Solutions, answers, and hints for selected problems

Complete solutions of some problems are given. Answers only are given for some other problems. For still others, only hints or partial solutions are given. Asterisks in "A Modern Approach to Probability Theory" by Fristedt and Gray identify the problems that are treated in this supplement.

For Chapter 18

18-5. *Hint:* Identify $\mathbf{C}[0,\infty)$ in a natural way with a closed subset of

$$\bigotimes_{n=0}^{\infty} \mathbf{C}[n, n+1]\,.$$

18-8. Let g be a continuous bounded R-valued function on Υ. Then $g \circ h$ is a continuous bounded R-valued function on Ψ. Therefore

$$\lim_{n\to\infty} \int_\Upsilon g\,dR_n = \lim_{n\to\infty} \int_\Psi (g\circ h)\,dQ_n = \int_\Psi (g \circ h)\,dQ = \int_\Psi g\,dR\,.$$

18-15. We first prove a related assertion—namely, the one obtained by replacing the hypothesis that A is open by the hypothesis that A is closed, in which case A is itself a Polish space by Proposition 3. If $Q(A) = 0$, this modified assertion (and also the original assertion) is clear, so assume that $Q(A) > 0$. For B a Borel subset of the Polish space A let

$$R(B) = \frac{Q(B)}{Q(A)}\,.$$

Clearly R is a probability measure. Let $\varepsilon > 0$. Corollary 18, applied to the Polish space A, shows that there exists a compact set K in the Polish space A such that $R(K) > 1 - \varepsilon$ and, thus,

$$Q(K) > (1-\varepsilon)Q(A) \geq Q(A) - \varepsilon\,.$$

The observation that, by Proposition 1 of Appendix C, K is compact in the Polish space Ψ completes the proof of the modification of the original assertion.

We return to the original assertion by now assuming that A is open in Ψ. We will prove that for every $\delta > 0$, there exists a subset C of A that is closed in Ψ and satisfies $Q(C) > Q(A) - \delta$. An application to C of the assertion proved above for closed sets then completes the proof.

Let S be a countable dense set in Ψ. It is easy to see that $S \cap A$ is a countable subset of A which, since A is open, is dense in A. For each $x \in S \cap A$, let B_x denote the closed ball centered at x whose radius is half the distance from x to A^c. It is easy to check that $A = \bigcup_{x \in S \cap A} B_x$. Replacing this union with a finite union over some finite subset of $S \cap A$ gives a closed set, a closed set whose Q-measure can, by continuity of measure, be chosen arbitrarily close to $Q(A)$, thus completing the proof.

Comment: The closed balls in the last paragraph of the proof need not be compact; this possibility is one reason the proof is so lengthy. Another reason is that an open subset of a Polish space is not necessarily a Polish space because it may not be complete. Thus, an intermediate result involving a closed subset is useful.

18-24. Let $w \in \mathrm{R}^{\mathrm{d}}$. By the Classical Central Limit Theorem,

$$\Big\langle w\,,\ \frac{\sum_{k=1}^n X_k - n\mu}{\sqrt{n}} \Big\rangle = \frac{\sum_{k=1}^n \langle w, X_k\rangle - nE(\langle w, X_1\rangle)}{\sqrt{n}} \xrightarrow{\mathcal{D}} Z_w\,,$$

where Z_w is a normally distributed R-valued random variable having mean 0 and variance $\mathrm{Var}\langle w, X_1\rangle$. By the Cramér-Wold Device,

$$\frac{\sum_{k=1}^n X_k - n\mu}{\sqrt{n}} \xrightarrow{\mathcal{D}} \text{some } Z$$

such that $\langle w, Z\rangle$ has the same distribution as Z_w for each $w \in \mathrm{R}^{\mathrm{d}}$, and so we may redefine Z_w to actually equal $\langle w, Z\rangle$. Since each Z_w is normally distributed, Z itself is, by definition, normally distributed.

Let $w = (0, \dots, 0, 1, 0, \dots)$, where 1 is in the j^{th} position. Then $Z_w = Z_j$, and hence $E(Z_j) = 0$. Also,

$$\mathrm{Var}\, Z_j = \mathrm{Var}\, Z_w = \mathrm{Var}\langle w, X_1\rangle\,,$$

which equals the variance of the j^{th} coordinate of X_1. Therefore the mean vector of Z is the zero vector and the diagonal members of the covariance matrix of Z are the diagonal members of Σ.

Now let $w = (0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$, where 1 is in both the j^{th} and k^{th} positions. Then $Z_w = Z_j + Z_k$ and so

$$\mathrm{Var}(w, X_1) = \mathrm{Var}(Z_w) = \mathrm{Var}(Z_j) + \mathrm{Var}(Z_k) + 2\,\mathrm{Cov}(Z_j, Z_k)\,.$$

The left side is the sum of the variances of the j^{th} and k^{th} coordinates of X_1 and twice the covariance of the j^{th} and k^{th} coordinates. By the preceding paragraph the sum of the variances of the j^{th} and k^{th} coordinates of X_1 equals the sum $\mathrm{Var}(Z_j) + \mathrm{Var}(Z_k)$. Thus twice the covariance of those two coordinates of X_1 must equal $2\,\mathrm{Cov}(Z_j, Z_k)$. Therefore the off-diagonal members of the covariance matrix of Z are the off-diagonal members of Σ.

18-26. *Hint:* Prove that $((A_\varepsilon)^c)_\varepsilon \subseteq A^c$.

18-29. first part: $\frac{|a|}{2} \wedge 1$.