Solutions, answers, and hints for selected problems

Complete solutions of some problems are given. Answers only are given for some other problems. For still others, only hints or partial solutions are given. Asterisks in "A Modern Approach to Probability Theory" by Fristedt and Gray identify the problems that are treated in this supplement.

For Chapter 33

33-2. *Hint:* Let $(\mathcal{F}_t : t \geq 0)$ denote the minimal filtration of the Wiener process W. Square both sides of (33.1) and then take expectations. Six terms result on the right side. The following calculation shows that one of them is equal to 0:

$$E\Big(Z_{n\varepsilon} a(Z_{n\varepsilon})(W_{(n+1)\varepsilon} - W_{n\varepsilon})\Big) = E\Big(E\big(Z_{n\varepsilon} a(Z_{n\varepsilon})(W_{(n+1)\varepsilon} - W_{n\varepsilon}) \bigm| \mathcal{F}_{n\varepsilon}\big)\Big)$$
$$= E\Big(Z_{n\varepsilon} a(Z_{n\varepsilon}) E\big((W_{(n+1)\varepsilon} - W_{n\varepsilon}) \bigm| \mathcal{F}_{n\varepsilon}\big)\Big) = 0\,.$$

Similarly,

$$E\big(b(Z_{n\varepsilon})\varepsilon a(Z_{n\varepsilon})(W_{(n+1)\varepsilon} - W_{n\varepsilon})\big) = 0\,.$$

The following calculation is relevant for another of the six terms:

$$E\Big([a(Z_\varepsilon)]^2(W_{(n+1)\varepsilon} - W_{n\varepsilon})^2\Big) = E\Big(E\big([a(Z_\varepsilon)]^2(W_{(n+1)\varepsilon} - W_{n\varepsilon})^2 \bigm| \mathcal{F}_{n\varepsilon}\big)\Big)$$
$$= E\Big([a(Z_\varepsilon)]^2 E\big((W_{(n+1)\varepsilon} - W_{n\varepsilon})^2 \bigm| \mathcal{F}_{n\varepsilon}\big)\Big) = \varepsilon E\Big([a(Z_\varepsilon)]^2\Big)\,.$$

33-5. yes

33-12. $d(e^{\alpha W}) = \alpha e^{\alpha W}\, dW + \frac{1}{2}\alpha^2 e^{\alpha W}\, dt$

33-15. TO BE DONE

33-17. TO BE DONE

33-29. TO BE DONE

For Appendix A

A-2. The derivative $x \rightsquigarrow 1 - \cos x$ is positive for $-2\pi < x < 0$ and also for $0 < x < 2\pi$. A theorem of calculus says that a continuous function on a closed interval that has a positive derivative at all interior points of that interval is strictly increasing on the closed interval. Therefore the given function is strictly increasing on the interval $[-2\pi, 0]$ and on the interval $[0, 2\pi]$. By the preceding problem it is strictly increasing on the interval $[-2\pi, 2\pi]$. (Notice that the argument can be extended to prove that the given function is strictly increasing on R.)

For Appendix B

B-1. Proof that a closed subset of a compact set is compact. Let B be a closed subset of a compact set C, and let $\mathcal{O}$ be an open covering of B. Consider $\mathcal{O} \cup \{B^c\}$, the collection obtained by adjoining the complement of B to the collection $\mathcal{O}$. This collection is an open covering of C. It contains a finite subcovering of C. The members of $\mathcal{O}$ in this finite subcovering of C constitute a finite subcovering (from $\mathcal{O}$) of B.

B-5. The 'only if' part is trivial. We will prove the contrapositive of the 'if part', so suppose that the sequence does not converge to y. Then there exists $\varepsilon > 0$ and an infinite subsequence $(x_{n_k} : k = 1, 2, \dots)$ of (x_n) such that $\rho(x_{n_k}, y) > \varepsilon$ for all k. No further subsequence of this subsequence can converge to y because the distance between y and every member of that further subsequence would be greater than ε.

For Appendix C

C-5. Suppose that $x \in \partial B$. Case 1, $x \in B$: Every neighborhood of x contains a member of B —namely x itself. If some neighborhood did not contain a member of B^c, then x would be a member an open subset of that neighborhood which itself would be a subset of B. Hence x would belong to the interior of B and thus not to ∂B.

Case 2, $x \notin B$: Now we must show that every neighborhood of x contains a member of B. If there were some neighborhood lying entirely inside B^c, there would be an open subset of that neighborhood containing x and having the same property. The complement of that open set would be a closed set containing B and thus containing the closure of B. Therefore x would not belong to ∂B.

For the converse suppose that every neighborhood of x contains at least one point of B and least one point in B^c. First we observe that x cannot be a member of the interior of B, for, if it were, this interior would be a neighborhood of x that contains no member of B^c. To finish the proof we must show that x belongs to the closure of B. If it did not, the complement of the closure of B would be a neighborhood of x containing no point of B, which is a contradiction.

C-6. *Hint:* Avoid doing work similar to that needed for the preceding problem.

C-9. $[a, b)$, both open and closed whether $b < \infty$ or $b = \infty$; $(a, b]$, neither open nor closed whether $a > -\infty$ or $a = -\infty$; $[a, b]$ closed but not open; (a, b) open but not closed whether a and b are finite or infinite; $[a, a]$ is only compact interval

C-10. Closure under arbitrary unions: clearly yes if all sets in the union belong to $\mathcal{O}$; if one of the sets in the union contains ∞ and has a complement that is a compact subset C of Ω, the union will contain ∞ and have a complement that is closed subset of the compact subset C of Ω. An appeal to Proposition 1 completes this portion of the proof.

Closure under finite intersections: clearly yes if one of the sets in the intersection does not contain ∞; if all do contain ∞, then so does the intersection and the complement of the intersection is the union of a finite number of compact subsets of Ω. The definition of compactness shows that a finite union of compact sets is compact.

Compactness: An open covering must have at least one set that contains ∞. Take any such set O. The remaining sets in the open covering cover the compact complement of O. Thus there is a finite subcovering of this complement. Adjoin O to this finite subcovering to obtain a finite subcovering of Ω^*.

C-14. The closed interval $[0, 1]$ of R with the usual topology is not open in that topology, but it is an open subset of the topological space $[0, 1]$ with the relative topology.

Now assume that $\Psi \in \mathcal{O}$ and that $O \subset \Psi$ is open in the relative topology on Ψ. Then $O = A \cap \Psi$ for some $A \in \mathcal{O}$. Hence, O, the intersection of two members of $\mathcal{O}$, is itself a member of $\mathcal{O}$.

For Appendix D

D-1. 30

D-2. $\frac{3}{11}$

D-14. According to Theorem 4 we only need prove that f is Riemann-Stieltjes integrable with respect to g, and for doing that, Proposition 2 says that we only need prove that f is bounded and fg' is Riemann integrable.

Suppose that f is unbounded. For each m there exists $x_m \in [a, b]$ such that $|f(x_m)| > m$. Let x denote a limit of a subsequence of (x_m). It cannot be that infinitely many members of the subsequence equal x. If infinitely many members are larger than x, then $f(x+)$ does not exist. If infinitely many members are smaller than x, then $f(x-)$ does not exist. Therefore the assumption that f is unbounded leads to a contradiction, and hence f is bounded.

For future use we show that for each $\delta > 0$, there exists only finitely many x such that

$$f(x-) \vee f(x) \vee f(x+) > \delta + f(x-) \wedge f(x) \wedge f(x+)\,.$$

If there were infinitely many, then at the limit y of a convergence sequence of distinct such x, either $f(y+)$ or $f(y-)$ would fail to exist.

Turning to the proof of Riemann integrability of fg', we let $\varepsilon > 0$. For each $x \in [a, b]$ let J_x be an open interval in $[a, b]$ such that

- $x \in J_x$,
- $|f(y) - f(x+)| < \frac{\varepsilon}{4(b-a)}$ if $x < y \in J_x$,
- $|f(y) - f(x-)| < \frac{\varepsilon}{4(b-a)}$ if $x > y \in J_x$.

(Reminder: Intervals in $[a, b]$ including the endpoint a or b can be open in the relative topology of $[a, b]$. Alternatively, we could have let J_a and J_b be open intervals in R containing members outside the interval $[a, b]$.) Since $[a, b]$ is compact there exists a finite collection of intervals J_x whose union equals $[a, b]$. Let $\widehat{P}$ be the point partition of $[a, b]$ consisting of the endpoints of the intervals in this finite collection and the points midway between two consecutive endpoints.

For each point x for which

$$f(x-) \vee f(x) \vee f(x+) > \tfrac{\varepsilon}{4(b-a)} + f(x-) \wedge f(x) \wedge f(x+)\,,$$

of which there are only finitely many—say q —introduce a close interval $K_x \subseteq [a, b]$ containing x as an interior point and having length less than $\frac{\varepsilon}{4qs}$, where s denotes the supremum of $|f(x)g'(x)|$ for $x \in [a, b]$. Let P denote the point partition of $[a, b]$ obtained by adjoining the endpoints of each such K_x to $\widehat{P}$.

Consider any refinement P' of P. For any Riemann sum of fg' corresponding to P', the total contribution arising from intervals lying in the various K_x is less than $\varepsilon/4$. The contributions to any two such Riemann sums arising from other intervals differ by less than $3\varepsilon/4$. Thus any two Riemann sums of any refinement of P differ by less than ε.

Now a straightforward argument using a sequence of refinements corresponding to a decreasing sequence (ε_k) gives a Cauchy sequence of Riemann sums. Then the above argument can be used again to show that the limit of this Cauchy sequence is the value of the Riemann integral, and thus in particular, that the Riemann integral of fg' exists.

Comment: For those whose definition of Riemann integrals involves upper and lower integrals and sums rather than Riemann sums, the above argument can be shortened a bit. We have not adopted the 'upper-lower' approach because it does not generalize nicely to the Riemann-Stieltjes setting.

For Appendix E

E-4. We consider the real part of $\exp \circ \lambda$:

$$(\mathrm{R} \circ \exp \circ \lambda) = (\exp \circ \mathrm{R} \circ \lambda) \cdot (\cos \circ \mathrm{I} \circ \lambda)\,.$$

Using the Product Rule and Chain Rule for R-valued functions we obtain

$$\begin{aligned}(\mathrm{R} \circ \beta)' &= (\exp \circ \mathrm{R} \circ \lambda) \cdot (\mathrm{R} \circ \lambda)' \cdot (\cos \circ \mathrm{I} \circ \lambda) \\ &\qquad - (\exp \circ \mathrm{R} \circ \lambda) \cdot (\sin \circ \mathrm{I} \circ \lambda) \cdot (\mathrm{I} \circ \lambda)' \\ &= (\mathrm{R} \circ \lambda') \cdot (\mathrm{R} \circ \exp \circ \lambda) - (\mathrm{I} \circ \lambda') \cdot (\mathrm{I} \circ \exp \circ \lambda) \\ &= \mathrm{R} \circ \big(\lambda' \cdot (\exp \circ \lambda)\big)\,,\end{aligned}$$

as desired. We omit the similar calculation relevant for the imaginary part.

E-9. no