Solutions, answers, and hints for selected problems

Complete solutions of some problems are given. Answers only are given for some other problems. For still others, only hints or partial solutions are given. Asterisks in "A Modern Approach to Probability Theory" by Fristedt and Gray identify the problems that are treated in this supplement.

For Chapter 15

15-1. $\sum_{k=1}^{\infty} |X_k| \le 5 \sum_{k=1}^{\infty} 6^{-k} = 1$. Hence the series converges absolutely a.s. and therefore, it converges a.s., in probability, and in distribution; this is true without the independence assumption. The remainder of this solution, which concerns the limiting distribution and its characteristic function does use the independence assumption. The characteristic function of X_k is the function

$$v \rightsquigarrow \frac{1}{3}\Big(\cos\frac{v}{6^k} + \cos\frac{3v}{6^k} + \cos\frac{5v}{6^k}\Big) .$$

Therefore the characteristic function of $\sum_{k=1}^{\infty} X_k$ is the function

$$v \rightsquigarrow \prod_{k=1}^{\infty}\Big[\frac{1}{3}\Big(\cos\frac{v}{6^k} + \cos\frac{3v}{6^k} + \cos\frac{5v}{6^k}\Big)\Big] . \tag{0.1}$$

A direct simplification of this formula is not easy, so we will obtain the distribution by a method that does not rely on characteristic functions.

Calculations for $n = 1, 2, 3$ lead to the conjecture that the distribution Q_n of $\sum_{k=1}^{n} X_k$ is given by

$$Q_n\{m6^{-n}\} = 6^{-n} \quad \text{for } m \text{ odd}, |m| < 6^n .$$

This is easily proved by induction once it is noted that

$$\frac{m}{6^n} + \frac{5}{6^{n+1}} < \frac{m+2}{6^n} - \frac{5}{6^{n+1}} .$$

Then it is easy to let $n \to \infty$ to conclude that the distribution of $\sum_{k=1}^{\infty} X_k$ is the uniform distribution on $(-1, 1)$.

A sidelight: we have proved that the infinite product (7.7) equals the characteristic function of the uniform distribution on $(-1, 1)$ —namely $\frac{\sin v}{v}$.

15-6. 0.10

15-9. are not (except for the delta distribution at 0 in case one regards it as a degenerate Poisson distribution)

15-14. strict type consisting of positive constants (note: negative constants constitute another strict type)

15-16. *Hint:* The function g given by

$$g(u) = \int_0^\infty \frac{1}{x^{3/2}} e^{-\frac{b}{x} - ux}\, dx$$

can be evaluated by relating $g'(u)$ to the integral that can be obtained for $g(u)$ by using the substitution $y = \frac{c}{x}$ with an appropriate c. $a = \frac{1}{\sqrt{2}}$

15-20. (i) $\left(\frac{1-p}{1-p-\varepsilon}\right)^{1-p-\varepsilon}\left(\frac{p}{p+\varepsilon}\right)^{p+\varepsilon}$ (ii) $\left(1 + \frac{\varepsilon}{E(X_1)}\right)e^{-\varepsilon/E(X_1)}$

15-28. $\sup_{\{z\colon\, z-n \text{ even}\}} \Big| P[S_n = z] - \frac{1}{\sqrt{2\pi np(1-p)}} \exp\Big(-\frac{[z-n(2p-1)]^2}{8np(1-p)}\Big)\Big| = o\big(n^{-1/2}\big)$ as $n \to \infty$