CHAPTER 1 Going Down the Drain

What have *Helianthus annuus* and *Helix pomatia* got in common? First of all, you probably need to know what these things are. *Helianthus annuus* is generally known as the (common) sunflower, while *Helix pomatia* is the common or garden French snail that finds its way onto dinner plates in fancy restaurants all around the world.

We suppose there's a sense in which both the sunflower and the escargot are edible. The one provides seeds to go in snacks and salads and edible oil which is used in margarine and for cooking, while the other provides what some people believe is a delectable source of protein. But the gastronomic connection is not what we had in mind.

1.1 Constructions

While you're working on that conundrum, try doing something more practical. In Figure 1 we have a spider web grid for you. You might like to photocopy or trace it, because we want you to start drawing all over it. While we're not against defacing books if it's in a good (mathematical) cause, you may want to use Figure 1 several more times. It's best to start with a clean version each time.

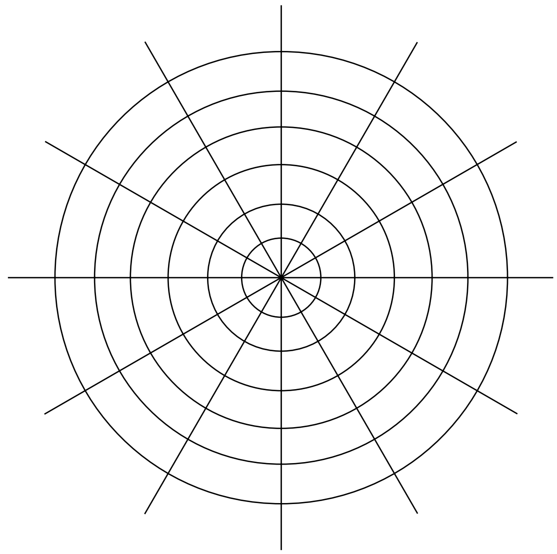
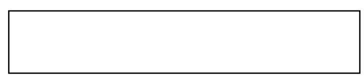


FIGURE 1.1



What do you see in Figure 1? There is a series of concentric circles whose radii are increasing at a constant rate. In fact, the radii are 1, 2, 3, 4, 5, and 6 units, respectively. Then there is a series of straight lines all of which pass through the central point. The angle between neighboring pairs of these straight lines is 30°. Actually, you'll notice that these lines go off to infinity in only one direction. We call such half-rays ***rays***.

Some of you may recognize Figure 1 as ***polar graph paper*** but we won't worry about that for a moment or two. What we are interested in is that you go off and find a rectangular piece of cardboard. You'll need a pencil too. We'll wait here while you go and get them.

Now look at Figure 2. Choose a point P_1, anywhere on one of the rays of Figure 1. Now put the cardboard on your polar graph paper so that one side touches P_1. Then slide the cardboard so that the adjacent side of the card touches the next ray (see Figure 2(a)). When you've done that, mark the point on this next ray which is at the corner of the right angle in your card. Call this new point P_2.

When you've got that organized, do the same thing again but this time start at the point P_2. So now one side of the card touches P_2 and the adjacent side of the card runs along the next ray around (see Figure 2(b)). Mark the point where the right angle touches this next ray and call it P_3.

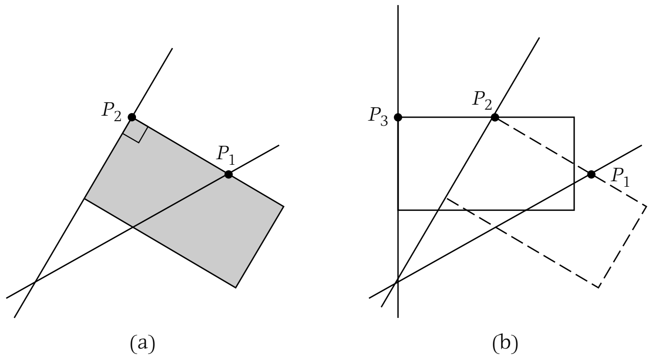


FIGURE 1.2

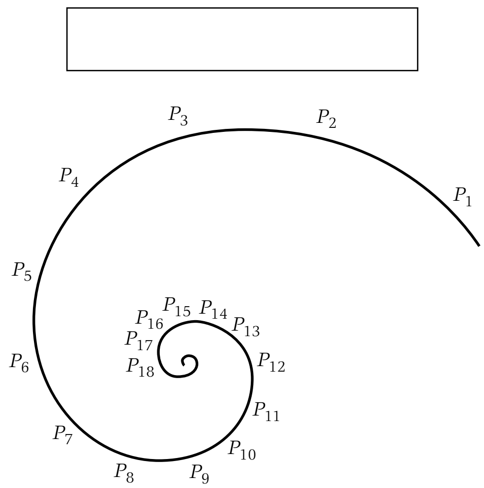


FIGURE 1.3

Once you've got the idea, continue until it's no longer physically possible to add any more points. Suppose that P_n is the last point that you were able to mark on your copy of Figure 1. Now join the points P_1, P_2, P_3, up to P_n in as smooth a curve as you can manage. You should produce a spiral similar to the one in Figure 3.

It's worth reflecting for a moment on what you have just done. You have just been involved in an *iterative* geometrical procedure which generates a sequence of points. This means that we perform an operation on one point (P_1 here) to get another (P_2). We then perform the same operation again but this time on the new point (P_2), to get the next point (P_3). We keep doing this over and over again.

In Chapter 3, you'll see us playing around with Fibonacci and Lucas numbers. There we will be iterating *numbers.* Here we are iterating *points.* Later on in this chapter, we'll tie up these two ideas.

In the meantime, we just want to stop for a minute because some of you may have got a different spiral from the one we've drawn in Figure 3. Our curve is spiraling inward in a *counterclockwise* (anticlockwise if you don't have a North American) direction. The different spiral that we've just mentioned would be spiraling in toward the center in a *clockwise* fashion!

• • • **BREAK**
You might like to think for a minute how that could possibly have happened, given the exquisitely accurate directions that we described above. • • •

Well, while you were thinking, we have looked back at our iterative instructions and have discovered that, although we pointed you to Figure 2, we didn't *actually* say that the ray that the right angle touched had to be the one in a counterclockwise direction from the ray the point P_1 was on. The misinterpretation that we noticed clearly put P_2 on the ray that was the next *clockwise* around from P_1. Obviously, this was the work of a left-handed person!

OK, so things can be done that way. For those of you who followed the implied counterclockwise direction of Figure 2, have another go, but this time do it clockwise. And for the people who did it clockwise the first time, would you mind having a try in the other direction now, please?

• • • **BREAK**
Can you manage to make your spiral go the other way? • • •

Fine! So now everybody should have *two* spirals, one with a clockwise decline and the other with a counterclockwise decline. This left-handed version we've shown in Figure 4.

But what we would dearly like to know is: Why is the spiral heading for the *center*? What are the alternatives? The points P_1, P_2, etc., could spiral *in* to the center, they could keep the *same distance* from the center, they could spiral *away* from the center, or they could exhibit erratic, exotic behavior not yet described in the pages of this *magnum opus*.

• • • **BREAK**
Why do the points spiral *in*? • • •

Before we start our erudicious explanation, you must write down a quick reason of your own. Nothing too elaborate, mind. Something like

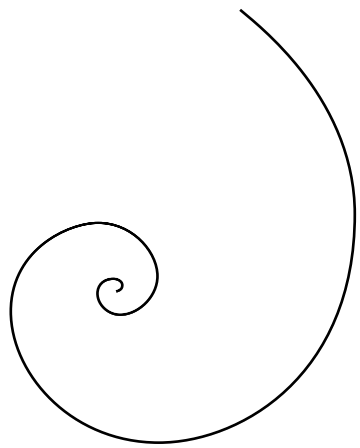


FIGURE 1.4

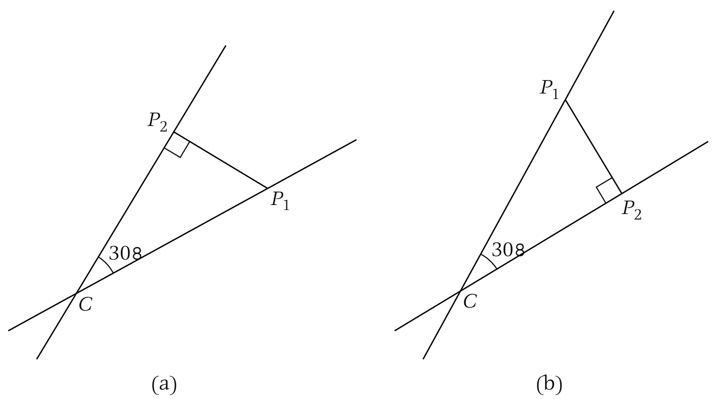


FIGURE 1.5

"the hypotenuse of a right-angled triangle is longer than either of the other sides" will do. In fact, if that's what you wrote, then you're on top of the game. That's exactly what's going on. Look at the counterclockwise iteration shown in Figure 5(a) and let C be the center of the polar graph paper. Then you'll see that ΔCP_1P_2 is right-angled at P_2. The hypotenuse of this triangle is CP_1. So clearly $CP_2 < CP_1$. This means that the point P_2 is closer to the center C than P_1. Hence the points go spiraling in as we move in a counterclockwise direction.

For the left-handed among us, the clockwise situation is dealt with in Figure 5(b). Of course, we haven't yet used all of the information available from the precise rules of the construction.

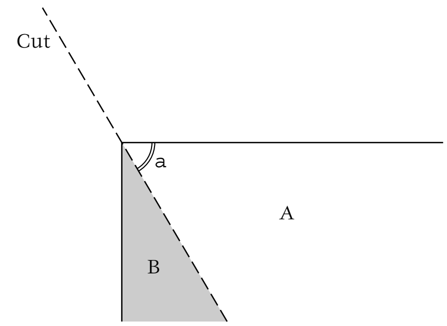


FIGURE 1.6

Now when you're on a good thing, stick to it. We'll just vary the iteration slightly. Take your card and a pair of scissors and cut off a right angle as shown in Figure 6. Make the angle α any size you want. Keep one part of the card to use straightaway. Call this part A, and the other part B, and put B aside somewhere. We won't need it for the moment but we will use it later on.

Now get hold of another copy of Figure 1 and use the A part of your card to go through the iterative process described above, all over again. The only difference now is that this time the angle α goes where the right angle went before. If you take an arbitrary point P_1 in any ray, and have one side of the card touching P_1 and the neighboring side of the card along the next ray, then P_2 is at the vertex of the angle α. You should be able to see how to continue from here. It's the same old routine.

• • • **BREAK**

The big question now is: "What sort of a curve did you get when you put a smooth curve through the points $P_1, P_2, \ldots$?" Did you get another spiral? Did it spiral in or not? Did it stay a constant distance from the center? Did it exhibit some exotic, erotic behavior? If so, what sort of behavior? • • •

So what happened? First of all, we'll assume that you all adopted the Figure 2 approach so that P_2 was counterclockwise from P_1 and so on. (Perhaps there is still the odd person who went the other way!) We've listed some possible outcomes in Figure 7. Which, if any, did you get?

The thing that interests us is that we can get *any* of the shapes in Figure 7! Those with some other sort of erratic behavior should go back

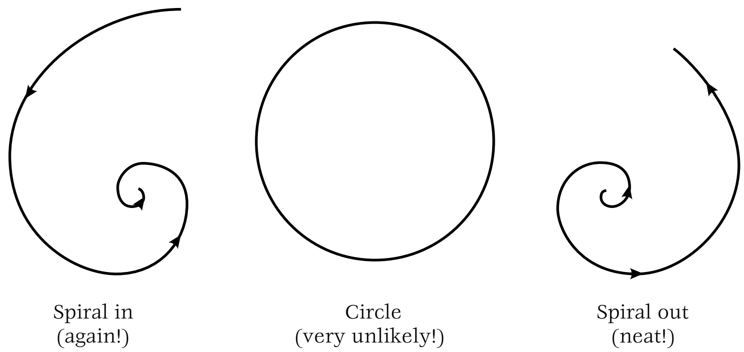


FIGURE 1.7

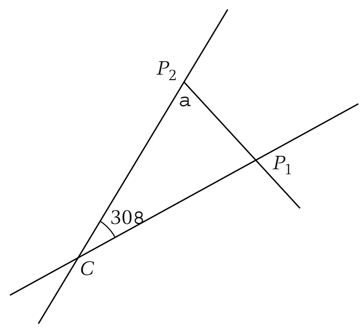


FIGURE 1.8

to the drawing board. The answer is definitely one of the curves in Figure 7, as we will now show.

Perhaps a diagram like Figure 5 will be of some help. We may be able to sort it all out with a simple right-angled triangle. Except when we look at Figure 8 there don't appear to be *any* right-angled triangles!

How can we compare CP_1 and CP_2? Is it possible that CP_1 could be *bigger* than CP_2 for some value of α? Could CP_1 actually *equal* CP_2? We know already that if $\alpha = 90°$, then CP_1 is *smaller* than CP_2 so that ought to be a possibility too.

Ah! Is that the clue? What do we know about the relative sizes of sides and their opposite angles? Surely the bigger side is opposite the bigger angle. So if the angle at P_2 is *bigger* than the angle at P_1, then CP_1 is *bigger* than CP_2.

Now if the angle at P_2 is α (as we know it is), then $180° - 30° - \alpha$ is the angle at P_1. So whether the iterative curve spirals in or out, depends on whether α is bigger or smaller than $150^\circ - \alpha$! So when is α bigger than $150^\circ - \alpha$?

Now	$\alpha > 150^\circ - \alpha$,
is equivalent to	$2\alpha > 150^\circ$,
or	$\alpha > 75^\circ$.

Those of you who had cut your card so that α was *bigger* than 75° found that your curve spiraled *in* because $CP_2 < CP_1$. Those of you who had α *smaller* than 75° has a spiral going *out* ($CP_2 > CP_1$). And one of you may have fluked a circle by taking α exactly equal to 75°.

α	curve
bigger than 75°	spiral in
equal to 75°	circle
smaller than 75°	spiral out

It's actually interesting to play around with α very close to 75° and see how long it takes for your spirals to move away from the circle.

• • • **BREAK**
Instead of using the A part of the card with the angle α, try using the angle $90^\circ - \alpha$ from the B part (see B in Figure 6). Is there any connection between the A and B curves? What about a right-handed A curve and a left-handed B curve? It's worth looking at Figure 1 again too. There we had rays that were 30° apart. What happens if you repeat the card construction with rays that are only 10° apart? What is the critical value of α for this case? • • •

If you use a 10° gap between rays you'll find it much easier to get a smooth curve than in the 30° case. However, it all takes a bit longer and you will have to be more careful with your construction because small errors mount up.

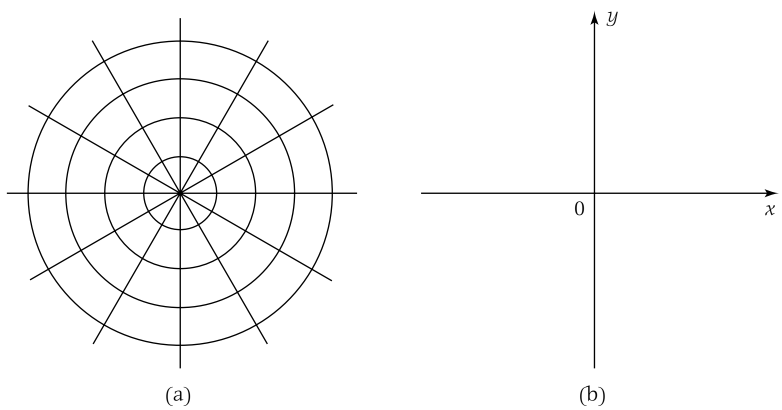


FIGURE 1.9

1.2 Cobwebs

We've drawn the cobweb of Figure 1 for you again in Figure 9(a). Compare it to the rectangular grid of Figure 9(b). In Figure 9(b) we've put in the x- and y-axes. You're probably used to this. It's easy to locate a point in the plane using the x- and y-coordinates. Anything that's x units horizontally away from the origin O and y units vertically away from O, is given the coordinates (x, y). The streets of many North American cities are laid out on such a rectangular grid, perhaps with the x-axis called Main Street and the y-axis called State Street. It makes it very easy to find your way around.

On the other hand, if you are a spider and you have just captured a particularly delicious *Musca domestica*, what you'll probably do is park it for a while to let it mature. Of course, you would like to remember where the *Musca domestica* is for future gastronomic purposes. It doesn't make any sense to superimpose a rectangular grid on your cobweb. Why not use what you've got directly? You've got a polar graph situation, why not use ***polar coordinates***? A fairly simple approach, using the web of Figure 9(a), would be to say, well, the *Musca domestica* is 15 units (probably centimeters but we won't bother to specify them precisely) from the center C and $60°$ around from the window ledge. (We're assuming here that the ledge has a ray that you, as the spider, are particularly fond of and that you have decided to use this as your reference point.) All you now have to do is to store the polar coordinates of the point M as $(15, 60°)$

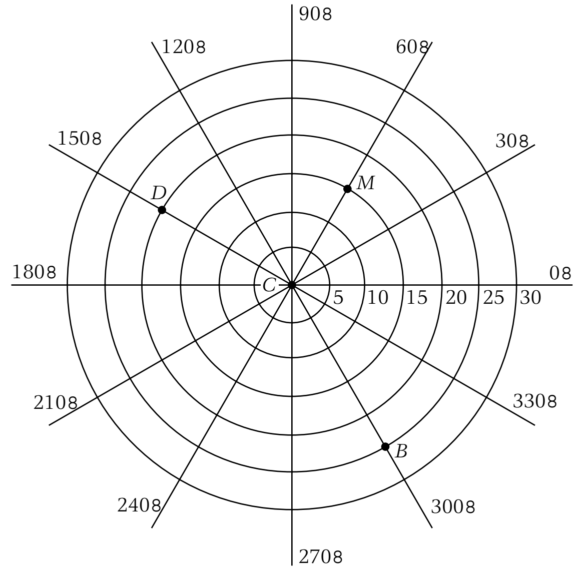


FIGURE 1.10

in your brain next to the *Musca domestica* and you'll know exactly where your next meal is coming from.

In Figure 10 we've shown the position of the *Musca domestica* as M. We also notice that you've gathered a few other interesting specimens in your web. For instance, there is a *Diptera culicidae* at D (reference $(20, 150^\circ)$) and a poor *Bombus bombus* at $B = (25, 300^\circ)$.

But there is one thing that you need to know straightaway. The more educated spiders amongst you use radians for angle measurement rather than degrees. This came about because you realized that when you walked once around your web one unit out from C, you actually traveled 2π units. So you thought of this as having turned through an angle of 2π radians. So, for spiders, 2π radians equals 360°. This means that $180^\circ = \pi$ radians, that $30^\circ = \frac{\pi}{6}$ radians, and so on.

• • • BREAK

Locate the positions of the *Diptera culicidae* and the *Bombus bombus* using polar coordinates (r, θ), where r is the distance from C and θ in radians is the angle turned through, starting from the ledge already mentioned. • • •

Actually when you think about it, the place where the *Musca domestica* is stored cannot only be described as $\left(15, \frac{\pi}{3}\right)$, but also as $\left(15, \frac{7\pi}{3}\right)$, and $\left(15, \frac{13\pi}{3}\right)$, and indeed $\left(15, \frac{\pi}{3} + 2n\pi\right)$, for any value of n, positive or negative! So unlike Cartesian coordinates, polar coordinates are not uniquely defined. It's always going to be possible to monkey around with the angle part to the tune of multiples of 2π. Now it's possibly a minor complication that there is more than one way to locate every point, but it does seem to be an easier way to locate objects on your web than using Cartesian coordinates. You never know, there may be some other advantages. Who knows?

Now if you were a particularly intelligent spider, you might be interested in the card construction of the last section. What's more, you might even start to draw spirals on your web. If you could manage a colored thread, then, no doubt, insects would be attracted from miles around you, and you and your descendants would therefore have an evolutionary advantage over the rest of your species. You might even take over the world eventually. We can just imagine huge webs, with colored spirals, attracting members of the species *homo sapiens* to their doom in droves.

But as you know, being a spider, it's a little hard to carry a card and pencils around with you to mark out the position of the next point in the spiral. It would be much easier to know the location of the next point so that you could lay out your colored spiral thread in that direction.

The big question then is, given the first point P_1, what is the location of the point P_2? Let's make life easier for you and put P_1 at $(5, 0)$ and use rays that are $\frac{\pi}{6}$ radians (or 30°) apart.

● ● ● **BREAK**

We'll also use the first card construction, where the card has a right angle at the corner as shown in Figure 11. If P_1 is at $(5, 0)$, where is P_2? ● ● ●

The coordinates of P_2 have to be found, right? Now we know that P_2 is on the $\frac{\pi}{6}$ ray. So $P_2 = \left(r, \frac{\pi}{6}\right)$. All we have to do is to find r. But ΔCP_1P_2 is a right-angled triangle. We know all the angles in this triangle (after all, $P_1CP_2 = \frac{\pi}{6}$, so $CP_1P_2 = \frac{\pi}{3}$). So we only need use a bit of trigonometry to see that $\frac{CP_2}{CP_1} = \cos\frac{\pi}{6}$. Therefore, $CP_2 = 5\cos\frac{\pi}{6} = \frac{5\sqrt{3}}{2} \simeq 4.3$. So P_2 is approximately $\left(4.3, \frac{\pi}{6}\right)$.

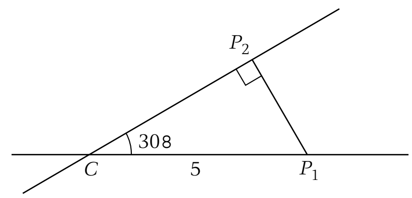


FIGURE 1.11

Using the same method, your spiderness could calculate the positions of a whole collection of points P_3, P_4, and so on. You could do this forever if you liked, though this might delay the colored spiral thread manufacture and your inheritance of the Earth.

But maybe you could find a formula which would give you all these points in one fell swoop. What a savings that would be! What a colossal evolutionary advantage. Soon spiders of all genera would be at your door for the rule that would provide the key to everlasting lashings of fast food, fully self-delivered to the table.

Before you get too many dreams of arachnidic grandeur you'd better find an equation for the spiral. What you need to be able to do is to find a relation between r and ϑ so that any point with coordinate (r, ϑ) lies on the spiral and no other points do. First, of course, we must find the relation satisfied by all the points P_n.

Let's have a look at the situation in Figure 12. This supposes that we know $P_n = (r_n, \vartheta_n)$ and we want to find $P_{n+1} = (r_{n+1}, \vartheta_{n+1})$. Once again, of course, $\vartheta_{n+1} = \vartheta_n + \frac{\pi}{6}$. So it's easy enough to find the angle part of the coordinate. But we have another right-angled triangle here. So $CP_{n+1} = CP_n \cos\frac{\pi}{6}$. This means that $r_{n+1} = r_n \cos\frac{\pi}{6}$. In other words, $P_{n+1} = \left(r_n \cos\frac{\pi}{6}, \vartheta_n + \frac{\pi}{6}\right)$.

Now that's all very well, and we know that you are only a spider, but if you want to get on in this world you are probably going to have to find an equation linking r and ϑ for the general point (r, ϑ). All you've been able to do is to give us an iterative relation between the coordinates of P_n and P_{n+1}.

Forget about that for a minute and let's see what we *can* work out. If we add $\frac{\pi}{6}$ to the angle every time we move on, then $P_1 = (r_1, 0)$, $P_2 = \left(r_2, \frac{\pi}{6}\right)$, $P_3 = \left(r_3, \frac{\pi}{3}\right)$, and so on. So the angle part of P_{n+1} should be just a multiple

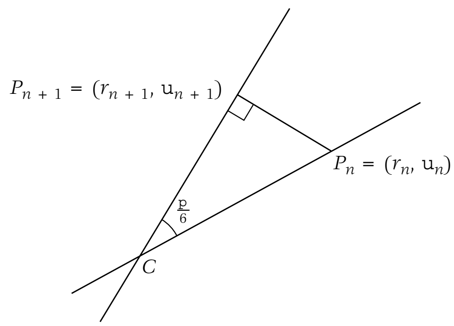


FIGURE 1.12

of $\frac{\pi}{6}$. Probably $P_{n+1} = \left(r_{n+1}, \frac{n\pi}{6}\right)$. Check that out to make sure it's OK. It is, so $\vartheta_{n+1} = \frac{n\pi}{6}$.

So can we calculate the distance from C in the same way? Let's tackle it the same way. We know that $P_1 = (5, \vartheta_1)$ and $P_2 = \left(r\frac{\sqrt{3}}{2}, \vartheta_2\right)$. From what we know about the way P_n and P_{n+1} are related

$$P_3 = \left(\left(5\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right), \vartheta_3\right), \quad P_4 = \left(5\left(\frac{\sqrt{3}}{2}\right)^3, \vartheta_4\right),$$

and so on. In general then, $P_{n+1} = \left(5\left(\frac{\sqrt{3}}{2}\right)^n, \vartheta_{n+1}\right)$. This means that we can at last give the complete polar coordinates for P_{n+1}. They are $\left(5\left(\frac{\sqrt{3}}{2}\right)^n, \frac{n\pi}{6}\right)$.

But how do we get a formula linking r and ϑ for the general point (r, ϑ)? Let's think what's going on for a minute. Suppose we let $r = 5\left(\frac{\sqrt{3}}{2}\right)^n$ and $\vartheta = \frac{n\pi}{6}$. Now both of these last equations have an n in them. What if we eliminate n? Won't we then have a relation between r and ϑ, satisfied by the coordinates of all points P_n?

Well, $\vartheta = \frac{n\pi}{6}$, so $n = \frac{6\vartheta}{\pi}$. Substituting for n in the r equation gives $r = 5\left(\frac{\sqrt{3}}{2}\right)^{\frac{6\vartheta}{\pi}}$. What a mess! Let's write it out large to see if it looks any better

$$r = 5\left(\frac{\sqrt{3}}{2}\right)^{\frac{6\vartheta}{\pi}}. \tag{1}$$

It certainly is a mess but it does seem correct. After all, the points P_n all satisfy it. The other points on the spiral are just what we get by smoothing between the P_n points. As well as that, we can easily see that as ϑ gets larger r gets smaller. This is because $\frac{\sqrt{3}}{2}$ is less than 1. As t approaches infinity $\left(\frac{\sqrt{3}}{2}\right)^t$ approaches zero. This means that r will approach zero as ϑ gets larger and larger. So this curve will definitely spiral *in* as we've already seen.

Young arachnid, we think you're on a winner here. You'll get so many insects in your new colored spiral web that you'll be able to sell them to all the spiders in the neighborhood. Just think of it. Arachdonalds! Selling fries and juicy Big Arachs!

1.3 Consolidation

The work of the last section has, of course, only opened up a can of worms. What equation would we get if we had used rays which were only $\frac{\pi}{18}$ apart? What equation would we find for cards which have corner angles α equal to $\frac{4\pi}{9}$, $\frac{7\pi}{18}$, $\frac{\pi}{3}$, and especially $\frac{5\pi}{12}$? How could we explain the right-handed and left-handed versions of all the spirals? There are clearly a lot of mathematical questions that are still unresolved here, not to mention the sunflower–escargot conundrum.

Now we have been looking for a relation between r and ϑ, starting from curves that we knew something about. We certainly knew how to construct them. Why don't we turn the questions around and look at (r, ϑ) relations to see what curves *they* produce? It's probably a good idea to start with something simple. We'll then say goodbye to you and let you explore to your heart's content.

So what could be simpler than $r = k$, a constant? In such a curve, the points are always a constant distance from the origin. Hence they must lie on a circle, center C.

Another simple equation that needs to be dealt with is $\vartheta = k$. Any point on the graph of this relation is always the same fixed angle from the initial direction. Hence we get a straight line which starts at C and heads off to infinity. Notice that we get a ray, not a complete line through C.

Another simple equation is $r = \vartheta$. What does the spider web graph of this relation look like? Well, there are at least three ways to go about answering that question. We could plot lots of points and join them all up, or we could use a graphing calculator, or we could think about what could happen.

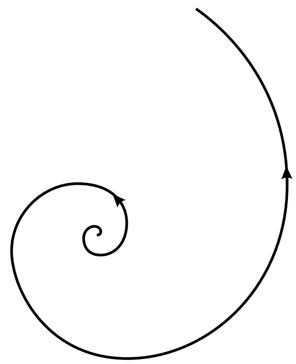


FIGURE 1.13

• • • **BREAK**
See if you can make any progress with the graph whose polar equation is $r = \vartheta$? • • •

If you've plotted points you may have found that they disappeared off your web pretty quickly. We hope that you changed your scale so that you were able to get points whose values of ϑ was larger than 2π.

Is there very much to say about $r = \vartheta$? As ϑ increases so does r. So the curve formed by the points (r, ϑ), where $r = \vartheta$, must spiral out from the center. It'll have to look like the curve in Figure 13.

Normally in polar coordinates, we only allow r to be positive or zero. After all, it is the *distance* of the point from the pole. However, in some books you will see r being allowed to be negative. We won't though, because we have an aversion to negative distances. Of course, ϑ is *generally* allowed to be *any* real number but, for any particular relation, we only allow those values of ϑ which make $r \geq 0$. Naturally, the point with coordinates (r, ϑ) is the same as the point with coordinates $(r, \vartheta + 2\pi)$.

Getting back to relations between r and ϑ, the next obvious things to try are the *linear* relations—things like $r = m\vartheta + c$, where m and c are fixed real numbers.

• • • **BREAK**
Why don't you see what sort of curves have equation
$r = m\vartheta + c$? You may need to use a combination of point-plotting and thinking. But thinking is always preferable if there's a choice.
You might like to try the special cases $m = 0$ and $c = 0$. • • •

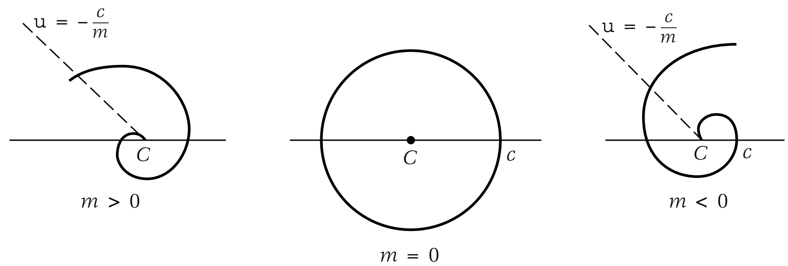


FIGURE 1.14

You've probably realized by now that all the relations with m positive give spirals. Since we allow ϑ to be negative, the spirals start at $\vartheta = -\frac{c}{m}$, because r can't be negative, that is, we require $\vartheta \geq -\frac{c}{m}$. We already know that if $m = 0$, we get a circle of radius c. For m negative, we require $\vartheta \leq -\frac{c}{m}$. All of these situations are shown in Figure 14.

Let's have a little deeper look into a special case of these ***linear*** spirals. So let $r = 2\vartheta$. The polar curve with this equation is given in Figure 15.

The interesting thing that we want to point out here is the constant nature of this curve. Look what happens every time it crosses the initial line.

TABLE 1. $r = 2\vartheta$.

ϑ	0	2π	4π	6π	8π
r	0	4π	8π	12π	16π

From the table you can see that the value of r increases each time by 4π. But the same thing happens no matter what ray we look at. As the curve spirals out, every time it crosses a fixed ray, it is 4π further out than the last time. To see this constant increase for r, take any ray, $\vartheta = \vartheta_1$, say. When the curve crosses that ray again, ϑ has increased by 2π to $\vartheta_1 + 2\pi$. At the first crossing, $r_1 = 2\vartheta_1$ and at the second $r_2 = 2(\vartheta_1 + 2\pi) = 2\vartheta_1 + 4\pi$. Clearly, the difference between r_1 and r_2 is 4π. And that constant difference applies no matter which ray the curve crosses.

You probably also managed to show that the same thing happened for any polar curve of the form

$$r = m\vartheta + c. \tag{2}$$

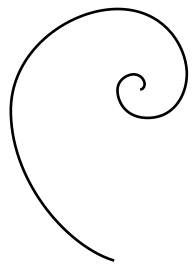


FIGURE 1.15

The argument is the same. At the ray $\vartheta = \vartheta_1$, we get $r_1 = m\vartheta_1 + c$ at first. The next time past this ray $\vartheta = \vartheta_1 + 2\pi$ and $r_2 = m(\vartheta_1 + 2\pi) + c$. So the difference between the two values of r is

$$r_2 - r_1 = (m\vartheta_1 + m2\pi + c) - (m\vartheta_1 + c) = 2\pi m.$$

Again, a constant increase. Again, the same increase occurs for *every* ray. Such curves are known as ***Archimedean spirals*** (see [2] and [3], for example).

● ● ● **BREAK**
Can you think where you might have seen Archimedean spirals?
● ● ●

If you have a non-zero constant c, in your Archimedean spiral, see (2), the curve looks as if it might follow the surface of some sort of material on a roll—dress material, for instance. But this isn't quite right. Material certainly winds around the roll adding a constant width once every time round. However, the start isn't quite right. On the other hand, if the constant is zero, the spiral above is just the kind of curve you get when you roll a length of something tightly up onto itself. Tape measures are sometimes rolled this way.

Now when we're dealing with Cartesian coordinates, polynomial relations give some interesting curves. But if we allow, say, $r = \vartheta^2 + 2\vartheta + 2$, then we find we don't get anything very exciting—just more spirals. So we'll try something different.

● ● ● **BREAK**
What do you think the polar curve with equation $r = \sin\vartheta$ looks like? Have a guess and then try to sketch it. ● ● ●

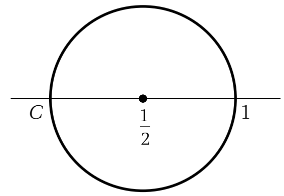


FIGURE 1.16

Before we do $r = \sin\vartheta$, let's have a look at $r = \cos\vartheta$. In fact, we'll show it in Figure 16. It's a circle, with center at $\left(\frac{1}{2}, 0\right)$ and radius $\frac{1}{2}$. How does that come about?

If you can't see how we got this, draw up a table of values. You should find that as ϑ goes from 0 to $\frac{\pi}{2}$, r goes from 1 to 0. There are no values for r with ϑ between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, but from $\frac{3\pi}{2}$ to 2π, r increases from 0 to 1 and the circle is completed. (If you take increasing values of ϑ from here, you just go round the circle again and again with suitable gaps every π radians, in the same way that you do from $\vartheta = 0$ to 2π). Alternatively, there is a straightforward proof using Cartesian coordinates. Starting with $r = \cos\vartheta$, multiply both sides by r to get $r^2 = r\cos\vartheta$. Since $x = r\cos\vartheta$ and $y = r\sin\vartheta$, we then see that $x^2 + y^2 = x$. Completing the square gives $\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$. Do you recognize this as a circle?

If you're still worried about $r = \sin\vartheta$, you should now be able to show that it too is a circle. This one, though, has center at $\left(\frac{1}{2}, \frac{\pi}{2}\right)$ in polar coordinates and again the radius is $\frac{1}{2}$.

• • • BREAK

Actually, there is an easy way of obtaining the graph of $r = \sin\vartheta$ from that of $r = \cos\vartheta$. Recall that $\cos\left(\frac{\pi}{2} - \vartheta\right) = \sin\vartheta$? How does that help you sketch $r = \sin\vartheta$? What is the effect on the curve $r = \cos\vartheta$ of changing ϑ to $\frac{\pi}{2} - \vartheta$? • • •

Looking at trigonometric functions opens up floodgates. You should find a lot of interesting shapes of the form $r = \cos 2\vartheta$, $r = \cos 3\vartheta$, and so on. Something like $r = 1 - \sin\vartheta$ is interesting too. If you're hooked on these polar curves, we suggest you try to graph a few more of them.

If we can remind you of your spider days in the previous section, remember that we came up with a polar equation of the form $r = ka^{\vartheta}$. (In actual fact k was 5 and a was $\frac{\sqrt{3}}{2}^{\frac{6}{\pi}}$. Now this curve has an interesting property. Look at the values of r for two values of ϑ.

If $\vartheta = \vartheta_1$, $r_1 = ka^{\vartheta_1}$ and if $\vartheta = \vartheta_2$, $r_2 = ka^{\vartheta_2}$.

"So what?" we hear you ask. OK, so take the ratio $r_2 : r_1$. Then

$$\frac{r_2}{r_1} = \frac{ka^{\vartheta_2}}{ka^{\vartheta_1}} = a^{\vartheta_2 - \vartheta_1}. \tag{3}$$

So here's the insight. If we take any two values of ϑ which differ by a given amount (($\vartheta_2 - \vartheta_1$) is constant), the resulting ratio $\frac{r_2}{r_1}$ is the *same*, no matter where you are on the spiral. Because of this property, curves with polar equation $r = ka^{\vartheta}$ are called ***equiangular spirals*** (for more details see [2] and [3]).

The property also means that, in some sense, the curve is self-similar. The distance from the origin increases by the same amount (by the same ratio) for every constant angle that the spiral goes through. Every section of the spiral is then a replication of the previous section. Zooming in (or out) on the spiral you see the same shape. Hence the spiral is much like a fractal (see Chapter 8).

1.4 Fibonacci Strikes

We have constructed spirals with cards on cobwebs but there are other methods of construction. Take your favorite sequence—the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, ... comes to mind (see Chapter 3)—and use a polar grid with rays at angles of $\frac{\pi}{6}$ for a start. When $\vartheta = 0$, let $r = F_1 = 1$. When $\vartheta = \frac{\pi}{6}$, let $r = F_2 = 1$. Keep going so that when $\vartheta = n\frac{\pi}{6}$, $r = F_{n+1}$.

• • • **BREAK**
Plot the Fibonacci points as indicated above and draw a smooth curve between them. You should get a respectable spiral. Check it out. • • •

You have probably ended up with something like the graph of Figure 17. Actually, with a little work, you can give a relation between r and ϑ for this curve. Have a go and see what you come up with.

But you don't *have* to use the Fibonacci sequence. Something like 1, 2, 4, 8, 16, 32, ... or 1, 4, 9, 16, 25, ..., will give you a spiral too.

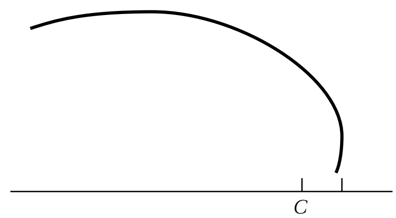


FIGURE 1.17

● ● ● **BREAK**
Experiment with different sequences of numbers and see what your spirals look like. ● ● ●

Just to wrap this one up, let's find the relation between r and ϑ for the curve formed from 1, 2, 4, 8, 16, ... Using rays $\frac{\pi}{6}$ apart, when $\vartheta = n\frac{\pi}{6}$, r must be 2^{n-1}. So $r = 2^{\frac{6\vartheta}{\pi}-1} = \frac{1}{2}\left(2^{\frac{6\vartheta}{\pi}}\right)$. It looks as if we've ended up with another equiangular spiral. Did you get an equiangular spiral for the Fibonacci curve?

● ● ● **BREAK**
Use the Binet formula (see Chapter 3) to express the Fibonacci curve by an equation in polar coordinates. ● ● ●

1.5 Dénouement

We started off this chapter asking what there is in common between the sunflower and the snail. The answer is that the seeds of the sunflower and the shell of the snail both exhibit a spiral structure. If you look at the snail's shell, you'll see a clear spiral. What may not be obvious, at first, is that the spiral is equiangular. "The whorls continually increase in breadth and do so in a steady and unchanging ratio" (see [1, Volume 2, p. 753]).

The same kind of behavior is to be found in the Nautilus shell and in many other shells, too. But it is not to be found in sunflowers. Instead, sunflowers exhibit the Fibonacci spiral behavior. This is illustrated in Figure 18.

In this chapter we have only skimmed the surface of the study of spirals and polar curves. There is a lot more out there to investigate. You

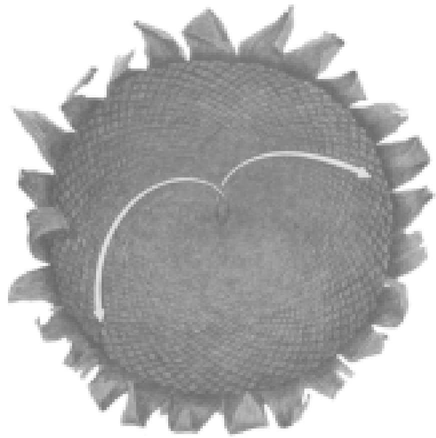


FIGURE 1.18

might actually like to do some of that investigating. If you do, don't forget about three-dimensional spirals. The common or garden helix not only occurs in circular staircases and in bed-springs but also seems to have something to do with DNA too. But what on earth is an Archimedean screw?

Oh! It's suddenly occurred to us that we haven't mentioned drains and that was in the title of the chapter. What's the path traced out by a fleck of fat as it goes down the drain? And does it matter whether the fat is in Sydney, Southampton, or Seattle?

• • • **FINAL BREAK**

Here are a few problems for you to try out your new skills on.

1. In Section 1, when using the card construction of a spiral we found that 75° (or $\frac{5\pi}{12}$) was a critical angle. Would the same angle be critical if the angle between the rays was changed?

2. Draw four equally spaced equiangular spirals on a piece of card. Pin the point C to the center of a turntable. What effect do you get when the turntable rotates?

3. Consider the polar curve whose equation is $r = \sin\vartheta$. What would happen if ϑ were allowed to be negative?

4. Find the polar equation for the circle, center (a, α) and radius b. [*Hint*: First get the Cartesian equation.] • • •

References

1. D'Arcy W. Thompson, *On Growth and Form*, Cambridge University Press, London, 1952.
2. E.H. Lockwood, *A Book of Curves*, Cambridge University Press, London, 1963.
3. H.S.M. Coxeter, *Introduction to Geometry*, Wiley, New York, 1961.

Answers

1. Suppose the angle between rays is ϑ. Then the critical case occurs when $\pi - \vartheta - \alpha = \alpha$. Hence the critical value of α is $\frac{1}{2}(\pi - \vartheta)$, which certainly depends on the angle ϑ between the rays.
2. You should get an interesting optical effect.
3. You get the same circle again and again.
4. In Cartesian coordinates the center is $(a\cos\alpha, a\sin\alpha)$ and the radius is b. So the equation is

$$(x - a\cos\alpha)^2 + (y - a\sin\alpha)^2 = b^2.$$

Simplifying, this becomes

$$x^2 + y^2 - 2xa\cos\alpha - 2ya\sin\alpha = b^2 - a^2.$$

Converting to polar coordinates, we obtain

$$r^2 - 2ra\cos\vartheta\cos\alpha - 2ra\sin\vartheta\sin\alpha = b^2 - a^2,$$

or

$$r^2 - 2ra\cos(\vartheta - \alpha) = b^2 - a^2.$$

The moral is that, when dealing with circles whose centers are not at the origin, it's easier to use Cartesian coordinates than polar coordinates.

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