Chapter 2

Elements of Linear System Theory

2.1 State-Space Description of Linear Systems

A linear system is described in the state space as

$$\dot{x} = Ax + Bu, \tag{2.1a}$$
$$y = Cx + Du, \tag{2.1b}$$

where $u \in R^r$, $y \in R^m$, and $x \in R^n$ are the input, the output, and the state, respectively. The differential equation (2.1a) is usually referred to as a *state equation*, which is solved to be

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau, \tag{2.2}$$

where

$$e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \cdots. \tag{2.3}$$

Differentiation of (2.3) yields

$$\frac{d}{dt}e^{At} = Ae^{At}. \tag{2.4}$$

In the description (2.1), the input $u(t)$ and the output $y(t)$ are *visible* externally, whereas the state $x(t)$ is not. The state is only accessible indirectly through the input $u(t)$ and the output $y(t)$. In this sense,

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the state has less physical reality than the input and the output. On the other hand, as was shown in (2.2), the state $x(0)$ at time $t = 0$ and the future input $\{u(t), t \geq 0\}$ determine the whole future behavior of the state $\{x(t), t \geq 0\}$. This implies that the state carries all the information about the past behavior of the system which is necessary to determine the future system behavior. If we know $x(0)$, we don't have to know anything else about the past behavior of the system, for example, $x(-1)$, to determine $x(t)$, $t \geq 0$. In this sense, the state is regarded as an *interface between the past and the future* with respect to the information carried over from the past to the future. From this point of view, we can say that the state is concerned with the *information* aspect rather than the *physical* aspect. Hence, we can choose the state in many different ways to represent physical aspects of the system, as long as it preserves the information. For instance, we can describe (2.1a) alternatively by a different frame of the state

$$x' = Tx, \tag{2.5}$$

where T is an arbitrary nonsingular matrix. The system (2.1) is then described by

$$\begin{aligned} \dot{x}' &= A'x' + B'u, && (2.6a) \\ y &= C'x + D'u, && (2.6b) \end{aligned}$$

where

$$A' = TAT^{-1}, \quad B' = TB, \quad C' = CT^{-1}, \quad D' = D. \tag{2.7}$$

The transformation (2.5) introduces a transformation (2.7) between the coefficient matrices of the state space descriptions. This transformation is called the *similarity transformation*, and the system (2.1) and the system (2.6) are said to be *similar* to each other. We sometimes write

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \overset{T}{\sim} \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix},$$

if (2.7) holds.

Using an appropriate similarity transformation, we can represent the system (2.1) in various ways which might be convenient for some specific purposes. For instance, A' in (2.7) can be chosen as the Jordan canonical form:

$$A' = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \cdots & & & \\ 0 & 0 & \cdots & J_k \end{bmatrix}, \quad J_i = \begin{bmatrix} \lambda_i & 1 & 0 & 0 \\ 0 & \lambda_i & 1 & 0 \\ \cdots & & & \\ 0 & 0 & 0 & \lambda_i \end{bmatrix},$$

or

$$A' = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

where $A_i, i = 1, 2$, have specific spectrum configurations. One of the central themes of linear system theory is the canonical structure of linear systems, which amounts to finding the invariants of similarity transformations.

A linear system is described alternatively by the transfer function. Laplace transformation of both sides of (2.1) with $x(0) = 0$ yields

$$sX(s) = AX(s) + BU(s),$$
$$Y(s) = CX(s) + DU(s),$$

where the Laplace transform of each signal in (2.1) is denoted by the corresponding capital letter. Eliminating $X(s)$ in the preceding relations yields

$$Y(s) = G(s)U(s), \tag{2.8}$$
$$G(s) : = C(sI - A)^{-1}B + D. \tag{2.9}$$

The matrix $G(s)$ relates $Y(s)$ with $U(s)$ linearly and is called a *transfer function* of the system (2.1). The relation (2.9) is usually represented as

$$G(s) := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \tag{2.10}$$

The preceding notation implies that $G(s)$ has a *realization* (2.1).

The transfer function represents the input/output relation generated by the system (2.1). The transfer function is an invariant of similarity transformation (2.7).

LEMMA 2.1 *The transfer function remains invariant with respect to the similarity transformation.*

Proof. Let $G'(s)$ be the transfer function of the system (2.6), Then, from (2.7), it follows that

$$\begin{aligned} G'(s) &= C'(sI - A')^{-1}B' + D' \\ &= CT^{-1}(sI - TAT^{-1})^{-1}TB + D \\ &= C(sI - A)^{-1}B + D = G(s) \end{aligned}$$

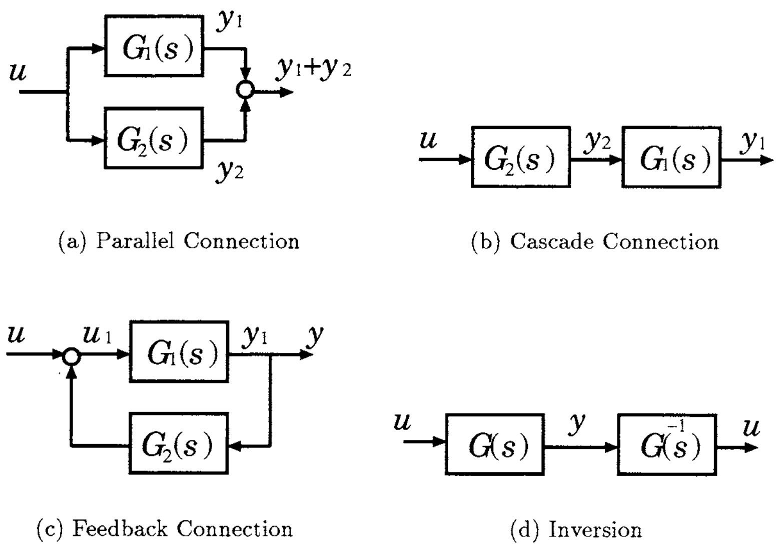
which establishes the assertion.

Now, a natural question arises whether the converse of the preceding lemma holds, that is, whether the two systems with the identical transfer function are similar to each other. This question is answered in the next section.

The invariance of the transfer functions with respect to similarity transformations is expressed as

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right]. \tag{2.11}$$

Before concluding the section, we collect state-space descriptions of various operations for transfer functions described in Figure 2.1.



(a) Parallel Connection (b) Cascade Connection

(c) Feedback Connection (d) Inversion

Figure 2.1 Connections of Transfer Functions.

We assume

$$G_i(s) = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right], i = 1, 2.$$

(a) *Parallel Connection*

$$G(s) := G_1(s) + G_2(s) = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]. \tag{2.12}$$

(b) *Cascade Connection*

$$\begin{aligned} G(s) := G_1(s)G_2(s) &= \left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_2 & 0 & B_2 \\ B_1C_2 & A_1 & B_1D_2 \\ \hline D_1C_2 & C_1 & D_1D_2 \end{array}\right]. \end{aligned} \tag{2.13}$$

(c) *Feedback Connection*

$$\begin{aligned} G(s) &:= (I - G_1(s)G_2(s))^{-1}G_1(s) \\ &= \left[\begin{array}{cc|c} A_1 + B_1D_2VC_1 & B_1WC_2 & B_1W \\ B_2VC_1 & A_2 + B_2VD_1C_2 & B_2VD_1 \\ \hline VC_1 & VD_1C_2 & VD_1 \end{array}\right], \end{aligned} \tag{2.14}$$

where $V := (I - D_1D_2)^{-1}$ and $W := (I - D_2D_1)^{-1}$.
(d) *Inversion*

$$G(s) := G(s)^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right]. \tag{2.15}$$

(e) *Conjugate*

$$G^{\sim}(s) := G^T(-s) = \left[\begin{array}{c|c} -A^T & C^T \\ \hline -B^T & D^T \end{array}\right]. \tag{2.16}$$

2.2 Controllability and Observability

The notions of controllability and observability are of central importance in linear system theory and connect the state-space description (2.1) and transfer function (2.9). The duality between controllability and observability appears in various ways throughout linear system theory and manifests itself as one of the most salient characteristic features of linear systems.

Consider a linear system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u, \tag{2.17a}$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2.17b}$$

Since $\dot{x}_2 = A_2 x_2$, this portion of the state is not affected by control input $u(t)$. In other words, the input $u(t)$ cannot control x_2 at all. Therefore, the system (2.17a) contains an uncontrollable part x_2 and such a system is called *uncontrollable.* A system that does not include an uncontrollable part is called *controllable.* More precisely, a system that cannot be expressed in the form (2.17a) by any similarity transformation is called controllable.

The preceding definition of controllability is indirect in the sense that it is defined by its negation. A more direct definition is as follows.

DEFINITION 2.2 *A system*

$$\dot{x} = Ax + Bu \tag{2.18}$$

is said to be controllable, *or the pair* (A, B) *is said to be* controllable, *if, for any initial condition* $x(0)$ *and* $t_1 > 0$, *there exists an input* $u(t)$ $(0 \leq t \leq t_1)$ *which brings the state at the origin, that is,* $x(t_1) = 0$.

The well-known characterizations of controllability are as follows.

LEMMA 2.3 *The following statements are equivalent.*

(i) (A, B) *is controllable.*

(ii) Rank $\begin{bmatrix} \lambda I - A & B \end{bmatrix} = n, \quad \forall \lambda.$

(iii) *(A, B) is not similar to any pair of the type*

$$\left(\begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix}\right). \tag{2.19}$$

(iv) Rank $[B \quad AB \quad \cdots \quad A^{n-1}B] = n.$

(v) *If $\xi^T(sI - A)^{-1}B = 0$ for each s, then $\xi = 0$.*

(vi) *If $\xi^T e^{At}B = 0$ for $0 \le t < t_1$ with arbitrary $t_1 > 0$, then $\xi = 0$.*

Proof. (i) $\rightarrow$ (ii).
If (ii) does not hold, then rank $[\lambda I - A \quad B] < n$ for some λ. Therefore, there exists $\xi \neq 0$ such that

$$\xi^T[\lambda I - A \quad B] = 0.$$

From (2.18), it follows that

$$\xi^T \dot{x}(t) = \lambda \xi^T x(t),$$

for any input $u(t)$. Thus, $\xi^T x(t) = e^{\lambda t}\xi^T x(0)$. If $x(0)$ is chosen such that $\xi^T x(0) \neq 0$, then $\xi^T x(t) \neq 0$ for any $t \geq 0$. This implies that $x(t) \neq 0$ for any $u(t)$. Hence, (A, B) is not controllable.

(ii) $\rightarrow$ (iii).
Assume that there exists a similarity transformation T such that

$$TAT^{-1} = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}, \quad TB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

Let λ be an eigenvalue of A_2^T with eigenvector ξ_2. Clearly, we have

$$\begin{aligned} &[0 \quad \xi_2^T]T[\lambda I - A \quad B]\begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} \\ &= [0 \quad \xi_2^T]\begin{bmatrix} \lambda I - A_1 & -A_{12} & B_1 \\ 0 & \lambda I - A_2 & 0 \end{bmatrix} = 0, \end{aligned}$$

which contradicts (ii).

(iii) $\rightarrow$ (iv).
Assume that rank $[B \quad AB \quad \cdots \quad A^{n-1}B] < n$. Then there exists a nonzero vector ξ such that $\xi^T B = \xi^T AB = \cdots = \xi^T A^{n-1} B = 0$. Let k be the maximum number such that $\{\xi, A^T\xi, \cdots, (A^T)^{k-1}\xi\}$ are linearly independent and let

$$T_2 = \begin{bmatrix} \xi^T \\ \xi^T A \\ \vdots \\ \xi^T A^{k-1} \end{bmatrix}.$$

Obviously, $k < n$, and $\xi^T A^k = \alpha_1 \xi^T + \alpha_2 \xi^T A + \cdots + \alpha_k \xi^T A^{k-1}$ for some real numbers $\alpha_1, \alpha_2, \cdots, \alpha_k$. Thus, we have

$$T_2 A = A_2 T_2,$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_k \end{bmatrix}.$$

Let T_1 be any matrix in $R^{(n-k)\times n}$ such that

$$T := \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

is nonsingular. Since $T_2 B = 0$, we can easily see that

$$TA = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} T, \quad TB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

for some A_1, A_{12} and B_1. This contradicts (iii).

(iv) $\rightarrow$ (v).
If $\xi^T(sI - A)^{-1}B = 0 \quad \forall s$ for some $\xi \neq 0$, the Laurent expansion of $(sI - A)^{-1}$ implies

$$(\xi^T s^{-1} + \xi^T A s^{-2} + \cdots + \xi^T A^k s^{-(k+1)} + \cdots)B = 0, \quad \forall s.$$

This obviously implies that

$$\xi^T A^k B = 0, \quad \forall k,$$

which contradicts (iv).

(v) $\rightarrow$ (vi).
Assume that $\xi^T e^{At} B \equiv 0$ for $0 \leq t < t_1$. Then, $d^k(e^{At}B)/dt^k = 0$ at $t = 0$ implies

$$\xi^T A^k B = 0, \quad \forall k.$$

This obviously implies $\xi^T(sI - A)^{-1}B = 0$ for each s.

(vi) $\rightarrow$ (i).
Let

$$M(t_1) := \int_0^{t_1} e^{-At} B B^T e^{-A^T t} dt.$$

If (vi) holds, $M(t_1)$ is nonsingular for any $t_1 > 0$. Indeed, if $M(t_1)\xi = 0$, we have

$$\xi^T M(t_1) \xi = \int_0^{t_1} \|\xi^T e^{-At} B\|^2 dt = 0.$$

This implies that $\xi^T e^{-At} B = 0$ for $0 \leq t < t_1$. Hence, $\xi = 0$.
Let

$$u(t) := -B^T e^{-A^T t} M(t_1)^{-1} x(0). \tag{2.20}$$

This input brings the state of (2.1a) to the origin at $t = t_1$. Indeed, we see that

$$\begin{aligned} x(t_1) &= e^{At_1} x(0) + \int_0^{t_1} e^{A(t_1 - t)} B u(t) dt \\ &= e^{At_1} \left(x(0) - \int_0^{t_1} e^{-At} B B^T e^{-A^T t} dt \cdot M(t_1)^{-1} x(0) \right) \\ &= 0, \end{aligned}$$

which establishes the assertion. ∎

If (A, B) is controllable, we can bring the state of the system (2.1a) to any position in the state-space at an arbitrary time starting at the origin. The term controllability was originated from this fact.

The dual notion of controllability is observability, which is defined as follows.

DEFINITION 2.4 *A system*

$$\dot{x} = Ax, \quad y = Cx \tag{2.21}$$

is said to be observable, or the pair (A, C) is said to be observable, if the output segment $\{y(t); 0 \le t < t_1\}$ of arbitrary length uniquely determines the initial state $x(0)$ of (2.21).

In (2.21), the input $u(t)$ is not included. The preceding definition can be extended in an obvious way to the case where input is incorporated.

A typical example of an unobservable system is described as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_2 & 0 \\ A_{21} & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \tag{2.22a}$$

$$y = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2.22b}$$

It is clear that the component x_2 of the state cannot affect $y(t)$ and hence cannot be estimated from the observation of $y(t)$.

The duality between the representations (2.17) and (2.22) is obvious. Actually, it is remarkable that the observability is characterized in the way completely dual to the controllability, as is shown in the following results:

LEMMA 2.5 *The following statements are all equivalent:*

(i) (A, C) *is controllable;*

(ii) rank $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n, \quad \forall \lambda;$

(iii) (A, C) *is not similar to any pair of the type*

$$\left(\begin{bmatrix} A_1 & 0 \\ A_{21} & A_2 \end{bmatrix}, [C_1 \quad 0] \right);$$

(iv) rank $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n;$

(v) *if* $C(sI - A)^{-1}\eta = 0$ *for each* s*, then* $\eta = 0$*;*

(vi) *if* $Ce^{At}\eta = 0$ *for* $0 \le t < t_1$ *with arbitrary* $t_1 > 0$, *then* $\eta = 0$.

The proof can be done in a similar way to that of Lemma 2.3 and is left to the reader.

If (A_1, B_1) is controllable in (2.17), then x_1 represents the controllable portion of the state, whereas x_2 denotes its uncontrollable portion. In this sense, the system (2.1a) is decomposed into the controllable portion and the uncontrollable one in (2.17a). The representation (2.17a) is schematically represented in Figure 2.2.

In the dual way, the representation (2.22) decomposes the state into the observable and the unobservable portions, provided that the pair (A_1, C_1) is observable. The schematic representation of the system (2.22) is given in Figure 2.3.

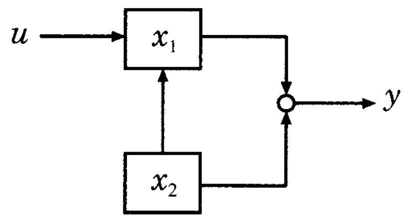


Figure 2.2 Controllable and Uncontrollable Portions.

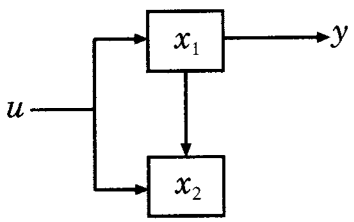


Figure 2.3 Observable and Unobservable Portions.

If (A, B) is controllable and (A, C) is observable, then the system (2.1) is said to be *minimal* or *irreducible*. The converse of Lemma 2.1 holds under the assumption that the systems under consideration are minimal. The following lemma is one of the most fundamental results in linear system theory.

LEMMA 2.6 *If two minimal realizations have an identical transfer* *function, then they are similar to each other. In other words, if*

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] \tag{2.23}$$

and both realizations are minimal, then

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \overset{T}{\sim} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}. \tag{2.24}$$

for some nonsingular T.

Proof. Let n be the size of A_1 and A_2 and define

$$M_i = [B_i \quad A_iB_i \quad A_i^2B_i \quad \cdots \quad A_i^{n-1}B_i], \quad i = 1, 2$$

$$W_i = \begin{bmatrix} C_i \\ C_iA_i \\ \vdots \\ C_iA_i^{n-1} \end{bmatrix}, \quad i = 1, 2.$$

Since $D_1 + C_1(sI - A_1)^{-1}B_1 = D_2 + C_2(sI - A_2)^{-1}B_2$ for each s, we have

$$D_1 = D_2, \quad C_1A_1^kB_1 = C_2A_2^kB_2, \quad \forall k,$$

by comparing each coefficient of the Laurent expansion of $(sI - A_i)^{-1}$, $i = 1, 2$. In other words,

$$W_1M_1 = W_2M_2. \tag{2.25}$$

Since both realizations in (2.23) are minimal, both $M_1M_1^T$ and $W_1^TW_1$ are invertible. Let

$$T_1 := M_2M_1^T(M_1M_1^T)^{-1}, \quad T_2 := (W_1^TW_1)^{-1}W_1^TW_2.$$

Due to (2.25), $T_2T_1 = (W_1^TW_1)^{-1}W_1^TW_2M_2M_1^T(M_1M_1^T)^{-1} = I$. Therefore, $T_2 = T_1^{-1}$. Also,

$$T_2M_2 = M_1, \quad W_2T_1 = W_1,$$

which implies that

$$B_1 = T_2B_2, \quad C_1 = C_2T_2^{-1}.$$

Since $W_1A_1M_1 = W_2A_2M_2 = W_1T_1^{-1}A_2T_1M_1$, we have

$$A_1 = T_1^{-1}A_2T_1,$$

which establishes (2.24) with $T = T_1$.

The description (2.17a) decomposes the state-space into the controllable portion and the uncontrollable one, whereas the description (2.22a) decomposes the state-space into the observable portion and the unobservable one. We can integrate these two decompositions to get the *canonical decomposition*, based on the controllability matrix M and the observability matrix W given, respectively, by

$$M := [B \quad AB \quad \cdots \quad A^{n-1}B],$$

$$W := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix},$$

which were used in the proof of Lemma 2.6.

We introduce the following four subspaces in $\mathbf{R}^n$:

$$R_a := M\ \mathrm{Ker}WM = \mathrm{Im}M \cap \mathrm{Ker}W, \tag{2.26}$$

$$R_b := \mathrm{Im}M \cap (\mathrm{Im}W^T + \mathrm{Ker}M^T), \tag{2.27}$$

$$R_c := \mathrm{Ker}W \cap (\mathrm{Im}W^T + \mathrm{Ker}M^T), \tag{2.28}$$

$$R_d := W^T\ \mathrm{Ker}M^TW^T = \mathrm{Im}W^T \cap \mathrm{Ker}M^T. \tag{2.29}$$

We see that

$$\mathrm{Im}M = R_a + R_b, \tag{2.30}$$

$$\mathrm{Ker}W = R_a + R_c. \tag{2.31}$$

Indeed, since $\mathbf{R}^n = \mathrm{Im}W^T + \mathrm{Ker}M^T + \mathrm{Ker}W \cap \mathrm{Im}M$, $\mathrm{Im}M = \mathrm{Im}M \cap (\mathrm{Im}W^T + \mathrm{Ker}M^T + \mathrm{Ker}W \cap \mathrm{Im}M) = \mathrm{Im}M \cap (\mathrm{Im}W^T + \mathrm{Ker}M^T) + \mathrm{Im}M \cap (\mathrm{Ker}W \cap \mathrm{Im}M) = R_b + R_a$. The relation (2.31) can be proven similarly. We also see that $R_a + R_b + R_c = \mathrm{Im}M + \mathrm{Ker}W$ and R_d is the orthogonal complement of $\mathrm{Im}M + \mathrm{Ker}W$. Hence,

$$\mathbf{R}^n = R_a + R_b + R_c + R_d. \tag{2.32}$$

Obviously, ImM and Ker$W$ are A-invariant. Hence,

$$AR_a \subset R_a, \tag{2.33}$$
$$AR_b \subset \mathrm{Im}M = R_a + R_b, \tag{2.34}$$
$$AR_c \subset \mathrm{Ker}W = R_a + R_c. \tag{2.35}$$

Also, we have

$$\mathrm{Im}B \subset \mathrm{Im}M = R_a + R_b, \tag{2.36}$$
$$\mathrm{Ker}C \supset \mathrm{Ker}W = R_a + R_c\ . \tag{2.37}$$

Let T_a, T_b, T_c, and T_d be matrices such that their columns span R_a, R_b, R_c, and R_d, respectively. The relations (2.33)~(2.37) imply the following representations:

$$A\begin{bmatrix} T_a & T_b & T_c & T_d \end{bmatrix} = \begin{bmatrix} T_a & T_b & T_c & T_d \end{bmatrix}\begin{bmatrix} A_{aa} & A_{ab} & A_{ac} & A_{ad} \\ 0 & A_{bb} & 0 & A_{bd} \\ 0 & 0 & A_{cc} & A_{cd} \\ 0 & 0 & 0 & A_{dd} \end{bmatrix},$$

$$B = \begin{bmatrix} T_a & T_b & T_c & T_d \end{bmatrix}\begin{bmatrix} B_a \\ B_b \\ 0 \\ 0 \end{bmatrix},$$

$$C\begin{bmatrix} T_a & T_b & T_c & T_d \end{bmatrix} = \begin{bmatrix} 0 & C_b & 0 & C_d \end{bmatrix}.$$

Due to (2.32), $T := \begin{bmatrix} T_1 & T_2 & T_3 & T_4 \end{bmatrix} \in \mathbf{R}^{n\times n}$ is invertible. Therefore, we have the following fundamental result.

THEOREM 2.7 [47] *Every state-space representation is similar to the following structure*

$$\frac{d}{dt}\begin{bmatrix} x_a \\ x_b \\ x_c \\ x_d \end{bmatrix} = \begin{bmatrix} A_{aa} & A_{ab} & A_{ac} & A_{ad} \\ 0 & A_{bb} & 0 & A_{bd} \\ 0 & 0 & A_{cc} & A_{cd} \\ 0 & 0 & 0 & A_{dd} \end{bmatrix}\begin{bmatrix} x_a \\ x_b \\ x_c \\ x_d \end{bmatrix} + \begin{bmatrix} B_a \\ B_b \\ 0 \\ 0 \end{bmatrix} u \tag{2.38a}$$

$$y = \begin{bmatrix} 0 & C_b & 0 & C_d \end{bmatrix} \begin{bmatrix} x_a \\ x_b \\ x_c \\ x_d \end{bmatrix} + Du. \tag{2.38b}$$

Each component of the state has the following meaning,

x_a : *controllable but unobservable portion,*

x_b : *controllable and observable portion,*

x_c : *uncontrollable and unobservable portion,*

x_d : *uncontrollable but observable portion.*

The representation (2.38) is often referred to as a *canonical decomposition.* Its schematic diagram is shown in Figure 2.4.

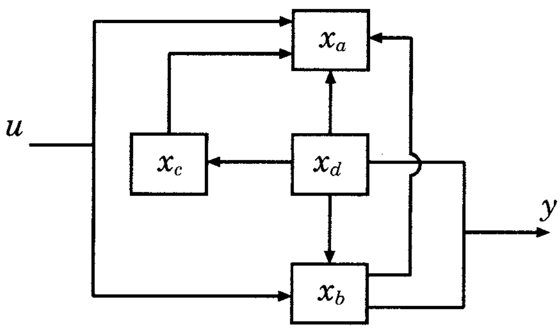


Figure 2.4 Canonical Decomposition.

In the preceding diagram, the only pass from the input u to the output y is through x_b. This implies that the transfer function $G(s)$ which represents the input/output relation of the system involves only the x_b-portion, that is, the controllable and observable portion. This is confirmed by the identity

$$\left[\begin{array}{cccc|c} A_{aa} & A_{ab} & A_{ac} & A_{ad} & B_a \\ 0 & A_{bb} & 0 & A_{bd} & B_b \\ 0 & 0 & A_{cc} & A_{cd} & 0 \\ 0 & 0 & 0 & A_{dd} & 0 \\ \hline 0 & C_b & 0 & C_d & D \end{array}\right] = \left[\begin{array}{c|c} A_{bb} & B_b \\ \hline C_b & D \end{array}\right]$$

$$= D + C_b(sI - A_{bb})^{-1}B_b.$$

2.3 State Feedback and Output Insertion

If the input u in (2.1a) is determined as an affine function of the state, that is,

$$u = Fx + u', \tag{2.39}$$

then u is called the *state feedback*. The additional control input u' can be determined for other purposes. Substitution of (2.39) in (2.1a) yields

$$\dot{x} = (A + BF)x + Bu'. \tag{2.40}$$

Thus, the state feedback (2.39) changes the A-matrix of the system from A to $A + BF$. Figure 2.5 shows a block diagram of the state feedback (2.39). As was stated in Section 2.1, the state vector contains enough information about the past to determine the future behavior of the system. In this sense, the state feedback represents an ideal situation because it can utilize the full information for feedback.

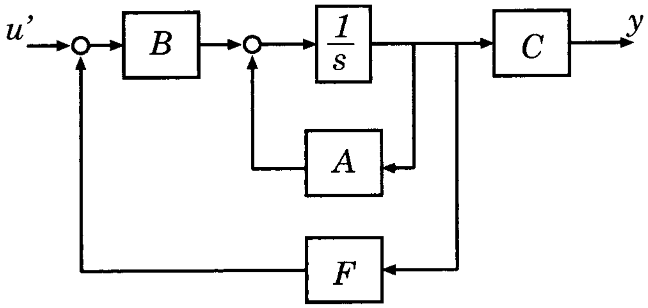


Figure 2.5 Block-Diagram of State Feedback.

The eigenvalues of A (denoted by $\sigma(A)$) are important factors that determine the behavior of the system (2.1). They are the singular points of the transfer function $G(s)$ and are called the *poles* of $G(s)$. The eigenvalues of $A+BF$ are the so-called *closed-loop poles* with respect to a state feedback (2.39). To choose the configuration of the closed-loop poles is one of the primal objectives of state feedback.

DEFINITION 2.8 *The pair (A, B) is said to be pole-assignable, if, for any set of complex numbers $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, there exists a state feedback (2.39) such that $\sigma(A + BF) = \Lambda$.*

The following fact holds which is one of the most fundamental results in linear system theory.

THEOREM 2.9 *The following statements are equivalent.*

(i) *(A, B) is pole-assignable.*

(ii) *(A, B) is controllable*

Proof. (i) $\rightarrow$ (ii).
If (A, B) is not controllable, rank $\begin{bmatrix} \lambda I - A & B \end{bmatrix} < n$ for some λ due to Lemma 2.3. Hence, $\xi^T(\lambda I - A) = 0$, $\xi^T B = 0$ for some $\xi \neq 0$. Therefore, for any F, $\xi^T(\lambda I - A - BF) = 0$. This implies that λ is an eigenvalue of $A + BF$ for each F. Hence, (A, B) is not pole-assignable.

(ii)$\rightarrow$(i).
We first prove the assertion for the case with scalar input; that is, $B = b \in \mathbf{R}^{n\times 1}$. In that case, (ii) implies that $\{b, Ab, \cdots, A^{n-1}b\}$ is linearly independent. Let $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ be an arbitrary set of n distinct complex numbers which are not the eigenvalues of A. We show that

$$f_i := (\lambda_i I - A)^{-1} b, \quad i = 1, 2, \cdots, n \tag{2.41}$$

is linearly independent. Indeed, we see that

$$(\lambda I - A)^{-1} b = (\psi_n(\lambda) A^{n-1} + \psi_{n-1}(\lambda) A^{n-2} + \cdots + \psi_1(\lambda) I) b / \psi_0(\lambda), \tag{2.42}$$

where $\psi_i(\lambda), i = 0, \cdots, n$ is the sequence of polynomials of degree i defined sequentially as follows:

$$\begin{aligned} \psi_0(\lambda) &= \det(\lambda I - A), \\ \psi_i(\lambda) &= (\psi_{i-1}(\lambda) - \psi_{i-1}(0))/\lambda, \quad i = 1, \cdots, n, \\ \psi_n(\lambda) &= 1. \end{aligned} \tag{2.43}$$

This implies

$$\begin{bmatrix} f_1 & f_2 & \cdots & f_n \end{bmatrix} = \begin{bmatrix} A^{n-1}b & A^{n-2}b & \cdots & Ab & b \end{bmatrix} \Theta \Psi,$$

$$\Theta := \begin{bmatrix} \psi_n(\lambda_1) & \psi_n(\lambda_2) & \cdots & \psi_n(\lambda_n) \\ \psi_{n-1}(\lambda_1) & \psi_{n-1}(\lambda_2) & \cdots & \psi_{n-1}(\lambda_n) \\ \vdots & \vdots & & \vdots \\ \psi_1(\lambda_1) & \psi_1(\lambda_2) & \cdots & \psi_1(\lambda_n) \end{bmatrix},$$

$$\Psi := \begin{bmatrix} 1/\psi_0(\lambda_1) & 0 & \cdots & 0 \\ 0 & 1/\psi_0(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\psi_0(\lambda_n) \end{bmatrix}.$$

Actually, the matrix Θ is calculated to be

$$\Theta = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \beta_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n-1} & \beta_{n-2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}, \qquad (2.44)$$

where $\psi_0(\lambda) = \det(\lambda I - A) = \lambda^n + \beta_1\lambda^{n-1} + \cdots + \beta_n$. Thus, Θ is nonsingular for distinct λ_i. Therefore, we conclude that f_i is independent. Let F be a $1 \times n$ matrix such that

$$Ff_i = 1, \quad i = 1, 2, \cdots, n$$

or, equivalently,

$$F = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & \cdots & f_n \end{bmatrix}^{-1}.$$

Due to (2.41),

$$(\lambda_i I - A - bF)f_i = b - b = 0.$$

This implies that λ_i is an eigenvalue of $A + bF$. Since λ_i is arbitrary, the assertion has been proven for the scalar input case.

The multi-input case can be reduced to the scalar-input case by utilizing the fact that if (A, B) is controllable, we can find $\overline{F}$ and $\overline{g}$ such that $(A + B\overline{F}, B\overline{g})$ is controllable. ∎

In the case where the system is not controllable, it is desirable that the uncontrollable mode is stable, that is, the matrix A_2 in (2.17a) is stable. In that case, the system is made stable by application of suitable state feedback.

DEFINITION 2.10 *The pair (A, B) is said to be stabilizable, if there exists a matrix F such that $A+BF$ is stable.*

Obviously, a controllable pair is stabilizable, but the converse is not true.

The dual of the state feedback (2.39) is the *output insertion* which is represented as

$$\dot{x} = Ax + Bu + Ly + \xi', \tag{2.45}$$

where ξ' is an auxiliary signal to be used for other purposes. A well-known identity state observer corresponds to the case where $\xi' = LCx$. The "closed-loop" poles are given by the eigenvalues of $A + LC$. Figure 2.6 shows a block diagram of output insertion (2.45). The dualization of Theorem 2.9 shows that the eigenvalues of $A + LC$ can be chosen arbitrarily by choosing L, if and only if (A, C) is observable.

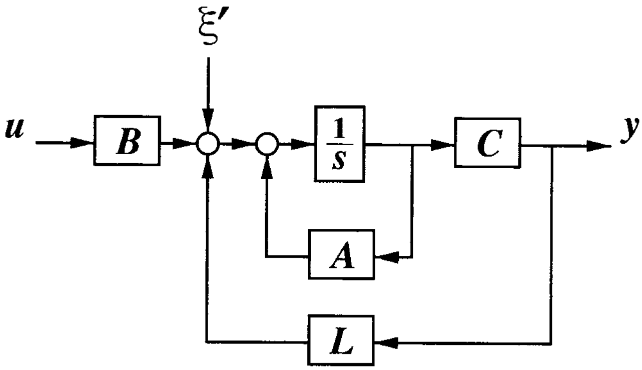


Figure 2.6 Block-Diagram of Output Insertion.

Through the output insertion, we can define the dual of stabilizable pair.

DEFINITION 2.11 *The pair (A, C) is said to be detectable if there exists a matrix L such that $A+LC$ is stable.*

Obviously, an observable pair is detectable, but the converse is not true.

The values of s at which $G(s)$ loses its normal rank are called the *transmission zeros* or the *zeros* of the system (2.1). If the realization (2.1) is minimal, z is a transmission zero of $G(s)$ iff

$$\operatorname{rank} \begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} < n + \min(m, r). \tag{2.46}$$

The transmission zeros are important factors for control system design. Sometimes, they are more important than the poles. For instance, the existence of right-half plane zeros severely limits the achievable closed-loop performance. It is important to notice that the transmission zeros are invariant with respect to the state feedback and the input transformation

$$u = Fx + Uu', \qquad U : \text{invertible}. \tag{2.47}$$

Indeed, application of (2.47) to (2.1) yields

$$\dot{x} = (A + BF)x + BUu', \tag{2.48a}$$
$$y = (C + DF)x + DUu'. \tag{2.48b}$$

It follows that

$$\begin{aligned} &\operatorname{rank}\begin{bmatrix} A + BF - \lambda I & BU \\ C + DF & DU \end{bmatrix} \\ &= \operatorname{rank}\left(\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}\begin{bmatrix} I & 0 \\ F & U \end{bmatrix}\right) \\ &= \operatorname{rank}\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}, \end{aligned}$$

which shows the invariance of zeros.

Also, the transmission zeros are invariant under the output insertion and output transformation

$$\begin{aligned} \dot{x} &= Ax + Bu + Ly \\ y' &= Vy, \qquad V : \text{invertible}, \end{aligned}$$

which gives rise to the state-space representation

$$\dot{x} = (A + LC)x + (B + LD)u, \tag{2.49a}$$
$$y' = V(Cx + Du). \tag{2.49b}$$

It follows that

$$\begin{aligned} &\operatorname{rank}\begin{bmatrix} A + LC - \lambda I & B + LD \\ VC & VD \end{bmatrix} \\ &= \operatorname{rank}\begin{bmatrix} I & L \\ 0 & V \end{bmatrix}\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \\ &= \operatorname{rank}\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}, \end{aligned}$$

which shows the invariance of zeros.

We summarize the preceding reasoning as follows,

LEMMA 2.12 *The transmission zeros of the transfer function are invariant with respect to state feedback, input transformation, output insertion, and output transformation.*

2.4 Stability of Linear Systems

Consider an autonomous linear system obtained by putting $u \equiv 0$ in (2.1a),

$$\dot{x} = Ax. \tag{2.50}$$

This system is said to be *asymptotically stable* if the solution $x(t)$ converges to the origin as $t \to \infty$ for each initial state $x(0)$. Since the solution of (2.50) is given by $x(t) = e^{At}x(0)$, the system (2.50) is asymptotically stable iff

$$\|e^{At}\| \to 0, \quad \text{as} \quad t \to \infty. \tag{2.51}$$

If (2.51) holds, A is said to be *stable* or *Hurwitz*. It is easily shown that A is stable iff each eigenvalue of A has a negative real part, that is,

$$\text{Re}\,[\lambda_i(A)] < 0, \quad i = 1, \cdots, n. \tag{2.52}$$

A simple method of checking the stability of A is known as the Lyapunov theorem which reduces the stability problem to solving a linear algebraic equation with respect to P given by

$$PA + A^T P = -Q. \tag{2.53}$$

The preceding equation is called a *Lyapunov equation.*

LEMMA 2.13 *The following statements are equivalent.*

(i) *A is stable.*

(ii) *There exists a positive number σ and M such that*

$$\|e^{At}\| \leq Me^{-\sigma t} \tag{2.54}$$

(iii) *For each matrix $Q > 0$, the Lyapunov equation (2.53) has a solution $P > 0$.*

Proof. (i) $\rightarrow$ (ii).
Since A is stable, there exists $\sigma > 0$ such that $-\sigma > \mathrm{Re}[\lambda_i(A)]$ for $i = 1, 2, \cdots, n$. In other words, $\sigma I + A$ is stable. Since $\|e^{(\sigma I + A)t}\| \rightarrow 0$, $\|e^{(\sigma I + A)t}\|$ is bounded for each t. Therefore, the inequality

$$\|e^{(\sigma I + A)t}\| \leq M \tag{2.55}$$

holds for some $M > 0$, which implies (2.54).

(ii) $\rightarrow$ (iii).
Due to (2.54), we can define

$$P := \int_0^\infty e^{A^T t} Q e^{At} dt.$$

Since $Q > 0$, $P > 0$. It is easy to see that P is a solution to Equation (2.53). Indeed,

$$\begin{aligned} PA + A^T P &= \int_0^\infty e^{A^T t} Q e^{At} A dt + \int_0^\infty A^T e^{A^T t} Q e^{At} dt \\ &= \int_0^\infty \frac{d}{dt}(e^{A^T t} Q e^{At}) dt = [e^{A^T t} Q e^{At}]_0^\infty \\ &= -Q. \end{aligned}$$

(iii) $\rightarrow$ (i).
Let λ be an eigenvalue of A and ξ the corresponding eigenvector, that is, $A\xi = \lambda\xi$. The premultiplication by ξ^* and the postmultiplication by ξ of (2.53) yield

$$(\lambda + \overline{\lambda})\xi^* P \xi = -\xi^* Q \xi.$$

Since both P and Q are positive definite from the assumption, we conclude that $\lambda + \overline{\lambda} < 0$. Hence, $\mathrm{Re}[\lambda] < 0$, which establishes the assertion. ∎

We sometimes use a positive semidefinite Q in (2.53), instead of a positive definite one. A typical example is the case $Q = C^T C$.

LEMMA 2.14 *Assume that (A, C) is observable. The Lyapunov equation*

$$PA + A^T P = -C^T C \tag{2.56}$$

has a positive definite solution iff A is stable.

Proof. If A is stable, the solution to (2.56) is given by

$$P = \int_0^\infty e^{A^T t} C^T C e^{At} dt \geq 0.$$

If $Px = 0$ for some x, we see that

$$\int_0^\infty \|Ce^{At}x\|^2 dt = 0.$$

Hence, $Ce^{At}x = 0, \forall t$. Due to Lemma 2.5(vi), $x = 0$. Hence, $P > 0$. Conversely, assume that the solution P of (2.56) is positive definite. Let λ be an eigenvalue of A with corresponding eigenvector ξ, that is, $A\xi = \lambda\xi$. As in the proof of Lemma 2.13, we have

$$(\lambda + \bar{\lambda})x^* Px = -\|Cx\|^2.$$

Since (A, C) is observable, $Cx \neq 0$. Hence, $\lambda + \bar{\lambda} < 0$, which verifies the assertion. ∎

We can state the converse of the preceding lemma.

LEMMA 2.15 *If the Lyapunov equation (2.56) has a positive definite solution for a stable A, then (A, C) is observable.*

Proof. Assume that (A, C) is not observable. Then there exist λ and x such that $Ax = \lambda x$, $Cx = 0$. From (2.56), it follows that $\bar{x}^T Px \cdot (\lambda + \bar{\lambda}) = 0$. If A is stable, $\lambda + \bar{\lambda} < 0$. Since $P > 0$, we conclude that $x = 0$. ∎

A stronger version of Lemma 2.14 is given as follows.

LEMMA 2.16 *Assume that (A, C) is detectable. The Lyapunov equation (2.56) has a positive semi-definite solution iff A is stable.*

The proof is similar to that of Lemma 2.14 and is omitted here.

The solution P of (2.56) is called the *observability Gramian* of the system (2.1). Its dual is the *controllability Gramian* defined as the solution of

$$PA^T + AP = -BB^T. \tag{2.57}$$

Assuming that A is stable, the controllability Gramian is positive definite, iff (A, B) is controllable.

Problems

[1] Assume that the eigenvalues of A are all distinct. Prove that if (A, B) is controllable, there exists a vector $g \in \mathbf{R}^r$ such that (A, Bg) is controllable.

[2] Show that if (A, B) is controllable, then there exist a matrix F and a vector g such that $(A+BF, Bg)$ is controllable.

[3] In the proof of Lemma 2.6, the sizes of A_1 and A_2 are assumed to be the same from the outset. Justify this assumption.

[4] Show that the sequence $\psi_i(\lambda)$ defined in (2.43) satisfies the identity

$$\psi_0(\lambda) = \psi_n(0)\lambda^n + \psi_{n-1}(0)\lambda^{n-1} + \cdots + \psi_0(0).$$

[5] Using the identity $\psi_0(A) = 0$ (Caley-Hamilton theorem), prove the identity (2.42).

[6] Show that the following two statements are identical.

(i) (A, B) is stabilizable.

(ii) rank $\begin{bmatrix} \lambda I - A & B \end{bmatrix} = n, \qquad \forall \text{Re } \lambda \geq 0.$

[7] Write down a state-space model of the electrical circuit in Figure 2.7 taking the terminal current as the input and the voltage as the output. Show that the state space form is uncontrollable if

$$R_1 R_2 = \frac{L}{C}.$$

In this case, derive the canonical decomposition of the state-space form. Compute the transfer function $Z(s)$ of the state-space form and show that $Z(s) = R_1$ in the case

$$R_1 = R_2 = \sqrt{\frac{L}{C}}.$$

Give a dynamical interpretation of this condition.

[8] In the proof of (2.30), the following facts were used. Show them.

(i) For any matrix U with n rows,

$$\mathbf{R}^n = \mathrm{Im}U + \mathrm{Ker}U^T.$$

(ii) For any subspaces R_1, R_2, and R_3 with $R_1 \supset R_3$,

$$R_1 \cap (R_2 + R_3) = R_1 \cap R_2 + R_3.$$

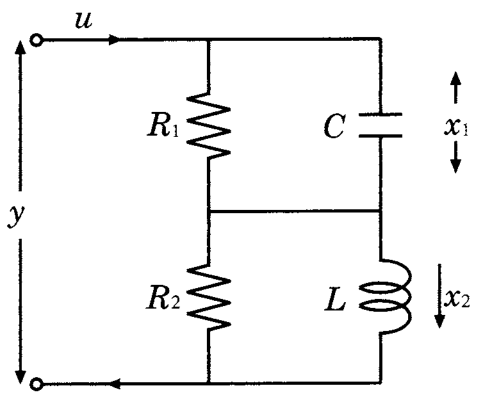


Figure 2.7 A one-terminal electrical circuit.

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