Solutions, answers, and hints for selected problems

Complete solutions of some problems are given. Answers only are given for some other problems. For still others, only hints or partial solutions are given. Asterisks in "A Modern Approach to Probability Theory" by Fristedt and Gray identify the problems that are treated in this supplement.

For Chapter 11

11-12. The one-point sets $\{0\}$ and $\{\pi\}$ each have probability $2^{n-1}3^{-n}$. The probability of any measurable B disjoint from each of these one-point sets is the product of $\frac{1}{2\pi}(1-2^n3^{-n})$ and the Lebesgue measure of B.

11-13.

$$P\Big(\big\{\omega\colon \big(N(\omega)-1, S_{N(\omega)-1}(\omega)\big)=(m,k)\big\}\Big) = r\binom{m}{k-m}q^{k-m}p^{2m-k}$$

for $m \le k \le 2m$ and 0 otherwise. $E(S_{N-1}) = \frac{p+2q}{r}$

11-14. for B a Borel subset of R^+,

$$P\big(\{\omega\colon N(\omega)-1=m, S_{N(\omega)-1}(\omega)\in B\}\big) = Q(\{\infty\})Q^{*m}(B)\,;$$

$$E(S_{N-1}) = \frac{1}{Q(\{\infty\})}E(S_1\,;\,\{\omega\colon S_1(\omega)<\infty\})$$

11-17. Suppose that N is a stopping time. Then, for all $n \in \overline{\mathrm{Z}}^+$,

$$\{\omega\colon N(\omega)\le n\}\in\mathcal{F}_n\,,$$

which for $n=0$ is the desired conclusion $\{\omega\colon N(\omega)=0\}\in\mathcal{F}_0$. Suppose $0<n<\infty$. Then

$$\{\omega\colon N(\omega)<n\}\in\mathcal{F}_{n-1}\subseteq\mathcal{F}_n\,.$$

Therefore,

$$\{\omega\colon N(\omega)=n\} = \{\omega\colon N(\omega)\le n\}\setminus\{\omega\colon N(\omega)<n\}\in\mathcal{F}_n\,.$$

We complete the proof in this direction by noting that

$$\{\omega\colon N(\omega)=\infty\} = \{\omega\colon N(\omega)\le\infty\}\setminus\bigcup_{m=0}^{\infty}\{\omega\colon N(\omega)\le m\}$$

and that all the events on the right side are members of $\mathcal{F}_\infty$.

For the converse we assume that $\{\omega\colon N(\omega) = n\} \in \mathcal{F}_n$ for all $n \in \overline{\mathrm{Z}}^+$. Then, whether $n < \infty$ or $n = \infty$,

$$\{\omega\colon N(\omega) \le n\} = \bigcup_{m \le n} \{\omega\colon N(\omega) = m\}\,.$$

All events on the right are members of $\mathcal{F}_n$ because filtrations are increasing. Therefore, the event on the left is a member of $\mathcal{F}_n$, as desired.

11-24. Let $A \in \mathcal{F}_M$. Then

$$\begin{aligned} A \cap \{\omega\colon N(\omega) \le n\} &= A \cap \big[\{\omega\colon M(\omega) \le n\} \cap \{\omega\colon N(\omega) \le n\}\big] \\ &= \big[A \cap \{\omega\colon M(\omega) \le n\}\big] \cap \{\omega\colon N(\omega) \le n\}\,, \end{aligned}$$

which, being the intersection of two members of $\mathcal{F}_n$, is a member of $\mathcal{F}_n$. Hence $A \in \mathcal{F}_N$. Therefore $\mathcal{F}_M \subseteq \mathcal{F}_N$.

11-28.

$$\sigma_1(s) = \frac{1 - \sqrt{1 - 4p(1-p)s^2}}{2(1-p)s}\,, \quad 0 \le s < 1$$

For n finite and even, the probability is 0 that n equals the hitting time of $\{1\}$. For $n = 2m - 1$, the hitting time of $\{1\}$ equals n with probability

$$\frac{1}{2m-1}\binom{2m-1}{m} p^m (1-p)^{m-1}\,.$$

The hitting time of $\{1\}$ equals ∞ with probability 0 or $(1-2p)/(1-p)$ according as $p \ge \frac{1}{2}$ or not.

If $p \ge \frac{1}{2}$, the global supremum equals ∞ with probability 1. If $p < \frac{1}{2}$, the global maximum exists a.s. and is geometrically distributed; the global maximum equals x with probability $\frac{1-2p}{1-p}(\frac{p}{1-p})^x$.

11-30. *Hint:* Use the Stirling Formula.

11-32. Let $(Z_j\colon j \ge 1)$ be a sequence of independent random variables with common distribution R (as used in the theorem). From the theorem we see that $(0, T_1, T_2, \dots)$ is distributed like a random walk with steps Z_j. Thus,

$$\begin{aligned} P(\{\omega\colon V(\omega) = k\}) &= P(\{\omega\colon Z_k(\omega) = \infty\,,\ Z_j(\omega) < \infty \text{ for } j < k\}) \\ &= P(\{\omega\colon T_1(\omega) = \infty\})\,\big[P(\{\omega\colon T_1(\omega) < \infty\})\big]^{k-1}\,. \end{aligned}$$

Set $k = 1$ to obtain the first equality in (11.6). The above calculation also shows that V is geometrically distributed unless $P(\{\omega\colon V(\omega) = \infty\}) = 1$. Thus, it only remains to prove the second equality in (11.6).

Notice that

$$V = \sum_{n=0}^{\infty} I_{\{\omega\colon S_n(\omega) = 0\}}\,.$$

Take expected values of both sides to obtain

$$E(V) = \sum_{n=0}^{\infty} Q^{*n}(\{0\})\,.$$

If the right side equals ∞, then $V = \infty$ a.s., for otherwise it would be geometrically distributed and have finite mean. If the right side is finite, then $E(V) < \infty$, and, so, V is geometrically distributed and, as for all geometrically distributed random variables with smallest value 1, $\frac{1}{E(V)} = P(\{\omega\colon V(\omega) = 1\})$.

11-40. $m!/m^m$

11-41. For $m = 3$, let Q_n denote the distribution of S_n.

$$Q_n(\{\emptyset\}) = \begin{cases} \frac{1}{4}(1 + 3^{-(n-1)}) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

$$Q_n(\{\{1\}\}) = Q_n(\{\{2\}\}) = Q_n(\{\{3\}\}) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{1}{4}(1 + 3^{-n}) & \text{if } n \text{ is odd} \end{cases}$$

$$Q_n(\{\{1,2\}\}) = Q_n(\{\{1,3\}\}) = Q_n(\{\{2,3\}\}) = \begin{cases} \frac{1}{4}(1 - 3^{-n}) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

$$Q_n(\{\{1,2,3\}\}) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{1}{4}(1 - 3^{-(n-1)}) & \text{if } n \text{ is odd} \end{cases}$$

11-42. probability that $\{\emptyset\}$ is hit at time n or sooner: $[1-(\frac{1}{2})^n]^m$; probability that $\{\{1, 2, \dots, k\}\}$ is hit at the positive time $n < \infty$:

$$(\tfrac{1}{2})^{nk}\big[\big(1 - (\tfrac{1}{2})^n\big)^{m-k} - \big(1 - (\tfrac{1}{2})^{n-1}\big)^{m-k}\big]\,;$$

probability that hitting time of $\{1, \dots, m-1\}$ equals ∞: $(2^m - 2)/(2^m - 1)$

11-45. For $n \geq 1$ the distribution of S_n assigns equal probability to each one-point event. The sequence S is an independent sequence of random variables. For $n \geq 1$, the probability that the first return time to 0 equals n is $(\frac{1}{m})(1 - \frac{1}{m})^{n-1}$, where m is the number of members of the group.