

Chapter 5: Space and Shape

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ABSTRACT

Three perspectives about the possible roles of shape and space are discussed:

- 1) Interacting with real shapes in space,
- 2) Shape and space as fundamental ingredients for constructing a theory, and
- 3) Shapes or visual representations as a means for better understanding concepts, processes, and phenomena in different areas of mathematics and science.

Shapes are considered as dynamic entities where their ability to change is one of their main characteristics. The meaning of shapes is also changing, both between and within, each of the above perspectives.

Student's ability to visualize real objects, mathematical concepts, processes, and phenomena is considered now as a mathematical activity like computing or symbolizing. Nevertheless, unlike numerical or algebraic education, visual education is often a neglected area in curricula, in particular in relation with the first perspective. This is the reason why the authors choose to describe three projects that invested efforts in a systematic development of visual education.

1. BACKGROUND: ABOUT SHAPE AND SPACE

What do mathematicians and mathematics educators mean when they speak about shape and space, or their role in education, or even more explicitly in mathematics education? This is not a very easy question and possible answers are not clear-cut.

Marjorie Senechal (1990), in her triggering chapter about shapes relates to this question, claims that shapes are patterns and that the concept of shape, like many other concepts, is an undefinable term. On the other hand she said: 'We assume we know what shapes are more or less: we know one when we see one, whether we see it with our eyes or in our imaginations' (p. 140). If we have difficulties with defining what shape is, we definitely have difficulties to describe what we mean, when we talk about space – The world around

us? The world of shapes and relations between them? And what does this mean? What is the kind of thinking we have to develop in order to interact successfully with this world, and what does it mean to interact successfully?

If one tries to come closer to these kind of questions, and to get some ideas of possible answers, one has to discuss the following three perspectives:

- Interaction with real shapes in space (see 1.1)
- Shape and space as fundamental ingredients in the construction of a theory (see 1.2)
- Shapes or visual representations as means for better understanding concepts, processes, and phenomena in different areas of mathematics and science. (see 1.3)

As can be inferred from the following, the boundaries between the three are quite vague.

1.1 First perspective: Interacting with real shapes in space

Interacting with real shapes involves understanding the visual world around us (mostly what can be seen), its description, encoding and decoding of visual information that flows plentifully in this world. It also means interpretation of visual information. And more important, integrating all the above to handle, to cause and to understand changes on the shapes that populate our space. This is the space in which we have to deal with realistic problems typical to many areas; from simple everyday problems to problems in architecture, art, engineering etc.

According to Senechal (1990) meaningful interaction with real shapes in our space has three main goals: ‘To discover similarities and differences among objects, to analyse the components of form, and to recognize shapes in different representations’ (p. 140). She discusses and suggests a strand of topics related to shapes, through all school years, guided by three main tools: identification and classification of shapes, analysis of forms and representations and visualization of shapes.

In addition to Senechal’s claims, we would like also to emphasize *dynamic aspects*, such as the relative position of several shapes to each other, the relative position of an observer and the things that the observer looks at, and the processes of changing shapes.

1.2 Second perspective: Shape and space as fundamental ingredients for constructing a theory

The traditional geometry, which was codified by Euclid more than 2000 years ago, had started from what can be seen with the eyes. Space and the shapes

provide the environment in which the learner can get the feeling for a mathematical theory (Freudenthal 1973). This kind of environment, at a more advanced stage, acquires a broader and more abstract aspect, without the necessity of a real environment as a basis. (e.g. multi-dimensional Euclidean geometry or non-Euclidean geometries). But, even in the most abstract case we still deal with shapes and some sort of space, even when their visual representations cannot be seen with the eyes. What is left are representations in the minds' eyes (mental images), or in other words the theoretical representations.

1.3 Third perspective: Shapes or visual representations, as means for better understanding concepts, processes, and phenomena in different areas of mathematics, science and related subjects.

This perspective was discussed intensively by educators in the last decade (Eisenberg & Dreyfus 1991; Clements & Battista 1992). In the introduction to their book: *Visualization in teaching and learning mathematics*, Zimmerman & Cunningham (1991), even speak about 'the visualization renaissance' (p. 1). (We understand visualization as the transfer of objects, concepts, phenomenon, processes and their representations to some kind of visual representations and vice versa. This includes also the transfer from one type of visual representations to another.)

Researchers usually speak of one main reason for this renaissance. In our modern life the presentation of phenomena has changed from tables and formulae heavy with numbers and symbols, to a dynamic visual presentation on the computer monitor. The computer visualization becomes a scientific and mathematical tool. In order to understand, analyse and predict, we will have to engage in some visual thinking.

The second reason for the renaissance is expressed by Zimmerman & Cunningham (1991), saying that currently there are changes in the view of the nature of mathematics, whereby mathematics is seen as an on-going 'search for patterns' (Steen 1988, 1990), and this metaphor is surely a visual one. Therefore: 'It is natural to try to find the most effective ways to visualize these patterns and to learn to use visualization creatively as a tool for understanding' (p. 3).

1.4 The changeable nature of shapes and how it relates to visual thinking

In the following we will relate to shapes not only as static entities but rather as dynamic entities where their ability to change (not only in meaning) is one of their main characteristics. Shapes have different meaning and their role is to keep changing between and within the above perspectives. In the first per-

spective, where we relate to shape and space for its own sake, we are interested mostly in real objects which populate the real world, their shapes, and relations between them. There are many levels of abstraction concerning the meaning of shape in this perspective – for example, when we classify objects according to some chosen properties we eventually come to the essence of a shape as a representative of a whole class of objects. The abilities to perform different kind of classifications according to different kinds of characterizations (e.g. similarity and self similarity, combinatorial etc., (Senechal 1990)), to analyse different objects into their primitive shapes and above all to interpret and describe information from our real world are only some of the ingredients of visual thinking needed here.

The second perspective stresses a dual nature of shapes (here mainly geometrical figures); as Laborde (1993) expressed it: ‘Figures can be viewed as playing the role of reality with respect to theory as well as playing the role of model for a geometrical theory’ (p. 49).

The word ‘figure’ itself is ambiguous, because it can refer either to a geometrical object (i.e. conceptual, or ‘idealis’ in a platonic sense) or to a graphical representation (i.e. material) of such an object. This ambiguity is a well known source of difficulty for younger students because they do not understand that the objects referred to by their teacher are not the drawings (diagrams) which they can see in their textbooks, or on the blackboard, or that they realize themselves.

In this case the ‘figures’ (diagrams) are in fact tools for solving problems or to construct a geometrical theory (Euclidean geometry), and the process can be schematized by:

Diagram → Figure → Geometrical Theory.

But, in a dual way, a geometrical theory may be made more accessible by using a ‘model’ which can be visualized as for instance the Poincaré disc model with regard to the hyperbolic plane.

In this case the ‘reality’ (i.e. diagrams made on a sheet of paper) is not the starting point of the cognitive process, but of a process that is the reverse of the previous one:

Geometrical Theory → Figure (Model) → Diagram.

In the third perspective shapes are considered as the visual representations of mathematical or scientific entities (concepts, processes, phenomena).

Here in addition to the ingredients of visual thinking mentioned above, we have to develop the ability to interpret, understand, and create, the mutual relationships and analogies between the visual representations and other kinds of representations (Kaput 1992).

The various perspectives of shape and space on the one hand, and the interplay between concrete and abstraction on the other hand, are what make visual thinking both very rich but quite complicated; they are at the same time intuitive, global and analytical.

2. WHY VISUAL EDUCATION?

Visual education is needed for effective and correct interaction with shapes, relationships between them, transformations on shapes, relationships between shapes and other entities etc. This is true for the three perspectives mentioned above because whenever students have to relate to shapes and space, whether in their eyes or in their minds' eyes they have to apply some sort of visual thinking. Visual ways of thinking and reasoning may be acquired through a well planned visual education.

There is now a consensus that in the student's search for mathematical patterns and relationships within the various perspective mentioned above, visualizing is to be considered a mathematical action like computing, or symbolizing, (NCTM 1989; Senechal 1990). Nevertheless visual education is often a neglected area in curricula. This is especially true if we speak about the first perspective: interacting with the real world.

The reality of mathematics education in many cases only touches visualization and visual shapes indirectly:

- i) This happens in the traditional Euclid geometry course, which is still the most important school geometry course in many parts of the world. In such a course, where deductive reasoning is the main goal and shapes are the theoretical objects, knowing shapes and visual reasoning are considered, on one hand, as the intuitive basis for the development of higher levels of reasoning (See the Van Hiele's theory in Van Hiele 1987, Hoffer 1983, Crowley 1987). On the other hand, visual reasoning is considered as a low level reasoning that can even distort students' logical reasoning (Schoenfeld 1986). For example, in France, in the '1970's, this way of thinking about geometry was brought to its extreme consequences, when diagrams were positively prohibited from the teaching of geometry.
- ii) Lately, with the boom in changes in the views about mathematics education, where visualization in mathematics (the third perspective) becomes a popular subject, the development of visualization and visual thinking is expressed in the learning of the visual representations of mathematical concepts. Here also the main goal of teaching and learning is not the visual shapes for their own sake. In addition, for many years teaching multi-representational processes in mathematics meant a more or less systematic way of looking for dynamic processes in the analysis of algebraic and numeric representations. There was

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