Preface: Focusing Your Attention

The purpose of this book is (at least) two-fold. First, we want to show you what mathematics *is*, what it is *about*, and how it is *done*—by those who do it successfully. We are, in fact, trying to give effect to what we call, in Section 9.3, our *basic principle of mathematical instruction*, asserting that 'mathematics must be taught so that students comprehend how and why mathematics is done by those who do it successfully.'

However, our second purpose is quite as important. We want to attract you—and, through you, future readers—to mathematics. There is general agreement in the (so-called) civilized world that mathematics is important, but only a very small minority of those who make contact with mathematics in their early education would describe it as delightful. We want to correct the false impression of mathematics as a combination of skill and drugery, and to reinforce for our readers a picture of mathematics as an exciting, stimulating and engrossing activity, as a world of accessible ideas rather than a world of incomprehensible techniques, as an area of continued interest and investigation and not a set of procedures set in stone.

To achieve these two purposes, and to make available to you some good mathematics in the process, we have chosen to present you eight topics, organized into the first eight chapters. These topics are drawn both from what is traditionally described as applied mathematics and what is traditionally described as pure mathematics. On the other hand, these are not topics which are often presented to secondary students of mathematics, undergraduate students of mathematics, or adults wishing to update and upgrade their mathematical competence—and these are the three constituencies we are most anxious to serve. Naturally we hope that the teachers of the students referred to will enjoy the content of our text and adopt its pedagogical strategy.

Thus we have chapters on spirals in nature and mathematics, on the designing of quilts, on the modern topic of fractals and the ancient topic of Fibonacci numbers—topics which can be given either an applied or a pure flavor—on Pascal's Triangle and on paper-folding—where geometry, combinatorics, algebra, and number-theory meet—on modular arithmetic, which is a fascinating arithmetic of finite systems, and on infinity itself, that is, on the arithmetic of infinite sets. We have tried to cater to all mathematical tastes; but, of course, we do not claim to be able, through this or any other text, to reach those unfortunate people for whom all mathematical reasoning is utterly distasteful. Pythagoras inscribed on the entrance to his academy, 'Let nobody who is ignorant of geometry enter.' We might say, at this point, 'Let nobody who abhors all mathematics read any further.' We see this book as a positive encouragement to those who have already derived some satisfaction from some contact they have had with mathematics. We do not see it as performing a therapeutic function on the 'mathophobic'—unless, as is ofte the case, their mathophobia springs purely from a mistaken view of what mathematics *is.*

You will see that the eight chapters described above are largely independent of each other. We are not at all insisting that you read them in the order in which we have written them. On the other hand, we do also want to stress the *unity* of mathematics, so that there is bound to be a considerable measure of independence in the material we present. Cross-referencing will help you to find material from another chapter relevant to the chapter you are currently studying. We should add that Chapter 2 is particularly rich in ideas which play a part in the other chapters of the book.

We set our views of what mathematics *should be* in action, that is, how it should best be done, in our final chapter, Chapter 9. Naturally, this chapter is quite different in nature from those which precede it. It is not, however, different in tone; we continue the informal, friendly approach which, we very much hope, shows up clearly throughout our text. But we do believe that our readers may find it helpful to have available to them, in easily digestible form, some suggestions as to how to raise their expectations of success in tackling a mathematical exercise. That is our main purpose in including Chapter 9.

We would like to say a few words here about our notation and terminology and our expository conventions. First, we number the items in each chapter separately but as a unity; that is, we 'start again' with each chapter but, within a chapter, we do not take account of the various sections in our numbering system. Moreover, we have two numbering systems within a chapter; one is the numbering system for theorems, corollaries, examples, etc., and the other, appearing on the right side of the page, indicates the sequence of displayed formulae of special importance.

Second, we adopt certain conventions and practices in this text to help you to appreciate the significance of the material. From time to time within a chapter we introduce **BREAK**s which give you the opportunity to test your understanding of the material just presented. At the end of some chapters there is a **FINAL BREAK**, testing your understanding of the entire chapter, which is followed, where appropriate, by a list of **REFERENCES** and the **ANSWERS** to the problems in the final break. As to the references, they are numbered 1, 2, 3,..., and referred to in the text as [1], [2], [3],.... (You are warned that, in Chapter 2, [3] may be the residue class of the integer 3! The context will make this quite clear and, as we say at the end of the Preface, no notation can be reserved for eternity for one single idea.) From time to time we introduce harder material, which you may prefer to ignore, or to save for a second reading; the beginning and end of such material are marked by stars in the left-hand margin and the extent of the difficult material is indicated by a wavy line, also in the left-hand margin. Certain key statements and questions appear displayed in boxes; the purpose of this display is to draw your attention to ideas whose pursuit is going to determine the direction of the subsequent development.

Third, despite our informal approach, we always introduce you to the *correct mathematical term.* In particular, *theorems* are important assertions (not necessarily of a geometrical nature) which are going to be *proved.* *Conjectures*, on the other hand, are hypotheses which may or may not be true, but which we have some rational grounds for believing. It is the standard means of making progress in mathematics to survey what one knows, to make conjectures based on such a survey, to prove these (by logical argument) or disprove them (very often by finding *counterexamples*), to find consequences of the theorems thereby established, and thus to formulate new conjectures (compare Principle 4 in Chapter 9, Section 1). We employ the phrase 'if and only if' when we are claiming, or proving, that two propositions are *equivalent.* Thus 'proposition A if proposition B' means that 'B implies A', usually written '$B \Rightarrow A$'; while 'proposition A only if proposition B' means[1] that 'A implies B' or '$A \Rightarrow B$'. The obvious, simple notation for 'proposition A if and only if proposition B' is then '$A \Leftrightarrow B$'. As a final example, we may use the term *lemma* to refer to an assertion which is to be established for the explicit purpose of providing a crucial step in proving a theorem.[2]

It may also be helpful to say a word about the use of the letters of our alphabet, or the Greek alphabet, to represent mathematical concepts like numbers, functions, points, angles, etc.[3] There are a few—very few—cases in which, throughout mathematical writing, a certain fixed letter is always used for a particular concept; thus the ratio of the circumference of a circle to its diameter is always π, the base of natural logarithms is always e, the square root of -1 (regarded as a complex number) is always i, a triangle is always Δ, a sum is represented by $\sum$, and so on. However, this does *not* mean that every time these letters appear they refer to the concept indicated above; for example, i may be the subscript for the ith term a_i of a (finite or infinite) sequence of numbers. By the same token we cannot reserve, in the strict sense, any letter throughout an article, much less a book, so that it always refers to the same concept. Of course, different usages of the same letter should be kept far apart, so that confusion is not created; but the reader should remember that any particular usage has a *local* nature—local in place and time. There are far

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[1]Proposition A is proposition B' appears sometimes as 'B is a *sufficient* condition for A'; while 'proposition A only if proposition B' may appear as 'B is a *necessary* condition for A'. These terms, however, often create difficulties for students.

[2]In fact, the literature of mathematics contains many examples of lemmas which have become more famous than the theorems they were originally designed to prove (e.g., Zorn's Lemma in set theory, Dehn's Lemma in topology, and the Snake Lemma in homological algebra).

[3]You will find one use of a letter of the Hebrew alphabet. This is the standard use of the first letter $\aleph$, pronounced 'aleph', to represent the cardinality (size) of an infinite set in Chapter 7.

too many ideas in mathematics for each of them to be associated, for all time, with a particular letter of one of four alphabets (small or capital, standard or Greek)—or 40 alphabets, for that matter.

Now you're ready to start, and we hope you have an interesting and enjoyable journey through our text.

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