Preface

One of the landmarks in the history of mathematics is the proof of the nonexistence of algorithms based solely on radicals and elementary arithmetic operations (addition, subtraction, multiplication, and division) for solutions of general algebraic equations of degrees higher than four. This proof by the French mathematician Evariste Galois in the early nineteenth century used the then novel concept of the permutation symmetry of the roots of algebraic equations and led to the invention of group theory, an area of mathematics now nearly two centuries old that has had extensive applications in the physical sciences in recent decades.

The radical-based algorithms for solutions of general algebraic equations of degrees 2 (quadratic equations), 3 (cubic equations), and 4 (quartic equations) have been well-known for a number of centuries. The quadratic equation algorithm uses a single square root, the cubic equation algorithm uses a square root inside a cube root, and the quartic equation algorithm combines the cubic and quadratic equation algorithms with no new features. The details of the formulas for these equations of degree $d(d = 2,3,4)$ relate to the properties of the corresponding symmetric groups S_d which are isomorphic to the symmetries of the equilateral triangle for $d = 3$ and the regular tetrahedron for $d = 4$.

Related ideas can be used to generate an algorithm for solution of the general algebraic equation of degree 5 (the quintic equation). Such a quintic equation algorithm does not violate the classical theorem proved by Galois since it contains more complicated mathematical functions than the radicals that suffice for the algorithms for roots of the general quadratic, cubic, and quartic equations.

The underlying mathematics for an algorithm to solve the quintic equation was developed by nineteenth century European mathematicians shortly after Galois' discovery of the insolubility of the general quintic equation using only radicals. The initial work in this area was done by Hermite and then developed further by Gordan. This culminated in two key publications, an 1878 article by Kiepert[1] describing a quintic equation algorithm and the classic 1884 book by Klein[2] describing the relationship between the icosahedron and the solution of the quintic equation. At that stage this work lay fallow for more than a century since the algorithm for roots of the general quintic equation appeared intractable before the era of computers. Many of the key ideas appear to have been forgotten by the subsequent generations of mathematicians during the past century so that some of the underlying mathematics has the status of a lost art.

The work discussed in this book arose when I discovered the 1878 Kiepert paper and wished to see if the quintic equation algorithm described therein would work on a modern computer. I enlisted the help of a computer scientist, Prof. E. R. Canfield of the University of Georgia, to write a program that would use the Kiepert algorithm to solve the quintic equation (i.e., to calculate the roots of the quintic equation from its coefficients). This proved to be much more difficult than anticipated because of errors and lack of detail in the Kiepert paper. However, the quintic equation algorithm was finally made to work on an IBM-compatible personal computer but only after studying a considerable amount of related mathematics in several languages mainly in nineteenth century journals. This work led to the discovery of some unanticipated features of the quintic equation algorithm.

This book presents for the first time a complete algorithm for the roots of the general quintic equation with enough background information to make the key ideas accessible to nonspecialists and even to mathematically oriented readers who are not professional mathematicians. Two relatively short papers[3,4], have been published on this work on the quintic equation algorithm but the whole story is far too long to fit into a journal article of reasonable length. The book includes initial introductory chapters on group theory and symmetry, the Galois theory of equations, and some elementary properties of elliptic functions and associated theta functions in an attempt to make some of the key ideas accessible to less sophisticated readers. The book also includes discussion of the much simpler algorithms for roots of the general quadratic, cubic, and quartic equations before discussing the algorithm for the roots of the general quintic equation. The book concludes with a brief discussion of attempts to extend these ideas to algorithms for the roots of general equations of degrees higher than five.

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March, 1996

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