8 Stability analysis by the direct method

8.1 Introduction

In this chapter we shall discuss a method for studying the stability of a solution, which is very different from the method of linearisation of the preceding chapter. When linearising one starts off with small perturbations of the equilibrium or periodic solution and one studies the effect of these *local* perturbations. In the so-called direct method one characterises the solution in a way with respect to stability which is not necessarily local.
The method originates from the field of classical mechanics where this non-local characterisation arises from the laws of statics and dynamics. A basic idea can be found in the work of Torricelli (1608-1647), a student of Galileï. From Torricelli we have the following "axiom", as he calls it: "Connected heavy bodies cannot start moving by themselves if their common centre of gravity does not move downward." Torricelli's axiom finds its application in statics; Huygens (1629-1695) developed this idea, together with other insights, for the dynamics of particles, bodies and fluids. Later this was taken up by Lagrange (1736-1813) who formulated his well-known principle for the stability of a mechanical system:
"A mechanical system which is in a state where its potential energy has an isolated minimum, is in a state of stable equilibrium."
We shall prove the validity of the principle of Lagrange in section 8.3; we shall also illustrate the concept of potential energy there. The reader who is interested in the historical development of mechanics should consult the classic by Dijksterhuis (1950).

Around 1900 these ideas of stability in mechanics were generalised strongly by Lyapunov. In his work differential equations are studied which have not been characterised apriori by a potential energy or a quantity energy in general. To introduce Lyapunov's ideas we discuss an example in which also the geometric aspects of the method are transparent.
Consider the system of equations

$$\begin{aligned} \dot{x} &= ax - y + kx(x^2+y^2) \\ \dot{y} &= x - ay + ky(x^2+y^2) \end{aligned} \tag{8.1}$$

with constants a and k, $a > 0$. We are interested in the stability of the trivial solution.
Linearisation in a neighbourhood of $(0,0)$ yields the eigenvalues

$$\pm(a^2-1)^{\frac{1}{2}},$$

so in the linear approximation we find for the critical point $(0,0)$

$$\begin{array}{rl} a^2 > 1 & \text{saddle} \\ a^2 = 1 & \text{degenerate case} \\ 0 < a^2 < 1 & \text{centre.} \end{array}$$

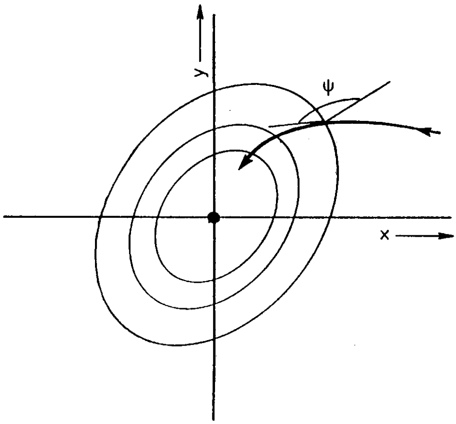

Figure 8.1.

We conclude with theorem 7.3 that if $a^2 > 1$, the trivial solution is unstable. If $0 < a^2 \leq 1$ the method of linearisation of chapter 7 is not conclusive.
We now consider a one-parameter family of ellipses around $(0,0)$:

$$x^2 - 2axy + y^2 = c.$$

The parameter is c, a is fixed with $0 < a^2 < 1$.
An orbit in the x, y-phaseplane, corresponding with a solution of system 8.1, will intersect an ellipse with angle ψ (this angle is taken between the tangent vector of the orbit and the outward directed normal vector in the point of intersection). If $\frac{\pi}{2} < \psi < 3\frac{\pi}{2}$, $\cos\psi < 0$, the orbit enters the interior of this particular ellipse. If $\cos\psi < 0$ for all solutions and all ellipses in a neighbourhood of $(0,0)$, the trivial solution is asymptotically stable. We compute $\cos\psi$.
The tangent vector $\vec{\tau}$ of the orbit is given by $\vec{\tau} = (\dot{x}, \dot{y})$. The gradient vector of the function

$$V = x^2 - 2axy + y^2$$

produces the normal vector field

$$\nabla V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}\right) = (2x - 2ay, -2ax + 2y).$$

We compute $\cos\psi$ using the vector product

$$\cos\psi = \frac{(\nabla V.\vec{\tau})}{\|\nabla V\|.\|\vec{\tau}\|}$$

$$= \frac{\frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}}{\|\nabla V\|.\|\vec{\tau}\|}$$

The sign of $\cos\psi$ is determined by the numerator

$$\frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} = L_t V.$$

In section 2.4 we have called L_t the orbital derivative of the function V. Using equations 8.1 we find in this case

$$\begin{aligned} L_t V &= (2x - 2ay)\dot{x} + (-2ax + 2y)\dot{y} \\ &= 2k(x^2 + y^2)(x^2 + y^2 - 2axy). \end{aligned}$$

$$\text{So} \quad \begin{aligned} &\cos\psi < 0 \quad if \quad k < 0 \\ &\cos\psi > 0 \quad if \quad k > 0. \end{aligned}$$

The result holds for all ellipses and all orbits in a neighbourhood of $(0,0)$ so we conclude that the trivial solution is asymptotically stable if $k < 0$, unstable if $k > 0$. The result holds for all orbits so that for $k < 0$ and $0 < a^2 < 1$ we have even global stability of $(0,0)$.
The case $a^2 = 1$ is left to the reader.

In this chapter we shall present the most important results of the direct method of Lyapunov; many more results can be found in Hahn (1967). The direct method has also been applied and extended considerably in the theory of optimal control.

8.2 Lyapunov functions

Consider the equation

(8.2) $$\dot{x} = f(t,x), \quad t \geq t_0, x \in D \subset \mathbb{R}^n.$$

and assume that the trivial solution satisfies the equation, so $f(t,0) = 0$, $t \geq t_0$, $0 \in D$.

Definition $V(t,x)$
In this chapter the scalar function $V(t,x)$ is defined and continuously differentiable in $[t_0,\infty) \times D$, $D \subset \mathbb{R}^n$. Moreover $x = 0$ is an interior point of D and

$$V(t,0) = 0.$$

In some cases the function $V(t,x)$ does not depend explicitly on t and we write for short $V(x)$. The function $V(t,x)$ being positively definite or negatively definite is introduced as follows.

Definition
The function $V(x)$ (with $V(0) = 0$) is called positively (negatively) definite in D if $V(x) > 0$ (< 0) for $x \in D$, $x \neq 0$.

There are cases in which the function $V(x)$ takes the value zero in a subset of D but has otherwise a definite sign.

Definition
The function $V(x)$ (with $V(0) = 0$) is called positively (negatively) semidefinite in D if $V(x) \geq 0$ (≤ 0) for $x \in D$.

If the function $V(t, x)$ depends explicitly on t, these definitions are adjusted as follows.

Definition
The function $V(t, x)$ is called positively (negatively) definite in D if there exists a function $W(x)$ with the following properties: $W(x)$ is defined and continuous in D, $W(0) = 0$, $0 < W(x) \leq V(t, x)$ ($V(t, x) \leq W(x) < 0$) for $x \neq 0$, $t \geq t_0$.
To define semidefinite functions $V(t, x)$ we replace $<$ ($>$) by $\leq$ ($\geq$).

Example 8.1.
Definite functions which are used very often are quadratic functions with positive coefficients. Consider in $\mathbb{R}^3$ $D = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ and for $t \geq 0$ the functions

$x^2 + 2y^2 + 3z^2 + z^3$	positive definite
$x^2 + z^2$	positive semidefinite
$-x^2 \sin^2 t - y^2 - 4z^2$	negative semidefinite
$x^2 + y^2 + \cos^3 t z^2$	not sign definite

In the sequel we shall use a simple extension of the concept of orbital derivative, cf. section 2.4.

Definition
The orbital derivative L_t of the function $V(t, x)$ in the direction of the vectorfield $f(t, x)$, where x is a solution of equation 8.2 $\dot{x} = f(t, x)$ is

$$\begin{aligned} L_t V = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\dot{x} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \\ &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1} f_1(t, x) + \cdots + \frac{\partial V}{\partial x_n} f_n(t, x) \end{aligned}$$

with $x = (x_1, \cdots, x_n)$ and $f = (f_1, \cdots, f_n)$.

Now we can formulate and prove the basic theorems.

Theorem 8.1.
Consider equation 8.2 $\dot{x} = f(t, x)$ with $f(t, 0) = 0$, $x \in D \subset \mathbb{R}^n$, $t \geq t_0$. If a function $V(t, x)$ can be found, defined in a neighbourhood of $x = 0$ and positively definite for $t \geq t_0$ with orbital derivative negatively semidefinite, the solution $x = 0$ is stable in the sense of Lyapunov.

Proof
In a neighbourhood of $x = 0$ we have for certain $R > 0$ and $\|x\| \leq R$

$$V(t, x) \geq W(x) > 0, \ x \neq 0, \ t \geq t_0$$

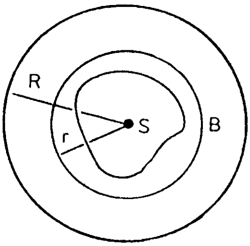

Figure 8.2

$$L_t V \leq 0.$$

Consider a spherical shell B, given by $0 < r \leq \|x\| \leq R$ and put

$$m = \min_{x \in B} W(x).$$

Consider now a neighbourhood S of $x = 0$ with the property that if $x \in S$, $V(t,x) < m$. Such a neighbourhood exists as $V(t,x)$ is continuous and positively definite while $V(t,0) = 0$. Starting a solution in S at $t = t_0$, the solution can never enter B, as we have for $t \geq t_0$

$$V(t, x(t)) - V(t_0, x(t_0)) = \int_{t_0}^{t} L_\tau V(\tau, x(\tau)) d\tau \leq 0.$$

In other words, the function $V(t, x(t))$ cannot increase along a solution and this would be necessary to enter B as initially $V(t_0, x(t_0)) < m$.
We can repeat the argument for arbitrary small R, from which follows the stability.
□

The scalar function $V(t,x)$ which we employed in this theorem is called a Lyapunov-function. Whether such functions exist and how one should construct them is known in a number of cases but not in general. For each class of problems we have to start again if we want to use the concept of Lyapunov-function.
Note that in the formulation of theorem 8.1 we have assumed that the orbital derivative is semidefinite. This includes the case that $L_t V = 0$, $t \geq t_0$, $x \in D$, in other words: $V(t,x)$ is a first integral of the equation; see also section 2.4. We shall return to the part played by first integrals in the applications of section 8.3.
In requiring more of the orbital derivative, we obtain a stronger form of stability.

Theorem 8.2.
Consider equation 8.2 $\dot{x} = f(t,x)$ with $f(t,0) = 0$, $x \in D \subset \mathbb{R}^n$, $t \geq t_0$. If a function $V(t,x)$ can be found, defined in a neighbourhood of $x = 0$, which for $t \geq t_0$ is

positively definite in this neighbourhood with negative definite orbital derivative, the solution $x = 0$ is asymptotically stable.

Proof

It follows from theorem 8.1 that $x = 0$ is a stable solution. Is it possible that for each $R > 0$ there exists a solution $x(t)$ which starts in the domain given by $\|x\| \leq R$ and which does not tend to zero? Put differently: is there a solution $x(t)$ and a constant $a > 0$ such that $\|x(t)\| \geq a$ for $t \geq t_0$, when starting arbitrarily close to zero?

Suppose this is the case, the solution remains in the spherical shell B: $a \leq \|x(t)\| \leq R$, $t \geq t_0$. We have $L_t V(t, x) \leq W(x) < 0$, $x \neq 0$. So we have in B

$$L_t V \leq -\mu \; , \; \mu > 0$$

so that

$$V(t, x(t)) - V(t_0, x(t_0)) = \int_{t_0}^{t} L_\tau V(\tau, x(\tau)) d\tau \leq -\mu(t - t_0).$$

On the other hand, we know that $V(t, x)$ is positively sign definite, whereas it follows from this estimate that after some time $V(t, x)$ becomes negative. This is a contradiction. □

In the proofs of theorems 8.1 and 8.2 we have developed, while using the Lyapunov-function $V(t, x)$, a picture of the behaviour of the solutions. In particular in the proof of theorem 8.2 we have indicated when a solution has left the spherical shell B and one can compute how long this will take at most.

If the trivial solution is asymptotically stable, we are interested of course in the set of all initial values corresponding with solutions which go to zero. This set is called the domain of attraction; we shall define this set for autonomous equations.

Definition

Consider the equation $\dot{x} = f(x)$ and suppose that $x = 0$ is an asymptotically stable solution. A set of points x_0 with the property that for the solution of

$$\dot{x} = f(x) \; , \; x(0) = x_0$$

we have $x(t) \to 0$ for $t \to \infty$, is called a domain of attraction of $x = 0$.

Using a Lyapunov-function, we are now able to characterise domains of attraction. The following result follows directly from the proof of theorem 8.2.

Corollary theorem 8.2.

Consider equation $\dot{x} = f(x)$ in $\mathbb{R}^n$ with $f(0) = 0$. The Lyapunov- function $V(x)$ is positively definite for $\|x\| \leq R$. S is a closed $(n-1)$-dimensional manifold which encloses $x = 0$ and which is contained in the ball with radius R. Suppose that

a. $L_t V < 0$, x in the interior of S
b. $L_t V = 0$, $x \in S$
c. $L_t V > 0$, x outside S.
d. The $(n-1)$-dimensional manifold $V(x) = c$ (c a positive constant) is entirely contained in the interior of S.

Then the set defined by $V(x) \le c$ in the ball $\|x\| \le R$ is a domain of attraction of $x = 0$.

A more extensive treatment of results for domains of attraction can be found in Hahn (1967), sections 4.26 and 4.33.
Using a Lyapunov-function one can also establish the instability of a solution.

Theorem 8.3.
Consider equation 8.2 $\dot{x} = f(t,x)$ with $f(t,0) = 0$, $x \in D \subset \mathbb{R}^n$, $t \ge t_0$. If there exists a function $V(t,x)$ in a neighbourhood of $x = 0$ such that:

a. $V(t,x) \to 0$ for $\|x\| \to 0$, uniformly in t;

b. $L_t V$ is positively definite in a neighbourhood of $x = 0$;

c. from a certain value $t = t_1 \ge t_0$, $V(t,x)$ takes positive values in each sufficiently small neighbourhood of $x = 0$;

then the trivial solution is unstable.

Proof
For certain positive constants a and b we have with $x \neq 0$ and $\|x\| \le a$: $L_t V(t,x) \ge W(x) > 0$ and $|V(t,x)| \le b$; the last estimate follows from assumption a.
Suppose that $x = 0$ is a stable solution. Then there exists a $\varepsilon > 0$ with $0 < \varepsilon < a$ such that when starting in x_0 with $\|x_0\| \le \varepsilon$, we have $\|x(t)\| \le a$ for $t \ge t_1$. Using assumption c we can choose x_0 such that $V(t_1, x_0) > 0$. We find for the solution $x(t)$ which starts in x_0 at $t = t_1$:

$$V(t, x(t)) - V(t_1, x_0) = \int_{t_1}^{t} L_\tau V(\tau, x(\tau)) > 0.$$

So $V(t, x(t))$ is non-decreasing. Consider now the set of points x with the property that $V(t,x) \ge V(t_1, x_0)$ and $\|x\| \le a$. This set is contained in the spherical shell S given by $0 < r \le \|x\| \le a$. We have

$$\mu = \inf_S W(x) > 0$$

so that

$$V(t, x(t)) - V(t_1, x_0) \ge \mu (t - t_1).$$

So for $\|x\| \le a$, $V(t,x)$ can become arbitrarily large; this is a contradiction.

Theorem 8.3 has been generalized by Chetaev in such a way that its applicability has increased; see again Hahn (1967).
In the next sections we study some applications and examples.

8.3 Hamiltonian systems and systems with first integrals

In example 2.12 we introduced Hamilton's equations:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}\ ,\ \dot{q}_i = \frac{\partial H}{\partial p_i}\ ,\ i = 1, \cdots, n, \tag{8.3}$$

in which H is a twice continuously differentiable function of the $2n$ variables p_i and q_i, $H : \mathbb{R}^{2n} \to \mathbb{R}$. The Hamilton function H is a first integral of the equations, i.e. the orbital derivative produces

$$L_t H = 0.$$

Suppose that the trivial solution satisfies system 8.3 and that we are interested in the stability of this equilibrium solution. Without loss of generality we may assume $H(0,0) = 0$ for we can add a constant to $H(p,q)$ without changing system 8.3. If now we require $H(p,q)$ to be sign definite in a neighbourhood of $x = 0$, the stability of the trivial solution follows by application of Lyapunov's theorem 8.1. The Hamilton function H is also a Lyapunov function. So we have the following result.

Theorem 8.4.
Consider Hamilton's equations 8.3, of which we assume that they admit the trivial solution. If $H(p,q) - H(0,0)$ is sign definite in a neighbourhood of $(p,q) = (0,0)$, the trivial solution is stable in the sense of Lyapunov.

Remark
In section 2.4 we discussed special but frequently occurring Hamiltonian systems, in which $(0,0)$ is a nondegenerate (Morse) critical point of the Hamilton function. We made use of the Morse lemma (see appendix 1); this application leads off to the existence of invariant, closed manifolds around the critical point in phase space. From this geometric picture one can also deduce stability.

Mechanical systems in which the force field can be derived from a potential $\phi(q)$ are characterised in many cases by the Hamiltonian

$$H(p,q) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \phi(q). \tag{8.4}$$

The equations of motion 8.3 can be written as

$$\ddot{q}_i = -\frac{\partial \phi}{\partial q_i}\ ,\ i = 1, \cdots, n. \tag{8.5}$$

Equilibrium solutions correspond with critical points determined by

$$p_i (= \dot{q}_i) = 0\ ,\ \frac{\partial \phi}{\partial q_i} = 0\ ,\ i = 1, \cdots, n.$$

It follows from theorem 8.4 that the equilibrium solution is stable if $\phi(q)$ has for the corresponding value of q an isolated minimum. This is the *principle of Lagrange.*
We shall now discuss the stability of equilibrium solutions which correspond with isolated maxima of the potential function $\phi(q)$. It is clear, that in this case the Hamiltonian is not a Lyapunov-function in the sense of theorem 8.1 or 8.3.
Assume again that the critical point is $(p, q) = (0, 0)$ and that $H(0, 0) = 0$, so $\phi(0) = 0$. Also we assume that $\phi(q)$ can be expanded in a Taylor series to degree $2m + 1$ $(m \in IN)$ in a neighbourhood of $q = 0$, with mixed terms of degree $2m$ transformed away, so that

$$\phi(q) = -\sum_{i=1}^{n} a_i q_i^{2m} + O\|q\|^{2m+1} \text{ as } \|q\| \to 0, \tag{8.6}$$

with $a_i > 0$, $i = 1, \cdots, n$. If $m = 1$ we can use linearisation and theorem 7.3 to show that the trivial solution is unstable. If $m > 1$ we have to proceed in a different way.
We introduce the function

$$V(p, q) = \sum_{i=1}^{n} p_i q_i.$$

In each neighbourhood of $(0, 0)$ the function V takes positive (and negative) values. We compute the orbital derivative

$$\begin{aligned} L_t V(p, q) &= \textstyle\sum_{i=1}^{n}(\dot{p}_i q_i + p_i \dot{q}_i) \\ &= \textstyle\sum_{i=1}^{n}(-\frac{\partial \phi}{\partial q_i} q_i + p_i^2) \\ &= \textstyle\sum_{i=1}^{n}(2m a_i q_i^{2m} + p_i^2) + O\|q\|^{2m+1} \end{aligned}$$

For the last step we used the expansion 8.6. We conclude that $L_t V$ is positively definite in a neighbourhood of $(0, 0)$ whereas V takes positive values. Using theorem 8.3 we conclude that $(0, 0)$ is unstable.
We summarise as follows.

Theorem 8.5.
Consider the Hamilton function

$$H(p, q) = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \phi(q) \;,\; (p, q) \in \mathbb{R}^{2n}$$

with potential $\phi(q)$ which can be expanded in a Taylor series in a neighbourhood of each critical point. An isolated minimum of the potential corresponds with a stable equilibrium solution, an isolated maximum corresponds with an unstable equilibrium solution.

Remark
Of course one can weaken the assumption on the Taylor series of $\phi(q)$; the expansion 8.6 is often met in applications but it has been used only to facilitate the demonstration.

Example 8.2.
In example 2.11 we studied the equation

$$\ddot{x} + f(x) = 0$$

which can be viewed as the equation of motion corresponding to the Hamilton function

$$H(p, q) = \frac{1}{2}p^2 + \int^q f(\tau)d\tau.$$

To obtain the equation of motion we put $(p, q) = (\dot{x}, x)$. The function $\int^q f(\tau)d\tau$ can be identified with the potential $\phi(q)$ in theorem 8.5. The isolated minima and maxima of $\phi(q)$ correspond with respectively stable and unstable equilibrium solutions, see figure 8.3.

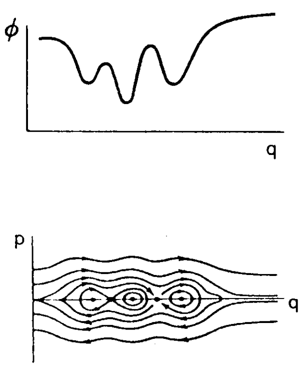

Figure 8.3. Potential ϕ and corresponding phase flow.

Example 8.3.
A potential problem which plays an important part in the modern theory of Hamiltonian systems was formulated and studied by Hénon and Heiles. The Hamilton function is

$$\begin{aligned} H(p, q) &= \tfrac{1}{2}(p_1^2 + p_2^2) + \phi(q_1, q_2) \\ &= \tfrac{1}{2}(p_1^2 + p_2^2) + \tfrac{1}{2}(q_1^2 + q_2^2) + q_1^2 q_2 - \tfrac{1}{3}q_2^3. \end{aligned}$$

The potential function $\phi(q)$ has the following critical points:
$(0, 0)$ isolated minimum
$(0, 1)$ and $(\pm\frac{1}{2}\sqrt{3}, -\frac{1}{2})$ are points where no maximum or minimum value is assumed.
Theorem 8.5, applied to this system, tells us that $(p_1, p_2, q_1, q_2) = (0, 0, 0, 0)$ is a stable equilibrium solution. In a neighbourhood of the three critical points $(0, 0, 0, 1)$ and $(0, 0, \pm\frac{1}{2}\sqrt{3}, -\frac{1}{2})$ we perform linearisation after which we apply theorem 7.3 to conclude instability.

We shall now consider an example of a system with several first integrals. These integrals will be used to construct a Lyapunov function. In this construction we shall use the fact that a continuously differentiable function of a first integral is again a first integral. To see this consider the equation

$$\dot{x} = f(x) \text{ in } \mathbb{R}^n$$

with k first integrals $I_1, \cdots, I_k$ $(k < n)$, so $L_t I_i(x) = 0$, $i = 1, \cdots, k$. The function $F : \mathbb{R}^k \to \mathbb{R}$ is continuously differentiable and $F(I_1(x), \cdots, I_k(x))$ is a first integral of the equation:

$$L_t F = \sum_{i=1}^{k} \frac{\partial F}{\partial I_i} L_t I_i(x) = 0.$$

Example 8.4.
Consider a solid body rotating around a fixed point, which coincides with its centre of gravity. One can think for instance of a triaxial homogeneous ellipsoid which can rotate around its centre. A, B and C are the three moments of inertia; x, y and z are the projections of the angular velocity vector on the three coordinate axis. The equations of motion are

$$\begin{aligned} A\dot{x} &= (B - C)yz \\ B\dot{y} &= (C - A)zx \\ C\dot{z} &= (A - B)xy \end{aligned} \tag{8.7}$$

The axis corresponding with the moments of inertia concide with the coordinate axis. This means that a critical point represents an equilibrium solution corresponding with rotation of the solid body around one axis with a certain speed. We shall study the stability of the equilibrium solution $x = x_0 > 0$, $y = z = 0$ which corresponds with the smallest moment of inertia A as we shall assume $0 < A < B < C$. The matrix arising after linearisation in $(x_0, 0, 0)$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{C-A}{B}x_0 \\ 0 & \frac{A-B}{C}x_0 & 0 \end{pmatrix}$$

It is clear that this matrix has one eigen-value zero so that the Poincaré-Lyapunov theorems of chapter 7 do not apply. This is not unexpected as the critical points of system 8.7 are not isolated.
We can find first integrals of system 8.7 by deriving equations for dx/dy and dy/dz and by integrating these. We find the first integrals

$$\begin{aligned} V_1(x, y) &= A(C - A)x^2 + B(C - B)y^2 \\ V_2(y, z) &= B(B - A)y^2 + C(C - A)z^2. \end{aligned}$$

V_1, or better $V_1(x, y) - V_1(x_0, 0)$, is not sign definite in a neighbourhood of $(x_0, 0, 0)$, V_2 is semidefinite. One can construct a Lyapunov function in various ways, for instance

$$V(x, y, z) = [V_1(x, y) - V_1(x_0, 0)]^2 + V_2(y, z).$$

The function V is positively definite in a neighbourhood of $(x_0, 0, 0)$ while $L_t V = 0$. So the equilibrium solution $(x_0, 0, 0)$ is Lyapunov stable.
The reader should study the stability of the critical points which are found on the z-axis, corresponding with the largest moment of inertia C. With a similar computation one finds again stability.

8.4 Applications and examples

We mentioned already that, generally speaking, we do not know much about the existence and form of Lyapunov-functions. In each particular problem we shall have to use our intuition and to trust our luck. It is however easy to indicate the relation between the method of linearisation in chapter 7 and the direct method of Lyapunov.
To demonstrate this relation we shall not consider the general case, but we shall restrict ourselves to the system

$$\dot{x} = Ax + f(x)$$

with

$$\lim_{\|x\| \to 0} \frac{\|f(x)\|}{\|x\|} = 0, \ , \ x = (x_1, \cdots, x_n).$$

The constant $n \times n$-matrix A has the diagonal form, the eigenvalues $\lambda_1, \cdots, \lambda_n$ are all real and negative. With the Poincaré-Lyapunov theorem 7.1 we conclude that the trivial solution $x = 0$ is asymptotically stable.
Consider on the other hand the function

$$V(x) = x_1^2 + x_2^2 + \cdots + x_n^2.$$

This function is positively definite. We compute the orbital derivative

$$\begin{aligned} L_t V &= 2\sum_{i=1}^n x_i \dot{x}_i \\ &= 2\sum_{i=1}^n \lambda_i x_i^2 + 2\sum_{i=1}^n x_i f_i(x) \end{aligned}$$

For sufficiently small $\varepsilon > 0$ we have: if $\|x\| \leq \delta(\varepsilon)$ then $\|f(x)\| \leq \varepsilon \|x\|$ where $\delta \to 0$ if $\varepsilon \to 0$. As the eigenvalues λ_i are all negative we find that $L_t V$ is negatively definite in a neighbourhood of $x = 0$. $V(x)$ is a Lyapunov-function and application of theorem 8.2 yields the asymptotic stability of $x = 0$.

We shall consider some other examples.

Example 8.5.
In a neighbourhood of $(0, 0)$ we consider the system

$$\begin{aligned} \dot{x} &= a(t)y + b(t)x(x^2 + y^2) \\ \dot{y} &= -a(t)x + b(t)y(x^2 + y^2). \end{aligned}$$

The functions $a(t)$ and $b(t)$ are continuous for $t \geq t_0$. The trivial solution $(0, 0)$ is stable if $b(t) \leq 0$, unstable if $b(t) > 0$ for $t \geq t_0$. It is easy to demonstrate this;

take

$$V(x, y) = x^2 + y^2.$$

as a Lyapunov-function. V is positively definite whereas

$$L_t V = 2b(t)(x^2 + y^2)^2.$$

Application of the theorems 8.1 and 8.3 produces the required result.
The ideas which play a part in the proofs of Lyapunov's theorems can also be used to show the boundedness of solutions and attraction in a domain.

Example 8.6.
Consider the equation for a nonlinear oscillator with linear damping

$$\ddot{x} + \mu\dot{x} + x + ax^2 + bx^3 = 0$$

with constants $\mu, a, b; \mu > 0$. The damping $\mu\dot{x}$ causes the trivial solution to be asymptotically stable, see also example 7.1. We introduce the energy of the non-linear oscillator without damping:

$$V(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{1}{3}ax^3 + \frac{1}{4}bx^4.$$

We can find a neighbourhood D of $(0, 0)$, dependent in size on a and b, in which V is positively definite. Furthermore we have

$$\begin{aligned} L_t V &= \dot{x}\ddot{x} + x\dot{x} + ax^2\dot{x} + bx^3\dot{x} \\ &= -\mu\dot{x}^2. \end{aligned}$$

We can apply theorem 8.1 to find that $(0, 0)$ is Lyapunov-stable but we cannot apply theorem 8.2 to obtain asymptotic stability as $L_t V \leq 0$. As $L_t V \leq 0$ in D, the solutions cannot leave D (see the proof of theorem 8.1), also we have that the equality $L_t V = 0$ is only valid if $\dot{x} = 0$. However, if $x \neq 0$, $\dot{x} = 0$ is a transversal of the phase-flow so we conclude that D is a domain of attraction of the trivial solution.

8.5 Exercises

8-1. Consider the two-dimensional system

$$\begin{aligned} \dot{x} &= -y + f(x, y) \\ \dot{y} &= \sin x \end{aligned}$$

The function $f(x, y)$ is smooth and nonlinear.
Give sufficient conditions for $f(x, y)$ so that $(0, 0)$ is a stable equilibrium solution.

8-2. Equations of the form $\ddot{y} + p(t)\dot{y} + q(t)y = 0$ with $p(t), q(t)$ C^1 functions can be put in the form

$$\ddot{x} + w(t)x = 0 \quad , w(t) \text{ a } C^1 \text{ function,}$$

by the transformation of Liouville:

$$y(t) = x(t)e^{-\frac{1}{2}\int_{t_0}^{t} p(\tau)d\tau}.$$

Does Lyapunov-stability of $(x, \dot{x}) = (0, 0)$ imply the Lyapunov- stability of $(y, \dot{y}) = (0, 0)$?

8-3. Determine the stability of the solution $(x, \dot{x}) = (0, 0)$ of the set of equations $\ddot{x} + x^n = 0 \;\; (n \in \mathbb{N})$.

8-4. Consider the system

$$\dot{x} = 2y(z-1) \;,\; \dot{y} = -x(z-1) \;,\; \dot{z} = xy.$$

a. Show that the solution $(0, 0, 0)$ is stable

b. Is this solution asymptotically stable?

8-5. Determine the stability of the solution $(x, y) = (0, 0)$ of the system $\dot{x} = 2xy + x^3$, $\dot{y} = x^2 - y^5$.

8-6. Consider the equation

$$\ddot{x} + \phi(t)x = 0$$

with $\phi \in C^1(\mathbb{R})$, $\phi(t)$ is monotonic and

$$\lim_{t\to\infty} \phi(t) = c > 0.$$

a. Can we apply theorem 6.2 to prove stability of $(x, \dot{x}) = (0, 0)$. Consider also the counter-example 6.2.

b. Prove that $(0, 0)$ is stable.

8-7. The system $\dot{x} = Ax + f(x), x \in \mathbb{R}^n$ has the following properties. We have

$$\lim_{\|x\|\to 0} \frac{\|f(x)\|}{\|x\|} = 0$$

and the constant $n \times n$-matrix A has n distinct real eigenvalues $\lambda_1, \ldots, \lambda_n$ with $\lambda_1, \ldots, \lambda_p > 0$ and $\lambda_{p+1}, \ldots, \lambda_n < 0$, $0 < p < n$. Find a Lyapunov function to prove the instability of $x = 0$.

8-8. We are interested in conditions which quarantee that the following third-order oscillator

$$\dddot{x} + f(\dot{x})\ddot{x} + a\dot{x} + bx = 0$$

has a globally attracting trivial solution; a and b are positive constants.

8-9. Determine the stability of the trivial solution of

$$\dot{x} = xy^2 - \frac{1}{2}x^3 \;,\; \dot{y} = -\frac{1}{2}y^3 + \frac{1}{5}x^2y$$

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