious action $\hat{\alpha}^{\mathfrak{M}}: \mathbf{n} \mapsto \hat{\alpha}_{\mathbf{n}}^{\mathfrak{M}}$ on \mathfrak{M} given by

$$\hat{\alpha}_{\mathbf{n}}^{\mathfrak{M}}(a) = u^{\mathbf{n}} \cdot a \quad (5.5)$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $a \in \mathfrak{M}$. We write $X = \widehat{\mathfrak{M}}$ for the dual group of \mathfrak{M} and obtain a dual action

$$\alpha^{\mathfrak{M}}: \mathbf{n} \mapsto \alpha_{\mathbf{n}}^{\mathfrak{M}} \in \text{Aut}(X) \quad (5.6)$$

of \mathbb{Z}^d on X . For future reference we collect these observations in a lemma.

LEMMA 5.1. *Let $\alpha: \mathbf{n} \mapsto \alpha_{\mathbf{n}}$ be a \mathbb{Z}^d -action by automorphisms of a compact, abelian group X , and let $\hat{\alpha}: \mathbf{n} \mapsto \hat{\alpha}_{\mathbf{n}}$ be the dual action of \mathbb{Z}^d on the dual group \hat{X} of X . If \mathfrak{R}_d is the ring defined in (5.1) then \hat{X} is an \mathfrak{R}_d -module under the \mathfrak{R}_d -action (5.3). Conversely, if \mathfrak{M} is an \mathfrak{R}_d -module, then (5.5) and (5.6) define \mathbb{Z}^d -actions $\hat{\alpha}^{\mathfrak{M}} = \hat{\alpha}$ and $\alpha^{\mathfrak{M}} = \alpha$ by automorphisms of \mathfrak{M} and $X^{\mathfrak{M}} = \widehat{\mathfrak{M}}$, respectively.*

EXAMPLES 5.2. Let $d \geq 1$.

(1) Let $\mathfrak{M} = \mathfrak{R}_d$. Since \mathfrak{R}_d is isomorphic to the direct sum $\sum_{\mathbb{Z}^d} \mathbb{Z}$ of copies of \mathbb{Z} indexed by \mathbb{Z}^d , the dual group $X = \widehat{\mathfrak{R}_d}$ is isomorphic to the cartesian product $\mathbb{T}^{\mathbb{Z}^d}$ of copies of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We write a typical element $x \in \mathbb{T}^{\mathbb{Z}^d}$ as $x = (x_{\mathbf{n}}) = (x_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}^d)$ with $x_{\mathbf{n}} \in \mathbb{T}$ for every $\mathbf{n} \in \mathbb{Z}^d$ and choose the following identification of $X^{\mathfrak{R}_d} = \widehat{\mathfrak{R}_d}$ and $\mathbb{T}^{\mathbb{Z}^d}$: for every $x = (x_{\mathbf{n}})$ in $\mathbb{T}^{\mathbb{Z}^d}$ and $f \in \mathfrak{R}_d$,

$$\langle x, f \rangle = e^{2\pi i \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) x_{\mathbf{n}}}, \quad (5.7)$$

where f is given by (5.2). Under this identification the \mathbb{Z}^d -action $\alpha^{\mathfrak{R}_d}$ on $X^{\mathfrak{R}_d} = \mathbb{T}^{\mathbb{Z}^d}$ becomes the shift-action

$$\alpha_{\mathbf{n}}^{\mathfrak{R}_d}(x)_{\mathbf{m}} = (\sigma_{\mathbf{n}}(x))_{\mathbf{m}} = x_{\mathbf{m}+\mathbf{n}}, \quad (5.8)$$

with $\mathbf{n} \in \mathbb{Z}^d$ and $x = (x_{\mathbf{m}}) \in X^{\mathfrak{R}_d} = \mathbb{T}^{\mathbb{Z}^d}$.

(2) Let $\mathfrak{a} \subset \mathfrak{R}_d$ be an ideal, and let $\mathfrak{M} = \mathfrak{R}_d/\mathfrak{a}$. Since \mathfrak{M} is a quotient of the additive group \mathfrak{R}_d by a $\hat{\alpha}^{\mathfrak{R}_d}$ -invariant subgroup, the dual group $X^{\mathfrak{M}}$ is the $\alpha^{\mathfrak{R}_d}$ -invariant subgroup

$$\begin{aligned} X^{\mathfrak{R}_d/\mathfrak{a}} &= \{x \in X^{\mathfrak{R}_d} = \mathbb{T}^{\mathbb{Z}^d} : \langle x, f \rangle = 1 \text{ for every } f \in \mathfrak{a}\} \\ &= \left\{ x \in \mathbb{T}^{\mathbb{Z}^d} : \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) x_{\mathbf{m}+\mathbf{n}} = 0 \pmod{1} \right. \\ &\quad \left. \text{for every } f \in \mathfrak{a} \text{ and } \mathbf{m} \in \mathbb{Z}^d \right\}, \end{aligned} \quad (5.9)$$

and $\alpha^{\mathfrak{R}_d/a}$ is the restriction of $\alpha^{\mathfrak{R}_d}$ to $X^{\mathfrak{M}} \subset \mathbb{T}^{\mathbb{Z}^d}$, i.e.

$$\alpha_{\mathbf{n}}^{\mathfrak{R}_d/a} = \sigma_{\mathbf{n}}^{X^{\mathfrak{R}_d/a}} \quad (5.10)$$

for every $\mathbf{n} \in \mathbb{Z}^d$.

(3) Let $X \subset \mathbb{T}^{\mathbb{Z}^d} = \widehat{\mathfrak{R}_d}$ be a closed subgroup, and let $X^\perp = \{f \in \mathfrak{R}_d : \langle x, f \rangle = 1 \text{ for every } x \in X\}$ be the annihilator of X in \mathfrak{R}_d . Then X is shift-invariant if and only if X^\perp is an ideal in \mathfrak{R}_d : indeed, if X^\perp is an ideal, it is obviously invariant under multiplication by the group of units $\{u^{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\} \subset \mathfrak{R}_d$, i.e. X^\perp is $\hat{\alpha}^{\mathfrak{R}_d}$ -invariant; conversely, if X^\perp is $\hat{\alpha}^{\mathfrak{R}_d}$ -invariant, then (5.3) shows that $f \cdot a \in X^\perp$ for every $f \in \mathfrak{R}_d$ and $a \in X^\perp$. In other words, X^\perp is an ideal.

(4) Let \mathfrak{M} be a Noetherian \mathfrak{R}_d -module, and let $\{a_1, \dots, a_k\}$ be a set of generators for \mathfrak{M} , i.e. $\mathfrak{M} = \mathfrak{R}_d \cdot a_1 + \dots + \mathfrak{R}_d \cdot a_k$. The surjective homomorphism $(f_1, \dots, f_k) \mapsto f_1 \cdot a_1 + \dots + f_k \cdot a_k$ from \mathfrak{R}_d^k to \mathfrak{M} induces a dual injective homomorphism $\phi: X^{\mathfrak{M}} \hookrightarrow X^{\mathfrak{R}_d^k} \cong (\mathbb{T}^k)^{\mathbb{Z}^d} = Y$ such that $\alpha_{\mathbf{n}}^{\mathfrak{M}} \cdot \phi = \sigma_{\mathbf{n}} \cdot \phi$ for every $\mathbf{n} \in \mathbb{Z}^d$, where $\sigma_{\mathbf{n}}$ is the shift on $(\mathbb{T}^k)^{\mathbb{Z}^d}$ defined in (5.8). In particular, ϕ embeds $X^{\mathfrak{M}}$ as a closed, shift-invariant subgroup of $(\mathbb{T}^k)^{\mathbb{Z}^d}$. Conversely, if $X \subset (\mathbb{T}^k)^{\mathbb{Z}^d}$ is a closed, shift-invariant subgroup, then $\hat{X} = \mathfrak{R}_d^k / X^\perp$, and X^\perp is a submodule of \mathfrak{R}_d^k . \square

EXAMPLES 5.3. (1) Let α be the automorphism of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ determined by the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. In Example 2.18 (2) we have seen that α (or, more precisely, the \mathbb{Z} -action on \mathbb{T}^2 defined by α) is conjugate to $(X^{\mathfrak{R}_1/(f)}, \alpha^{\mathfrak{R}_1/(f)})$, where $(f) \subset \mathfrak{R}_1$ is the principal ideal generated by the characteristic polynomial $f(u_1) = 1 + u_1 - u_1^2$ of A . Indeed, an element $x \in X = \widehat{\mathfrak{R}_1} = \mathbb{T}^{\mathbb{Z}}$ satisfies that $\langle x, u_1^n f \rangle = 1$ if and only if $x_n + x_{n+1} - x_{n+2} = 0 \pmod{1}$, and hence

$$X^{\mathfrak{R}_1/(f)} = \{x \in \mathbb{T}^{\mathbb{Z}} : x_n + x_{n+1} - x_{n+2} = 0 \pmod{1} \text{ for all } n \in \mathbb{Z}\}$$

(cf. (5.7) and (5.9)). The continuous group isomorphism $\phi = \pi_{\{0,1\}}: X^{\mathfrak{R}_1/(f)} \hookrightarrow \mathbb{T}^2$ makes the diagram

$$\begin{array}{ccc} X^{\mathfrak{R}_1/(f)} & \xrightarrow{\alpha^{\mathfrak{R}_1/(f)}} & X^{\mathfrak{R}_1/(f)} \\ \phi \downarrow & & \downarrow \phi \\ \mathbb{T}^2 & \xrightarrow{\alpha} & \mathbb{T}^2 \end{array} \quad (5.11)$$

commute, and the automorphism $\alpha^{\mathfrak{R}_1/(f)}$ is equal to the shift on $X^{\mathfrak{R}_1/(f)}$.

(2) Example (1) depends on the fact that the matrix A is conjugate (over \mathbb{Z}) to the companion matrix of its characteristic polynomial. If α is the automorphism of \mathbb{T}^2 defined by $A = \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix}$, then the characteristic polynomial of A is $f(u_1) = -1 - 4u_1 + u_1^2$, and $AM = MB$, where $B = \begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix}$ and $M = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$. The

map $\phi: X^{\mathfrak{R}_1/(f)} \mapsto \mathbb{T}^2$ given by $\phi(x) = (x_0 + 3x_1, x_1)$ for all $x \in X^{\mathfrak{R}_1/(f)} \subset \mathbb{T}^{\mathbb{Z}}$ is a group isomorphism, and the diagram (5.11) commutes.

If $A' = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$, then the characteristic polynomial of A' is again equal to $f(u_1) = -1 - 4u_1 + u_1^2$, $A'M = MB$ with $M = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$, but there is no matrix M' with integer entries and determinant 1 such that $A'M' = M'B$. The homomorphism $\phi': X^{\mathfrak{R}_1/(f)} \mapsto \mathbb{T}^2$ with $\phi'(x) = (x_0 + 3x_1, 2x_1)$ for all $x \in X^{\mathfrak{R}_1/(f)} \subset \mathbb{T}^{\mathbb{Z}}$ is surjective, and we write $\psi' = \hat{\phi}: \mathbb{Z}^2 \mapsto \mathfrak{R}_1/(f)$ for the dual homomorphism, which is injective, but not bijective. The \mathfrak{R}_1 -module $\mathfrak{M} = \hat{X}$ arising from the \mathbb{Z} -action $n \mapsto (A')^n$ via Lemma 5.1 is (isomorphic to) the submodule $\psi'(\mathbb{Z}^2)$ of $\mathfrak{R}_1/(f)$. We claim that \mathfrak{M} is not isomorphic to $\mathfrak{R}_1/(f)$ —in fact, \mathfrak{M} is not even cyclic, i.e. not of the form $\mathfrak{M} = \mathfrak{R}_1 \cdot a$ for some $a \in \mathfrak{M}$. Indeed, if \mathfrak{M} were cyclic, there would exist an element $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$ such that $\{(A')^n \mathbf{m} : n \in \mathbb{Z}\}$ generates \mathbb{Z}^2 , which is equivalent to the condition that

$$\{\mathbf{m}, A'\mathbf{m}\} = \{(m_1, m_2), (3m_1 + 2m_2, 2m_1 + m_2)\}$$

generates \mathbb{Z}^2 . Hence

$$\det \begin{pmatrix} m_1 & 3m_1 + 2m_2 \\ m_2 & 2m_1 + m_2 \end{pmatrix} = 2m_1^2 - 2m_1m_2 - 2m_2^2 = 1,$$

which is obviously impossible.

(3) Let $f = 2 - u_1 \in \mathfrak{R}_1$, and let (f) be the principal ideal generated by f . According to (5.7) and (5.9),

$$X = X^{\mathfrak{R}_1/(f)} = \{x = (x_n) \in \mathbb{T}^{\mathbb{Z}} : 2x_n = x_{n+1} \pmod{1} \text{ for all } n \in \mathbb{Z}\},$$

and $\alpha^{\mathfrak{R}_1/(f)}$ is equal to the shift-action σ of \mathbb{Z} on X . The zero coordinate projection $\phi = \pi_{\{0\}}: X \mapsto \mathbb{T}$ is surjective and satisfies that $\phi \cdot \sigma_1 = T \cdot \phi$, where $T: \mathbb{T} \mapsto \mathbb{T}$ is the surjective homomorphism consisting of multiplication by 2 modulo 1.

(4) Let $f_1 = 2 - u_1$, $f_2 = 3 - u_2$, and let $\mathfrak{a} = (f_1, f_2) = f_1\mathfrak{R}_2 + f_2\mathfrak{R}_2 \subset \mathfrak{R}_2$. Then

$$X = X^{\mathfrak{R}_2/\mathfrak{a}} = \{x = (x_{m,n}) \in \mathbb{T}^{\mathbb{Z}^2} : 2x_{(m,n)} = x_{(m+1,n)} \pmod{1} \text{ and } 3x_{(m,n)} = x_{(m,n+1)} \pmod{1} \text{ for every } (m,n) \in \mathbb{Z}^2\},$$

and $\alpha^{\mathfrak{R}_2/\mathfrak{a}} = \sigma$ is the shift-action of \mathbb{Z}^2 on $X^{\mathfrak{R}_2/\mathfrak{a}}$. The zero coordinate projection $\phi = \pi_{\{(0,0)\}}: X \mapsto \mathbb{T}$ is again surjective and satisfies that $\phi \cdot \sigma_{\mathbf{n}} = T_{\mathbf{n}} \cdot \phi$ for every $\mathbf{n} \in \mathbb{Z}^2$, where T is the \mathbb{N}^2 -action on \mathbb{T} defined by $T_{(m,n)}(t) = 2^m 3^n t \pmod{1}$ for every $(m,n) \in \mathbb{Z}^2$ and $t \in \mathbb{T}$.

(5) Let

$$X = \{x = (x_{\mathbf{n}}) \in \mathbb{Z}/2 : x_{(m_1, m_2)} + x_{(m_1+1, m_2)} + x_{(m_1, m_2+1)} = 0 \pmod{2} \text{ for all } \mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2\}.$$

From (5.7) and (5.9) we see that the shift-action σ of \mathbb{Z}^2 on the full, shift-invariant subgroup $X \subset \mathbb{Z}_{/2}^{\mathbb{Z}^2}$ is conjugate to $(X^{\mathfrak{R}_2/\mathfrak{a}}, \alpha^{\mathfrak{R}_2/\mathfrak{a}})$, where $\mathfrak{a} = (2, 1 + u_1 + u_2) \subset \mathfrak{R}_2$ is the ideal generated by 2 and $1 + u_1 + u_2$.

(6) Let $d \geq 1$. A Laurent polynomial $f \in \mathfrak{R}_d$ is *primitive* if the highest common factor of its coefficients is equal to 1. Suppose that f is primitive and $m > 1$ an integer, and let (f) and (mf) be the principal ideals in \mathfrak{R}_d generated by f and mf , respectively. The map $h \mapsto mh$ from \mathfrak{R}_d to \mathfrak{R}_d induces an injective homomorphism $\xi: \mathfrak{R}_d/(f) \hookrightarrow \mathfrak{R}_d/(mf)$, the dual homomorphism $\phi: X^{\mathfrak{R}_d/(mf)} \hookrightarrow X^{\mathfrak{R}_d/(f)}$ is surjective, and $\ker(\phi) \cong \mathbb{Z}_{/m}^{\mathbb{Z}^d}$. The group $X^{\mathfrak{R}_d/(f)}$ is connected, and the connected component of the identity in $X^{\mathfrak{R}_d/(mf)}$ is isomorphic to $X^{\mathfrak{R}_d/(f)}$.

More generally, if $\mathfrak{a} \subset \mathfrak{R}_d$ is an arbitrary ideal such that the additive group $\mathfrak{R}_d/\mathfrak{a}$ is torsion-free (or, equivalently, such that $X^{\mathfrak{R}_d/\mathfrak{a}}$ is connected), and if $m \geq 1$ is an integer, then we obtain an exact sequence

$$0 \rightarrow \mathbb{Z}_{/m}^{\mathbb{Z}^d} \xrightarrow{\psi} X^{\mathfrak{R}_d/m\mathfrak{a}} \xrightarrow{\phi} X^{\mathfrak{R}_d/\mathfrak{a}} \rightarrow 0,$$

where $\phi: X^{\mathfrak{R}_d/m\mathfrak{a}} \hookrightarrow X^{\mathfrak{R}_d/\mathfrak{a}}$ is the surjection dual to the injective homomorphism $\xi: \mathfrak{R}_d/\mathfrak{a} \hookrightarrow \mathfrak{R}_d/m\mathfrak{a}$ consisting of multiplication by m , and where ψ is the inclusion map. Note that $\psi \cdot \sigma_{\mathbf{n}}(x) = \alpha_{\mathbf{n}}^{\mathfrak{R}_d/m\mathfrak{a}} \cdot \psi(x)$ and $\phi \cdot \alpha_{\mathbf{n}}^{\mathfrak{R}_d/m\mathfrak{a}}(y) = \alpha_{\mathbf{n}}^{\mathfrak{R}_d/\mathfrak{a}} \cdot \phi(y)$ for all $\mathbf{n} \in \mathbb{Z}^d$, $x \in \mathbb{Z}_{/m}^{\mathbb{Z}^d}$, and $y \in X^{\mathfrak{R}_d/m\mathfrak{a}}$, where σ is the shift-action of \mathbb{Z}^d on $\mathbb{Z}_{/m}^{\mathbb{Z}^d}$, and that the map ϕ induces an isomorphism of the connected component of the identity in $X^{\mathfrak{R}_d/m\mathfrak{a}}$ with $X^{\mathfrak{R}_d/\mathfrak{a}}$. \square

The next proposition is a straightforward consequence of Theorem 4.2 and Pontryagin duality (cf. also Example 5.2 (4)).

PROPOSITION 5.4. *Let X be a compact, abelian group, α a \mathbb{Z}^d -action by automorphisms of X . The following conditions are equivalent.*

- (1) *The \mathfrak{R}_d -module $\mathfrak{M} = \hat{X}$ obtained via Lemma 5.1 is Noetherian;*
- (2) *(X, α) satisfies the d.c.c.;*
- (3) *(X, α) is conjugate to a subshift of $(\mathbb{T}^n)^{\mathbb{Z}^d}$ for some $n \geq 1$.*

The Noetherian \mathfrak{R}_d -modules form a particularly well-behaved class of \mathfrak{R}_d -modules, and it is therefore not surprising that \mathbb{Z}^d -actions by automorphisms of compact, abelian groups satisfying the d.c.c. have many exceptional properties. As a first illustration of the rôle played by the descending chain condition, let us consider the set of periodic points for a \mathbb{Z}^d -action α on a compact, abelian group X .

DEFINITION 5.5. Let Γ be a countable group and let α be a Γ -action by automorphisms of a compact group X . A point $x \in X$ is *periodic* under α (or *α -periodic*) if its orbit $\alpha_{\Gamma}(x) = \{\alpha_{\gamma}(x) : \gamma \in \Gamma\}$ is finite. If $\beta \in \text{Aut}(X)$ then a point $x \in X$ is *periodic* under β if $\beta^n(x) = x$ for some $n \geq 1$.

The following examples show that a \mathbb{Z}^d -action by automorphisms of a compact, abelian group need not have any periodic points other than the fixed point $\mathbf{0}_X$, but in Theorem 5.7 we shall see that the set of α -periodic points is dense if (X, α) satisfies the d.c.c.

EXAMPLES 5.6. (1) Let $X = \widehat{\mathbb{Q}}$ be the dual group of the additive group \mathbb{Q} , and consider the automorphism α of X dual to multiplication by $\frac{3}{2}$ on \mathbb{Q} . If $x \in X$ is a periodic point of α , i.e. if $\alpha^n(x) = x$ for some $n \geq 1$, then $\langle \alpha^n(x) - x, a \rangle = \langle x, (\frac{3^n}{2^n} - 1)a \rangle = 1$ for every $a \in \mathbb{Q}$. However, $(\frac{3^n}{2^n} - 1) \neq 0$, so that $\langle x, a \rangle = 1$ for every $a \in \mathbb{Q}$. This shows that $x = \mathbf{0}_X$.

(2) Let $Y = \mathbb{Z}_{/2}^{\mathbb{Z}}$. For every $n \geq 2$ we define a continuous, shift commuting, surjective homomorphism $\phi_n: Y \rightarrow Y$ by setting $(\phi_n(y))_m = \sum_{k=m}^{m+n-1} y_k$ for every $m \in \mathbb{Z}$ and $y = (y_k, k \in \mathbb{Z}) \in Y$. We put $\psi_n = \phi_n$ for every $n \geq 2$ and denote by X the projective limit

$$Y \xleftarrow{\psi_2} Y \xleftarrow{\psi_3} \dots \xleftarrow{\psi_n} Y \xleftarrow{\psi_{n+1}} \dots \quad (5.12)$$

The shift σ on Y commutes with the maps ψ_n and induces an automorphism α of the projective limit X in (5.12). Suppose that α has a periodic point $x \in X$ with period n , say. We can write x as $(x^{(k)}, k \geq 1)$ with $x^{(k)} \in Y$ and $\psi_k(x^{(k)}) = x^{(k-1)}$ for every $k \geq 2$. Since x has period n , $\sigma^n(x^{(k)}) = x^{(k)}$ for every $k \geq 1$. However, $\psi_{nk}(x^{(nk)}) = \phi_{nk}(x^{(nk)}) = x^{(nk-1)} \in \{\mathbf{0}, \mathbf{1}\}$ for every $k \geq 1$, where $\mathbf{0} = (\dots, 0, 0, 0, \dots)$ and $\mathbf{1} = (\dots, 1, 1, 1, \dots)$ are the fixed points of σ in Y . As k can be arbitrarily large we see that $x^{(k)} \in \{\mathbf{0}, \mathbf{1}\}$ for every $k \geq 0$. Finally we observe that, if $k \geq 2$ is even, then $x^{(k-1)} = \psi_k(x^{(k)}) = \mathbf{0}$. This shows that $x^{(k)} = \mathbf{0}$ for every $k \geq 1$, i.e. that $x = \mathbf{0}_X$.

(3) We stay with the notation of Example (2) and set $\psi_n = \phi_2$ for every $n \geq 2$ in (5.12). The projective limit X in (5.12) can be written as $X = \{x = (x_{(m,n)}) \in \mathbb{Z}_{/2}^{\mathbb{Z} \times \mathbb{N}} : x_{(m,n)} = x_{(m,n+1)} + x_{(m+1,n+1)} \pmod{2} \text{ for every } m \in \mathbb{Z} \text{ and } n \geq 1\}$, and α is the horizontal shift on X defined by $(\alpha(x))_{(m,n)} = x_{(m+1,n)}$ for all $x \in X$ and $(m,n) \in \mathbb{Z} \times \mathbb{N}^*$. The same argument as in Example (2) shows that every point $x \in X$ with period 2^k , $k \geq 0$ is equal to the identity element $\mathbf{0}_X$, but that there exist 2^{k-1} points of period k if $k \geq 1$ is odd (for every sequence $y = (y_m) \in Y$ with $y_{(m+k)} = y_m$ and $\sum_{j=0}^{k-1} x_{m+j} = 0 \pmod{2}$ for all $m \in \mathbb{Z}$ there exists a unique point $x \in X$ with $\alpha^k(x) = x$ and $x_{(m,1)} = y_m$ for all $m \in \mathbb{Z}$).

If $\mathfrak{a} \subset \mathfrak{R}_2$ is the ideal $(2, 1 + u_2 + u_1 u_2) = 2\mathfrak{R}_2 + (1 + u_2 + u_1 u_2)\mathfrak{R}_2$, then (5.7) and (5.9) show that $(X^{\mathfrak{R}_2/\mathfrak{a}}, \alpha^{\mathfrak{R}_2/\mathfrak{a}})$ is (conjugate to) the shift-action of \mathbb{Z}^2 on

$$\begin{aligned} X' &= \{x = (x_{(m,n)}) \in \mathbb{Z}_{/2}^{\mathbb{Z}^2} : x_{(m,n)} + x_{(m,n+1)} + x_{(m+1,n+1)} \\ &= 0 \pmod{2} \text{ for every } (m,n) \in \mathbb{Z}^2\}, \end{aligned}$$

and a comparison of X' with the definition of X in the preceding paragraph reveals that X is equal to the projection of X' onto its coordinates in the upper half plane of \mathbb{Z}^2 , and that this projection sends the horizontal shift $\sigma_{(1,0)}$ of X' to the automorphism α of X . In particular we see that the shift-action σ of \mathbb{Z}^2 on X' has only one point with horizontal period 2^k for every $k \geq 0$ (the identity element). We also refer to Example 5.3 (5): the \mathbb{Z}^2 -action $\alpha^{\mathfrak{R}_2/\mathfrak{a}}$ appearing there obviously has the same property.

(4) Let $\psi_n = \phi_3$ for every $n \geq 1$ in (5.12). Then the resulting automorphism α of the projective limit X in (5.12) has only one point with period 3^k , $k \geq 0$, but there exist 2^k points with period k for every k which is not divisible by 3.

(5) Let $(p_n, n \geq 2)$ be a sequence of rational primes in which every prime occurs infinitely often, and let $(q_n, n \geq 2)$ be a sequence of odd primes in which every odd prime occurs infinitely often. If $\psi_n = \phi_{p_n}$ for every $n \geq 2$, then the automorphism α of the projective limit X in (5.12) has no periodic points other than the fixed point $\mathbf{0}_X$. However, if $\psi_n = \phi_{q_n}$, $n \geq 2$, then the resulting automorphism α will have 2^{2^k} periodic points with period 2^k for every $k \geq 0$, but only one point with period $2l + 1$ for every $l \geq 0$ (the fixed point $\mathbf{0}_X$).

None of the automorphisms α in Examples (1)–(5) satisfies the d.c.c. \square

THEOREM 5.7. *Let X be a compact, abelian group, and let α be a \mathbb{Z}^d -action by automorphisms of X . If (X, α) satisfies the d.c.c. then the set of α -periodic points is dense in X .*

PROOF. Let $\mathfrak{M} = \hat{X}$ be the \mathfrak{R}_d -module arising from Lemma 5.1. Fix a non-zero element $a \in \mathfrak{M}$ and choose a submodule $\mathfrak{M}_a \subset \mathfrak{M}$ which is maximal with respect to the property that $a \notin \mathfrak{M}_a$. Then the \mathfrak{R}_d -module $\mathfrak{M}' = \mathfrak{M}/\mathfrak{M}_a$ has the minimal non-zero submodule $\mathfrak{M}'_1 = (\mathfrak{R}_d \cdot a + \mathfrak{M}_a)/\mathfrak{M}_a$. Consider the ideal $\mathfrak{a} = \{f \in \mathfrak{R}_d : f \cdot \mathfrak{M}'_1 = \{0\}\}$, and let \mathfrak{b} be an ideal with $\mathfrak{a} \subsetneq \mathfrak{b} \subsetneq \mathfrak{R}_d$. The minimality of \mathfrak{M}'_1 implies that $\mathfrak{b} \cdot \mathfrak{M}'_1 = \mathfrak{M}'_1$, and Corollary 2.5 in [5] shows that there exists an element $x \in 1 + \mathfrak{b}$ such that $x \cdot \mathfrak{M}'_1 = \{0\}$. This contradicts our definition of \mathfrak{a} , and we conclude that the ideal $\mathfrak{a} \subset \mathfrak{R}_d$ is maximal, and that $\mathfrak{k} = \mathfrak{R}_d/\mathfrak{a}$ is a (necessarily finite) field.

For every $m \geq 1$ we write $\mathfrak{a}^m \subset \mathfrak{R}_d$ for the ideal generated by $\{f_1 \cdots f_m : f_i \in \mathfrak{a} \text{ for } i = 1, \dots, m\}$. If $a' = a + \mathfrak{M}_a \in \mathfrak{a}^m \cdot \mathfrak{M}'$ for every $m \geq 1$, then $a \in \mathfrak{M}'' = \bigcap_{m \geq 1} \mathfrak{a}^m \cdot \mathfrak{M}''$, and $\mathfrak{a} \cdot \mathfrak{M}''/\mathfrak{M}''$. The argument in the preceding paragraph shows that there exists an element $y \in 1 + \mathfrak{a}$ with $y \cdot \mathfrak{M}'' = \{0\}$, and the maximality of \mathfrak{a} implies that $\mathfrak{M}'' = \{0\}$, which is absurd. Hence there exists an integer $m \geq 1$ with $a' \notin \mathfrak{a}^m \cdot \mathfrak{M}'$, and the maximality of \mathfrak{M}_a implies that $\mathfrak{a}^m \cdot \mathfrak{M}' = \{0\}$.

Each of the successive quotients $\mathfrak{a}^r \cdot \mathfrak{M}'/\mathfrak{a}^{r+1} \cdot \mathfrak{M}'$ in the decreasing sequence of \mathfrak{R}_d -modules $\mathfrak{M}' \supset \mathfrak{a} \cdot \mathfrak{M}' \supset \cdots \supset \mathfrak{a}^m \cdot \mathfrak{M}' = \{0\}$ is a Noetherian module over \mathfrak{k} . Since \mathfrak{k} is finite we conclude that \mathfrak{M}' is finite.

We have found, for every non-zero $a \in \mathfrak{M} = \hat{X}$, a submodule $\mathfrak{M}_a \subset \mathfrak{M}$ such that $a \notin \mathfrak{M}_a$ and $\mathfrak{M}/\mathfrak{M}_a$ is finite. The subgroup $X_a = \mathfrak{M}_a^\perp \subset X$ is finite, α -invariant, and is not annihilated by (the character corresponding to) a . Since every point in X_a must be α -periodic, and since the α -periodic points form a subgroup of X , this shows that the set of α -periodic points is dense in X . \square

Before turning to the problem of relating the algebraic properties of a Noetherian \mathfrak{R}_d -module \mathfrak{M} to the dynamical properties of $(X^\mathfrak{M}, \alpha^\mathfrak{M})$ we should discuss the extent to which \mathfrak{M} and $(X^\mathfrak{M}, \alpha^\mathfrak{M})$ determine each other. Let $d \geq 1$, and let \mathfrak{M} be a Noetherian \mathfrak{R}_d -module which is torsion-free when regarded as an additive group or, equivalently, as a \mathbb{Z} -module (this is equivalent to the assumption that $X^\mathfrak{M} = \widehat{\mathfrak{M}}$ is connected). We define the \mathbb{Z}^d -action $\alpha^\mathfrak{M}$ on $X^\mathfrak{M}$ by (5.5) and (5.6) and consider the action induced by $\alpha^\mathfrak{M}$ on the Čech homology group $H_1(X^\mathfrak{M}, \mathbb{T})$ (cf. [20]).

LEMMA 5.8. *The group $H_1(X^\mathfrak{M}, \mathbb{T})$ is isomorphic to $X^\mathfrak{M}$, and the automorphism induced by $\alpha_\mathbf{n}^\mathfrak{M}$ on $H_1(X^\mathfrak{M}, \mathbb{T})$ is equal to $\alpha_\mathbf{n}^\mathfrak{M}$ for every $\mathbf{n} \in \mathbb{Z}^d$.*

PROOF. In view of Example 5.2 (4) we may assume that $X = X^\mathfrak{M}$ is a closed, shift-invariant subgroup of $(\mathbb{T}^k)^{\mathbb{Z}^d}$, and the connectedness of X allows us to assume that X is full. If $F(n) = \{-n, \dots, n\}^d \subset \mathbb{Z}^d$ then $\pi_{F(n)}(X) \subset (\mathbb{T}^k)^{F(n)}$ is a finite-dimensional torus, and X is equal to the projective limit

$$\pi_{F(1)}(X) \xleftarrow{\pi_{F(1)}} \pi_{F(2)}(X) \xleftarrow{\pi_{F(2)}} \pi_{F(3)}(X) \xleftarrow{\pi_{F(3)}} \dots \quad (5.13)$$

Since $H_1(\pi_{F(k)}(X), \mathbb{T}) \cong \pi_{F(k)}(X)$ ([20]), we see from (5.13) that $H_1(X, \mathbb{T}) \cong X$, and that the automorphism induced by $\alpha_\mathbf{n}^\mathfrak{M} = \sigma_\mathbf{n}$ on $H_1(X, \mathbb{T})$ is equal to $\sigma_\mathbf{n}$ for every $\mathbf{n} \in \mathbb{Z}^d$. \square

THEOREM 5.9. *Let X and X' be compact, connected, abelian groups, and let α and α' be \mathbb{Z}^d -actions by automorphisms of X and X' which satisfy the d.c.c. The following statements are equivalent.*

- (1) *The \mathbb{Z}^d -actions α and α' are topologically conjugate, i.e. there exists a homeomorphism $\phi: X \rightarrow X'$ with $\phi \cdot \alpha_\mathbf{n} = \alpha'_\mathbf{n} \cdot \phi$ for every $\mathbf{n} \in \mathbb{Z}^d$;*
- (2) *The \mathbb{Z}^d -actions α and α' are algebraically conjugate, i.e. there exists a continuous group isomorphism $\psi: X \rightarrow X'$ such that $\psi \cdot \alpha_\mathbf{n} = \alpha'_\mathbf{n} \cdot \psi$ for every $\mathbf{n} \in \mathbb{Z}^d$.*

PROOF. The implication (2) \Rightarrow (1) is obvious. If (1) is satisfied we use Lemma 5.1 and Proposition 5.4 to find Noetherian \mathfrak{R}_d -modules \mathfrak{M} and \mathfrak{M}' such that (X, α) and (X', α') are conjugate to $(X^\mathfrak{M}, \alpha^\mathfrak{M})$ and $(X^{\mathfrak{M}'}, \alpha^{\mathfrak{M}'})$, respectively. By Lemma 5.8, $H_1(X^\mathfrak{M}, \mathbb{T}) \cong X^\mathfrak{M}$, $H_1(X^{\mathfrak{M}'}, \mathbb{T}) \cong X^{\mathfrak{M}'}$, and for every $\mathbf{n} \in \mathbb{Z}^d$ the isomorphisms of $H_1(X^\mathfrak{M}, \mathbb{T})$ and $H_1(X^{\mathfrak{M}'}, \mathbb{T})$ defined by $\alpha_\mathbf{n}^\mathfrak{M}$ and $\alpha_\mathbf{n}^{\mathfrak{M}'}$ are equal to $\alpha_\mathbf{n}^\mathfrak{M}$ and $\alpha_\mathbf{n}^{\mathfrak{M}'}$, respectively. The continuous group isomorphism $\psi': H_1(X^\mathfrak{M}, \mathbb{T}) \rightarrow H_1(X^{\mathfrak{M}'}, \mathbb{T})$ induced by $\phi: X \rightarrow X'$ satisfies that $\psi' \cdot \alpha_\mathbf{n}^\mathfrak{M} = \alpha_\mathbf{n}^{\mathfrak{M}'} \cdot \psi'$ for every $\mathbf{n} \in \mathbb{Z}^d$, and this implies (2). \square

COROLLARY 5.10. *Let $d \geq 1$, and let \mathfrak{M} and \mathfrak{M}' be finitely generated \mathfrak{R}_d -modules which are torsion-free (as additive groups). The following statements are equivalent.*

- (1) *The \mathbb{Z}^d -actions $\alpha^{\mathfrak{M}}$ and $\alpha^{\mathfrak{M}'}$ are topologically conjugate;*
- (2) *The \mathbb{Z}^d -actions $\alpha^{\mathfrak{M}}$ and $\alpha^{\mathfrak{M}'}$ are algebraically conjugate;*
- (3) *There exists an \mathfrak{R}_d -module isomorphism $\chi: \mathfrak{M} \rightarrow \mathfrak{M}'$.*

PROOF. The equivalence of (1) and (2) is stated in Theorem 5.9. If (2) is satisfied, then any group isomorphism $\psi: X^{\mathfrak{M}} \rightarrow X^{\mathfrak{M}'}$ with $\psi \cdot \alpha_{\mathbf{n}}^{\mathfrak{M}} = \alpha_{\mathbf{n}}^{\mathfrak{M}'} \cdot \psi$ for all $\mathbf{n} \in \mathbb{Z}^d$ induces a dual isomorphism $\hat{\psi}: \mathfrak{M}' \rightarrow \mathfrak{M}$ which is easily seen to be an \mathfrak{R}_d -module isomorphism. The implication (3) \Rightarrow (2) is obvious. \square

CONCLUDING REMARKS 5.11. (1) Most of the material of this section comes from [45], except for Lemma 5.8, Theorem 5.9, and Corollary 5.10, which come from [94]. Example 5.3 (2) is taken from [110], Example 5.3 (4) features in [23] and [89], Example 5.3 (5) comes from [56] (cf. (0.1)), and Example 5.6 (1) appears to be oral tradition attributed to Furstenberg. For \mathbb{Z} -actions Theorem 5.7 was first proved in [55], and the general proof presented here is due to Hartley. A more general version of Theorem 5.7 will be proved in Section 10 (Theorem 10.2).

(2) If X and X' are not connected, Theorem 5.9 (or the equivalence of (1) and (2) in Corollary 5.10) is not true in general. The shifts on the groups $\mathbb{Z}_{/4}^{\mathbb{Z}}$ and $(\mathbb{Z}_{/2}^2)^{\mathbb{Z}}$ are topologically, but not algebraically conjugate. However, the equivalence of (2) and (3) in Corollary 5.10 holds for any pair of \mathfrak{R}_d -modules \mathfrak{M} and \mathfrak{M}' , whether they are torsion-free (as additive groups) or not.

6. The dynamical system defined by a Noetherian module

We begin with a little bit of algebra. Let $d \geq 1$, and let \mathcal{R} be a commutative ring. We denote by \mathcal{R}^{\times} the set of invertible elements (or units) in \mathcal{R} , write $\mathcal{R}[u_1, \dots, u_d]$ and $\mathcal{R}[u_1^{\pm 1}, \dots, u_d^{\pm d}]$ for the rings of polynomials and Laurent polynomials in the commuting variables u_1, \dots, u_d with coefficients in \mathcal{R} , and we define \mathfrak{R}_d by (5.1). For every rational prime p we denote by $\overline{\mathbb{F}}_p$ the algebraic closure of the prime field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}_{/p}$ and define a homomorphism $f \mapsto f_{/p}$ from \mathfrak{R}_d to

$$\mathfrak{R}_d^{(p)} = \overline{\mathbb{F}}_p[u_1^{\pm 1}, \dots, u_d^{\pm d}] \quad (6.1)$$

by reducing the coefficients of $f \in \mathfrak{R}_d$ modulo p . An element $f \in \mathfrak{R}_d^{(p)}$ will again be written in the form (5.1) with $c_f(\mathbf{n}) \in \overline{\mathbb{F}}_p$ for all $\mathbf{n} \in \mathbb{Z}^d$, where $c_f(\mathbf{n}) \neq 0$ for only finitely many $\mathbf{n} \in \mathbb{Z}^d$. For notational consistency we set $\overline{\mathbb{F}}_0$ equal to the algebraic closure $\overline{\mathbb{Q}}$ of \mathbb{Q} and put $\mathfrak{R}_d^{(0)} = \mathfrak{R}_d$ and $f_{/0} = f$ for every $f \in \mathfrak{R}_d$.

Let $\mathfrak{p} \subset \mathfrak{R}_d$ be a prime ideal. We identify \mathbb{Z} with the set of constant polynomials in \mathfrak{R}_d , denote by $p(\mathfrak{p})$ the characteristic $\text{char}(\mathfrak{R}_d/\mathfrak{p})$ of $\mathfrak{R}_d/\mathfrak{p}$, i.e.

the unique non-negative integer such that $\mathfrak{p} \cap \mathbb{Z} = p(\mathfrak{p})\mathbb{Z}$, and define the *variety* of \mathfrak{p} by

$$V(\mathfrak{p}) = \{c \in (\overline{\mathbb{F}}_{p(\mathfrak{p})}^\times)^d : f_{/p(\mathfrak{p})}(c) = 0 \text{ for every } f \in \mathfrak{p}\}. \quad (6.2)$$

If $\mathfrak{a} \subset \mathfrak{R}_d$ is an arbitrary ideal we set

$$V_{\mathbb{C}}(\mathfrak{a}) = \{c \in (\mathbb{C}^\times)^d : f(c) = 0 \text{ for every } f \in \mathfrak{a}\}. \quad (6.3)$$

Suppose that \mathfrak{M} is an \mathfrak{R}_d -module. For every $f \in \mathfrak{R}_d$ we write $f_{\mathfrak{M}} : \mathfrak{M} \rightarrow \mathfrak{M}$ for the map $a \mapsto f \cdot a$, $a \in \mathfrak{M}$, and we denote by $\text{ann}(a) = \{f \in \mathfrak{R}_d : f \cdot a = 0\}$ the annihilator of an element $a \in \mathfrak{M}$. A prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ is *associated with* \mathfrak{M} if $\mathfrak{p} = \text{ann}(a)$ for some $a \in \mathfrak{M}$, and the module \mathfrak{M} is *associated with* \mathfrak{p} if \mathfrak{p} is the only prime ideal in \mathfrak{R}_d associated with \mathfrak{M} . If \mathfrak{M} is Noetherian then it is associated with \mathfrak{p} if and only if

$$\mathfrak{p} = \{f \in \mathfrak{R}_d : f_{\mathfrak{M}} \text{ is not injective}\} = \{f \in \mathfrak{R}_d : f_{\mathfrak{M}} \text{ is nilpotent}\} \quad (6.4)$$

(cf. Corollary VI.4.11 in [51]). If \mathfrak{M} is associated with \mathfrak{p} and $\mathfrak{N} \subset \mathfrak{M}$ is a non-zero submodule, then \mathfrak{N} is again associated with \mathfrak{p} . The module \mathfrak{M} is a *torsion module* if the prime ideal $\{0\}$ is not associated with \mathfrak{M} . We shall have to be careful to distinguish between \mathfrak{R}_d -modules \mathfrak{M} which are not *torsion* and those which are *torsion-free* as additive groups (or \mathbb{Z} -modules): \mathfrak{M} is a torsion module if every associated prime ideal is non-zero, \mathfrak{M} is a torsion group if each of its associated primes contains a non-zero constant, and \mathfrak{M} is torsion-free (as an additive group) if none of its associated primes contains a non-zero constant.

A submodule $\mathfrak{W} \subset \mathfrak{M}$ is *\mathfrak{p} -primary* (or *\mathfrak{p} belongs to \mathfrak{W}*) if $\mathfrak{M}/\mathfrak{W}$ is associated with \mathfrak{p} . From now on we assume that \mathfrak{M} is Noetherian. By Theorem VI.5.3 in [51] there exist primary submodules $\mathfrak{W}_1, \dots, \mathfrak{W}_m$ of \mathfrak{M} with the following properties:

the primes \mathfrak{p}_i belonging to the submodules \mathfrak{W}_i are all distinct;

$$\mathfrak{W}_1 \cap \dots \cap \mathfrak{W}_m = \{0\}; \quad (6.5)$$

for every subset $S \subsetneq \{1, \dots, m\}$, $\bigcap_{i \in S} \mathfrak{W}_i \neq \{0\}$.

A family $\{\mathfrak{W}_1, \dots, \mathfrak{W}_m\}$ of primary submodules satisfying (6.5) is called a *reduced primary decomposition* of \mathfrak{M} , and $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ is the *set of associated primes* of \mathfrak{M} . According to the Theorems VI.5.2 and VI.5.5 in [51] the set of associated primes of \mathfrak{M} is independent of the specific decomposition (6.5), and

$$\{f \in \mathfrak{R}_d : f_{\mathfrak{M}} \text{ is not injective}\} = \bigcup_{i=1, \dots, m} \mathfrak{p}_i. \quad (6.6)$$

PROPOSITION 6.1. *Let $d \geq 1$, $\mathfrak{q} \subset \mathfrak{R}_d$ a prime ideal, and let \mathfrak{W} be a Noetherian \mathfrak{R}_d -module associated with \mathfrak{q} . Then there exist integers $1 \leq t \leq s$ and submodules $\{0\} = \mathfrak{N}_0 \subset \dots \subset \mathfrak{N}_s = \mathfrak{W}$ such that, for every $i = 1, \dots, s$,*

$\mathfrak{N}_i/\mathfrak{N}_{i-1} \cong \mathfrak{R}_d/\mathfrak{q}_i$ for some prime ideal $\mathfrak{q} \subset \mathfrak{q}_i \subset \mathfrak{R}_d$, $\mathfrak{q}_i = \mathfrak{q}$ for $i = 1, \dots, t$, and $\mathfrak{q}_i \supsetneq \mathfrak{q}$ for $i = t+1, \dots, s$.

PROOF. Note that, if $\mathfrak{N} \subset \mathfrak{W}$ is a submodule, and if $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal associated with $\mathfrak{W}/\mathfrak{N}$, then $\mathfrak{p} \supset \mathfrak{q}$. Indeed, if $\mathfrak{p} = \text{ann}(a)$ for some $a \in \mathfrak{W}/\mathfrak{N}$, choose $b \in \mathfrak{W}$ such that $a = b + \mathfrak{N}$, and set $\mathfrak{N}' = \mathfrak{p} \cdot b = \{f \cdot b : f \in \mathfrak{p}\} \subset \mathfrak{N}$. If $\mathfrak{N}' \neq \{0\}$ then \mathfrak{N}' is associated with \mathfrak{q} , and (6.4) shows that $g^n \in \mathfrak{p}$ for every $g \in \mathfrak{q}$ and every sufficiently large $n \geq 1$. Since \mathfrak{p} is prime we conclude that $\mathfrak{q} \subset \mathfrak{p}$.

Let Ω_1 be the set of submodules $\mathfrak{N} \subset \mathfrak{W}$ with the following property: there exists an integer $r \geq 1$ and submodules $\{0\} = \mathfrak{N}_0 \subset \dots \subset \mathfrak{N}_r = \mathfrak{N}$ such that $\mathfrak{N}_i/\mathfrak{N}_{i-1} \cong \mathfrak{R}_d/\mathfrak{q}$ for every $i = 1, \dots, r$. It is clear that $\Omega_1 \neq \emptyset$, since we can find an $a \in \mathfrak{W}$ with $\text{ann}(a) = \mathfrak{q}$ and $\mathfrak{N} = \mathfrak{R}_d a \cong \mathfrak{R}_d/\mathfrak{q}$. Since \mathfrak{W} is Noetherian, Ω_1 contains a maximal element \mathfrak{W}' , and we set $\mathfrak{V} = \mathfrak{W}/\mathfrak{W}'$ and consider the set of prime ideals $\{\mathfrak{q}_1, \dots, \mathfrak{q}_l\}$ associated with the \mathfrak{R}_d -module \mathfrak{V} . If $\mathfrak{q}_i = \mathfrak{q}$ for some $i \in \{1, \dots, l\}$, then there exists an element $b \in \mathfrak{W}$ with $b \notin \mathfrak{W}'$ and $\{f \in \mathfrak{R}_d : fb \in \mathfrak{W}'\} = \mathfrak{q}$, and this violates the maximality of \mathfrak{W}' .

Let Ω_2 be the set of submodules \mathfrak{N} with $\mathfrak{W}' \subset \mathfrak{N} \subset \mathfrak{W}$, for which there exist submodules $\mathfrak{W}' = \mathfrak{L}_0 \subset \dots \subset \mathfrak{L}_t = \mathfrak{N}$ such that, for every $i = 1, \dots, t$, $\mathfrak{L}_i/\mathfrak{L}_{i-1} \cong \mathfrak{R}_d/\mathfrak{q}_i$ for some prime ideal $\mathfrak{q}_i \supsetneq \mathfrak{q}$. Then Ω_2 again has a maximal element \mathfrak{W}'' . If $\mathfrak{W}'' \neq \mathfrak{W}$ we set $\mathfrak{V}' = \mathfrak{W}/\mathfrak{W}''$, consider the set of prime ideals associated with \mathfrak{V}' , all of which are strictly greater than \mathfrak{q} by the argument in the first paragraph of this proof, and obtain a contradiction to the maximality of \mathfrak{W}'' exactly as before, where we were dealing with \mathfrak{W}' . Hence $\mathfrak{W}'' = \mathfrak{W}$, and the proposition is proved by setting $\mathfrak{N}_0 \subset \dots \subset \mathfrak{N}_s$ equal to $\{0\} = \mathfrak{N}_0 \subset \dots \subset \mathfrak{N}_s = \mathfrak{L}_0 \subset \dots \subset \mathfrak{L}_t = \mathfrak{N}$. \square

COROLLARY 6.2. *Let $d \geq 1$, \mathfrak{M} a Noetherian \mathfrak{R}_d -module with associated primes $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ and a corresponding reduced primary decomposition $\{\mathfrak{W}_1, \dots, \mathfrak{W}_m\}$. Then there exist submodules $\mathfrak{M} = \mathfrak{N}_s \supset \dots \supset \mathfrak{N}_0 = \{0\}$ such that, for every $i = 1, \dots, s$, $\mathfrak{N}_i/\mathfrak{N}_{i-1} \cong \mathfrak{R}_d/\mathfrak{q}_i$ for some prime ideal $\mathfrak{q}_i \subset \mathfrak{R}_d$, and $\mathfrak{q}_i \supset \mathfrak{p}_j$ for some $j \in \{1, \dots, m\}$ (such a sequence $\mathfrak{M} = \mathfrak{N}_s \supset \dots \supset \mathfrak{N}_0 = \{0\}$ is called a prime filtration of \mathfrak{M}).*

PROOF. Apply Proposition 6.1 to the successive quotients of the sequence

$$\mathfrak{M} \supset \mathfrak{W}_1 \supset (\mathfrak{W}_1 \cap \mathfrak{W}_2) \supset \dots \supset (\mathfrak{W}_1 \cap \dots \cap \mathfrak{W}_m) = \{0\},$$

bearing in mind that

$$(\mathfrak{W}_1 \cap \dots \cap \mathfrak{W}_i)/(\mathfrak{W}_1 \cap \dots \cap \mathfrak{W}_{i+1}) \cong (\mathfrak{W}_1 \cap \dots \cap \mathfrak{W}_i)/\mathfrak{W}_{i+1} \subset \mathfrak{M}/\mathfrak{W}_{i+1}$$

is associated with \mathfrak{p}_{i+1} for every $i = 1, \dots, m-1$ (if B, C are subgroups of an abelian group A we use the symbol B/C to denote $(B+C)/C$). \square

Let \mathfrak{M} be a Noetherian \mathfrak{R}_d -module with a prime filtration $\mathfrak{M} = \mathfrak{N}_s \supset \dots \supset \mathfrak{N}_0 = \{0\}$, and define the \mathbb{Z}^d -action $\alpha = \alpha^{\mathfrak{M}}$ on $X = X^{\mathfrak{M}}$ by (5.5) and (5.6). For every $j = 0, \dots, s$, $Y_j = \mathfrak{N}_j^\perp$ is a closed, α -invariant subgroup of X , and the dual group of Y_{j-1}/Y_j is isomorphic to $\mathfrak{R}_d/\mathfrak{q}_j$, where $\mathfrak{q}_j \subset \mathfrak{R}_d$ is a prime ideal containing one of the associated primes of \mathfrak{M} . This allows one to build up (X, α) from the successive quotients $(Y_{j-1}/Y_j, \alpha^{Y_{j-1}/Y_j})$, which have the explicit realization (5.9)–(5.10) with $\mathfrak{a} = \mathfrak{q}_j$. However, although the prime ideals $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ are canonically associated with \mathfrak{M} , the ideals \mathfrak{q}_j appearing in Proposition 6.1 and Corollary 6.2 need no longer be canonical, and may depend on a specific prime filtration of \mathfrak{M} . The next corollary can help to overcome this problem.

COROLLARY 6.3. *Let $d \geq 1$, \mathfrak{M} a Noetherian \mathfrak{R}_d -module with associated primes $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$. Then there exists a Noetherian \mathfrak{R}_d -module $\mathfrak{N} = \mathfrak{N}^{(1)} \oplus \dots \oplus \mathfrak{N}^{(m)}$ and an injective \mathfrak{R}_d -module homomorphism $\phi: \mathfrak{M} \hookrightarrow \mathfrak{N}$ such that each of the modules $\mathfrak{N}^{(j)}$ has a prime filtration $\mathfrak{N}^{(j)} = \mathfrak{N}_{r_j}^{(j)} \supset \dots \supset \mathfrak{N}_0^{(j)} = \{0\}$ with $\mathfrak{N}_k^{(j)}/\mathfrak{N}_{k-1}^{(j)} \cong \mathfrak{R}_d/\mathfrak{p}_j$ for $k = 1, \dots, r_j$.*

If $X = X^{\mathfrak{M}}$ and $Y = X^{\mathfrak{N}} = X^{\mathfrak{N}^{(1)}} \times \dots \times X^{\mathfrak{N}^{(m)}}$, then the homomorphism $\psi: Y \hookrightarrow X$ dual to ϕ is surjective and satisfies that

$$\psi \cdot \alpha_{\mathbf{n}}^{\mathfrak{N}} = \psi \cdot (\alpha_{\mathbf{n}}^{\mathfrak{N}^{(1)}} \times \dots \times \alpha_{\mathbf{n}}^{\mathfrak{N}^{(m)}}) = \alpha_{\mathbf{n}}^{\mathfrak{M}} \cdot \psi \quad (6.7)$$

for every $\mathbf{n} \in \mathbb{Z}^d$.

PROOF. Choose a reduced primary decomposition $\mathfrak{W}_1, \dots, \mathfrak{W}_m$ of \mathfrak{M} as in (6.5). Then the map $\phi': a \mapsto (a + \mathfrak{W}_1, \dots, a + \mathfrak{W}_m)$ from \mathfrak{M} into $\mathfrak{K} = \bigoplus_{i=1}^m \mathfrak{M}/\mathfrak{W}_i$ is injective. We fix $j \in \{1, \dots, m\}$ for the moment and apply Proposition 6.1 to find a prime filtration $\{0\} = \mathfrak{N}_0 \subset \dots \subset \mathfrak{N}_s = \mathfrak{M}/\mathfrak{W}_j$ such that $\mathfrak{N}_k^{(j)}/\mathfrak{N}_{k-1}^{(j)} \cong \mathfrak{R}_d/\mathfrak{q}_k^{(j)}$ for every $k = 1, \dots, s_j$, where $\mathfrak{q}_k^{(j)} \subset \mathfrak{R}_d$ is a prime ideal containing \mathfrak{p}_j , and where there exists an $r_j \in \{1, \dots, s_j\}$ such that $\mathfrak{q}_k^{(j)} = \mathfrak{p}_j$ for $k = 1, \dots, r_j$, and $\mathfrak{q}_k^{(j)} \supsetneq \mathfrak{p}_j$ for $k = r_j + 1, \dots, s_j$. If $r_j < s_j$ we choose Laurent polynomials $g_k^{(j)} \in \mathfrak{q}_k^{(j)} \setminus \mathfrak{p}_j$ for $k = r_j + 1, \dots, s_j$, set $g^{(j)} = g_{r_j+1}^{(j)} \dots g_{s_j}^{(j)}$, and note that the map $\psi^{(j)}: \mathfrak{M}/\mathfrak{W}_j \hookrightarrow \mathfrak{N}_{r_j}^{(j)}$ consisting of multiplication by $g^{(j)}$ is injective. Since $\mathfrak{N}_{r_j}^{(j)}$ has the prime filtration $\{0\} = \mathfrak{N}_0^{(j)} \subset \dots \subset \mathfrak{N}_{r_j}^{(j)}$ whose successive quotients are all isomorphic to $\mathfrak{R}_d/\mathfrak{p}_j$, the module $\mathfrak{N} = \mathfrak{N}_{r_1}^{(1)} \oplus \dots \oplus \mathfrak{N}_{r_m}^{(m)}$ has the required properties. The last assertion follows from duality. \square

EXAMPLE 6.4. In Example 5.3 (2) we considered the automorphism of \mathbb{T}^2 given by the matrix $A' = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$ and obtained that the \mathbb{Z} -action on \mathbb{T}^2 defined by A' is conjugate to $(X^{\mathfrak{M}}, \alpha^{\mathfrak{M}})$, where \mathfrak{M} is the \mathfrak{R}_1 -module $\psi'(\mathbb{Z}^2) \subset \mathfrak{R}_1/(f)$ with $f(u_1) = -1 - 4u_1 + u_1^2$ and $\psi'(m_1, m_2) = m_1 + (3m_1 + 2m_2)u_1 \in \mathfrak{R}_1/(f)$ for every $(m_1, m_2) \in \mathbb{Z}^2$. As a submodule of $\mathfrak{R}_1/(f)$, \mathfrak{M} is associated with

(f). Let $a = \psi'(0, 1) = 2u_1 \in \mathfrak{R}_1/(f)$, and let $\mathfrak{N} = \mathfrak{R}_1 \cdot a = 2\mathfrak{R}_1/(f)$. Then $\mathfrak{M}/\mathfrak{N} = \mathfrak{R}_1/\mathfrak{a}$, where \mathfrak{a} is the prime ideal $(2, 1 + u_1) = 2\mathfrak{R}_1 + \mathfrak{R}_1(1 + u_1) \subset \mathfrak{R}_1$, and $\{0\} \subset \mathfrak{N} \subset \mathfrak{M}$ is a prime filtration of \mathfrak{M} with $\mathfrak{M}/\mathfrak{N} \cong \mathfrak{R}_1/\mathfrak{a}$ and $\mathfrak{N}/\{0\} \cong \mathfrak{R}_1/(f)$. \square

Our next result shows that certain dynamical properties of the \mathbb{Z}^d -action $\alpha^{\mathfrak{M}}$ on $X^{\mathfrak{M}}$ can be expressed purely in terms of the primes associated with \mathfrak{M} and do not require the much more difficult analysis of the primes which may occur in a prime filtration of \mathfrak{M} . Recall that an element $g \in \mathfrak{R}_d$ is a *generalized cyclotomic polynomial* if it is of the form $g(u_1, \dots, u_d) = u^{\mathbf{m}}c(u^{\mathbf{n}})$, where $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$, $\mathbf{n} \neq \mathbf{0}$, and c is a cyclotomic polynomial in a single variable.

THEOREM 6.5. *Let $d \geq 1$, let \mathfrak{M} a Noetherian \mathfrak{R}_d -module with associated primes $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$, and let $(X, \alpha) = (X^{\mathfrak{M}}, \alpha^{\mathfrak{M}})$ be defined by (5.5)–(5.6). For every $i = 1, \dots, m$ we denote by $p(\mathfrak{p}_i) \geq 0$ the characteristic of $\mathfrak{R}_d/\mathfrak{p}_i$.*

(1) *The following conditions are equivalent.*

- (a) α is ergodic;
- (b) $\alpha_{\mathbf{n}}$ is ergodic for some $\mathbf{n} \in \mathbb{Z}^d$;
- (c) $\alpha^{\mathfrak{R}_d/\mathfrak{p}_i}$ is ergodic for every $i \in \{1, \dots, m\}$;
- (d) *There do not exist integers $i \in \{1, \dots, m\}$ and $l \geq 1$ with*

$$\{u^{l\mathbf{n}} - 1 : \mathbf{n} \in \mathbb{Z}^d\} \subset \mathfrak{p}_i;$$

- (e) *There do not exist integers $i \in \{1, \dots, m\}$ and $l \geq 1$ with*

$$V(\mathfrak{p}_i) \subset \{c = (c_1, \dots, c_d) \in (\overline{\mathbb{F}}_{p(\mathfrak{p}_i)}^\times)^d : c_1^l = \dots = c_d^l = 1\}.$$

(2) *The following conditions are equivalent.*

- (a) α is mixing;
- (b) *For every $i = 1, \dots, m$, $\alpha^{\mathfrak{R}_d/\mathfrak{p}_i}$ is mixing;*
- (c) *None of the prime ideals associated with \mathfrak{M} contains a generalized cyclotomic polynomial, i.e. $\{u^{\mathbf{n}} - 1 : \mathbf{n} \in \mathbb{Z}^d\} \cap \mathfrak{p}_i = \{0\}$ for $i = 1, \dots, m$.*

(3) *Let $\Lambda \subset \mathbb{Z}^d$ be a subgroup with finite index. The following conditions are equivalent.*

- (a) *The set*

$$\text{Fix}_\Lambda(\alpha) = \{x \in X : \alpha_{\mathbf{n}}(x) = x \text{ for every } \mathbf{n} \in \Lambda\}$$

is finite;

- (b) *For every $i = 1, \dots, m$, the set $\text{Fix}_\Lambda(\alpha^{\mathfrak{R}_d/\mathfrak{p}_i})$ is finite;*
- (c) *For every $i = 1, \dots, m$, $V_{\mathbb{C}}(\mathfrak{p}_i) \cap \Omega(\Lambda) = \emptyset$, where*

$$\Omega(\Lambda) = \{c \in \mathbb{C}^d : c^{\mathbf{n}} = 1 \text{ for every } \mathbf{n} \in \Lambda\}$$

with $c = (c_1, \dots, c_d)$, $\mathbf{n} = (n_1, \dots, n_d)$, and $c^{\mathbf{n}} = c_1^{n_1} \cdot \dots \cdot c_d^{n_d}$.

(4) *The following conditions are equivalent.*

- (a) α is expansive;

- (b) For every $i = 1, \dots, m$, $\alpha^{\mathfrak{R}_d/\mathfrak{p}_i}$ is expansive;
- (c) For every $i = 1, \dots, m$, $V_{\mathbb{C}}(\mathfrak{p}_i) \cap \mathbb{S}^d = \emptyset$;
- (d) For every $i = 1, \dots, m$ with $p(\mathfrak{p}_i) = 0$, $V(\mathfrak{p}_i) \cap \mathbb{S}^d = \emptyset$.

We begin the proof of Theorem 6.5 with a general proposition.

PROPOSITION 6.6. *Let \mathfrak{M} a countable \mathfrak{R}_d -module.*

- (1) *For any $\mathbf{n} \in \mathbb{Z}^d$ the following conditions are equivalent.*
 - (a) $\alpha_{\mathbf{n}}^{\mathfrak{M}}$ is ergodic;
 - (b) $\alpha_{\mathbf{n}}^{\mathfrak{R}_d/\mathfrak{p}}$ is ergodic for every prime ideal \mathfrak{p} associated with \mathfrak{M} ;
 - (c) No prime ideal \mathfrak{p} associated with \mathfrak{M} contains a polynomial of the form $u^{l\mathbf{n}} - 1$ with $l \geq 1$.
- (2) *The following conditions are equivalent.*
 - (a) $\alpha^{\mathfrak{M}}$ is ergodic;
 - (b) $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is ergodic for every prime ideal \mathfrak{p} associated with \mathfrak{M} ;
 - (c) No prime ideal \mathfrak{p} associated with \mathfrak{M} contains a set of the form $\{u^{l\mathbf{n}} - 1 : \mathbf{n} \in \mathbb{Z}^d\}$ with $l \geq 1$.
- (3) *The following conditions are equivalent.*
 - (a) $\alpha^{\mathfrak{M}}$ is mixing;
 - (b) $\alpha_{\mathbf{n}}^{\mathfrak{M}}$ is ergodic for every non-zero element $\mathbf{n} \in \mathbb{Z}^d$;
 - (c) $\alpha_{\mathbf{n}}^{\mathfrak{M}}$ is mixing for every non-zero element $\mathbf{n} \in \mathbb{Z}^d$;
 - (d) $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is mixing for every prime ideal \mathfrak{p} associated with \mathfrak{M} ;
 - (e) None of the prime ideals associated with \mathfrak{M} contains a generalized cyclotomic polynomial.

PROOF. From Lemma 1.2 and (5.5)–(5.6) it is clear that the \mathbb{Z} -action $k \mapsto \alpha_{k\mathbf{n}}^{\mathfrak{M}}$ is non-ergodic if and only if there exists a non-zero element $a \in \mathfrak{M}$ such that $(u^{l\mathbf{n}} - 1)a = 0$ for some $l \geq 1$. Let $\mathfrak{N} = \mathfrak{R}_d \cdot a$, and let $b \in \mathfrak{N}$ be a non-zero element such that $\mathfrak{p} = \text{ann}(b)$ is maximal in the set of annihilators of elements in \mathfrak{N} . Then \mathfrak{p} is a prime ideal associated with \mathfrak{M} which contains $u^{l\mathbf{n}} - 1$. This shows that (1.c) \Rightarrow (1.a). Conversely, if there exists a prime ideal \mathfrak{p} associated with \mathfrak{M} which contains $u^{l\mathbf{n}} - 1 \in \mathfrak{p}$ for some $l \geq 1$, we choose $a \in \mathfrak{M}$ with $\text{ann}(a) = \mathfrak{p}$, note that $(u^{l\mathbf{n}} - 1)a = 0$, and obtain that (1.a) \Rightarrow (1.c).

If we apply the equivalence (1.a) \iff (1.c) to the \mathfrak{R}_d -module $\mathfrak{R}_d/\mathfrak{p}$, whose only associated prime is \mathfrak{p} , we see that $\alpha_{\mathbf{n}}^{\mathfrak{R}_d/\mathfrak{p}}$ is non-ergodic if and only if $u^{l\mathbf{n}} - 1 \in \mathfrak{p}$ for some $l \geq 1$, which completes the proof of the first part of this lemma.

If $\alpha^{\mathfrak{M}}$ is non-ergodic, then Lemma 1.2 implies that there exists a non-zero element $a \in \mathfrak{M}$ such that the orbit $\{u^{\mathbf{m}} \cdot a : \mathbf{m} \in \mathbb{Z}^d\}$ of the \mathbb{Z}^d -action $\hat{\alpha}^{\mathfrak{M}}$ in (5.5) is finite. As in the proof of (1) we set $\mathfrak{N} = \mathfrak{R}_d \cdot a$, choose $0 \neq b \in \mathfrak{N}$ such that $\mathfrak{p} = \text{ann}(b)$ is maximal, and note that \mathfrak{p} is a prime ideal which contains $\{u^{l\mathbf{m}} - 1 : \mathbf{m} \in \mathbb{Z}^d\}$ for some $l \geq 1$. Conversely, if there exists a prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ associated with \mathfrak{M} which contains $\{u^{l\mathbf{m}} - 1 : \mathbf{m} \in \mathbb{Z}^d\}$ for some $l \geq 1$, and Lemma 1.2 shows that the \mathbb{Z}^d -action $\alpha^{\mathfrak{M}}$ cannot be ergodic. This

shows that (2.c) \iff (2.a), and the equivalence of (2.b) and (2.c) is obtained by applying the equivalence of (2.a) and (2.c) to the \mathfrak{R}_d -module $\mathfrak{R}_d/\mathfrak{p}$.

In order to prove (3) we note that the equivalence (3.a) \iff (3.b) \iff (3.c) follows from Theorem 1.6 (2), and the proof is completed by applying the part (1) of this lemma both to $\alpha^{\mathfrak{M}}$ and to $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$, where \mathfrak{p} ranges over the set of prime ideals associated with \mathfrak{M} . \square

PROOF OF THEOREM 6.5 (1). The implication (b) \Rightarrow (a) is obvious. If (b) does not hold there exists, for every $\mathbf{n} \in \mathbb{Z}^d$, an $l \geq 1$ with $u^{l\mathbf{n}} - 1 \in \bigcup_{1 \leq i \leq m} \mathfrak{p}_i$ (Proposition 6.6). For every $i = 1, \dots, m$, the set $\Gamma_i = \{\mathbf{n} \in \mathbb{Z}^d : u^{\mathbf{n}} - 1 \in \mathfrak{p}_i\}$ is a subgroup of \mathbb{Z}^d . As we have just observed, the set $\Gamma = \bigcup_{i=1}^m \Gamma_i$ contains some multiple of every element of \mathbb{Z}^d ; if every Γ_i has infinite index in \mathbb{Z}^d , then Γ is contained in the intersection with \mathbb{Z}^d of a union of m at most $d - 1$ -dimensional subspaces of \mathbb{R}^d , which is obviously impossible. Hence Γ_i must have finite index in \mathbb{Z}^d for some $i \in \{1, \dots, m\}$, and we can find an integer $l \geq 1$ such that $u^{l\mathbf{n}} - 1 \in \mathfrak{p}_i$ for every $\mathbf{n} \in \mathbb{Z}^d$. This proves the implication (d) \Rightarrow (b). The implications (a) \iff (c) \iff (d) were proved in Proposition 6.6, and the equivalence of (d) and (e) follows from Hilbert's Nullstellensatz. \square

PROOF OF THEOREM 6.5 (2). Use Proposition 6.6. \square

LEMMA 6.7. *Let $\mathfrak{a} \subset \mathfrak{R}_d$ be an ideal. Then $\mathfrak{a} \cap \mathbb{Z} \neq \{0\}$ if and only if $V_{\mathbb{C}}(\mathfrak{a}) = \emptyset$.*

PROOF. If $\mathfrak{a} \cap \mathbb{Z} \neq \{0\}$ then $V_{\mathbb{C}}(\mathfrak{a}) = \emptyset$. Conversely, if $V_{\mathbb{C}}(\mathfrak{a}) = \emptyset$, then the Nullstellensatz implies that $\overline{\mathbb{Q}}[u_1^{\pm 1}, \dots, u_d^{\pm 1}] \cdot \mathfrak{a} = \overline{\mathbb{Q}}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$, and there exist polynomials $f_i \in \mathfrak{a}$, $g_i \in \overline{\mathbb{Q}}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$, $i = 1, \dots, n$, with $1 = \sum_{i=1}^n f_i g_i$. The coefficients of the g_i generate a finite extension field $\mathbb{K} \supset \mathbb{Q}$, and $\mathfrak{R}_d^{(\mathbb{K})} = \mathbb{K}[u_1^{\pm 1}, \dots, u_d^{\pm 1}] = \sum_{j=1}^l v_j \mathfrak{R}_d^{(\mathbb{Q})}$ for suitably chosen elements $\{v_1, \dots, v_l\} \in \mathfrak{R}_d^{(\mathbb{K})}$, where $\mathfrak{R}_d^{(\mathbb{Q})} = \mathbb{Q}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$. Since $\mathfrak{a}^{(\mathbb{Q})} = \mathfrak{R}_d^{(\mathbb{Q})} \cdot \mathfrak{a}$ is an ideal in $\mathfrak{R}_d^{(\mathbb{Q})}$ and $\mathfrak{R}_d^{(\mathbb{K})} \cdot \mathfrak{a}^{(\mathbb{Q})} = \mathfrak{R}_d^{(\mathbb{K})}$, there exist elements $\{h_{j,k} : 1 \leq j, k \leq l\} \subset \mathfrak{a}^{(\mathbb{Q})}$ such that, for every $j = 1, \dots, l$, $v_j = \sum_{k=1}^l h_{j,k} v_k$. Hence $\det(\delta_{j,k} - h_{j,k}) = 0$, where $\delta_{j,k} = 1$ for $j = k$ and $\delta_{j,k} = 0$ otherwise, and we conclude that $1 \in \mathfrak{a}^{(\mathbb{Q})}$. This proves that $\mathfrak{a} \cap \mathbb{Z} \neq \{0\}$. \square

PROOF OF THEOREM 6.5 (3). If $\mathfrak{b}(\Lambda) \subset \mathfrak{R}_d$ is the ideal generated by $\{u^{\mathbf{n}} - 1 : \mathbf{n} \in \Lambda\}$, then

$$V_{\mathbb{C}}(\mathfrak{b}(\Lambda)) = \{c \in \mathbb{C}^d : c^{\mathbf{n}} = 1 \text{ for every } \mathbf{n} \in \Lambda\} = \Omega(\Lambda),$$

$\text{Fix}_{\Lambda}(\alpha)^{\perp} = \mathfrak{b}(\Lambda) \cdot \mathfrak{M}$, and $\widehat{\text{Fix}_{\Lambda}(\alpha)} = \mathfrak{M}/\mathfrak{b}(\Lambda) \cdot \mathfrak{M}$ (cf. (5.5)–(5.6)). In particular, $\text{Fix}_{\Lambda}(\alpha)$ is finite if and only if $\mathfrak{M}/\mathfrak{b}(\Lambda) \cdot \mathfrak{M}$ is finite.

Suppose that $\text{Fix}_{\Lambda}(\alpha)$ is finite. For every $i = 1, \dots, m$ we choose $a_i \in \mathfrak{M}$ such that $\mathfrak{p}_i = \text{ann}(a_i)$ and hence $\mathfrak{L}_i = \mathfrak{R}_d \cdot a_i \cong \mathfrak{R}_d/\mathfrak{p}_i$. The Artin-Rees Lemma

(Corollary 10.10 in [5]) implies that

$$\mathfrak{b}(\Lambda)^{(t)} \cdot \mathfrak{M} \cap \mathfrak{L}_i = \mathfrak{b}(\Lambda) \cdot (\mathfrak{b}(\Lambda)^{(t-1)} \cdot \mathfrak{M} \cap \mathfrak{L}_i) \subset \mathfrak{b}(\Lambda) \cdot \mathfrak{L}_i$$

for some $t \geq 1$, where $\mathfrak{b}(\Lambda)^{(t)} \subset \mathfrak{R}_d$ is the ideal generated by $\{f_1 \cdots f_t : f_i \in \mathfrak{b}(\Lambda) \text{ for } i = 1, \dots, t\}$. By assumption,

$$\widehat{\text{Fix}_\Lambda(\alpha)} = \mathfrak{M}/\mathfrak{b}(\Lambda) \cdot \mathfrak{M}$$

is finite. Since $\mathfrak{b}(\Lambda)$ is finitely generated we can choose f_1, \dots, f_r such that $\mathfrak{b}(\Lambda) = f_1 \mathfrak{R}_d + \cdots + f_r \mathfrak{R}_d$, and we conclude that

$$\begin{aligned} |\mathfrak{b}(\Lambda) \cdot \mathfrak{M}/\mathfrak{b}(\Lambda)^{(2)} \cdot \mathfrak{M}| &\leq \sum_{j=1}^r \left| f_j \cdot \mathfrak{M} \middle/ \left(\sum_{j,j'=1}^r f_j f_{j'} \cdot \mathfrak{M} \right) \right| \\ &\leq \sum_{j=1}^r \left| f_j \cdot \mathfrak{M} \middle/ \left(\sum_{j'=1}^r f_j f_{j'} \cdot \mathfrak{M} \right) \right| \\ &\leq r \left| \mathfrak{M} \middle/ \left(\sum_{j'=1}^r f_{j'} \cdot \mathfrak{M} \right) \right| = r |\mathfrak{M}/\mathfrak{b}(\Lambda) \cdot \mathfrak{M}| < \infty. \end{aligned}$$

An induction argument shows that $\mathfrak{b}(\Lambda)^{(k)} \mathfrak{M}/\mathfrak{b}(\Lambda)^{(k+1)} \cdot \mathfrak{M}$ is finite for every $k \geq 1$, and we conclude that $\mathfrak{M}/\mathfrak{b}(\Lambda)^{(k)} \cdot \mathfrak{M}$ is finite for every $k \geq 1$. In particular, the modules $\mathfrak{L}_i/\mathfrak{b}(\Lambda)^{(t)} \cdot \mathfrak{M} \cong \mathfrak{L}_i/(\mathfrak{b}(\Lambda)^{(t)} \cdot \mathfrak{M} \cap \mathfrak{L}_i)$ and $\mathfrak{L}_i/\mathfrak{b}(\Lambda) \cdot \mathfrak{L}_i \cong \mathfrak{R}_d/(\mathfrak{p}_i + \mathfrak{b}(\Lambda))$ are finite. From Lemma 6.7 we conclude that $V_{\mathbb{C}}(\mathfrak{p}_i + \mathfrak{b}(\Lambda)) = V_{\mathbb{C}}(\mathfrak{p}_i) \cap \Omega(\Lambda) = \emptyset$ for every $i = 1, \dots, m$, which proves (c).

Conversely, if (c) is satisfied, we choose a prime filtration $\mathfrak{M} = \mathfrak{N}_s \supset \cdots \supset \mathfrak{N}_0 = \{0\}$ of \mathfrak{M} such that, for every $j = 1, \dots, s$, $\mathfrak{N}_j/\mathfrak{N}_{j-1} \cong \mathfrak{R}_d/\mathfrak{q}_j$ for some prime ideal \mathfrak{q}_j which contains one of the associated primes \mathfrak{p}_i of \mathfrak{M} (cf. Corollary 6.2). Since

$$V_{\mathbb{C}}(\mathfrak{q}_j + \mathfrak{b}(\Lambda)) = V_{\mathbb{C}}(\mathfrak{q}_j) \cap V_{\mathbb{C}}(\mathfrak{b}(\Lambda)) \subset V_{\mathbb{C}}(\mathfrak{p}_i) \cap V_{\mathbb{C}}(\mathfrak{b}(\Lambda)) = \emptyset$$

for every $j = 1, \dots, s$, the module $\mathfrak{R}_d/(\mathfrak{q}_j + \mathfrak{b}(\Lambda))$ is finite for every j by Lemma 6.7. Hence $\mathfrak{N}_j/(\mathfrak{N}_{j-1} + \mathfrak{b}(\Lambda) \cdot \mathfrak{M})$ is finite for $j = 1, \dots, s$, since it is (isomorphic to) a quotient of $\mathfrak{R}_d/(\mathfrak{q}_j + \mathfrak{b}(\Lambda))$, and $\mathfrak{M}/\mathfrak{b}(\Lambda) \cdot \mathfrak{M}$ is finite. This implies the finiteness of $\text{Fix}_\Lambda(\alpha)$ and completes the proof of the implication (c) \Rightarrow (a). The equivalence of (b) and (c) is obtained by applying what we have just proved to the \mathbb{Z}^d -actions $\alpha^{\mathfrak{R}_d/\mathfrak{p}_i}$, $i = 1, \dots, m$. \square

LEMMA 6.8. *Let $\mathfrak{a} \subset \mathfrak{R}_d$ be an ideal with $V_{\mathbb{C}}(\mathfrak{a}) \cap \mathbb{S}^d = \emptyset$. Then $\alpha^{\mathfrak{R}_d/\mathfrak{a}}$ is expansive.*

PROOF. We assume that $X^{\mathfrak{R}_d/\mathfrak{a}} = \widehat{\mathfrak{R}_d/\mathfrak{a}}$ and $\alpha^{\mathfrak{R}_d/\mathfrak{a}}$ are given by (5.9)–(5.10). For every $f \in \mathfrak{R}_d$ of the form (5.2) we set $\|f\| = \sum_{\mathbf{n} \in \mathbb{Z}^d} |c_f(\mathbf{n})|$. Let $\{f_1, \dots, f_k\}$ be a set of generators for \mathfrak{a} , $\varepsilon = (10 \sum_{j=1}^k \|f_j\|)^{-1}$, and $N = \{x \in$

$X^{\mathfrak{A}_d/\mathfrak{a}} : \|x_0\| < \varepsilon\}$, where $\|t\| = \min\{|t - n| : n \in \mathbb{Z}\}$ for every $t \in \mathbb{T}$. We claim that N is an expansive neighbourhood of the identity $\mathbf{0}$ in $X^{\mathfrak{A}_d/\mathfrak{a}}$.

If N is not expansive, there exists a point $\mathbf{0} \neq x \in \bigcap_{\mathbf{n} \in \mathbb{Z}^d} \sigma_{\mathbf{n}}(N)$. Let $\mathbf{B} = \ell^\infty(\mathbb{Z}^d)$ be the Banach space of all bounded, complex valued functions $(z_{\mathbf{n}}) = (z_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}^d)$ on \mathbb{Z}^d in the supremum norm. Since $\|x_{\mathbf{n}}\| < \varepsilon$ for every $\mathbf{n} \in \mathbb{Z}^d$, there exists a unique non-zero point $y \in \mathbf{B}$ with $|y_{\mathbf{n}}| < \varepsilon$ and $y_{\mathbf{n}} \pmod{1} = x_{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{Z}^d$. From (5.7) and (5.9) we know that

$$\langle x, f_j \rangle = e^{2\pi i \sum_{\mathbf{n} \in \mathbb{Z}^d} c_{f_j}(\mathbf{n})x_{\mathbf{n}}} = 1$$

and hence

$$\sum_{\mathbf{n} \in \mathbb{Z}^d} c_{f_j}(\mathbf{n})y_{\mathbf{n}} \in \mathbb{Z}$$

for $j = 1, \dots, k$, and our choice of ε implies that

$$\sum_{\mathbf{n} \in \mathbb{Z}^d} c_{f_j}(\mathbf{n})y_{\mathbf{n}} = 0 \quad (6.8)$$

for all j . Consider the group of isometries $\{U_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ of \mathbf{B} defined by $(U_{\mathbf{n}}z)_{\mathbf{m}} = z_{\mathbf{m}+\mathbf{n}}$ for all $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$ and $z \in \mathbf{B}$, and put

$$\begin{aligned} \mathbf{S} &= \left\{ z \in \mathbf{B} : \sum_{\mathbf{n} \in \mathbb{Z}^d} c_{f_j}(\mathbf{n})z_{\mathbf{m}+\mathbf{n}} = 0 \text{ for all } \mathbf{m} \in \mathbb{Z}^d \text{ and } j = 1, \dots, k \right\} \\ &= \left\{ z \in \mathbf{B} : \left(\sum_{\mathbf{n} \in \mathbb{Z}^d} c_{f_j}(\mathbf{n})U_{\mathbf{n}} \right) z = 0 \text{ for } j = 1, \dots, k \right\}. \end{aligned} \quad (6.9)$$

From (6.8) we know that the closed linear subspace $\mathbf{S} \subset \mathbf{B}$ is non-zero. Let $\mathcal{B}(\mathbf{S})$ be the Banach algebra of all bounded, linear operators on \mathbf{S} , denote by $V_{\mathbf{n}}$ the restriction of $U_{\mathbf{n}}$ to \mathbf{S} , and let $\mathcal{A} \subset \mathcal{B}(\mathbf{S})$ be the Banach subalgebra generated by $\{V_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$. We write $\mathcal{M}(\mathcal{A})$ for the space of maximal ideals of \mathcal{A} in its usual topology. The Gelfand transform $A \mapsto \hat{A}$ from \mathcal{A} to the Banach algebra $\mathcal{C}(\mathcal{M}(\mathcal{A}), \mathbb{C})$ of continuous, complex valued functions on $\mathcal{M}(\mathcal{A})$ is a norm-non-increasing Banach algebra homomorphism (cf. §11 in [75]). For every $\mathbf{n} \in \mathbb{Z}^d$, both $V_{\mathbf{n}}$ and $V_{-\mathbf{n}} = V_{\mathbf{n}}^{-1}$ are isometries of \mathbf{S} , and hence $|\widehat{V_{\mathbf{n}}}(\omega)| = 1$ for every $\omega \in \mathcal{M}(\mathcal{A})$. Since $\sum_{\mathbf{n} \in \mathbb{Z}^d} c_{f_j}(\mathbf{n})V_{\mathbf{n}} = 0$ (cf. (6.9)) we obtain that $\sum_{\mathbf{n} \in \mathbb{Z}^d} c_{f_j}(\mathbf{n})\widehat{V_{\mathbf{n}}}(\omega) = 0$ for every $j = 1, \dots, k$ and $\omega \in \mathcal{M}(\mathcal{A})$. Fix $\omega \in \mathcal{M}(\mathcal{A})$ and put $c_i = \widehat{V_{\mathbf{e}^{(i)}}}(\omega)$ for every $i = 1, \dots, d$, where $\mathbf{e}^{(i)}$ is the i -th unit vector in \mathbb{Z}^d . Then $\sum_{\mathbf{n} \in \mathbb{Z}^d} c_{f_j}(\mathbf{n})c^{\mathbf{n}} = f_j(c) = 0$ for $j = 1, \dots, k$ with $c = (c_1, \dots, c_d) \in \mathbb{S}^d$. It follows that $c \in V_{\mathbb{C}}(\mathfrak{a}) \cap \mathbb{S}^d$, contrary to our initial assumption. This proves that $\alpha^{\mathfrak{A}_d/\mathfrak{a}}$ is expansive. \square

PROOF OF THEOREM 6.5 (4). We begin by proving the equivalence of (a) and (c). Suppose that (c) is satisfied, but that α is non-expansive. We apply Corollary 6.2 and choose a prime filtration $\mathfrak{M} = \mathfrak{N}_s \supset \dots \supset \mathfrak{N}_0 = \{0\}$ such

that, for every $j = 1, \dots, s$, $\mathfrak{N}_j/\mathfrak{N}_{j-1} \cong \mathfrak{R}_d/\mathfrak{q}_j$ for some prime ideal $\mathfrak{q}_j \subset \mathfrak{R}_d$ which contains one of the associated primes \mathfrak{p}_i . Put $X_j = \mathfrak{N}_j^\perp \subset X$ and observe that $X = X_0 \supset \dots \supset X_s = \{\mathbf{1}\}$, that X_j is a closed, α -invariant subgroup of X , and that $X_{j-1}/X_j \cong \widehat{\mathfrak{R}_d/\mathfrak{q}_j}$ for $j = 1, \dots, s$. Then $V_{\mathbb{C}}(\mathfrak{q}_1) \subset \bigcup_{i=1}^m V_{\mathbb{C}}(\mathfrak{p}_i)$, hence $V_{\mathbb{C}}(\mathfrak{q}_1) \cap \mathbb{S}^d = \emptyset$, and Lemma 6.8 shows that $\alpha^{\mathfrak{R}_d/\mathfrak{q}_1}$ is expansive. Since $\alpha^{\mathfrak{R}_d/\mathfrak{q}_1}$ is conjugate to $\alpha^{X/X_1} = \alpha^{X_0/X_1}$ we see that α^{X_0/X_1} is expansive. The non-expansiveness of α implies that α^{X_1} cannot be expansive, and by repeating this argument we eventually obtain that α^{X_s} is non-expansive, which is absurd. This contradiction proves the expansiveness of α .

In order to explain the idea behind the proof of the reverse implication we assume for the moment that \mathfrak{M} is of the form $\mathfrak{R}_d/\mathfrak{a}$ for some ideal $\mathfrak{a} \subset \mathfrak{R}_d$. If $c = (c_1, \dots, c_d) \in V_{\mathbb{C}}(\mathfrak{a})$ then the evaluation map $f \mapsto f(c)$ defines an \mathfrak{R}_d -module homomorphism $\eta_c: \mathfrak{R}_d/\mathfrak{a} \rightarrow \mathbb{C}$, where \mathbb{C} is an \mathfrak{R}_d -module under the action $(f, z) \mapsto f(c)z$, $f \in \mathfrak{R}_d$, $z \in \mathbb{C}$. If W is the closure of $\eta_c(\mathfrak{R}_d/\mathfrak{a}) \subset \mathbb{C}$, then η_c conjugates the \mathbb{Z}^d -action $\hat{\alpha}$ on \mathfrak{M} to the action θ on W , where $\theta_{\mathbf{n}}$ is multiplication by $c^{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{Z}^d$. If $c \in V_{\mathbb{C}}(\mathfrak{a}) \cap \mathbb{S}^d$ then θ is isometric (with respect to the usual metric on \mathbb{C}), and the homomorphism η_c induces an inclusion of $V = \hat{W}$ in $X^{\mathfrak{R}_d/\mathfrak{a}} = \widehat{\mathfrak{R}_d/\mathfrak{a}}$. Since θ is isometric on W , the dual action $\hat{\theta}$ on V is also equicontinuous, and coincides with the restriction of α to V . This shows that α cannot be expansive.

We return to our given module \mathfrak{M} with its associated primes $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ and a corresponding reduced primary decomposition $\mathfrak{W}_1, \dots, \mathfrak{W}_m$. If $V_{\mathbb{C}}(\mathfrak{p}_i) \cap \mathbb{S}^d \neq \emptyset$ for some $i \in \{1, \dots, m\}$ we set $\mathfrak{M}' = \mathfrak{M}/\mathfrak{W}_i$, choose $a_1, \dots, a_k \in \mathfrak{M}'$ such that $\mathfrak{M}' = \mathfrak{R}_d a_1 + \dots + \mathfrak{R}_d a_k$, and define a surjective homomorphism $\zeta: \mathfrak{R}_d^k \rightarrow \mathfrak{M}'$ by $\zeta(f_1, \dots, f_k) = f_1 a_1 + \dots + f_k a_k$.

Choose a point $c = (c_1, \dots, c_d) \in V_{\mathbb{C}}(\mathfrak{p}_i) \cap \mathbb{S}^d$, denote by $\eta_c: \mathfrak{R}_d \rightarrow \mathbb{C}$ the evaluation map at c , and observe that $\mathfrak{a} = \ker(\eta_c) \supset \mathfrak{p}_i$. Let $\mathfrak{L} = \ker(\zeta) + \mathfrak{a}^k \subset \mathfrak{R}_d^k$, and let $\mathfrak{N} = \{(0, \dots, 0, f) : f \in \mathfrak{R}_d\} \subset \mathfrak{R}_d^k$. From (6.6) (with \mathfrak{M} replaced by \mathfrak{M}') we see that $\text{ann}(a_k) \subset \mathfrak{p}_i$, so that

$$\mathfrak{L} \cap \mathfrak{N} \subset \{(0, \dots, 0, f) : f \in \mathfrak{p}_i\} \subset \{(0, \dots, 0, f) : f \in \mathfrak{a}\}.$$

This allows us to define an additive group homomorphism $\xi: \mathfrak{L} + \mathfrak{N} \rightarrow \mathbb{C}$ by $\xi(a + b) = \eta_c(f)$ for all $a \in \mathfrak{L}$ and $b = (0, \dots, 0, f) \in \mathfrak{N}$. Then

$$\xi(a) = 0 \text{ for } a \in \mathfrak{L}, \quad (6.10)$$

and

$$\xi \cdot \hat{\alpha}_{\mathbf{n}}^{\mathfrak{R}_d^k}(a) = c^{\mathbf{n}} \xi(a) \text{ for all } a \in \mathfrak{L} + \mathfrak{N}, \mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d, \quad (6.11)$$

where $c^{\mathbf{n}} = c_1^{n_1} \cdot \dots \cdot c_d^{n_d}$. We claim that ξ can be extended to a homomorphism $\bar{\xi}: \mathfrak{R}_d^k \rightarrow \mathbb{C}$ which still satisfies (6.10) and (6.11). Indeed, there exists a maximal extension ξ' of ξ to a submodule $\mathfrak{N}' \subset \mathfrak{R}_d^k$ satisfying (6.11) for every $a \in \mathfrak{N}'$. If $b \in \mathfrak{R}_d^k \setminus \mathfrak{N}'$ and $\xi'(b') = 0$ for every $b' \in \mathfrak{R}_d b \cap \mathfrak{N}'$, then we put $\rho = 0$. If there exists an element $f \in \mathfrak{R}_d$ with $fb \in \mathfrak{R}_d b \cap \mathfrak{N}'$ and $\xi'(fb) \neq 0$,

then $f(c) = \eta_c(f) \neq 0$: otherwise $f \in \mathfrak{a}$, $fb \in \mathfrak{a}^k \subset \mathfrak{L}$, and $\xi'(fb) = \xi(fb) = 0$ by (6.10), which is impossible. Hence we can set $\rho = \xi'(fb)/f(c)$. The map $\xi'': \mathfrak{N}'' = \mathfrak{R}_d b + \mathfrak{N}' \mapsto \mathbb{C}$, defined by $\xi''(fb + a) = f(c)\rho + \xi'(a)$ for $f \in \mathfrak{R}_d$ and $a \in \mathfrak{N}'$, is a homomorphism which extends ξ' and satisfies (6.11) for all $a \in \mathfrak{N}''$. This contradiction to the maximality of \mathfrak{N}' proves our claim.

We have obtained an extension $\bar{\xi}: \mathfrak{R}_d^k \mapsto \mathbb{C}$ of ξ satisfying (6.11) for all $a \in \mathfrak{R}_d^k$; this implies that $\ker(\bar{\xi})$ is a submodule of \mathfrak{R}_d^k which contains $\ker(\zeta)$, and that $\bar{\xi}$ induces an \mathfrak{R}_d -module homomorphism $\Xi: \mathfrak{M}' \cong \mathfrak{R}_d^k/\ker(\zeta) \mapsto \mathbb{C}$ with $\Xi(\mathfrak{M}') \supset \eta_c(\mathfrak{R}_d)$ and

$$\Xi \cdot \hat{\alpha}_{\mathbf{n}}^{\mathfrak{M}'} = \theta_{\mathbf{n}} \cdot \Xi \quad (6.12)$$

for every $\mathbf{n} \in \mathbb{Z}^d$, where $\theta_{\mathbf{n}}$ is multiplication by $c^{\mathbf{n}}$. We denote by W the closure of $\Xi(\mathfrak{M}')$ in \mathbb{C} and write $V = \hat{W}$ for the dual group of W . Since Ξ sends \mathfrak{M}' to a dense subgroup of W , there is a dual inclusion $V \subset \widehat{\mathfrak{M}'/\ker(\Xi)} \subset \widehat{\mathfrak{M}'} \subset X$, and (6.12) shows that, for every $v \in V$ and $\mathbf{n} \in \mathbb{Z}^d$,

$$\hat{\theta}_{\mathbf{n}}(v) = \alpha_{\mathbf{n}}(v). \quad (6.13)$$

If the closed subgroup $W \subset \mathbb{C}$ is countable, then the group $\Theta = \{\theta_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\} \subset \text{Aut}(W)$ is finite, since it consists of isometries of W , and hence $\hat{\Theta} = \{\hat{\theta}_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\} \subset \text{Aut}(V)$ is finite. From (6.13) it is clear that the restriction of α to the infinite subgroup $V \subset X$ cannot be expansive.

If W is uncountable, but disconnected, we replace W by its infinite, discrete quotient group $W' = W/W^\circ$, and obtain an α -invariant subgroup $V' = \widehat{W'/W^\circ} \subset V \subset X$ on which α is not expansive.

If W is connected, it is either equal to \mathbb{C} or isomorphic to \mathbb{R} , and the definition of Θ implies that W has a basis of Θ -invariant neighbourhoods of the identity. The dual group V is isomorphic to W , and again possesses a basis of $\hat{\Theta}$ -invariant neighbourhoods of the identity. Since the inclusion $V \hookrightarrow X$ is continuous, the \mathbb{Z}^d -action $\mathbf{n} \mapsto \hat{\theta}_{\mathbf{n}}$ on $V \subset X$ must also be non-expansive in the subspace topology, i.e. α is not expansive on V .

We have proved that there always exists an infinite, α -invariant, but not necessarily closed, subgroup $V \subset X$ on which α is non-expansive in the induced topology. This shows that α is not expansive and completes the proof that (a) \iff (c).

The equivalence of (b) and (c) is seen by applying the implications (a) \iff (c) already proved to the \mathbb{Z}^d -actions $\alpha^{\mathfrak{R}_d/\mathfrak{p}_i}$, $i = 1, \dots, m$.

It is clear that (c) \implies (d). Conversely, if $V_{\mathbb{C}}(\mathfrak{p}_i) \cap \mathbb{S}^d \neq \emptyset$ for some $i \in \{1, \dots, m\}$, choose f_1, \dots, f_k in \mathfrak{R}_d with $\mathfrak{p}_i = f_1\mathfrak{R}_d + \dots + f_k\mathfrak{R}_d$, and define polynomials g_j, h_j , $j = 1, \dots, k$, in

$$\mathcal{R}_d = \mathbb{Q}[x_1, \dots, x_d, y_1, \dots, y_d]$$

by

$$g_j(a_1, \dots, a_d, b_1, \dots, b_d) = \operatorname{Re}(f_j(a_1 + b_1\sqrt{-1}, \dots, a_d + b_d\sqrt{-1}))$$

and

$$h_j(a_1, \dots, a_d, b_1, \dots, b_d) = \operatorname{Im}(f_j(a_1 + b_1\sqrt{-1}, \dots, a_d + b_d\sqrt{-1}))$$

for all $j = 1, \dots, k$ and $(a_1, \dots, a_d, b_1, \dots, b_d) \in \mathbb{R}^{2d}$, where $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ denote the real and imaginary parts of $z \in \mathbb{C}$. For $l = 1, \dots, d$ we put

$$\chi_l(x_1, \dots, x_d, y_1, \dots, y_d) = x_l^2 + y_l^2 - 1 \in \mathcal{R}_d.$$

The ideal $\mathcal{J} \subset \mathcal{R}_d$ generated by $\{g_1, \dots, g_k, h_1, \dots, h_k, \chi_1, \dots, \chi_k\}$ satisfies that $V_{\mathbb{C}}(\mathcal{J}) \cap \mathbb{R}^{2d} \neq \emptyset$. Hence \mathcal{J} does not contain a polynomial of the form $1 + \sum_{j=1}^r \psi_j^2$ with $r \geq 1$ and $\psi_j \in \mathcal{R}_d$, and the real version of Hilbert's Nullstellensatz implies that $V_{\mathbb{C}}(\mathcal{J}) \cap \mathbb{R}^{2d} \cap \overline{\mathbb{Q}}^{2d} \neq \emptyset$ (proposition 4.1.7 and corollaire 4.1.8 in [11]). In particular we see that (d) cannot be satisfied, and this shows that (d) \Rightarrow (c) and completes the proof of Theorem 6.5 (4). \square

Before we start listing some useful corollaries of Theorem 6.5 we give an elementary characterization of the connectedness of a group X carrying a \mathbb{Z}^d -action by automorphisms in terms of the prime ideals associated with the \mathfrak{R}_d -module \hat{X} .

PROPOSITION 6.9. *Let α be a \mathbb{Z}^d -action by automorphisms of a compact, abelian group X , and let $\mathfrak{M} = \hat{X}$ be the \mathfrak{R}_d -module defined by Lemma 5.1. The following conditions are equivalent.*

- (1) X is connected;
- (2) $V_{\mathbb{C}}(\mathfrak{p}) \neq \emptyset$ for every prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ associated with \mathfrak{M} .

PROOF. Suppose that X is connected, and let $\mathfrak{p} \subset \mathfrak{R}_d$ be a prime ideal associated with \mathfrak{M} . Then there exists an element $a \in \mathfrak{M}$ with $\mathfrak{R}_d \cdot a \cong \mathfrak{R}_d/\mathfrak{p}$, which implies that $X^{\mathfrak{R}_d/\mathfrak{p}}$ is a quotient group of X . In particular, $X^{\mathfrak{R}_d/\mathfrak{p}}$ is connected, so that $\mathfrak{R}_d/\mathfrak{p}$ is a torsion-free, abelian group, and Lemma 6.7 implies that $V_{\mathbb{C}}(\mathfrak{p}) \neq \emptyset$. Conversely, if X is disconnected, then there exists—by duality theory—a non-zero element $a \in \mathfrak{M}$ and a positive integer m with $ma = 0$, and we set $\mathfrak{N} = \mathfrak{R}_d \cdot a$ and observe that \mathfrak{N} (and hence \mathfrak{M}) has an associated prime ideal \mathfrak{p} containing a non-zero constant (cf. (6.6)). In particular, $V_{\mathbb{C}}(\mathfrak{p}) = \emptyset$. \square

COROLLARY 6.10 (OF THEOREM 6.5). *If α is an ergodic \mathbb{Z}^d -action by automorphisms of a compact, abelian group X satisfying the d.c.c., then $\alpha_{\mathbf{n}}$ is ergodic for some $\mathbf{n} \in \mathbb{Z}^d$.*

PROOF. Lemma 5.1, Proposition 5.4, and Theorem 6.5 (1). \square

COROLLARY 6.11. *Let $d \geq 2$, and let $(f) \subset \mathfrak{R}_d$ be a principal ideal. Then $\alpha^{\mathfrak{R}_d/(f)}$ is ergodic.*

PROOF. By Theorem 6.5 (1), the non-ergodicity of α implies that $V(\mathfrak{p}_i)$ is finite for at least one of the associated primes of $\mathfrak{M} = \mathfrak{R}_d/(f)$. However, the associated primes of \mathfrak{M} are all principal (they are given by the prime factors of f in \mathfrak{R}_d), and have infinite varieties. \square

COROLLARY 6.12. *Let $d \geq 1$ and $f \in \mathfrak{R}_d$. If f is not divisible by any generalized cyclotomic polynomial then $\alpha^{\mathfrak{R}_d/(f)}$ is mixing.*

PROOF. If \mathfrak{p} is one of the associated primes of $\mathfrak{R}_d/(f)$ then $\mathfrak{p} = (h)$ for a prime factor h of f in \mathfrak{R}_d , and \mathfrak{p} contains a polynomial of the form $u^n - 1$ for some (non-zero) $\mathbf{n} \in \mathbb{Z}^d$ if and only if $h = c(u^n)$ for some cyclotomic polynomial c (cf. Theorem 6.5 (2)). \square

COROLLARY 6.13. *Let X be a compact, abelian group, and let α be an expansive \mathbb{Z}^d -action by automorphisms of X . Then the \mathfrak{R}_d -module $\mathfrak{M} = \hat{X}$ is a Noetherian torsion module.*

PROOF. According to (4.10) and Proposition 5.4, \mathfrak{M} is Noetherian, and by Theorem 6.5 (4), $\{0\}$ cannot be an associated prime ideal of \mathfrak{M} . \square

COROLLARY 6.14. *Let X be a compact, connected group, and let α be an expansive \mathbb{Z}^d -action by automorphisms of X . Then X is abelian and α is ergodic.*

PROOF. Theorem 2.4 shows that X is abelian, and (4.10) and Proposition 5.4 allow us to assume that $(X, \alpha) = (X^{\mathfrak{M}}, \alpha^{\mathfrak{M}})$ for some Noetherian \mathfrak{R}_d -module \mathfrak{M} . By recalling Proposition 6.9 and comparing the conditions (1.e) and (4.c) in Theorem 6.5 we see that α is ergodic. \square

COROLLARY 6.15. *Let X be a compact group, and let α be an expansive \mathbb{Z}^d -action by automorphisms of X . If $Y \subset X$ is a closed, normal, α -invariant subgroup, then α^Y and $\alpha^{X/Y}$ are both expansive.*

PROOF. The expansiveness of α^Y is obvious. In order to see that $\alpha^{X/Y}$ is expansive we note that the connected component of the identity $X^\circ \subset X$ is abelian by Corollary 2.5. The group X/X° is zero-dimensional, and $X/(Y+X^\circ)$ is a quotient of a zero-dimensional group and hence again zero dimensional. Since the \mathbb{Z}^d -action $\alpha^{X/(Y+X^\circ)}$ satisfies the d.c.c., Corollary 3.4 implies that $\alpha^{X/(Y+X^\circ)}$ is expansive.

The group $(Y + X^\circ)/Y$ is isomorphic to $X^\circ/(Y \cap X^\circ)$, and this isomorphism carries $\alpha^{(Y+X^\circ)/Y}$ to $\alpha^{X^\circ/(Y \cap X^\circ)}$. We apply Lemma 5.1 to the abelian groups X° and $X^\circ/(Y \cap X^\circ)$, and obtain \mathfrak{R}_d -modules $\hat{X}^\circ = \mathfrak{M}$ and $X^\circ/\widehat{(Y \cap X^\circ)} = \mathfrak{N} \subset \mathfrak{M}$ satisfying (5.3)–(5.4). Since α^{X° is expansive, Theorem 6.5 (4) implies that $V_C(\mathfrak{p}) \cap \mathbb{S}^d = \emptyset$ for every prime ideal \mathfrak{p} associated with \mathfrak{M} . Every prime ideal associated with \mathfrak{N} is also associated with \mathfrak{M} , and Theorem 6.5 (4) implies that $\alpha^{\mathfrak{N}}$ is expansive. This implies the expansiveness of both $\alpha^{X^\circ/(Y \cap X^\circ)}$ and $\alpha^{(Y+X^\circ)/Y}$.

Suppose that $x \in X \setminus Y$. If $x \notin Y + X^\circ$ then the expansiveness of $\alpha^{X/(Y+X^\circ)}$ guarantees the existence of an open neighbourhood $N'(\mathbf{1}_X)$ of the identity in X such that $\alpha_{\mathbf{m}}(x) \notin N'(\mathbf{1}_X) + Y + X^\circ \supset N'(\mathbf{1}_X) + Y$ for some $\mathbf{m} \in \mathbb{Z}^d$. If $x \in Y + X^\circ$ then the expansiveness of $\alpha^{(Y+X^\circ)/Y}$ allows us to choose a neighbourhood $N''(\mathbf{1}_X)$ of the identity in X with $\alpha_{\mathbf{m}}(x) \notin N''(\mathbf{1}_X) + Y$ for some $\mathbf{m} \in \mathbb{Z}^d$. Put $N(\mathbf{1}_X) = N'(\mathbf{1}_X) \cap N''(\mathbf{1}_X)$. Then there exists, for every $x \in X \setminus Y$, an $\mathbf{m} \in \mathbb{Z}^d$ with $\alpha_{\mathbf{m}}(x) \notin N(\mathbf{1}_X) + Y$, which shows that $\alpha^{X/Y}$ is expansive. \square

In view of Theorem 6.5 we introduce the following definition, which will help to simplify terminology.

DEFINITION 6.16. Let $d \geq 1$, and let $\mathfrak{p} \subset \mathfrak{R}_d$ be a prime ideal. The ideal \mathfrak{p} will be called *ergodic*, *mixing*, or *expansive* if the \mathbb{Z}^d -action $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is ergodic, mixing, or expansive.

EXAMPLES 6.17. (1) Let $n \geq 1$, $\alpha = A \in \mathrm{GL}(n, \mathbb{Z}) = \mathrm{Aut}(\mathbb{T}^n)$, and let $\beta = \hat{A} = A^\top \in \mathrm{Aut}(\mathbb{Z}^n)$. The \mathfrak{R}_1 -module $\mathfrak{M} = \mathbb{Z}^n$ arising from α via Lemma 5.1 is Noetherian, and $\mathrm{ann}(\mathfrak{M}) = \{f \in \mathfrak{R}_1 : f(A^\top)\mathbf{m} = 0\}$ for every $\mathbf{m} \in \mathbb{Z}^n$. In particular, the associated primes of \mathfrak{M} are the principal ideals (h) , where h runs through the prime factors of the characteristic polynomial $\chi_A = \chi_{A^\top}$ of A (or A^\top) in \mathfrak{R}_1 . In this setting Theorem 6.5 (1) reduces to the following well known facts about toral automorphisms: (i) α is ergodic if and only if no root of χ_A is a root of unity; (ii) α is expansive if and only if no root of χ_A has modulus 1.

(2) The automorphism α in Example 5.6 (1) does not satisfy the d.c.c. (cf. Theorem 5.7), and is therefore non-expansive by (6.10). However, if we replace \mathbb{Q} by $\mathbb{Z}[\frac{1}{6}] = \{k/6^l : k \in \mathbb{Z}, l \geq 0\} \cong \mathfrak{R}_1/(2u_1 - 3) = \mathfrak{M}$, where the isomorphism between $\mathfrak{R}_1/(2u_1 - 3)$ and $\mathbb{Z}[\frac{1}{6}]$ is the evaluation $f \mapsto f(\frac{3}{2})$, then the automorphism β' of $\mathbb{Z}[\frac{1}{6}]$ consisting of multiplication by $\frac{3}{2}$ is conjugate to multiplication by u_1 on \mathfrak{M} . Since $\mathfrak{p} = (2u_1 - 3) \subset \mathfrak{R}_1$ is a prime ideal, \mathfrak{M} is associated with \mathfrak{p} , $V_{\mathbb{C}}(\mathfrak{p}) = \{\frac{3}{2}\}$, and the automorphism α' on $X = \widehat{\mathbb{Z}[\frac{1}{6}]}$ dual to β' is expansive by Theorem 6.5 (4). An explicit realization of α' can be obtained from Example 5.2 (2) by setting α' equal to the shift σ on $X' = \{(x_k) \in \mathbb{T}^{\mathbb{Z}} : 3x_k = 2x_{k+1} \text{ for every } k \in \mathbb{Z}\}$.

(3) Let $\mathfrak{p} \subset \mathfrak{R}_1$ be a prime ideal. Since the ring $\mathfrak{R}_1^{(\mathbb{Q})} = \mathbb{Q}[u_1^{\pm 1}]$ of Laurent polynomials with rational coefficients is a principal ideal domain, $\mathfrak{R}_1/\mathfrak{p}$ must be finite if \mathfrak{p} is non-principal. In order to see this, assume that $\mathfrak{p} \subsetneq \mathfrak{R}_1$ is a non-principal prime ideal, and choose two irreducible elements $g, h \in \mathfrak{p}$ with $g\mathfrak{R}_1 \neq h\mathfrak{R}_1$. We assume without loss in generality that $g\mathfrak{R}_1 \neq m\mathfrak{R}_1$ for any $m \in \mathbb{Z}$. Then $\mathfrak{q} = \{\frac{1}{n}f : n \geq 1, f \in \mathfrak{p}\} \subset \mathfrak{R}_1^{(\mathbb{Q})}$ is an ideal strictly containing the maximal ideal $g\mathfrak{R}_1^{(\mathbb{Q})}$, and therefore equal to $\mathfrak{R}_1^{(\mathbb{Q})}$. We conclude that \mathfrak{p} contains a prime constant p , and hence the ideal $(p, g) = p\mathfrak{R}_1 + g\mathfrak{R}_1$. It follows that $\mathfrak{R}_1/\mathfrak{p}$ is a quotient of the finite ring $\mathfrak{R}_1/(p, g) \cong \mathfrak{R}_1^{(p)}/g_p\mathfrak{R}_1^{(p)}$ (cf. (6.1)). In

particular, if $\mathfrak{p} \subset \mathfrak{R}_1$ is a non-principal prime ideal, then $X^{\mathfrak{R}_1/\mathfrak{p}} = \widehat{\mathfrak{R}_1/\mathfrak{p}}$ is finite, and $\alpha^{\mathfrak{R}_1/\mathfrak{p}}$ is non-ergodic.

If $\mathfrak{p} = (f)$ for some $f \in \mathfrak{R}_1$, the automorphism $\alpha = \alpha^{\mathfrak{R}_1/\mathfrak{p}}$ is non-ergodic if and only if f divides $u_1^n - 1$ for some $n \geq 1$ (Theorem 6.5 (1)) (as f is irreducible this means that $\pm u_1^n f$ is cyclotomic for some $n \in \mathbb{Z}$), and α is expansive if and only if f is non-zero and has no roots of modulus 1 (Theorem 6.5 (4)). Since we can write $X = X^{\mathfrak{R}_1/\mathfrak{p}}$ in the form (5.9) we see that X is (isomorphic to) a finite-dimensional torus if and only if there exists $n \in \mathbb{Z}$ and $s \geq 1$ such that $u_1^n f(u_1) = c_0 + c_1 u_1 + \cdots + c_s u_1^s$ with $|c_0 c_s| = 1$. If $|c_0 c_s| > 1$, then X is a *finite-dimensional solenoid*, i.e. \hat{X} is isomorphic to a subgroup of \mathbb{Q}^s (Example (2) and Example 5.3 (3)).

(4) Let α be an ergodic automorphism of a compact, abelian group X , and let $\mathfrak{M} = \hat{X}$ be the \mathfrak{R}_1 -module arising from α via Lemma 5.1. Then every prime ideal $\mathfrak{p} \subset \mathfrak{R}_1$ associated with \mathfrak{M} is principal, and $\mathfrak{p} \neq (f)$ for any cyclotomic polynomial $f \in \mathfrak{R}_1$ (Proposition 6.6 and Example (3)). \square

Further examples of expansive automorphisms of compact, abelian groups will appear in Chapter 3.

EXAMPLES 6.18. In the following illustrations of Theorem 6.5 we consider \mathfrak{R}_2 -modules of the form $\mathfrak{M} = \mathfrak{R}_2/\mathfrak{a}$, where $\mathfrak{a} \subset \mathfrak{R}_2$ is an ideal, realize $X = X^{\mathfrak{M}} \subset \mathbb{T}^2$ as in Example 5.2 (2), and denote by $\alpha = \alpha^{\mathfrak{M}}$ the shift-action of \mathbb{Z}^2 on X .

(1) Let $\mathfrak{a} = (1 + u_1 + u_2)$. Since \mathfrak{a} is prime, \mathfrak{M} is associated with \mathfrak{a} . Corollary 6.11 shows that α is ergodic, and Corollary 6.12 implies that α is mixing. Since $((-1 + i\sqrt{-3})/2, (-1 - i\sqrt{-3})/2) \in V_{\mathbb{C}}(\mathfrak{a}) \cap \mathbb{S}^2$, α is not expansive by Theorem 6.5 (4). Moreover, $V_{\mathbb{C}}(\mathfrak{a}) \cap \Omega(3\mathbb{Z}^2) \neq \emptyset$, so that $\text{Fix}_{3\mathbb{Z}^2}(\alpha)$ is infinite by Theorem 6.5 (3). note that $\text{Fix}_{3\mathbb{Z}^2}(\alpha)$ consists of all points

$$\begin{array}{cccccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \overset{\cdot}{a} & \overset{\cdot}{b} & \overset{\cdot}{c} & \overset{\cdot}{a} & \cdot \\ \cdot & a+2b+c & a+b+2c & 2a+b+c & a+2b+c & \cdot \\ \cdot & -a-b & -b-c & -a-c & -a-b & \cdot \\ \cdot & \overset{\cdot}{a} & \overset{\cdot}{b} & \overset{\cdot}{c} & \overset{\cdot}{a} & \cdot \end{array}$$

with $a, b, c \in \mathbb{T}$ and $3a + 3b + 3c = 0 \pmod{1}$. In particular, the connected component of the identity $\text{Fix}_{3\mathbb{Z}^2}(\alpha)^\circ \subset \text{Fix}_{3\mathbb{Z}^2}(\alpha)$ is isomorphic to \mathbb{T}^2 .

(2) Let $\mathfrak{a} = (2 + u_1 + u_2) \subset \mathfrak{R}_2$. The action α is ergodic, mixing, non-expansive, and $(-1, -1) \in V_{\mathbb{C}}(\mathfrak{a}) \cap \Omega(2\mathbb{Z}^2) \neq \emptyset$. The points in $\text{Fix}_{2\mathbb{Z}^2}(\alpha)$ are of the form

$$\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \overset{\cdot}{a} & \overset{\cdot}{b} & \overset{\cdot}{a} \\ \cdot & -2a-b & -a-2b & -2a-b \\ \cdot & \overset{\cdot}{a} & \overset{\cdot}{b} & \overset{\cdot}{a} \end{array}$$

with $4a + 4b = 1 \pmod{1}$, and $\text{Fix}_{2\mathbb{Z}^2}(\alpha)^\circ$ is isomorphic to \mathbb{T} .

(3) Let $\mathfrak{a} = (2 - u_1 - u_2) \subset \mathfrak{R}_2$. Then α is again ergodic, mixing, and non-expansive. Since $(1, 1) \in V_{\mathbb{C}}(\mathfrak{a})$, α has uncountably many fixed points, and hence $\text{Fix}_{\Lambda}(\alpha)$ is uncountable for every subgroup $\Lambda \subset \mathbb{Z}^d$.

(4) If $\mathfrak{a} = (3 + u_1 + u_2) \subset \mathfrak{R}_2$, then α is ergodic, mixing, expansive, and the expansiveness of α implies directly that $\text{Fix}_{\Lambda}(\alpha)$ is finite for every subgroup $\Lambda \subset \mathbb{Z}^d$ of finite index.

(5) In Example 5.3 (5) we considered the ideal $\mathfrak{a} = (2, 1 + u_1 + u_2) \subset \mathfrak{R}_2$. Then $V_{\mathbb{C}}(\mathfrak{a}) = \emptyset$, and Theorem 6.5 (4) re-establishes the fact that α is expansive. Since the polynomial $1 + u_1 + u_2$ is prime in $\mathfrak{R}_2^{(2)} = \mathbb{Z}/2[u_1^{\pm 1}, u_2^{\pm 2}]$, the ideal \mathfrak{a} is prime, and as in Corollary 6.12 we see that α is mixing (since every prime polynomial in $\mathbb{Z}/2[u]$ divides a polynomial of the form $u^l - 1$ for some $l \geq 1$, (the analogue of) Corollary 6.12 reduces to checking that $1 + u_1 + u_2 \in \mathfrak{R}_2^{(2)}$ is not a polynomial in the single variable $u^{\mathbf{n}}$ for some $\mathbf{0} \neq \mathbf{n} \in \mathbb{Z}^2$).

(6) Let $\mathfrak{a} = (4, 1 + u_1 - u_2 + 2u_2^2 + u_1u_2) \subset \mathfrak{R}_2$. Since every prime ideal \mathfrak{p} associated with $\mathfrak{M} = \mathfrak{R}_d/\mathfrak{a}$ must contain both the polynomial $1 + u_1 - u_2 + 2u_2^2 + u_1u_2$ and the constant 2, the prime ideals associated with \mathfrak{M} are given by $\mathfrak{p}_1 = (2, 1 - u_1)$ and $\mathfrak{p}_2 = (2, 1 - u_2)$. In particular, α is ergodic and expansive, but not mixing: the automorphisms $\alpha_{(1,0)}$ and $\alpha_{(0,1)}$ are non-ergodic, whereas $\alpha_{(1,1)}$ is ergodic.

(7) Let $\mathfrak{a} = (6 - 2u_1, 2 - 3u_1 - 5u_2^2)$. The prime ideals associated with $\mathfrak{M} = \mathfrak{R}_2/\mathfrak{a}$ are given by $\mathfrak{p}_1 = (3 - u_1, 7 + 5u_2^2)$, $\mathfrak{p}_2 = (3, 1 + u_2)$, $\mathfrak{p}_3 = (3, 1 - u_2)$, and the \mathbb{Z}^2 -action α is ergodic and expansive, but non-mixing. In this example $\alpha_{(0,1)}$ is non-ergodic (because of \mathfrak{p}_3), but $\alpha_{(1,0)}$ is ergodic.

(8) If $\mathfrak{a} = (1 + u_1 + u_1^2, 1 - u_2)$ then α is non-ergodic, since \mathfrak{a} is prime and contains $\{u^{3\mathbf{n}} - 1 : \mathbf{n} \in \mathbb{Z}^2\}$. \square

CONCLUDING REMARKS 6.19. (1) Most of the material in this section is taken from [94]. For Example 6.17 (2) we refer to [71].

(2) If $d \geq 2$, Corollary 6.10 is incorrect without the assumption that (X, α) satisfies the d.c.c.: indeed, let, for every $\mathbf{n} \in \mathbb{Z}^d$, $\mathfrak{N}_{\mathbf{n}} = \mathfrak{R}_d/(u^{\mathbf{n}} - 1)$. Then $\mathfrak{N}_{\mathbf{n}}$ is an \mathfrak{R}_d -module, and the \mathbb{Z}^d -action $\alpha^{\mathfrak{N}_{\mathbf{n}}}$ is ergodic by Corollary 6.11. We denote by $\mathfrak{M} = \sum_{\mathbf{n} \in \mathbb{Z}^d} \mathfrak{N}_{\mathbf{n}}$ the direct sum of the modules $\mathfrak{N}_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^d$, and write a typical element $a \in \mathfrak{M}$ as $a = (a_{\mathbf{n}})$ with $a_{\mathbf{n}} \in \mathfrak{N}_{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{Z}^d$. The \mathbb{Z}^d -action $\alpha = \alpha^{\mathfrak{M}}$ arising from the \mathfrak{R}_d -module \mathfrak{M} via Lemma 5.1 is ergodic by Lemma 1.2. However, $\alpha_{\mathbf{n}}$ is non-ergodic for every $\mathbf{n} \in \mathbb{Z}^d$: if $\mathbf{n} = \mathbf{0}$, this assertion is obvious, and if $\mathbf{n} \neq \mathbf{0}$, then the non-zero element $a(\mathbf{n}) \in \mathfrak{M}$ defined by

$$a(\mathbf{n})_{\mathbf{m}} = \begin{cases} 1 & \text{for } \mathbf{m} = \mathbf{n} \\ 0 & \text{for } \mathbf{m} \neq \mathbf{n} \end{cases}$$

satisfies that $u^{\mathbf{n}}a(\mathbf{n}) = a(\mathbf{n})$, and hence $\alpha_{\mathbf{n}}$ is non-ergodic by Lemma 1.2 (applied to the \mathbb{Z} -action $k \mapsto \alpha_{k\mathbf{n}}$).

(3) Let \mathfrak{M} be a countable \mathfrak{R}_d -module, and define $(X^{\mathfrak{M}}, \alpha^{\mathfrak{M}})$ by Lemma 5.1. For every $f = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) \in \mathfrak{R}_d$ we define a group homomorphism

$$\alpha_f^{\mathfrak{M}} = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) \alpha_{\mathbf{n}}^{\mathfrak{M}} : X^{\mathfrak{M}} \longrightarrow X^{\mathfrak{M}} \quad (6.14)$$

by setting

$$\alpha_f^{\mathfrak{M}}(x) = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) \alpha_{\mathbf{n}}^{\mathfrak{M}}(x)$$

for every $x \in X^{\mathfrak{M}}$, and note that $\alpha_f^{\mathfrak{M}}$ commutes with $\alpha^{\mathfrak{M}}$ (i.e. $\alpha_f^{\mathfrak{M}} \cdot \alpha_{\mathbf{n}}^{\mathfrak{M}} = \alpha_{\mathbf{n}}^{\mathfrak{M}} \cdot \alpha_f^{\mathfrak{M}}$ for every $\mathbf{n} \in \mathbb{Z}^d$), and that $\alpha_f^{\mathfrak{M}}$ is dual to the homomorphism

$$f_{\mathfrak{M}} : \mathfrak{M} \longrightarrow \mathfrak{M} \quad (6.15)$$

consisting of multiplication by f . In particular, $\alpha_f^{\mathfrak{M}}$ is surjective if and only if $f_{\mathfrak{M}}$ is injective, i.e. if and only if f does not lie in any prime ideal associated with \mathfrak{M} (cf. (6.4)). If $\mathfrak{M} = \mathfrak{R}_d/\mathfrak{a}$ for some ideal $\mathfrak{a} \subset \mathfrak{R}_d$, then (5.9) shows that

$$X^{\mathfrak{R}_d/\mathfrak{a}} = \{x \in \mathbb{T}^{\mathbb{Z}^d} = X^{\mathfrak{R}_d} : \alpha_f^{\mathfrak{R}_d}(x) = \mathbf{0}_X \text{ for every } f \in \mathfrak{a}\}, \quad (6.16)$$

and every α -commuting homomorphism $\psi : X^{\mathfrak{M}} \longrightarrow X^{\mathfrak{M}}$ is of the form $\psi = \alpha_f^{\mathfrak{M}}$ for some $f \in \mathfrak{R}_d$: indeed, if $\hat{\psi} : \mathfrak{R}_d/\mathfrak{a} \longrightarrow \mathfrak{R}_d/\mathfrak{a}$ is the homomorphism dual to ψ , then $\hat{\psi}(1) = f + \mathfrak{a}$ for some $f \in \mathfrak{R}_d$, and $\psi = \alpha_f^{\mathfrak{R}_d/\mathfrak{a}}$. For every ideal $\mathfrak{a} \subset \mathfrak{R}_d$ we set $\mathfrak{a}^{\perp} = \widehat{X^{\mathfrak{R}_d/\mathfrak{a}}} = \widehat{\mathfrak{R}_d/\mathfrak{a}} \subset \widehat{\mathfrak{R}_d} = \mathbb{T}^{\mathbb{Z}^d}$, and observe that $\alpha^{\mathfrak{R}_d/\mathfrak{a}}$ is the restriction of the shift-action σ of \mathbb{Z}^d on $\mathbb{T}^{\mathbb{Z}^d}$ to \mathfrak{a}^{\perp} . For every $f \in \mathfrak{R}_d$ the sequence

$$0 \longrightarrow (\mathfrak{a} + (f))^{\perp} \longrightarrow \mathfrak{a}^{\perp} \xrightarrow{\alpha_f^{\mathfrak{R}_d}} \mathfrak{b}^{\perp} \longrightarrow 0, \quad (6.17)$$

is exact, where

$$\mathfrak{b} = \{g \in \mathfrak{R}_d : fg \in \mathfrak{a}\}. \quad (6.18)$$

In particular, $\alpha_f^{\mathfrak{R}_d/\mathfrak{a}} : \mathfrak{a}^{\perp} \longrightarrow \mathfrak{a}^{\perp}$ is surjective if and only if $\mathfrak{a} = \mathfrak{b}$.

(4) Let $p > 1$ be a rational prime, and let α be a \mathbb{Z}^d -action by automorphisms of a compact, abelian group X with the property that $px = 0$ for every $x \in X$. If $\mathfrak{M} = \hat{X}$ is the \mathfrak{R}_d -module arising from lemma 5.1, then $pa = 0$ for every $a \in \mathfrak{M}$, so that \mathfrak{M} may be viewed as an $\mathfrak{R}_d^{(p)}$ -module. Conversely, suppose that \mathfrak{N} is a countable $\mathfrak{R}_d^{(p)}$ -module. Exactly as in (5.1)–(5.6) we can define a \mathbb{Z}^d -action $\alpha = \alpha^{\mathfrak{N}}$ on the dual group $X = X^{\mathfrak{N}} = \widehat{\mathfrak{N}}$ of \mathfrak{N} . Since $pa = 0$ for every $a \in \mathfrak{N}$, the group X is totally disconnected, and $x^p = \mathbf{1}_X$ for every $x \in X$. Since $\mathfrak{R}_d^{(p)}$ is a quotient ring of \mathfrak{R}_d , \mathfrak{N} is also an \mathfrak{R}_d -module, and we write \mathfrak{N}' instead of \mathfrak{N} if we wish to emphasize that \mathfrak{N} is viewed as an \mathfrak{R}_d -module. If \mathfrak{N} is Noetherian (either as an \mathfrak{R}_d -module or as an $\mathfrak{R}_d^{(p)}$ -module—the two conditions

are obviously equivalent), then we can realize $(X^{\mathfrak{N}}, \alpha^{\mathfrak{N}}) = (X^{\mathfrak{N}'}, \alpha^{\mathfrak{N}'})$ as the shift-action σ of \mathbb{Z}^d on a closed, shift-invariant subgroup $X \subset (\mathbb{T}^k)^{\mathbb{Z}^d}$ for some $k \geq 1$ (Example 5.2 (3)–(4)). Since $px = \mathbf{0}_X$ for every $x \in X$, we know that $x_{\mathbf{n}} \in (F_p)^k$ for every $\mathbf{n} \in \mathbb{Z}^d$, where $F_p = \{\frac{k}{p} : k = 0, \dots, p-1\} \subset \mathbb{T}$, and the obvious identification of F_p with the prime field \mathbb{F}_p allows us to regard X (and hence $X^{\mathfrak{N}}$) as a closed, shift-invariant subgroup of $(\mathbb{F}_p^k)^{\mathbb{Z}^d}$, and $\alpha^{\mathfrak{N}}$ as the shift-action on X .

In particular, if $\mathfrak{a} \subset \mathfrak{R}_d^{(p)}$ is an ideal, and if $\mathfrak{N} = \mathfrak{R}_d^{(p)}/\mathfrak{a}$, then we may regard $\alpha^{\mathfrak{N}} = \alpha^{\mathfrak{R}_d^{(p)}/\mathfrak{a}}$ as the shift-action of \mathbb{Z}^d on the subgroup

$$X^{\mathfrak{R}_d^{(p)}/\mathfrak{a}} = \left\{ x = (x_{\mathbf{m}}) \in \mathbb{F}_p^{\mathbb{Z}^d} : \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) x_{\mathbf{m}+\mathbf{n}} = \mathbf{0}_{\mathbb{F}_p} \text{ for all } f \in \mathfrak{a}, \mathbf{m} \in \mathbb{Z}^d \right\} \quad (6.19)$$

of $\mathbb{F}_p^{\mathbb{Z}^d}$. Conversely, if $X \subset \mathbb{F}_p^{\mathbb{Z}^d}$ is a closed, shift-invariant subgroup, then

$$X^\perp = \mathfrak{a} \subset \mathfrak{R}_d^{(p)} \cong \widehat{\mathbb{F}_p^{\mathbb{Z}^d}} \quad (6.20)$$

is an ideal, $X \cong X^{\mathfrak{R}_d^{(p)}/\mathfrak{a}}$, and the isomorphism between X and $X^{\mathfrak{R}_d^{(p)}/\mathfrak{a}}$ carries the shift-action σ of \mathbb{Z}^d on X to $\alpha^{\mathfrak{R}_d^{(p)}/\mathfrak{a}}$.

Every prime ideal $\mathfrak{p} \subset \mathfrak{R}_d^{(p)}$ associated with an $\mathfrak{R}_d^{(p)}$ -module \mathfrak{N} defines a prime ideal $\mathfrak{p}' = \{f \in \mathfrak{R}_d : f/p \in \mathfrak{p}\} \subset \mathfrak{R}_d$, and \mathfrak{p}' varies over the set of prime ideals in \mathfrak{R}_d associated with \mathfrak{N}' as \mathfrak{p} varies over the prime ideals in $\mathfrak{R}_d^{(p)}$ associated with \mathfrak{N} . As we have seen in Example 6.18 (5), the dynamical properties of $\alpha^{\mathfrak{N}'}$ expressed in terms of the associated primes $\mathfrak{p}' \subset \mathfrak{R}_d$ of \mathfrak{N}' have an analogous expression in terms of the prime ideals $\mathfrak{p} \subset \mathfrak{R}_d^{(p)}$ associated with \mathfrak{N} . In particular, $\alpha = \alpha^{\mathfrak{N}} = \alpha^{\mathfrak{N}'}$ is non-ergodic if and only if $V(\mathfrak{p})$ is finite for some prime ideal $\mathfrak{p} \subset \mathfrak{R}_d^{(p)}$ associated with \mathfrak{N} , and α is mixing if and only if no prime ideal $\mathfrak{p} \subset \mathfrak{R}_d^{(p)}$ associated with \mathfrak{N} contains a polynomial in a single variable $u^{\mathbf{n}}$, $\mathbf{0} \neq \mathbf{n} \in \mathbb{Z}^d$. Furthermore, if \mathfrak{N} is Noetherian, then $\text{Fix}_\Lambda(\alpha)$ is finite for every subgroup $\Lambda \subset \mathbb{Z}^d$ of finite index, and α is expansive.

The algebraic advantage in viewing an \mathfrak{R}_d -module \mathfrak{M} with $pa = 0$ for all $a \in \mathfrak{M}$ as an $\mathfrak{R}_d^{(p)}$ -module is that $\mathfrak{R}_d^{(p)}$ is a ring of polynomials with coefficients in the field \mathbb{F}_p , which simplifies the ideal structure of $\mathfrak{R}_d^{(p)}$ when compared with that of \mathfrak{R}_d . As far as the dynamics are concerned there is, of course, no difference between viewing \mathfrak{M} as a module over either of the rings \mathfrak{R}_d or $\mathfrak{R}_d^{(p)}$.

7. The dynamical system defined by a point

The results in Section 6 show that many questions about \mathbb{Z}^d -actions by automorphisms of compact, abelian groups can be reduced to questions about \mathbb{Z}^d -actions of the form $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$, where $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal. In this section we consider prime ideals of the form $\mathfrak{p} = \mathfrak{j}_c = \{f \in \mathfrak{R}_d : f(c) = 0\}$ with $c =$

$(c_1, \dots, c_d) \in (\overline{\mathbb{Q}}^\times)^d$. The groups $X^{\mathfrak{R}_d/\mathfrak{j}_c}$ arising from these ideals via Lemma 5.1 turn out to be connected and finite-dimensional (i.e. finite-dimensional tori or solenoids); conversely, if $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal such that $X^{\mathfrak{R}_d/\mathfrak{p}}$ is connected and finite-dimensional, then $\mathfrak{p} = \mathfrak{j}_c$ for some $c \in (\overline{\mathbb{Q}}^\times)^d$ (Corollary 7.4).

Let \mathbb{K} be an algebraic number field, i.e. a finite extension of \mathbb{Q} . A *valuation* of \mathbb{K} is a homomorphism $\phi: \mathbb{K} \rightarrow \mathbb{R}^+$ with the property that $\phi(a) = 0$ if and only if $a = 0$, $\phi(ab) = \phi(a)\phi(b)$, and $\phi(a+b) \leq c \cdot \max\{\phi(a), \phi(b)\}$ for all $a, b \in \mathbb{K}$ and some $c \in \mathbb{R}$ with $c \geq 1$. The valuation ϕ is *non-trivial* if $\phi(\mathbb{K}) \supsetneq \{0, 1\}$, *non-archimedean* if ϕ is non-trivial and we can set $c = 1$, and *archimedean* otherwise. Two valuations ϕ, ψ of \mathbb{K} are *equivalent* if there exists an $s > 0$ with $\phi(a) = \psi(a)^s$ for all $a \in \mathbb{K}$. An equivalence class v of non-trivial valuations of \mathbb{K} is called a *place* of \mathbb{K} , and v is *finite* if v contains a non-archimedean valuation, and *infinite* otherwise. If v is finite, all valuations $\phi \in v$ are non-archimedean.

Let v be a place of \mathbb{K} , and let $\phi \in v$ be a valuation. A sequence $(a_n, n \geq 1)$ is *Cauchy* with respect to ϕ if there exists, for every $\varepsilon > 0$, an integer $N \geq 1$ such that $\phi(a_m - a_n) < \varepsilon$ whenever $m, n \geq N$. It is clear that this definition does not depend on the valuation $\phi \in v$, so that we may call (a_n) a Cauchy sequence for v . Two Cauchy sequences (a_n) and (b_n) for v are *equivalent* if $\lim_{n \rightarrow \infty} \phi(a_n - b_n) = 0$, and this notion of equivalence again only depends on v and not on ϕ . With respect to the obvious operations the set of equivalence classes of Cauchy sequences for v is a field, denoted by \mathbb{K}_v , which contains \mathbb{K} as a dense subfield (every $a \in \mathbb{K}$ is identified with the equivalence class of the constant Cauchy sequence (a, a, a, \dots) in \mathbb{K}_v). The field \mathbb{K}_v is the *completion* of \mathbb{K} in the v -adic topology.

Ostrowski's Theorem (Theorem 2.2.1 in [16]) states that every non-trivial valuation ϕ of \mathbb{Q} is either equivalent to the absolute value (i.e. there exists a $t > 0$ with $\phi(a)^t = |a|$ for every $a \in \mathbb{Q}$), or to the p -adic valuation for some rational prime $p \geq 2$ (i.e. there exists a $t > 0$ such that $\phi(\frac{m}{n})^t = p^{(n'-m')} = |\frac{m}{n}|_p$ for all $\frac{m}{n} \in \mathbb{Q}$, where $m = p^{m'}m''$, $n = p^{n'}n''$, and neither m'' nor n'' are divisible by p). It is easy to see that the valuations $|\cdot|_\infty, |\cdot|_p, |\cdot|_q$ are mutually inequivalent whenever p, q are distinct rational primes, i.e. that the places of \mathbb{Q} are indexed by the set $\Pi \cup \{\infty\}$, where $\Pi \subset \mathbb{N}$ denotes the set of rational primes. The completion \mathbb{Q}_∞ of \mathbb{Q} is equal to \mathbb{R} , and for every rational prime p the completion \mathbb{Q}_p of \mathbb{Q} is the field of p -adic rationals.

For every valuation ϕ of \mathbb{K} , the restriction of ϕ to $\mathbb{Q} \subset \mathbb{K}$ is a valuation of \mathbb{Q} and is equivalent either to $|\cdot|_\infty$ or to $|\cdot|_p$ for some rational prime p . In the first case the place $v \ni \phi$ is infinite (or *lies above* ∞), and in the second case v *lies above* p (or p *lies below* v). We denote by w the place of \mathbb{Q} below v and observe that \mathbb{K}_v is a finite-dimensional vector space over the locally compact, metrizable field \mathbb{Q}_w and hence locally compact and metrizable in its own right. Choose a Haar measure λ_v on \mathbb{K}_v (with respect to addition), fix a compact set $C \subset \mathbb{K}_v$ with non-empty interior, and write $\text{mod}_{\mathbb{K}_v}(a) = \lambda_v(aC)/\lambda_v(C)$ for the *module* of an element $a \in \mathbb{K}_v$. The map $\text{mod}_{\mathbb{K}_v}: \mathbb{K} \rightarrow \mathbb{R}^+$ is continuous,

independent of the choice of λ_v , and its restriction to \mathbb{K} is a valuation in v which is denoted by $|\cdot|_v$.

Above every place v of \mathbb{Q} there are at least one and at most finitely many places of \mathbb{K} . Indeed, if $\mathbb{K} = \mathbb{Q}(a_1, \dots, a_n)$ with $\{a_1, \dots, a_n\} \subset \mathbb{Q}$, and if f is the minimal polynomial of a_1 over \mathbb{Q} , then f is irreducible over \mathbb{Q} , but f may be reducible over \mathbb{Q}_v ; we write $f = f_1 \cdot \dots \cdot f_k$ for the decomposition of f into irreducible factors over \mathbb{Q}_v and consider the field $\mathbb{Q}_v[x]/(f_i)$, where (f_i) denotes the principal ideal in the ring $\mathbb{Q}_v[x]$ generated by f_i . We define an injective field homomorphism $\zeta: \mathbb{K}^{(1)} = \mathbb{Q}_v(a_1) \hookrightarrow \mathbb{Q}_v[x]/(f_i)$ by setting $\zeta(a_1) = x$ and $\zeta(b) = b$ for every $b \in \mathbb{Q}_v$ and put $\phi_i(a) = \text{mod}_{\mathbb{Q}_v[x]/(f_i)}(\zeta(a))$ for every $a \in \mathbb{K}^{(1)}$. Then ϕ_i is a valuation of $\mathbb{K}^{(1)}$ whose place w_i lies above v . The places w_1, \dots, w_k are all distinct, and they are the only places of $\mathbb{K}^{(1)}$ above v (Theorem III.1 in [109]). In exactly the same way we find finitely many places of $\mathbb{K}^{(2)} = \mathbb{K}^{(1)}(a_2) = \mathbb{Q}(a_1, a_2)$ above each place of $\mathbb{K}^{(1)}$, and after n steps we obtain that there are at least one and at most finitely many places of \mathbb{K} above each place of \mathbb{Q} . A place v of \mathbb{K} is infinite if and only if it lies above ∞ ; in this case v is either *real* (if $\mathbb{K}_v = \mathbb{R}$) or *complex* (if $\mathbb{K}_v = \mathbb{C}$).

We write $P^\mathbb{K}$, $P_\mathfrak{f}^\mathbb{K}$, and $P_\infty^\mathbb{K}$, for the sets of places, finite places, and infinite places of \mathbb{K} . For every $v \in P^\mathbb{K}$, $\mathcal{R}_v = \{r \in \mathbb{K}_v : |r|_v \leq 1\}$ is a compact subset of \mathbb{K}_v . If $v \in P_\mathfrak{f}^\mathbb{K}$, then \mathcal{R}_v is, in addition, open, and is the unique maximal compact subring of \mathbb{K}_v ; furthermore there exists a *prime element* $\pi_v \in \mathcal{R}_v$ such that $\pi_v \mathcal{R}_v$ is the unique maximal ideal of \mathcal{R}_v . For every $v \in P_\mathfrak{f}^\mathbb{K}$ we set $\mathfrak{o}_v = \mathbb{K} \cap \mathcal{R}_v$, and we note that $\mathfrak{o}_\mathbb{K} = \bigcap_{v \in P_\mathfrak{f}^\mathbb{K}} \mathfrak{o}_v$ is the ring of integral elements in \mathbb{K} (Theorem V.1 in [109]). The set

$$\mathbb{K}_\mathbb{A} = \left\{ \omega = (\omega_v, v \in P^\mathbb{K}) \in \prod_{v \in P^\mathbb{K}} \mathbb{K}_v : \begin{aligned} &|\omega_v|_v \leq 1 \text{ for all but finitely many } v \in P^\mathbb{K} \end{aligned} \right\}, \quad (7.1)$$

furnished with that topology in which the subgroup

$$\begin{aligned} &\{\omega = (\omega_v, v \in P^\mathbb{K}) \in \mathbb{K}_\mathbb{A} : |\omega_v|_v \leq 1 \text{ for every } v \in P_\mathfrak{f}^\mathbb{K}\} \\ &\cong \prod_{v \in P_\infty^\mathbb{K}} \mathbb{K}_v \times \prod_{v \in P_\mathfrak{f}^\mathbb{K}} \mathcal{R}_v \end{aligned}$$

carries the product topology and is open in $\mathbb{K}_\mathbb{A}$, is the locally compact *adele ring* of \mathbb{K} . The diagonal embedding $i: \xi \mapsto (\xi, \xi, \dots)$ of \mathbb{K} in $\mathbb{A}_\mathbb{K}$ maps \mathbb{K} to a discrete, co-compact subring of $\mathbb{K}_\mathbb{A}$ (cf. [16], [109]).

We fix a non-trivial character $\chi \in i(\mathbb{K})^\perp \subset \widehat{\mathbb{K}_\mathbb{A}}$ and define, for every $a \in \mathbb{K}$, a character $\chi_a \in i(\mathbb{K})^\perp \subset \widehat{\mathbb{K}_\mathbb{A}}$ by setting

$$\chi_a(\omega) = \chi(i(a)\omega)$$

for every $\omega \in \mathbb{K}_{\mathbb{A}}$. By [16] or [109], the map $a \mapsto \chi_a$ is an isomorphism of the discrete, additive group \mathbb{K} onto $i(\mathbb{K})^\perp \subset \widehat{\mathbb{K}_{\mathbb{A}}}$. The resulting identification

$$\widehat{\mathbb{K}} \cong \mathbb{K}_{\mathbb{A}}/i(\mathbb{K}) \quad (7.2)$$

depends, of course, on the chosen character χ . In order to make the isomorphism (7.2) a little more canonical we consider, for every $w \in P_{\mathfrak{f}}^{\mathbb{K}}$, the subgroup

$$\Omega(\{w\})' = \{\omega = (\omega_v) \in \mathbb{K}_{\mathbb{A}} : \omega_v = 0 \text{ for every } v \neq w\} \cong \mathbb{K}_w$$

of $\mathbb{K}_{\mathbb{A}}$ and denote by $\chi^{(w)} \in \widehat{\mathbb{K}_w}$ the character induced by the restriction of χ to $\Omega(\{w\})'$. After replacing χ by a suitable χ_a , $a \in \mathbb{K}$, if necessary, we may assume that the induced characters $\chi^{(w)} \in \widehat{\mathbb{K}_w}$, $w \in P_{\mathfrak{f}}^{\mathbb{K}}$, satisfy that

$$\begin{aligned} \mathcal{R}_w \subset \ker(\chi^{(w)}) &= \{\omega \in \mathbb{A}_w : \chi^{(w)}(\omega) = 1\}, \\ \pi_w^{-1}\mathcal{R}_w &\not\subset \ker(\chi^{(w)}) \end{aligned} \quad (7.3)$$

for every $w \in P_{\mathfrak{f}}^{\mathbb{K}}$, where $\pi_w \in \mathcal{R}_w$ is the prime element appearing in the preceding paragraph (cf. [109]). With this choice of χ we have that

$$\chi \in (i(\mathbb{K}) + \Omega(P_{\mathfrak{f}}^{\mathbb{K}}))^\perp,$$

where

$$\Omega(P_{\mathfrak{f}}^{\mathbb{K}})' = \{\omega = (\omega_v) \in \mathbb{K}_{\mathbb{A}} : \omega_v = 0 \text{ for every } v \in P_{\infty}^{\mathbb{K}} = P^{\mathbb{K}} \setminus P_{\mathfrak{f}}^{\mathbb{K}}\}.$$

Now consider a finite subset $F \subset P^{\mathbb{K}}$ which contains $P_{\infty}^{\mathbb{K}}$, denote by

$$i_F: \mathbb{K} \longmapsto \prod_{v \in F} \mathbb{K}_v \quad (7.4)$$

the diagonal embedding $r \mapsto (r, \dots, r)$, $r \in \mathbb{K}$, put

$$R_F = \{a \in \mathbb{K} : |a|_v \leq 1 \text{ for every } v \notin F\}, \quad (7.5)$$

and observe that $i_F(R_F)$ is a discrete, additive subgroup of $\prod_{v \in F} \mathbb{K}_v$. If

$$\begin{aligned} \Omega &= \Omega(F) = \{\omega = (\omega_v) \in \mathbb{K}_{\mathbb{A}} : |\omega_v|_v \leq 1 \text{ for every } v \in P^{\mathbb{K}} \setminus F\}, \\ \Omega' &= \Omega(P^{\mathbb{K}} \setminus F)' = \{\omega = (\omega_v) \in \mathbb{K}_{\mathbb{A}} : \omega_v = 0 \text{ for every } v \in F\}, \\ \Omega'' &= \Omega \cap \Omega', \end{aligned}$$

then $i(\mathbb{K}) + \Omega'' = i(\mathbb{K}) + \Omega'$, and (7.3) implies that $\chi \in (i(\mathbb{K}) + \Omega'')^\perp = (i(\mathbb{K}) + \Omega')^\perp$ and

$$R_F = \{a \in \mathbb{K} : \chi_a \in (i(\mathbb{K}) + \Omega')^\perp\}.$$

Hence

$$\widehat{R_F} = \mathbb{K}_{\mathbb{A}}/(i(\mathbb{K}) + \Omega') \cong \left(\prod_{v \in F} \mathbb{K}_v \right) / i_F(R_F). \quad (7.6)$$

Let $d \geq 1$, $c = (c_1, \dots, c_d) \in (\overline{\mathbb{Q}}^\times)^d$, and $j_c = \{f \in \mathfrak{R}_d : f(c) = 0\}$. We wish to investigate the dynamical system $(X, \alpha) = (X^{\mathfrak{R}_d/j_c}, \alpha^{\mathfrak{R}_d/j_c})$ determined by c . Denote by $\mathbb{K} = \mathbb{Q}(c)$ the algebraic number field generated by $\{c_1, \dots, c_d\}$ and put

$$F(c) = \{v \in P_{\mathfrak{f}}^{\mathbb{K}} : |c_i|_v \neq 1 \text{ for some } i \in \{1, \dots, d\}\}, \quad (7.7)$$

which is finite by Theorem III.3 in [109], and

$$R_c = R_{P(c)}, \quad (7.8)$$

where $P(c) = P_{\infty}^{\mathbb{K}} \cup F(c)$. Then R_c is an \mathfrak{R}_d -module under the action $(f, a) \mapsto f(c)a$, and we define the \mathbb{Z}^d -action

$$\alpha^{(c)} = \alpha^{R_c} \quad (7.9)$$

on the compact group

$$Y^{(c)} = \widehat{R_c} = \left(\prod_{v \in P(c)} \mathbb{K}_v \right) / i_F(R_c) \quad (7.10)$$

by (5.5)–(5.6), where we use (7.6) to identify $\widehat{R_c}$ and $(\prod_{v \in P(c)} \mathbb{K}_v) / i_F(R_c)$.

THEOREM 7.1. *There exists a continuous, surjective, finite-to-one homomorphism $\phi: Y^{(c)} \mapsto X^{\mathfrak{R}_d/j_c}$ such that the diagram*

$$\begin{array}{ccc} Y^{(c)} & \xrightarrow{\alpha_{\mathbf{m}}^{(c)}} & Y^{(c)} \\ \phi \downarrow & & \downarrow \phi \\ X^{\mathfrak{R}_d/j_c} & \xrightarrow[\alpha_{\mathbf{m}}^{\mathfrak{R}_d/j_c}]{} & X^{\mathfrak{R}_d/j_c} \end{array} \quad (7.11)$$

commutes for every $\mathbf{m} \in \mathbb{Z}^d$.

PROOF. The evaluation map $\eta_c: f \mapsto f(c)$ induces an isomorphism η of the \mathfrak{R}_d -module \mathfrak{R}_d/j_c with the submodule $\eta_c(\mathfrak{R}_d) \subset R_c \subset \mathbb{K}$; in particular

$$\eta(\hat{\alpha}_{\mathbf{m}}^{\mathfrak{R}_d/j_c}(a)) = \hat{\alpha}_{\mathbf{m}}^{\eta_c(\mathfrak{R}_d)}(\eta(a)) = \hat{\alpha}_{\mathbf{m}}^{R_c}(\eta(a)) \quad (7.12)$$

for every $a \in \mathfrak{R}_d/j_c$ and $\mathbf{m} \in \mathbb{Z}^d$.

We claim that $R_c/\eta_c(\mathfrak{R}_d)$ is finite. Indeed, since $\mathbb{K} = \mathbb{Q}(c)$ is algebraic, every $a \in \mathbb{K}$ can be written as $a = b/m$ with $b \in \mathbb{Z}[c] = \mathbb{Z}[c_1, \dots, c_d]$ and $m \geq 1$. In particular, since the ring of integers $\mathfrak{o}(c) = \mathfrak{o}_{\mathbb{K}} \subset \mathbb{K}$ is a finitely generated \mathbb{Z} -module, there exist positive integers m_0, M_0 with $m_0\mathfrak{o}(c) \subset \mathbb{Z}[c] \subset \eta_c(\mathfrak{R}_d)$ and $|\mathbb{J}_c/\eta_c(\mathfrak{R}_d)| \leq |\mathfrak{o}(c)/m_0\mathfrak{o}(c)| = M_0 < \infty$.

According to the definition of $F(c)$ there exists, for every $v \in F(c)$, an element $a_v \in \eta_c(\mathfrak{R}_d)$ such that $|a_v|_v > 1$ and $|a_v|_w = 1$ for all $w \in P_{\mathfrak{f}}^{\mathbb{K}} \setminus F(c)$. Then $|a_v^n \mathfrak{o}(c)/\eta_c(\mathfrak{R}_d)| \leq M_0$ and $|(\sum_{v \in F(c)} a_v^n \mathfrak{o}(c))/\eta_c(\mathfrak{R}_d)| \leq M_0^{|F(c)|}$ for all

$n > 0$. As $n \rightarrow \infty$, $\sum_{v \in F(c)} a_v^n \mathfrak{o}(c)$ increases to R_c , and we conclude that $|R_c/\eta_c(\mathfrak{R}_d)| \leq M_0^{|F(c)|} < \infty$.

The inclusion map $\mathfrak{R}_d/\mathfrak{j}_c \cong \eta_c(\mathfrak{R}_d) \hookrightarrow R_c$ induces a dual, surjective, finite-to-one homomorphism $\phi: Y^{(c)} \twoheadrightarrow X = \widehat{\mathfrak{R}_d/\mathfrak{j}_c}$, and the diagram (7.11) commutes by (7.12). \square

This comparison between R_c and $\eta_c(\mathfrak{R}_d)$ shows that the \mathbb{Z}^d -actions $\alpha^{(c)}$ and $\alpha^{\mathfrak{R}_d/\mathfrak{j}_c}$ are closely related. The group R_c can be determined much more easily than $\eta_c(\mathfrak{R}_d)$ and has other advantages, e.g. for the computation of entropy in Section 7; on the other hand R_c may not be a cyclic \mathfrak{R}_d -module, in contrast to $\eta_c(\mathfrak{R}_d) \cong \mathfrak{R}_d/\mathfrak{j}_c$. Since R_c is torsion-free (as an additive group), $Y^{(c)}$ and $X^{\mathfrak{R}_d/\mathfrak{j}_c}$ are both connected.

PROPOSITION 7.2. *Let $d \geq 1$, $c = (c_1, \dots, c_d) \in (\overline{\mathbb{Q}}^\times)^d$, and let $(X^{\mathfrak{R}_d/\mathfrak{j}_c}, \alpha^{\mathfrak{R}_d/\mathfrak{j}_c})$ and $(Y^{(c)}, \alpha^{(c)})$ be defined as in Theorem 7.1.*

- (1) *For every $\mathbf{m} \in \mathbb{Z}^d$, the following conditions are equivalent.*
 - (a) $\alpha_{\mathbf{m}}^{(c)}$ *is ergodic;*
 - (b) $\alpha_{\mathbf{m}}^{\mathfrak{R}_d/\mathfrak{j}_c}$ *is ergodic;*
 - (c) $c^{\mathbf{m}}$ *is not a root of unity.*
- (2) *The following conditions are equivalent.*
 - (a) $\alpha^{(c)}$ *is ergodic;*
 - (b) $\alpha^{\mathfrak{R}_d/\mathfrak{j}_c}$ *is ergodic;*
 - (c) *At least one coordinate of c is not a root of unity.*
- (3) *The following conditions are equivalent.*
 - (a) $\alpha^{(c)}$ *is mixing;*
 - (b) $\alpha^{\mathfrak{R}_d/\mathfrak{j}_c}$ *is mixing;*
 - (c) $c^{\mathbf{m}} \neq 1$ *for all non-zero $\mathbf{m} \in \mathbb{Z}^d$.*
- (4) *If $\alpha^{(c)}$ is ergodic then the groups $\text{Fix}_\Lambda(\alpha^{(c)})$ and $\text{Fix}_\Lambda(\alpha^{\mathfrak{R}_d/\mathfrak{j}_c})$ are finite for every subgroup $\Lambda \subset \mathbb{Z}^d$ with finite index.*
- (5) *The following conditions are equivalent.*
 - (a) $\alpha^{(c)}$ *is expansive;*
 - (b) $\alpha^{\mathfrak{R}_d/\mathfrak{j}_c}$ *is expansive;*
 - (c) *The orbit of c under the diagonal action of the Galois group $\text{Gal}(\overline{\mathbb{Q}} : \mathbb{Q})$ on $(\overline{\mathbb{Q}}^\times)^d$ does not intersect \mathbb{S}^d .*

PROOF. The \mathfrak{R}_d -modules R_c and $\mathfrak{R}_d/\mathfrak{j}_c$ are both associated with the prime ideal \mathfrak{j}_c , $V_{\mathbb{C}}(\mathfrak{j}_c) = \text{Gal}[\overline{\mathbb{Q}} : \mathbb{Q}](c)$, and all assertions follow from Theorem 6.5. \square

PROPOSITION 7.3. *Let $N(c)$ be the cardinality of the orbit $\text{Gal}[\overline{\mathbb{Q}} : \mathbb{Q}](c)$ of c under the Galois group. Then $Y^{(c)} \cong \mathbb{T}^{N(c)}$ if and only if c_i is an algebraic unit for every $i = 1, \dots, d$ (i.e. c_i and c_i^{-1} are integral in $\mathbb{Q}(c)$ for $i = 1, \dots, d$). If at least one of the coordinates of c is not a unit, then $Y^{(c)}$ is a projective limit of copies of $\mathbb{T}^{N(c)}$.*

PROOF. We use the notation established in (7.1)–(7.8). The number $N(c)$ is equal to the degree $[\mathbb{Q}(c) : \mathbb{Q}]$. If $N_{\mathbb{R}}(c)$ and $N_{\mathbb{C}}(c)$ are the numbers of real and complex (infinite) places of $\mathbb{Q}(c)$ then $N(c) = N_{\mathbb{R}}(c) + 2N_{\mathbb{C}}(c)$, and the connected component of the identity in $\prod_{v \in P(c)} \mathbb{K}_v$ is isomorphic to $\mathbb{R}^{N(c)}$. The condition that every coordinate of c be a unit is equivalent to the assumption that $F(c) = \emptyset$; in this case $Y^{(c)}$ is isomorphic to the quotient of $\mathbb{R}^{N(c)}$ by the discrete, co-compact subgroup $i_{P(c)}(R_c)$, i.e. $Y^{(c)} \cong \mathbb{T}^{N(c)}$. If $F(c) \neq \emptyset$ then $Y^{(c)}$ is isomorphic to the quotient of $\mathbb{R}^{N(c)} \times \prod_{v \in F(c)} \mathbb{K}_v$ by $i_{P(c)}(R_c)$. In order to prove the assertion about the projective limit we choose, for every $v \in F(c)$, a prime element $p_v \in \mathbb{K}_v$ (i.e. an element with $p_v \mathcal{R}_v = \{a \in \mathbb{K}_v : |a|_v < 1\}$), and set $\Delta_n = i_{P(c)}(R_c) + \prod_{v \in F(c)} p_v^n \mathcal{R}_v$ for every $n \geq 1$. Then $\bigcap_{n \geq 1} \Delta_n = i_{P(c)}(R_c)$, and $Y^{(c)}$ is the projective limit of the groups $Y_n = Y^{(c)} / \Delta_n \cong \mathbb{T}^{N(c)}$, $n \geq 1$, where the last isomorphism is established by meditation. \square

If X is a compact, connected, abelian group with dual group \hat{X} , then \hat{X} is torsion-free, and the map $a \mapsto 1 \otimes a$ from \hat{X} into the tensor product $\mathbb{Q} \otimes_{\mathbb{Z}} \hat{X}$ is therefore injective. We denote by $\dim X$ the dimension of the vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \hat{X}$ over \mathbb{Q} and note that this definition of $\dim X$ is consistent with the usual topological dimension of X : in particular, $0 < \dim Y^{(c)} = N(c) < \infty$ in Proposition 7.3. With this terminology we obtain the following corollary of Theorem 7.1 and Proposition 7.3.

COROLLARY 7.4. *Let $\mathfrak{p} \subset \mathfrak{R}_d$ be a prime ideal, and let $(X^{\mathfrak{R}_d/\mathfrak{p}}, \alpha^{\mathfrak{R}_d/\mathfrak{p}})$ be defined as in Lemma 5.1. The following conditions are equivalent.*

- (1) $X^{\mathfrak{R}_d/\mathfrak{p}}$ is a connected, finite-dimensional, abelian group;
- (2) $\mathfrak{p} = \mathfrak{j}_c$ for some $c \in (\overline{\mathbb{Q}}^\times)^d$.

Furthermore, if α is an ergodic \mathbb{Z}^d -action by automorphisms of a compact, connected, finite-dimensional abelian group X , then the \mathfrak{R}_d -module $\mathfrak{M} = \hat{X}$ has only finitely many associated prime ideals, each of which is of the form $\mathfrak{p} = \mathfrak{j}_c$ for some $c \in (\overline{\mathbb{Q}}^\times)^d$.

PROOF. The implication (2) \Rightarrow (1) is clear from Theorem 7.1, Proposition 7.3, and the definition of $\dim X$. Conversely, if $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal such that $X^{\mathfrak{R}_d/\mathfrak{p}} = \widehat{\mathfrak{R}_d/\mathfrak{p}}$ is connected, then \mathfrak{p} does not contain any non-zero constants, and the map $a \mapsto 1 \otimes a$ from $\mathfrak{R}_d/\mathfrak{p}$ into the tensor product $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathfrak{R}_d/\mathfrak{p})$ is injective. This allows us to regard $\mathfrak{R}_d/\mathfrak{p}$ as a subring of $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathfrak{R}_d/\mathfrak{p})$. The variety $V(\mathfrak{p})$ is non-empty by Proposition 6.9, and is finite if and only if each of the elements $u_i + \mathfrak{p} \in \mathbb{Q} \otimes_{\mathbb{Z}} (\mathfrak{R}_d/\mathfrak{p})$, $i = 1, \dots, d$, is algebraic over the subring $\mathbb{Q} \subset \mathbb{Q} \otimes_{\mathbb{Z}} (\mathfrak{R}_d/\mathfrak{p})$. In particular, if $V(\mathfrak{p})$ is finite, then $\mathfrak{p} = \mathfrak{j}_c$ for every $c \in V(\mathfrak{p})$, which implies (2). If $V(\mathfrak{p})$ is infinite, then at least one of the elements $u_j + \mathfrak{p}$ is transcendental over $\mathbb{Q} \subset \mathbb{Q} \otimes_{\mathbb{Z}} (\mathfrak{R}_d/\mathfrak{p})$, and the powers $u_j^k + \mathfrak{p}$, $k \in \mathbb{Z}$, are rationally independent. This is easily seen to imply that $\dim X^{\mathfrak{R}_d/\mathfrak{p}} = \infty$.

In order to prove the last assertion we assume that $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal associated with \mathfrak{M} . Then $X^{\mathfrak{R}_d/\mathfrak{p}}$ is (isomorphic to) a quotient group of

X , hence connected and finite-dimensional, and Proposition 6.9 and the first part of this corollary together imply that $\mathfrak{p} = \mathfrak{j}_c$ for some $c \in (\overline{\mathbb{Q}}^\times)^d$. If \mathfrak{M} has infinitely many distinct associated prime ideals $\{\mathfrak{j}_{c(1)}, \mathfrak{j}_{c(2)}, \dots\}$, then we can find, for every $i \geq 1$, an element $a_i \in \mathfrak{M}$ with $\mathfrak{R}_d \cdot a_i \cong \mathfrak{R}_d/\mathfrak{j}_{c(i)}$. If $b \in (\sum_{i=1}^{j-1} \mathfrak{R}_d \cdot a_i) \cap \mathfrak{R}_d \cdot a_j \neq \{0\}$ for some $j > 1$, then the submodule $\mathfrak{R}_d \cdot b \subset \mathfrak{M}$ has an associated prime ideal \mathfrak{j} which strictly contains $\mathfrak{j}_{c(j)}$; in particular, \mathfrak{j} must contain a non-zero constant, in violation of the fact that every prime ideal \mathfrak{p} associated with $\mathfrak{R}_d \cdot b$ (and hence with \mathfrak{M}) must satisfy that $V_{\mathbb{C}}(\mathfrak{p}) \neq \emptyset$. It follows that \mathfrak{M} has a submodule isomorphic to $\mathfrak{R}_d/\mathfrak{j}_{c(1)} \oplus \mathfrak{R}_d/\mathfrak{j}_{c(2)} \oplus \dots$, and hence that $\dim X = \infty$. This contradiction proves that there are only finitely many distinct prime ideals associated with \mathfrak{M} . \square

EXAMPLE 7.5. If α is a \mathbb{Z}^d -action by automorphisms of a compact, connected, finite-dimensional, abelian group, then the \mathfrak{R}_d -module $\mathfrak{M} = \hat{X}$ need not be Noetherian (cf. Corollary 7.4): if α is the automorphism of $X = \hat{\mathbb{Q}}$ in Example 5.6 (1) consisting of multiplication by $\frac{3}{2}$, then $\dim(X) = 1$, but $\mathfrak{M} = \hat{X} = \mathbb{Q}$ is not Noetherian (cf. Example 6.17 (2)). \square

The following Examples 7.6 show that the \mathbb{Z}^d -actions $\alpha^{(c)}$ and $\alpha^{\mathfrak{R}/\mathfrak{j}_c}$ may be, but need not be, topologically conjugate.

EXAMPLES 7.6. (1) If $c = 2$ then $F(c) = \{2\}$, $R_c = \mathbb{Z}[\frac{1}{2}]$, and we claim that the automorphism $\alpha_1^{(c)}$ on $Y^{(c)} = \widehat{R}_c = (\mathbb{R} \times \mathbb{Q}_2)/i_{F(c)}(\mathbb{Z}[\frac{1}{2}])$, which is multiplication by 2, is conjugate to the shift $\alpha_1^{\mathfrak{R}_1/(2-u_1)}$ on the group $X^{\mathfrak{R}_1/(2-u_1)}$ described in Example 5.3 (3). In order to verify this we note that there exists, for every $(s, t) \in \mathbb{R} \times \mathbb{Q}_2$, a unique element $r \in \mathbb{Z}[\frac{1}{2}]$ with $r + s \in [0, 1)$ and $r + t \in \mathbb{Z}_2$. This allows us to identify $Y^{(c)} = \widehat{\mathbb{Z}[\frac{1}{2}]}$ with $(\mathbb{R} \times \mathbb{Z}_2)/i_{F(c)}(\mathbb{Z})$. An element $a = \frac{k}{2^r} \in \mathbb{Z}[\frac{1}{2}]$ defines a character on $Y^{(c)} = (\mathbb{R} \times \mathbb{Z}_2)/i_{F(c)}(\mathbb{Z})$ by $\langle a, (s, t) + i_{F(c)}(\mathbb{Z}) \rangle = e^{2\pi i(\text{Int}(as) + \text{Frac}(at))}$ for every $s \in \mathbb{R}$ and $t \in \mathbb{Z}_2$, where $\text{Int}(as)$ is the integral part of $as \in \mathbb{R}$ and $\text{Frac}(at) \in [0, 1)$ is the (well-defined) fractional part of $at \in \mathbb{Q}_2$. Consider the homomorphism $\phi: Y^{(c)} \rightarrow \mathbb{T}^{\mathbb{Z}}$ defined by $e^{2\pi i(\phi(y))_m} = \langle 2^m, y \rangle$ for every $y \in Y^{(c)}$ and $m \in \mathbb{Z}$. Then ϕ is injective, $\phi(Y^{(c)}) \subset X^{\mathfrak{R}_1/(2-u_1)}$, and it is not difficult to see that $\phi: Y^{(c)} \rightarrow X^{\mathfrak{R}_1/(2-u_1)}$ is a continuous group isomorphism which makes the diagram (7.11) commute. In particular, if we write a typical element $y \in Y^{(c)}$ as $y = (s, t) + i_{F(c)}(\mathbb{Z})$ with $s \in \mathbb{R}$ and $t \in \mathbb{Z}_2$, then

$$(\phi((0, t) + i_{F(c)}(\mathbb{Z})))_m = 0 \quad \text{and} \quad (\phi((s, 0) + i_{F(c)}(\mathbb{Z})))_m = 2^m s \pmod{1}$$

for every $m \geq 0$.

Proposition 7.2 shows that the automorphism $\alpha^{(c)} = \alpha^{\mathfrak{R}_1/\mathfrak{j}_c}$ is expansive and hence ergodic.

(2) If $c = 3/2$ then $F(c) = \{2, 3\}$, $R_c = \mathbb{Z}[\frac{1}{6}]$, and we see as in Example (1) that multiplication by $\frac{3}{2}$ on

$$Y^{(c)} = \widehat{R_c} = (\mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3)/i_{F(c)}(\mathbb{Z}[\frac{1}{6}]) \cong (\mathbb{R} \times \mathbb{Z}_2 \times \mathbb{Z}_3)/i_{F(c)}(\mathbb{Z})$$

is conjugate to the shift $\alpha^{\mathfrak{R}_1/\mathfrak{j}_c}$ on $X^{\mathfrak{R}_1/\mathfrak{j}_c}$ in Example 6.18 (2). The \mathbb{Z} -action $\alpha^{(c)} = X^{\mathfrak{R}_1/\mathfrak{j}_c}$ is expansive and ergodic by Proposition 7.2.

(3) Let $c = 2 + \sqrt{5}$. Then $\eta_c(\mathfrak{R}_1) = \{k + l\sqrt{5} : k, l \in \mathbb{Z}\} \cong \mathbb{Z}^2$, $F(c) = \emptyset$, and R_c is equal to the set $\mathfrak{o}(c) = \mathfrak{o}_{\mathbb{Q}(c)}$ of integral elements in $\mathbb{Q}(c)$. Since $\mathfrak{o}_{\mathbb{Q}(c)} = \{k\frac{1+\sqrt{5}}{2} + l\frac{1-\sqrt{5}}{2} : k, l \in \mathbb{Z}\}$ (cf. Lemma 10.3.3 in [16]), $R_c \neq \eta_c(\mathfrak{R}_1)$. By Proposition 7.2, the \mathbb{Z} -actions $\alpha^{(c)}$ and $\alpha^{\mathfrak{R}_1/\mathfrak{j}_c}$ are both expansive (and hence ergodic), but we claim that they are not topologically conjugate. According to Corollary 5.10 this amounts to showing that R_c and $\mathfrak{R}_1/\mathfrak{j}_c$ are not isomorphic as \mathfrak{R}_1 -modules, and we establish this by showing that R_c is not cyclic. In terms of the \mathbb{Z} -basis $\{\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\}$ for R_c , multiplication by c is represented by the matrix $A = \begin{pmatrix} 5 & -2 \\ 2 & -1 \end{pmatrix}$. If the module R_c is cyclic, then there exists a vector $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$ such that $\{\mathbf{m}, A\mathbf{m}\} = \{(m_1, m_2), (5m_1 - 2m_2, 2m_1 - m_2)\}$ generates \mathbb{Z}^2 , and as in Example 5.3 (2) we see that this is impossible.

In this example $X^{\mathfrak{R}_1/\mathfrak{j}_c} \cong Y^{(c)} \cong \mathbb{T}^2$. The matrix $A' = \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}$ represents multiplication by c in terms of the \mathbb{Z} -basis $\{1, \sqrt{5}\}$ of $\eta_c(\mathfrak{R}_1)$, and the matrices A and A' define non-conjugate automorphisms of \mathbb{T}^2 with identical characteristic polynomials (cf. Example 5.3 (2)).

(4) Let $c = \frac{1+\sqrt{5}}{2}$. Then $\eta_c(\mathfrak{R}_1) = \mathfrak{o}(\mathbb{Q}(c)) = R_c$, and the \mathbb{Z} -actions $\alpha^{(c)}$ and $\alpha^{\mathfrak{R}_1/\mathfrak{j}_c}$ are algebraically conjugate. However, a little care is needed in identifying $\widehat{R_c}$ with $Y^{(c)}$ in (7.10). The set $P(c) = P_{\infty}^{\mathbb{Q}(c)}$ consists of the two real places determined by the embeddings $\sqrt{5} \mapsto \sqrt{5}$ and $\sqrt{5} \mapsto -\sqrt{5}$ of $\mathbb{Q}(c) = \mathbb{Q}(\sqrt{5})$ in \mathbb{R} , so that $Y^{(c)} = \mathbb{R}^2/i_{P(c)}(R_c)$ with $i_{P(c)}(R_c) = \{(k + l\frac{1+\sqrt{5}}{2}, k + l\frac{1-\sqrt{5}}{2}) : (k, l) \in \mathbb{Z}^2\} \subset \mathbb{R}^2$. Under the usual identification of $\widehat{\mathbb{R}^2}$ with \mathbb{R}^2 given by $\langle (t_1, t_2), (s_1, s_2) \rangle = e^{2\pi i(s_1 t_1 + s_2 t_2)}$ for every $(s_1, s_2), (t_1, t_2) \in \mathbb{R}^2$, the annihilator $i_{P(c)}(R_c)^\perp \subset \widehat{\mathbb{R}^2} = \mathbb{R}^2$ is of the form $i_{P(c)}(R_c)^\perp = \frac{1}{\sqrt{5}} \cdot i_{P(c)}(R_c)$, and

$$\widehat{Y^{(c)}} = i_{P(c)}(R_c)^\perp = \frac{1}{\sqrt{5}} \cdot i_{P(c)}(R_c) = i_{P(c)}\left(\frac{1}{\sqrt{5}} \cdot R_c\right) \cong \frac{1}{\sqrt{5}} \cdot R_c \cong R_c.$$

(5) Let $\omega = (-1 + \sqrt{-3})/2$ and $c = 1 + 3\omega \in \overline{\mathbb{Q}}$. Then $\mathbb{K} = \mathbb{Q}(\omega)$ and $F(c) = \{7\}$. We claim that $R_c \neq \eta_c(\mathfrak{R}_1)$. Indeed, since the minimal polynomial $f(u) = u^2 + u + 1$ of ω is irreducible over the field \mathbb{Q}_3 of triadic rationals, there exists a unique place v of \mathbb{K} above 3, and $\mathbb{K}_v = \mathbb{Q}_3(\omega)$. Let $\mathcal{R}_v = \{a \in \mathbb{K}_v : |a|_v \leq 1\}$ and $\mathfrak{o}_v = \mathbb{K} \cap \mathcal{R}_v$. As $|3|_v = 1/9$, every $a \in S = \mathbb{Z} + 3\mathfrak{o}_v \subset \mathfrak{o}_v$ with $|a|_v < 1$ satisfies that $|a|_v \leq 3^{-2}$. In particular, $\zeta = 1 - \omega \in \mathfrak{o}_v \setminus S$, since $\zeta^2 = (1 - \omega)^2 = -3\omega$ and hence $|\zeta|_v = 1/3$ (cf. p.139 in [16]). Since $\eta_c(\mathfrak{R}_1) \subset S$ and $\zeta \in \mathfrak{o}(c) \subset R_c$ we conclude that $\zeta \in R_c \setminus \eta_c(\mathfrak{R}_1) \neq \emptyset$.

In order to verify that $\eta_c(\mathfrak{R}_1) \cong \mathfrak{R}_1/j_c$ and R_c are non-isomorphic we take an arbitrary, non-zero element $a \in R_c$ and note that

$$\begin{aligned} \{|b|_v : b \in \eta_c(\mathfrak{R}_1) \cdot a\} &= \{|f(c)|_v |a|_v : f \in \mathfrak{R}_1\} \subset \{|a|_v |b|_v : b \in S\} \\ &\subsetneq \{3^{-n} : n \geq 0\} = \{|b|_v : b \in R_c\}. \end{aligned}$$

Hence R_c is not cyclic, in contrast to \mathfrak{R}_1/j_c . Corollary 5.10 shows that the \mathbb{Z}^d -actions $\alpha^{(c)}$ and $\alpha^{\mathfrak{R}_1/j_c}$ are not topologically conjugate. In this example the isomorphic groups $Y^{(c)}$ and $X^{\mathfrak{R}_1/j_c}$ are projective limits of two-dimensional tori, and the automorphisms $\alpha^{(c)}$ and $\alpha^{\mathfrak{R}_1/j_c}$ are expansive (and ergodic) by Proposition 7.2. \square

EXAMPLES 7.7. (1) Let $c = (2, 3) \in (\overline{\mathbb{Q}}^\times)^2$. Then $j_c = (u_1 - 2, u_2 - 3) \subset \mathfrak{R}_2$, $F(c) = \{2, 3\}$, $R_c = \mathbb{Z}[\frac{1}{6}]$, and as in Example 7.6 (1) one sees that the \mathbb{Z}^2 -action $\alpha^{(c)}$ on $Y^{(c)}$ is conjugate to shift-action $\alpha^{\mathfrak{R}_2/j_c}$ on the group $X^{\mathfrak{R}_2/j_c}$ appearing in in Example 5.3 (4). Note that $\alpha^{\mathfrak{R}_2/j_c}$ is expansive and mixing; in fact, $\alpha_{\mathbf{n}}^{\mathfrak{R}_2/j_c}$ is expansive for every non-zero $\mathbf{n} \in \mathbb{Z}^2$ (Proposition 7.2). The group $Y^{(c)} = (\mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3)/i_{F(c)}(\mathbb{Z}[\frac{1}{6}]) \cong (\mathbb{R} \times \mathbb{Z}_2 \times \mathbb{Z}_3)/i_{F(c)}(\mathbb{Z})$ is the same as in Example 7.6 (2), but $X^{\mathfrak{R}_2/j_c}$ is now a closed, shift-invariant subgroup of $\mathbb{T}^{\mathbb{Z}^2}$. In order to describe an explicit isomorphism $\phi: Y^{(c)} \rightarrow X^{\mathfrak{R}_2/j_c}$ we proceed as in Example 7.6 (1): identify $Y^{(c)}$ with $(\mathbb{R} \times \mathbb{Z}_2 \times \mathbb{Z}_3)/i_{F(c)}(\mathbb{Z})$, and write the character of $Y^{(c)}$ defined by an element $a = \frac{j}{2^k 3^l} \in \mathbb{Z}[\frac{1}{6}]$ as $\langle a, (r, s, t) + i_{F(c)}(\mathbb{Z}) \rangle = e^{2\pi i(\text{Int}(ar) + \text{Frac}(as) + \text{Frac}(at))}$ for every $r \in \mathbb{R}, s \in \mathbb{Z}_2$ and $t \in \mathbb{Z}_3$. If $\phi: Y^{(c)} \rightarrow \mathbb{T}^{\mathbb{Z}^2}$ is the map given by $e^{2\pi i(\phi(y))_{(n_1, n_2)}} = \langle 2^{n_1} 3^{n_2}, y \rangle$ for every $y \in Y$ and $(n_1, n_2) \in \mathbb{Z}^2$, then ϕ is injective, $\phi(Y^{(c)}) = X^{\mathfrak{R}_2/j_c}$, and ϕ makes the diagram (7.11) commute.

(2) Let $\mathbb{K} \supset \mathbb{Q}$ be an algebraic number field. We denote by $\mathfrak{o}_{\mathbb{K}} \subset \mathbb{K}$ the ring of integers and write $\mathcal{U}_{\mathbb{K}} \subset \mathfrak{o}_{\mathbb{K}}$ for the group of units (i.e. $\mathcal{U}_{\mathbb{K}} = \{a \in \mathfrak{o}_{\mathbb{K}} : a^{-1} \in \mathfrak{o}_{\mathbb{K}}\}$). By Theorem 10.8.1 in [16], $\mathcal{U}_{\mathbb{K}}$ is isomorphic to the cartesian product $F \times \mathbb{Z}^{r+s-1}$, where F is a finite, cyclic group consisting of all roots of unity in \mathbb{K} and r and s are the numbers of real and complex places of \mathbb{K} . We set $d = r + s - 1$, choose generators $c_1, \dots, c_d \in \mathcal{U}_{\mathbb{K}}$ such that every $a \in \mathcal{U}_{\mathbb{K}}$ can be written as $a = uc_1^{k_1} \dots c_d^{k_d}$ with $u \in F$ and $k_1, \dots, k_d \in \mathbb{Z}$, and set $c = (c_1, \dots, c_d)$. Then $X^{\mathfrak{R}_d/j_c} \cong Y^{(c)} \cong \mathbb{T}^{r+2s}$, and the \mathbb{Z}^d -actions $\alpha^{\mathfrak{R}_d/j_c}$ and $\alpha^{(c)}$ are mixing by Proposition 7.2.

(3) Let $d \geq 1$, and let $\mathfrak{a} \subset \mathfrak{R}_d$ be an ideal with $V(\mathfrak{a}) \neq \emptyset$ (or, equivalently, with $V_{\mathbb{C}}(\mathfrak{a}) \neq \emptyset$). For every $c \in V(\mathfrak{a})$ the evaluation map $\eta_c: f \mapsto f(c)$ from $\mathfrak{R}_d/\mathfrak{a}$ to $\mathbb{Q}(c)$ induces a dual, injective embedding of $X^{\mathfrak{R}_d/j_c}$ in $X^{\mathfrak{R}_d/\mathfrak{a}}$, so that we may regard $X^{\mathfrak{R}_d/j_c}$ as a subgroup of $X^{\mathfrak{R}_d/\mathfrak{a}}$; in this picture $\alpha^{\mathfrak{R}_d/j_c}$ is the restriction of $\alpha^{\mathfrak{R}_d/\mathfrak{a}}$ to $X^{\mathfrak{R}_d/j_c}$. In fact, if \mathfrak{a} is *radical*, i.e. if $\mathfrak{a} = \sqrt{\mathfrak{a}} = \{f \in \mathfrak{R}_d : f^k \in \mathfrak{a} \text{ for some } k \geq 1\}$, then $\mathfrak{a} = \{f \in \mathfrak{R}_d : f(c) = 0 \text{ for every } c \in V(\mathfrak{a})\}$, and the group generated by $X^{\mathfrak{R}_d/j_c}$, $c \in V_{\mathbb{C}}(\mathfrak{a})$, is dense in $X^{\mathfrak{R}_d/\mathfrak{a}}$. In general, $\alpha^{\mathfrak{R}_d/\mathfrak{a}}$ is expansive if and only if $\alpha^{\mathfrak{R}_d/j_c}$ is expansive for every $c \in V(\mathfrak{a})$, but

$\alpha^{\mathfrak{R}_d/\mathfrak{a}}$ may be mixing in spite of $\alpha^{\mathfrak{R}_d/j_c}$ being non-ergodic for some $c \in V(\mathfrak{a})$: take, for example, $d = 2$, $\mathfrak{a} = (1 + u_1 + u_2) \subset \mathfrak{R}_2$, and $c = ((-1 + i\sqrt{-3})/2, (-1 - i\sqrt{-3})/2) \in V(\mathfrak{a})$ (Theorem 6.5, Proposition 7.2, and Example 6.18 (1)). \square

CONCLUDING REMARK 7.8. Theorem 7.1, Proposition 7.2, and Example 7.6 (5) are taken from [94], and Example 7.6 (4) was pointed out to me by Jenkner. The possible difference between $\alpha^{(c)}$ and $\alpha^{\mathfrak{R}_d/j_c}$ for $c \in (\overline{\mathbb{Q}}^\times)^d$ allows the construction of analogues to Williams' Example 5.3 (2) for \mathbb{Z}^d -actions.

8. The dynamical system defined by a prime ideal

In this section we continue our investigation of the structure of the \mathbb{Z}^d -actions $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$, where $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal. For prime ideals of the form j_c , $c \in (\overline{\mathbb{Q}}^\times)^d$, the work was done in Section 7, and for $\mathfrak{p} = \{0\}$ we already know that $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is the shift-action of \mathbb{Z}^d on $X^{\mathfrak{R}_d/\mathfrak{p}} = \mathbb{T}^{\mathbb{Z}^d}$. Another case which can be dealt with easily are the non-ergodic prime ideals (Definition 6.16).

PROPOSITION 8.1. *Let $\mathfrak{p} \subset \mathfrak{R}_d$ be a prime ideal. Then \mathfrak{p} is non-ergodic if and only if \mathfrak{p} is either maximal, or of the form j_c for a point $c = (c_1, \dots, c_d) \in \overline{\mathbb{Q}}^d$ with $c_1^l = \dots = c_d^l = 1$ for some $l \geq 1$. Furthermore, if $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is non-ergodic, then $X^{\mathfrak{R}_d/\mathfrak{p}}$ is either finite or a finite-dimensional torus, and there exists an integer $L \geq 1$ such that $\alpha_{L\mathbf{n}}^{\mathfrak{R}_d/\mathfrak{p}} = \text{id}_{X^{\mathfrak{R}_d/\mathfrak{p}}}$ for every $\mathbf{n} \in \mathbb{Z}^d$.*

PROOF. This is just a re-wording of Theorem 6.5 (1). An ideal $\mathfrak{p} \subset \mathfrak{R}_d$ is maximal if and only if $\mathfrak{R}_d/\mathfrak{p}$ is a finite field; in particular, the characteristic $p(\mathfrak{p})$ is positive for any maximal ideal \mathfrak{p} .

Let $\mathfrak{p} \subset \mathfrak{R}_d$ be a prime ideal such that $\alpha = \alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is non-ergodic. If $p = p(\mathfrak{p}) > 0$, then Theorem 6.5 (1.e) implies that $V(\mathfrak{p}) \subset (\overline{\mathbb{F}}_{p(\mathfrak{p})}^\times)^d$ is finite and that \mathfrak{p} is therefore maximal. In particular, $\mathfrak{R}_d/\mathfrak{p} \cong \mathbb{F}_{p^l}$ for some $l \geq 1$, where \mathbb{F}_{p^l} is the finite field with p^l elements, and $\alpha_{(p^l-1)\mathbf{n}}$ is the identity map on $X^{\mathfrak{R}_d/\mathfrak{p}} \cong \mathbb{F}_{p^l}$ for every $\mathbf{n} \in \mathbb{Z}^d$. Conversely, if \mathfrak{p} is maximal, then $|X^{\mathfrak{R}_d/\mathfrak{p}}| = |\mathfrak{R}_d/\mathfrak{p}|$ is finite, and α is therefore non-ergodic.

If $p(\mathfrak{p}) = 0$, then Theorem 6.5 (1.e) guarantees the existence of an integer $l \geq 1$ with $c_1^l = \dots = c_d^l = 1$ for every $c = (c_1, \dots, c_d) \in V(\mathfrak{p}) = V_{\mathbb{C}}(\mathfrak{p})$, so that $V(\mathfrak{p})$ is finite, and the primality of \mathfrak{p} allows us to conclude that $\mathfrak{p} = j_c$ for some $c = (c_1, \dots, c_d) \in \overline{\mathbb{Q}}^d$ with $c_1^l = \dots = c_d^l = 1$. From the definition of $\alpha^{(c)}$ in (7.9)–(7.10), Theorem 7.1, and Proposition 7.3, it is clear that $X^{\mathfrak{R}_d/\mathfrak{p}}$ is a finite-dimensional torus, and that $\alpha_{L\mathbf{n}}$ is the identity map on $X^{\mathfrak{R}_d/\mathfrak{p}}$ for every $\mathbf{n} \in \mathbb{Z}^d$. Conversely, if $\mathfrak{p} = j_c$ for some $c = (c_1, \dots, c_d) \in \overline{\mathbb{Q}}^d$ with $c_1^l = \dots = c_d^l = 1$, then Theorem 6.5 (1.e) shows that α is non-ergodic. \square

Next we consider ergodic prime ideals $\mathfrak{p} \subset \mathfrak{R}_d$ with $p(\mathfrak{p}) > 0$. We call a subgroup $\Gamma \subset \mathbb{Z}^d$ *primitive* if \mathbb{Z}^d/Γ is torsion-free; a non-zero element $\mathbf{n} \in \mathbb{Z}^d$ is *primitive* if the subgroup $\{k\mathbf{n} : k \in \mathbb{Z}\} \subset \mathbb{Z}^d$ is primitive. The following proposition shows that there exists, for every ergodic prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ with

$p(\mathfrak{p}) > 0$, a maximal primitive subgroup $\Gamma \subset \mathbb{Z}^d$ and a finite, abelian group G such that the restriction α^Γ of $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ to Γ is topologically and algebraically conjugate to the shift-action of Γ on G^Γ .

PROPOSITION 8.2. *Let $\mathfrak{p} \subset \mathfrak{R}_d$ be an ergodic prime ideal with $p = p(\mathfrak{p}) > 0$, and assume that $\alpha = \alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is the shift-action of \mathbb{Z}^d on the closed, shift-invariant subgroup $X = X^{\mathfrak{R}_d/\mathfrak{p}} \subset \mathbb{F}_p^{\mathbb{Z}^d}$ defined by (6.19). Then there exists an integer $r = r(\mathfrak{p}) \in \{1, \dots, d\}$, a primitive subgroup $\Gamma = \Gamma(\mathfrak{p}) \subset \mathbb{Z}^d$, and a finite set $Q = Q(\mathfrak{p}) \subset \mathbb{Z}^d$ with the following properties.*

- (1) $\Gamma \cong \mathbb{Z}^r$;
- (2) $\mathbf{0} \in Q$, and $Q \cap (Q + \mathbf{m}) = \emptyset$ whenever $\mathbf{0} \neq \mathbf{m} \in \Gamma$;
- (3) If $\bar{\Gamma} = \Gamma + Q = \{\mathbf{m} + \mathbf{n} : \mathbf{m} \in \Gamma, \mathbf{n} \in Q\}$, then the coordinate projection $\pi_{\bar{\Gamma}} : X \mapsto \mathbb{F}_p^{\bar{\Gamma}}$, which restricts any point $x \in X \subset \mathbb{F}_p^{\mathbb{Z}^d}$ to its coordinates in $\bar{\Gamma}$, is a continuous group isomorphism; in particular, the Γ -action $\alpha^\Gamma : \mathbf{n} \mapsto \alpha_{\mathbf{n}}$, $\mathbf{n} \in \Gamma$, is (isomorphic to) the shift-action of Γ on $(\mathbb{F}_p^Q)^\Gamma$.

PROOF. This is Noether's normalization lemma in disguise. Consider the prime ideal $\mathfrak{p}' = \{f/p : f \in \mathfrak{p}\} \subset \mathfrak{R}_d^{(p)}$ defined in Remark 6.19 (4), and write $\mathbf{e}^{(i)}$ for the i -th unit vector in \mathbb{Z}^d . We claim that there exists a matrix $A \in \text{GL}(d, \mathbb{Z})$ and an integer r , $1 \leq r \leq d$, such that the elements $v_i = u^{\text{Ae}^{(i)}} + \mathfrak{p}'$ are algebraically independent in the ring $\mathcal{R} = \mathfrak{R}_d^{(p)}/\mathfrak{p}'$ for $i = 1, \dots, r$, and $v_j = u^{\text{Ae}^{(j)}} + \mathfrak{p}'$ is an algebraic unit over the subring $\mathbb{F}_p[v_1^{\pm 1}, \dots, v_{j-1}^{\pm 1}] \subset \mathcal{R}$ for $j = r+1, \dots, d$. Indeed, if $u'_1 = u_1 + \mathfrak{p}', \dots, u'_d = u_d + \mathfrak{p}'$ are algebraically independent elements of \mathcal{R} , then $\mathfrak{p}' = \{0\}$, and the assertion holds with $r = d$, and with A equal to the $d \times d$ identity matrix. Assume therefore (after renumbering the variables, if necessary) that there exists an irreducible Laurent polynomial $f \in \mathfrak{p}'$ of the form $f = g_0 + g_1 u_d + \dots + g_l u_d^l$, where $g_i \in \mathbb{F}_p[u_1^{\pm 1}, \dots, u_{d-1}^{\pm 1}]$ and $g_0 g_l \neq 0$. If the supports of g_0 and g_l are both singletons, then u_d and u_d^{-1} are both integral over the subring $\mathbb{F}_p[u_1^{\pm 1}, \dots, u_{d-1}^{\pm 1}] \subset \mathcal{R}$. If the support of either g_0 or g_l is not a singleton one can find integers k_1, \dots, k_d such that substitution of the variables $w_i = u_i u_d^{k_i}$, $i = 1, \dots, d-1$, in f leads to a Laurent polynomial $g(w_1, \dots, w_{d-1}, u_d) = u_d^{k_d} f(u_1, \dots, u_d)$ of the form $g = g'_0 + g'_1 u_d + \dots + g'_l u_d^l$, where $g'_i \in \mathbb{F}_p[w_1^{\pm 1}, \dots, w_{d-1}^{\pm 1}]$, and where the supports of g'_0 and g'_l are both singletons. We set

$$B = \begin{pmatrix} 1 & 0 & \dots & 0 & k_1 \\ 0 & 1 & \dots & 0 & k_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & k_{d-1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

$w'_i = w_i + \mathfrak{p}' = u^{\text{Be}^{(i)}} + \mathfrak{p}'$, $i = 1, \dots, d-1$, and note that w'_d and w'^{-1}_d are integral over $\mathbb{F}_p[w'^{\pm 1}_1, \dots, w'^{\pm 1}_{d-1}] \subset \mathcal{R}$. If the elements w'_1, \dots, w'_{d-1} are algebraically independent in \mathcal{R} , then our claim is proved; if not, then we can apply the same argument to w_1, \dots, w_{d-1} instead of u_1, \dots, u_d , and iteration

of this procedure leads to a matrix $A \in \mathrm{GL}(d, \mathbb{Z})$ and an integer $r \geq 0$ such that the elements $v'_j = u^{A\mathbf{e}^{(j)}} + \mathbf{p}' \in \mathcal{R}$ satisfy that v'_1, \dots, v'_r are algebraically independent, and v'_j and v'^{-1}_j are integral over $\mathcal{R}^{(j-1)} = \mathbb{F}_p[v_1^{\pm 1}, \dots, v_{j-1}^{\pm 1}] \subset \mathcal{R}$ for $j > r$, where $\mathcal{R}^{(0)} = \mathbb{F}_p$ if $r = 0$ (in which case \mathcal{R} must be finite). From Theorem 3.2 it is clear that the ergodicity of α implies that $r \geq 1$, and this completes the proof of our claim.

For the remainder of this proof we assume for simplicity that A is the $d \times d$ identity matrix, so that $v_i = u_i$ for $i = 1, \dots, d$ (this is—in effect—equivalent to replacing α by the \mathbb{Z}^d -action $\alpha': \mathbf{n} \mapsto \alpha'_{\mathbf{n}} = \alpha_{A\mathbf{n}}$). The argument in the preceding paragraph gives us, for each $j = r+1, \dots, d$, an irreducible polynomial $f_j(x) = \sum_{k=0}^{l_j} g_k^{(j)} x^k$ with coefficients in the ring $\mathbb{F}_p[u_1^{\pm 1}, \dots, u_{j-1}^{\pm 1}] \subset \mathfrak{R}_d$ such that $h_j(u_j) = h_j(u_1, \dots, u_{j-1}, u_j) \in \mathbf{p}'$ and the supports of $g_0^{(j)}$ and $g_{l_j}^{(j)}$ are singletons. Let $\Gamma \subset \mathbb{Z}^d$ be the group generated by $\{\mathbf{e}^{(1)}, \dots, \mathbf{e}^{(r)}\}$, $Q = \{0\} \times \dots \times \{0\} \times \{0, \dots, l_{r+1} - 1\} \times \{0, \dots, l_d - 1\} \subset \mathbb{Z}^d$, and let $\bar{\Gamma} = \Gamma + Q = \{\mathbf{m} + \mathbf{n} : \mathbf{m} \in \Gamma, \mathbf{n} \in Q\}$. We write $\pi_{\bar{\Gamma}}: X \mapsto \mathbb{F}_p^{\bar{\Gamma}}$ for the coordinate projection which restricts every $x \in X$ to its coordinates in $\bar{\Gamma}$ and note that $\pi_{\bar{\Gamma}}: X \mapsto \mathbb{F}_p^{\bar{\Gamma}}$ is a continuous group isomorphism. In other words, the restriction of α to the group $\Gamma \cong \mathbb{Z}^r$ is conjugate to the shift-action of Γ on $(\mathbb{F}_p^Q)^{\Gamma}$. \square

If the prime ideal $\mathbf{p} \subset \mathfrak{R}_d$ satisfies that $p(\mathbf{p}) = 0$, then the analysis of the action $\alpha^{\mathfrak{R}_d/\mathbf{p}}$ becomes somewhat more complicated. We denote by $\kappa: \hat{\mathbb{Q}} \mapsto \mathbb{T}$ the surjective group homomorphism dual to the inclusion $\hat{\kappa}: \mathbb{Z} \mapsto \mathbb{Q}$. If $\mathbf{p} \subset \mathfrak{R}_d$ is a prime ideal with $p(\mathbf{p}) = 0$ we regard $X^{\mathfrak{R}_d/\mathbf{p}}$ as the subgroup (5.9) of $\mathbb{T}^{\mathbb{Z}^d}$, and define a closed, shift-invariant subgroup $\bar{X}^{\mathfrak{R}_d/\mathbf{p}} \subset \hat{\mathbb{Q}}^{\mathbb{Z}^d}$ by

$$\bar{X}^{\mathfrak{R}_d/\mathbf{p}} = \left\{ x = (x_{\mathbf{n}}) \in \hat{\mathbb{Q}}^{\mathbb{Z}^d} : \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) x_{\mathbf{m}+\mathbf{n}} = \mathbf{0}_{\hat{\mathbb{Q}}^{\mathbb{Z}^d}} \text{ for every } f \in \mathbf{p} \right\}. \quad (8.1)$$

The restriction of the shift-action σ of \mathbb{Z}^d on $\hat{\mathbb{Q}}^{\mathbb{Z}^d}$ to $\bar{X}^{\mathfrak{R}_d/\mathbf{p}}$ will be denoted by $\bar{\alpha}^{\mathfrak{R}_d/\mathbf{p}}$ (cf. (2.1)). Define a continuous, surjective homomorphism $\kappa: \hat{\mathbb{Q}}^{\mathbb{Z}^d} \mapsto \mathbb{T}^{\mathbb{Z}^d}$ by $(\kappa(x))_{\mathbf{n}} = \kappa(x_{\mathbf{n}})$ for every $x = (x_{\mathbf{m}}) \in \hat{\mathbb{Q}}^{\mathbb{Z}^d}$ and $\mathbf{n} \in \mathbb{Z}^d$, and write

$$\kappa^{\mathfrak{R}_d/\mathbf{p}}: \bar{X}^{\mathfrak{R}_d/\mathbf{p}} \mapsto X^{\mathfrak{R}_d/\mathbf{p}} \quad (8.2)$$

for the restriction of κ to $\bar{X}^{\mathfrak{R}_d/\mathbf{p}}$. The map $\kappa^{\mathfrak{R}_d/\mathbf{p}}$ is surjective, and the diagram

$$\begin{array}{ccc} \bar{X}^{\mathfrak{R}_d/\mathbf{p}} & \xrightarrow{\bar{\alpha}_{\mathbf{n}}^{\mathfrak{R}_d/\mathbf{p}}} & \bar{X}^{\mathfrak{R}_d/\mathbf{p}} \\ \kappa \downarrow & & \downarrow \kappa \\ X^{\mathfrak{R}_d/\mathbf{p}} & \xrightarrow{\alpha_{\mathbf{n}}^{\mathfrak{R}_d/\mathbf{p}}} & X^{\mathfrak{R}_d/\mathbf{p}} \end{array} \quad (8.3)$$

commutes for every $\mathbf{n} \in \mathbb{Z}^d$.

In order to explain this construction in terms of the dual modules we consider the ring $\mathfrak{R}_d^{(\mathbb{Q})} = \mathbb{Q}[u_1^{\pm 1}, \dots, u_d^{\pm 1}] = \mathbb{Q} \otimes_{\mathbb{Z}} \mathfrak{R}_d$, regard \mathfrak{R}_d as the subring of $\mathfrak{R}_d^{(\mathbb{Q})}$ consisting of all polynomials with integral coefficients, and denote by $\mathfrak{p}^{(\mathbb{Q})} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathfrak{p} \subset \mathfrak{R}_d^{(\mathbb{Q})}$ the prime ideal in $\mathfrak{R}_d^{(\mathbb{Q})}$ corresponding to \mathfrak{p} . Since $p(\mathfrak{p}) = 0$, every \mathfrak{R}_d -module \mathfrak{N} associated with \mathfrak{p} is embedded injectively in the $\mathfrak{R}_d^{(\mathbb{Q})}$ -module $\mathfrak{N}^{(\mathbb{Q})} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathfrak{N}$ by

$$\hat{i}^{\mathfrak{N}}: a \mapsto 1 \otimes_{\mathbb{Z}} a, \quad a \in \mathfrak{N}, \quad (8.4)$$

and $\mathfrak{N}^{(\mathbb{Q})}$ is associated with $\mathfrak{p}^{(\mathbb{Q})}$. Since $\mathfrak{R}_d \subset \mathfrak{R}_d^{(\mathbb{Q})}$, $\mathfrak{N}^{(\mathbb{Q})}$ is an \mathfrak{R}_d -module, and we can define the \mathbb{Z}^d -action $\alpha^{\mathfrak{N}^{(\mathbb{Q})}}$ on $X^{\mathfrak{N}^{(\mathbb{Q})}}$ as in Lemma 5.1. Note that the set of prime ideals associated with the \mathfrak{R}_d -module $\mathfrak{N}^{(\mathbb{Q})}$ is the same as that of \mathfrak{N} ; in particular, $\alpha^{\mathfrak{N}^{(\mathbb{Q})}}$ is ergodic if and only if $\alpha^{\mathfrak{N}}$ is ergodic and, for every $\mathbf{n} \in \mathbb{Z}^d$, $\alpha_{\mathbf{n}}^{\mathfrak{N}^{(\mathbb{Q})}}$ is ergodic if and only if $\alpha_{\mathbf{n}}^{\mathfrak{N}}$ is ergodic. The homomorphism

$$\iota^{\mathfrak{N}}: X^{\mathfrak{N}^{(\mathbb{Q})}} \longrightarrow X^{\mathfrak{N}} \quad (8.5)$$

dual to

$$\hat{i}: \mathfrak{N} \longrightarrow \mathfrak{N}^{(\mathbb{Q})} \quad (8.6)$$

is surjective, and the diagram

$$\begin{array}{ccc} X^{\mathfrak{N}^{(\mathbb{Q})}} & \xrightarrow{\alpha_{\mathbf{n}}^{\mathfrak{N}^{(\mathbb{Q})}}} & \bar{X}^{\mathfrak{N}^{(\mathbb{Q})}} \\ \kappa \downarrow & & \downarrow \kappa \\ X^{\mathfrak{N}} & \xrightarrow{\alpha_{\mathbf{n}}^{\mathfrak{N}}} & X^{\mathfrak{N}} \end{array} \quad (8.7)$$

commutes for every $\mathbf{n} \in \mathbb{Z}^d$. For $\mathfrak{N} = \mathfrak{R}_d/\mathfrak{p}$ we obtain that

$$\begin{aligned} X^{(\mathfrak{R}_d/\mathfrak{p})^{(\mathbb{Q})}} &= \bar{X}^{\mathfrak{R}_d/\mathfrak{p}}, \\ \alpha^{(\mathfrak{R}_d/\mathfrak{p})^{(\mathbb{Q})}} &= \bar{\alpha}^{\mathfrak{R}_d/\mathfrak{p}}, \\ \iota^{\mathfrak{R}_d/\mathfrak{p}} &= \kappa^{\mathfrak{R}_d/\mathfrak{p}}. \end{aligned} \quad (8.8)$$

PROPOSITION 8.3. *Let $\mathfrak{p} \subset \mathfrak{R}_d$ be a prime ideal with $p(\mathfrak{p}) = 0$ which is not of the form $\mathfrak{p} = \mathfrak{j}_c$ for any $c \in \overline{\mathbb{Q}}^d$. Then the \mathbb{Z}^d -action $\alpha = \alpha^{\mathfrak{R}_d/\mathfrak{p}}$ on $X = X^{\mathfrak{R}_d/\mathfrak{p}}$ is ergodic, and there exists an integer $r = r(\mathfrak{p}) \in \{1, \dots, d\}$, a primitive subgroup $\Gamma = \Gamma(\mathfrak{p}) \subset \mathbb{Z}^d$, and a finite set $Q = Q(\mathfrak{p}) \subset \mathbb{Z}^d$ with the following properties.*

- (1) $\Gamma \cong \mathbb{Z}^r$;
- (2) $\mathbf{0} \in Q$, and $Q \cap (Q + \mathbf{m}) = \emptyset$ whenever $\mathbf{0} \neq \mathbf{m} \in \Gamma$;

- (3) If $\bar{\Gamma} = \Gamma + Q = \{\mathbf{m} + \mathbf{n} : \mathbf{m} \in \Gamma, \mathbf{n} \in Q\}$, then the coordinate projection $\pi_{\bar{\Gamma}} : \bar{X}^{\mathfrak{R}_d/\mathfrak{p}} \mapsto \hat{\mathbb{Q}}^{\bar{\Gamma}}$, which restricts any point $x \in \bar{X}^{\mathfrak{R}_d/\mathfrak{p}} \subset \hat{\mathbb{Q}}^{\mathbb{Z}^d}$ to its coordinates in $\bar{\Gamma}$, is a continuous group isomorphism; in particular, the Γ -action $\mathbf{n} \mapsto \bar{\alpha}_{\mathbf{n}}^{\mathfrak{R}_d/\mathfrak{p}}$, $\mathbf{n} \in \Gamma$, is (isomorphic to) the shift-action of Γ on $(\hat{\mathbb{Q}}^Q)^{\Gamma}$.

PROOF. The proof is completely analogous to that of Proposition 8.2. We find a matrix $A \in \mathrm{GL}(d, \mathbb{Z})$ and an integer $r \in \{1, \dots, d\}$ with the following properties: if $v_j = u^A \mathbf{e}^{(j)}$ and $v'_j = v_j + \mathfrak{p}$ for $j = 1, \dots, d$, then v'_1, \dots, v'_r are algebraically independent elements of $\mathcal{R} = \mathfrak{R}_d/\mathfrak{p}$, and there exists, for each $j = r+1, \dots, d$, an irreducible polynomial $f_j(x) = \sum_{k=0}^{l_j} g_k^{(j)}(x^k)$ with coefficients in the ring $\mathbb{Z}[v_1^{\pm 1}, \dots, v_{j-1}^{\pm 1}] \subset \mathfrak{R}_d$ such that $f_j(v_1, \dots, v_{j-1}, v_j) \in \mathfrak{q}$ and the supports of $g_0^{(j)}$ and $g_{l_j}^{(j)}$ are singletons.

We assume again that A is the $d \times d$ identity matrix, so that $v_j = u_j$ for $j = 1, \dots, d$ and $\Gamma \cong \mathbb{Z}^r$ is generated by $\mathbf{e}^{(1)}, \dots, \mathbf{e}^{(r)}$, set $Q = \{0\} \times \dots \times \{0\} \times \{0, \dots, l_{r+1} - 1\} \times \dots \times \{0, \dots, l_d - 1\} \subset \mathbb{Z}^d$, and complete the proof in the same way as that of Proposition 3.4, using (8.1) instead of (6.19). The ergodicity of $\bar{\alpha}^{\mathfrak{R}_d/\mathfrak{p}}$ is obvious from the conditions (1)–(3), and from (8.3) we conclude the ergodicity of $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$. \square

REMARKS 8.4. (1) We can extend the definition of $r(\mathfrak{p})$ in Proposition 8.2 and 8.3 to ergodic prime ideals of the form $\mathfrak{p} = \mathfrak{j}_c, c \in (\overline{\mathbb{Q}}^\times)^d$, by setting $r(\mathfrak{j}_c) = 0$. Then the integer $r(\mathfrak{p})$ is a well-defined property of the prime ideal \mathfrak{p} , and is in particular independent of the choice of the primitive subgroup $\Gamma \subset \mathbb{Z}^d$ in Proposition 8.2 or 8.3 (it is easy to see that there is considerable freedom in the choice of Γ): if r', Γ', Q' are a positive integer, a primitive subgroup of \mathbb{Z}^d , and a finite subset of \mathbb{Z}^d , satisfying the conditions (1)–(3) in either of the Propositions 8.2 or 8.3, then $r' = r(\mathfrak{p})$. This follows from Noether's normalization theorem; a dynamical proof using entropy will be given in Section 24.

(2) If $\mathfrak{p} \subset \mathfrak{R}_d$ is an ergodic prime ideal with $p(\mathfrak{p}) > 0$, then the subgroup $\Gamma \subset \mathbb{Z}^d$ in Proposition 8.2 is a maximal subgroup of \mathbb{Z}^d for which the restriction α^Γ of $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ to Γ is expansive. In particular, $r(\mathfrak{p})$ is the smallest integer for which there exists a subgroup $\Gamma \cong \mathbb{Z}^r$ in \mathbb{Z}^d such that α^Γ is expansive.

(3) Even if the \mathbb{Z}^d -action $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ in Proposition 8.3 is expansive, the action $\alpha^{(\mathfrak{R}_d/\mathfrak{p})^{(\mathbb{Q})}}$ is non-expansive. By proving a more intricate version of Proposition 8.3 one can analyze the structure of the group $X^{\mathfrak{R}_d/\mathfrak{p}}$ directly, without passing to $X^{(\mathfrak{R}_d/\mathfrak{p})^{(\mathbb{Q})}}$: if $X^{\mathfrak{R}_d/\mathfrak{p}}$ is written as a shift-invariant subgroup of $\mathbb{T}^{\mathbb{Z}^d}$ (cf. (5.9)), and if $r = r(\mathfrak{p})$, Γ, Q are given as in Proposition 8.3, then the projection $\pi_{\bar{\Gamma}} : X^{\mathfrak{R}_d/\mathfrak{p}} \mapsto \mathbb{T}^{\bar{\Gamma}}$ is still surjective, but need no longer be injective; the kernel of $\pi_{\bar{\Gamma}}$ is of the form Y^Γ for some compact, zero-dimensional group Y (cf. Example 8.5 (2)).

EXAMPLES 8.5. (1) Let $\mathfrak{p} = (2, 1 + u_1 + u_2) \subset \mathfrak{R}_2$ (cf. Example 5.3 (5)). Then $p(\mathfrak{p}) = 2$, $r(\mathfrak{p}) = 1$, and we may set $\Gamma = \{(k, k) : k \in \mathbb{Z}\} \cong \mathbb{Z}$ and $Q = \{(0, 0), (1, 0)\} \subset \mathbb{Z}^2$ in Proposition 8.2. If $X = X^{\mathfrak{R}_2/\mathfrak{p}}$ is written in the form (6.19) as

$$X = \{x = (x_{\mathbf{m}}) \in \mathbb{F}_2^{\mathbb{Z}^d} : x_{(m_1, m_2)} + x_{(m_1+1, m_2)} + x_{(m_1, m_2+1)} = \mathbf{0}_{\mathbb{F}_2} \text{ for all } (m_1, m_2) \in \mathbb{Z}^2\},$$

then the projection $\pi_{\bar{\Gamma}} : X \mapsto \mathbb{F}_2^{\bar{\Gamma}}$ sends the shift $\alpha_{(1,1)}^{\mathfrak{R}_2/\mathfrak{p}} = \alpha_{(1,1)}$ on X to the shift on $\mathbb{F}_2^{\bar{\Gamma}} \cong (\mathbb{Z}/2 \times \mathbb{Z}/2)^{\mathbb{Z}}$. Note that, although $\alpha_{(1,1)}$ acts expansively on X , other elements of \mathbb{Z}^2 may not be expansive; for example, $\alpha_{(1,0)}$ is non-expansive.

(2) Let $\mathfrak{p} = (3 + u_1 + 2u_2) \subset \mathfrak{R}_2$. Then $p(\mathfrak{p}) = 0$, $r(\mathfrak{p}) = 1$, and Γ and Q may be chosen as in Example (1). Note that $X^{\mathfrak{R}_2/\mathfrak{p}} = X = \{x = (x_{\mathbf{m}}) \in \mathbb{T}^{\mathbb{Z}^d} : x_{(m_1, m_2)} + x_{(m_1+1, m_2)} + x_{(m_1, m_2+1)} = \mathbf{0}_{\mathbb{T}} \text{ for all } (m_1, m_2) \in \mathbb{Z}^2\}$; the coordinate projection $\pi_{\bar{\Gamma}} : X \mapsto \mathbb{T}^{\bar{\Gamma}}$ in Proposition 8.3 is not injective; for every $x \in X$, the coordinates $x_{(m_1, m_2)}$ with $m_1 \geq m_2$ are completely determined by $\pi_{\bar{\Gamma}}(x)$, but each of the coordinates $x_{(k, k+1)}$, $k \in \mathbb{Z}$, has two possible values. Similarly, if we know the coordinates $x_{(m_1, m_2)}$, $m_1 \geq m_2 - r$ of a point $x = (x_{\mathbf{m}}) \in X$ for any $r \geq 0$, then there are exactly two (independent) choices for each of the coordinates $x_{(k, k+r+1)}$, $k \in \mathbb{Z}$. This shows that the kernel of the surjective homomorphism $\pi_{\bar{\Gamma}} : X \mapsto \mathbb{T}^{\bar{\Gamma}} \cong (\mathbb{T}^2)^{\mathbb{Z}}$ is isomorphic to $\mathbb{Z}_2^{\bar{\Gamma}}$, where $Y = \mathbb{Z}_2$ denotes the group of dyadic integers.

If \mathfrak{p} is replaced by the prime ideal $\mathfrak{p}' = (1 + 3u_1 + 2u_2) \subset \mathfrak{R}_2$, then Γ and Q remain unchanged, but the kernel of $\pi_{\bar{\Gamma}}$ becomes isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_3)^{\bar{\Gamma}}$, where \mathbb{Z}_3 is the group of tri-adic integers. Finally, if $\mathfrak{p}'' = (1 + u_1 + u_2) \subset \mathfrak{R}_2$, and if Γ and Q are as in Example (1), then $\pi_{\bar{\Gamma}} : X^{\mathfrak{R}_2/\mathfrak{p}''} \mapsto (\mathbb{T}^Q)^{\mathbb{Z}}$ is a group isomorphism. \square

CONCLUDING REMARK 8.6. The material in this section (with the exception of Proposition 8.1) is taken from [38].



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