(vii) yes; $U\{0\}=1$, $U\{n\}=0$ for n odd, $U\{n\}=\binom{n-1}{n/2}[p(1-p)]^{n/2}$ for $n\geq 2$ and even; measure generating function of U:

$$s \rightsquigarrow 1+\sum_{k=1}^{\infty}\binom{2k-1}{k}[p(1-p)]^k s^{2k} = \frac{1}{2}+\frac{1}{2}\sum_{k=0}^{\infty}\binom{-1/2}{k}[-4p(1-p)s^2]^k$$
$$= \frac{1}{2}\Big[1+\big(1-4p(1-p)s^2\big)^{-1/2}\Big]\,;$$

measure generating function of R:

$$s \rightsquigarrow \frac{1-2p(1-p)s^2-\big(1-4p(1-p)s^2\big)^{1/2}}{2p(1-p)s^2} = 2\sum_{k=1}^{\infty}\binom{1/2}{k+1}[-4p(1-p)s^2]^k$$
$$= \sum_{k=1}^{\infty}\frac{1}{k+1}\binom{2k}{k}[p(1-p)]^k s^{2k}\,;$$

$R\{n\}=0$ for n odd, $R\{n\}=\frac{2}{n+2}\binom{n}{n/2}[p(1-p)]^{n/2}$ for n even, $R\{\infty\}=\frac{|2p-1|}{p\vee(1-p)}$ [Notice that the coefficient $\frac{2}{n+2}\binom{n}{n/2}$ in the formula for $R\{n\}$, n even, is the $(n/2)^{\text{th}}$ Catalan number.]

25-20. for B a set of consecutive integers, $P(N(B)>0)=1-p^{\#B}$, in notation of Problem 12

25-29. $\frac{\sigma^2+\mu(\mu-1)}{2\mu}$, where μ is mean and σ^2 (possibly ∞) is variance

25-36. $R\{n\}=\frac{1}{2n-1}\binom{2n}{n}4^{-n}$, $U\{n\}=\binom{2n}{n}4^{-n}$

25-39. The solution of Problem 28 of Chapter 11 gives the measure generating function of the waiting time distribution for strict ascending ladder times:

$$\varphi^{++}(s)=\frac{1-\sqrt{1-4p(1-p)s^2}}{2(1-p)s}\,.$$

The measure generating function of the waiting time distribution for weak descending ladder times can then be obtained from Theorem 22:

$$\varphi^{-}(s)=\frac{1+2(1-p)s-\sqrt{1-4p(1-p)s^2}}{2}\,.$$

It is straightforward to use the Binomial Theorem to obtain the waiting time distributions and potential measures corresponding to these two measure generating functions. The other two types of ladder times can be treated by interchanging p and $1-p$.

For Chapter 26

26-5.

$$Q_{n+1}(B)=P[X_{n+1}\in B]=E\Big(P([X_{n+1}\in B]\mid\mathcal{F}_n)\Big)$$
$$=E\big(\mu_{X_n}(B)\big)=\int\mu_x(B)Q_n(dx)=(Q_nT)(B)$$

$$E(f\circ X_{n+1}\mid\mathcal{F}_n)=\int f(y)\mu_{X_n}(dy)=(Tf)\circ X_n$$

26-19. Let f be the identity function on $[0,1]$. Clearly f is bounded and measurable. By Theorem 6, Y is a martingale where

$$Y_n = X_n - \sum_{k=0}^{n-1} (Gf) \circ X_k \,.$$

Solving for X_n gives a representation for X in terms of the martingale Y and a previsible sequence having the value 0 when $n = 0$. To show that this sequence is increasing, as required for a Doob decomposition, we only need show that Gf is a nonnegative function when f is the identity function. The following calculation does this:

$$Gf(x) = Tf(x) - x = E^x(X_1) - x \geq 0 \,,$$

the last equality using the fact that X is a submartingale.

26-28. $x \rightsquigarrow \frac{2}{x+1}$

26-29. *Hint:* Reminder: There is one value of x that is not required to satisfy the difference equation.

26-31. $x \rightsquigarrow e^{-1} \sum_{k=x}^{\infty} \frac{1}{k!}$

26-39. Denote the two states by x and y. By the last part of Problem 38, if one of the two states is transient so is the other. Now suppose that y is null recurrent; our goal is to show that x is not positive recurrent.

By the Renewal Theorem the sequence of entries of T^n in position y along the main diagonal converges to 0 as $n \to \infty$. We will complete the proof by finding an integer k and a positive constant c such that the entry in position y along the main diagonal in T^n is larger than c times the entry in position x along the main diagonal in T^{n-k} for all $n \geq k$, for then it will follow that the sequence of entries in T^{n-k} in position x along the main diagonal will converge to 0 as $n \to \infty$, implying that x is not positive recurrent.

One way to start at y and to then be there again at time n is to first be at state x at some time r, then be at x again at some time $n-k+r$, and then be at state y at time n. By first choosing r and then k appropriately one can make the product of the probabilities of the first and third of these three tasks a positive constant c.

We omit the part of the solution treating the periodicity issue.

26-43. Suppose that, for some k, all entries of T^k are positive. For any states x and y there is positive probability of being at y at time k if the starting state is x. Hence, $\pi_{xy} > 0$. Therefore, T is irreducible. Clearly, $T^m T^k = T^{m+k}$ has only positive entries for all nonnegative integers m, and thus 1 is the greatest common divisor of the powers of T for which the upper left entry (or any other diagonal entry) is positive. Aperiodicity follows.

For the converse suppose that T is irreducible and aperiodic. The sequence of numbers in a fixed diagonal position of $T^0, T^1, T^2, \ldots$ is an aperiodic potential sequence, which, by Lemma 18 of Chapter 25, contains only finitely many zero terms. Thus, there exists an integer m such that all diagonal entries of T^m are positive. Hence, all diagonal entries of T^n are positive for $n \geq m$. Since T is irreducible, there is, for each x and y, an integer k_{xy} such that the entry in row x and column y of $T^{k_{xy}}$ is positive.

Let $k = m + \max\{k_{xy}\}$. Since T^k can be obtained by multiplying $T^{k_{xy}}$ by a power of T at least as large as m, the entry in row x and column y of T^k is positive. Thus, all entries of T^k are positive, as desired.

26-52. starting state of interest denoted by 0; probabilities of absorption at the absorbing states -2 and -1, respectively:

$$\sum_{k=0}^{\infty} \frac{2^{2k-1}}{(3\cdot 2^{2k-1}-1)(3\cdot 2^{2k}-1)} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{2^{2k}}{(3\cdot 2^{2k}-1)(3\cdot 2^{2k+1}-1)}$$

probability of no absorption: 1/3

26-55. We can introduce infinitely many extra transient states in order to obtain a birth-death sequence. The transition distributions μ_x are given by

$$\mu_x\{x-1\} = \frac{x}{b} \wedge 1$$
$$\mu_x\{x+1\} = \frac{b-x}{b} \vee 0\,.$$

From Problem 54 we see the relevance of the following product:

$$\prod_{z=1}^{x} \frac{\frac{b-z+1}{b} \vee 0}{\frac{z}{b} \wedge 1} = \binom{b}{x}.$$

The number r as defined in Problem 54 can now be calculated:

$$r = \sum_{x=0}^{\infty} \binom{b}{x} = \sum_{x=0}^{b} \binom{b}{x} = 2^b\,.$$

The equilibrium distribution Q for the Ehrenfest urn sequence is given by

$$Q\{x\} = \frac{1}{2^b}\binom{b}{x}, \quad 0 \le x \le b\,.$$

For Chapter 27

27-2. μ denotes De Finetti measure; for $i = 1,2,3$, $\mu\{y_i\} = \frac{1}{3}$, where $y_i\{1\} = y_i\{6-i\} = \frac{1}{2}$

27-4. De Finetti measure equals delta measure at uniform distribution on $\{x \in \mathbb{Z} \colon 1 \le x \le 12\}$

27-6. Yes. By letting p_i equal the value assigned to the one-point set $\{i\}$ by a probability measure on $\{1,2,3,4\}$, the probability measure itself is represented by an ordered 4-tuple (p_1, p_2, p_3, p_4) The De Finetti measure assigns probability

$\frac{1}{512}$ to $(1,0,0,0)$ and to each of the other 3 permutations thereof;
$\frac{1}{128}$ to $(\frac{3}{4}, \frac{1}{4}, 0, 0)$ and to each of the other 11 permutations thereof;
$\frac{3}{128}$ to $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0)$ and to each of the other 11 permutations thereof;
$\frac{3}{256}$ to $(\frac{1}{2}, \frac{1}{2}, 0, 0)$ and to each of the other 5 permutations thereof;
$\frac{35}{64}$ to $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.

27-10. μ denotes De Finetti measure; $\mu\{m/n\} = P[Z_1 + \cdots + Z_n = m]$.

27-15. $P[X_1 = X_2 = 1] = P[X_1 = X_2 = 0] = \frac{5n-6}{12(n-1)}$
$P[X_1 = -X_2 = 1] = P[X_1 = -X_2 = -1] = \frac{n}{12(n-1)}$

27-31. $\alpha+$ the numbers of 1's, $\beta +$ the number of 0's

27-32. conditional distribution of (Y, X_{m+1}, X_{m+2}) has density with respect to $\mu \times \gamma \times \gamma$, where γ denotes counting measure on $\{0,1\}$; density is

$$(p, z_1, z_2) \rightsquigarrow \frac{p^{(X_1+\cdots+X_m+z_1+z_2)}\,(1-p)^{(m+2)-(X_1+\cdots+X_m+z_1+z_2)}}{\int_{[0,1]} x^{(X_1+\cdots+X_m)}\,(1-x)^{m-(X_1+\cdots+X_m)}\,\mu(dx)}\,;$$

integration in p and z_2 gives conditional density with respect to γ of X_{m+1}:

$$z_1 \rightsquigarrow \frac{\int_{[0,1]} p^{(X_1+\cdots+X_m+z_1)}\,(1-p)^{(m+1)-(X_1+\cdots+X_m+z_1)}\,\mu(dp)}{\int_{[0,1]} x^{(X_1+\cdots+X_m)}\,(1-x)^{m-(X_1+\cdots+X_m)}\,\mu(dx)}\,;$$

multiplication by z_1 and integration in z_1 give the conditional expectation of X_{m+1}:

$$\frac{\int_{[0,1]} p^{(X_1+\cdots+X_m+1)}\,(1-p)^{m-(X_1+\cdots+X_m)}\,\mu(dp)}{\int_{[0,1]} x^{(X_1+\cdots+X_m)}\,(1-x)^{m-(X_1+\cdots+X_m)}\,\mu(dx)}\,,$$

which equals

$$\frac{X_1 + \cdots + X_m + 1}{m+2}$$

in case μ is the standard uniform distribution.

27-39. density of each of X_1 and X_2: $x \rightsquigarrow \frac{1}{2}e^{-x} + \frac{1}{4}e^{-x/2}$; density of (X_1, X_2): $(x_1, x_2) \rightsquigarrow \frac{1}{4}\bigl(e^{-x_1-x_2/2} + e^{-x_2-x_1/2}\bigr)$; De Finetti measure assigns probability 1 to the set of uniform two-point distributions, the density of the two points being $\{y_1, y_2\} \rightsquigarrow \frac{1}{2}\bigl(e^{-y_1-y_2/2} + e^{-y_2-y_1/2}\bigr)$, $0 < y_1 < y_2$.

27-47. conditional distribution of reciprocal of mean of Y given $(X_1, \ldots, X_m)$ is gamma with main parameter $m+1$ and scaling parameter $1 + \sum_{j=1}^m X_j$

27-52. The stick-breaking random walk breaks a stick into random pieces in such a way that, say, the sizes of the first three pieces determines how much of the stick is left for pieces 4, 5, ..., to share but gives no information about the relative sizes of these pieces. Certain information about $(X_1, \ldots, X_m)$ might, for example, give information about the sizes of pieces 1, 2, and 3, without giving information about the relative sizes of the remaining pieces. (Comment: The authors of this book find this explanation to be neither complete nor satisfactory, but it is the best that they could do.)

27-55. The formula is trivial when $k = 0$; it is $1 = 1/1$. Assume it is true for k and multiply both sides by

$$P[X_{k+1} = x_{k+1} \mid X_1 = x_1, \ldots, X_k = x_k] = \frac{c + \gamma_{x_{k+1}}}{k + \sum_{i=1}^d \gamma_i}\,,$$

where c equals the number of x_j, $j \le k$, for which $x_j = x_{k+1}$. The result follows.

For Chapter 28

28-4. It suffices to prove that

$$\begin{aligned} &P[(X_m, X_{m+k}, \dots, X_{m+(d-1)k}) \in A] \\ &= P[(X_{m+k}, X_{m+2k}, \dots, X_{m+dk}) \in A] \end{aligned} \tag{7.12}$$

for every positive integer d and every Borel set $A \subseteq \Psi^d$, where Ψ denotes the common target of the X_j. Set

$$B = \{(x_0, x_1, \dots, x_{m+(d-1)k}) \in \Psi^{m+(d-1)k+1} \colon (x_m, x_{m+k}, \dots, x_{m+(d-1)k}) \in A\}.$$

Then the left side of (7.12) equals

$$P[(X_0, X_1, \dots, X_{m+(d-1)k}) \in B]$$

and the right side equals

$$P[(X_k, X_{k+1}, \dots, X_{m+dk}) \in B].$$

These are equal by Problem 3.

28-6. *Hint:* From the given sequence obtain the desired joint distributions of every finite set of random variables. Use this information to construct a sequence $(Y_0, Y_{-1}, Y_{-2}, \dots)$ using Theorem 3 of Chapter 22. Then treat $(\dots, Y_{-2}, Y_{-1}, Y_0)$ as a single random object and use it as the first member of a random sequence to be constructed using Theorem 3 of Chapter 22 again, with the next members being $Y_1, Y_2, \dots$.

28-21. Let A be an set for which $R(A) \neq S(A)$. By Problem 18,

$$(I_A\,,\, I_A \circ \tau\,,\, I_A \circ \tau^2\,,\, \dots)$$

is ergodic. By the Birkhoff Ergodic Theorem the sequence

$$\Big(\frac{1}{n}\sum_{k=0}^{n-1} I_A \circ \tau^k \colon n = 1, 2, \dots\Big)$$

converges to $R(A)$ with R-probability 1 and also to $S(A)$ with S-probability 1. Since $S(A) \neq R(A)$, these two events are disjoint, and thus the mutual singularity is established.

28-23. Suppose first that a is rational, say p/q in lowest terms with q positive. Then the following set is easily seen to be shift-invariant and have Lebesgue measure $\frac{1}{2}$:

$$\{x \in [0,1) \colon x \in [\tfrac{p}{q}, \tfrac{2p+1}{2q}) \text{ for some } p\}.$$

Now suppose that a is irrational. Rotation through angle $2\pi a$ generates a shift transformation on $[0,1)^\infty$. It is clear that any shift-invariant distribution is determined by the initial distribution on $[0,1)$, but it may be that some choices for that distribution do not yield a shift-invariant measure on $[0,1)^\infty$. In fact, we will prove that the only initial distribution that does yield a shift-invariant measure on $[0,1)^\infty$ is Lebesgue measure.

For every $n \in \mathbb{Z}^+$ and 'left-closed, right-open subinterval' J of $[0,1)$, possibly with 'wrap-around', any shift-invariant measure assigns the same value to J and the interval

J_{na} obtained by adding na to each endpoint of $J \mod n$. For any left-closed, right-open interval K having the same length as J, a sequence $(n_k \in \mathbb{Z}^+ \colon k = 1, 2, \dots)$ can be chosen so that

$$K = \lim_{k\to\infty} J_{n_k\, a} \,.$$

Hence all open intervals having the same length have the same measure, and therefore the only initial distribution that yields a shift-invariant distribution is Lebesgue measure.

Since there is only one shift-invariant distribution, that distribution is extremal and by Theorem 4, therefore ergodic. The Weyl Equidistribution Theorem is then an immediate consequence of the Birkhoff Ergodic Theorem.

28-25. $Q\{i\}T(i,j)$

28-28. Suppose that X is strongly mixing and consider any $A \in \mathcal{T}$. For each n there exists B_n such that $A = \tau^{-n}(B_n)$. As $n \to \infty$,

$$\begin{aligned}\left|Q(A) - [Q(A)]^2\right| &= \left|Q(A \cap \tau^{-n}(B_n)) - Q(A)Q(\tau^{-n}(B_n))\right| \\ &= \left|Q(A \cap \tau^{-n}(B_n)) - Q(A)Q(B_n)\right| \to 0 \,.\end{aligned}$$

Therefore $Q(A)$, being a solution of $\left|Q(A) - [Q(A)]^2\right| = 0$, equals 0 or 1, as desired.

For the converse we assume that $\mathcal{T}$ is 0-1 trivial and fix a member A of $\mathcal{H}$. Then for all $B \in \mathcal{H}$ and all positive integers n,

$$\begin{aligned}\left|Q(A \cap \tau^{-n}(B)) - Q(A)Q(B)\right| &= \left|E_Q\left(I_A I_{\tau^{-n}(B)} - Q(A) I_{\tau^{-n}(B)}\right)\right| \\ &= \left|E_Q\left(\left[Q(A \mid \mathcal{H}_n) - Q(A)\right] I_{\tau^{-n}(B)}\right)\right| \\ &\le E_Q\left(\left|Q(A \mid \mathcal{H}_n) - Q(A)\right|\right) , \end{aligned} \tag{7.13}$$

where E_Q denotes expectation based on the distribution Q and

$$\mathcal{H}_n = \{\tau^{-n}(C) \colon C \in \mathcal{H}\} \,.$$

To finish the proof we only need show that (7.13) approaches 0 as $n \to \infty$, the uniformity in B resulting from the fact that (7.13) does not depend on B. By the Bounded Convergence Theorem, we only need show

$$\lim_{n\to\infty} \left[Q(A \mid \mathcal{H}_n) - Q(A)\right] = 0 \,.$$

By the Reverse Martingale Convergence Theorem, this limit does exist and equals $Q(A \mid \mathcal{T}) - Q(A)$, a random variable which has mean 0 and which, since $\mathcal{T}$ is 0-1 trivial, is a.s. constant. Therefore it must equal 0 a.s. as desired.

28-30. For each positive integer l, we say that a path $(x_0, \dots, x_n)$ is *l-restricted* if every point on the path lies between the vertical line through the point $x_0 - (l, 0)$ and the vertical line through the point $x_n + (l, 0)$. Define a collection of random variables $(Z^{(l)}_{m,n})$ in terms of l-restricted paths in a manner that is analogous to the way in which the collection $(Z_{m,n})$ was defined. It is easy to use the independence of the random variables $(T_{x,y})$ to see that for positive integers k, l, the sequence $(Z^{(l)}_{nk,(n+1)k} \colon n = 1, 2, 3, \dots)$ is mixing (in fact strongly mixing). It is also easy to see that for each fixed k,

$$\lim_{l\to\infty} P[Z^{(l)}_{nk,(n+1)k} = Z_{nk,(n+1)k}] = 1 \,,$$

uniformly in n. We leave it to the reader to conclude from this last fact that the sequence $(Z_{nk,(n+1)k} \colon n = 1, 2, 3, \dots)$ is mixing (and hence ergodic) for each k.

28-45. Let X denote a stationary Gaussian sequence with correlation function $(m, n) \rightsquigarrow \rho^{|m-n|}$. The result is obvious if $\rho = \pm 1$, so we assume $|\rho| < 1$. Following the hint, the conditional distribution of X_n given $(X_0, X_1, \dots, X_{n-1})$ is Gaussian with a constant variance and mean

$$\begin{pmatrix} \rho & \rho^2 & \dots & \rho^n \end{pmatrix} \begin{pmatrix} 1 & \rho & \dots & \rho^{n-1} \\ \rho & 1 & \dots & \rho^{n-2} \\ \vdots & \vdots & & \vdots \\ \rho^{n-1} & \rho^{n-2} & \dots & 1 \end{pmatrix}^{-1} \begin{pmatrix} X_{n-1} \\ X_{n-2} \\ \vdots \\ X_0 \end{pmatrix} . \tag{7.14}$$

Since the first matrix is a row matrix that is a multiple of the first column of the matrix whose inverse is in the formula, the matrix product (7.14) is some multiple of X_{n-1}, and this is all that is needed to show that X is Markov.

For Chapter 29

29-5. Let $r = \sum_{j=1}^{m} r_j$.

$$P[X\{1\} = k] = \frac{\binom{r_1}{k}\binom{r-r_1}{n-k}}{\binom{r}{n}}$$

$$P[X\{1\} = k, X\{2\} = l] = \frac{\binom{r_1}{k}\binom{r_2}{l}\binom{r-r_1-r_2}{n-k-l}}{\binom{r}{n}}$$

$$P[X\{1\} = k, X\{2\} = l, X\{3\} = m] = \frac{\binom{r_1}{k}\binom{r_2}{l}\binom{r_3}{m}\binom{r-r_1-r_2-r_3}{n-k-l-m}}{\binom{r}{n}}$$

29-8. $P[X(B) = z] = \frac{(\#B)^z(n-\#B)^{r-z}}{n^r}\binom{r}{z}$, $0 \le z \le r$. Thus the distribution of $X(B)$ is binomial with parameters $\frac{\#B}{n}$ and r.

29-13. Let $(V_n \colon n \ge 0)$ be a renewal sequence. Define a random measure X on $\mathbb{Z}^+$ by $X\{n\} = V_n$. Clearly X is a point process and its intensity measure equals the potential measure of the renewal sequence.

29-18. We use the formula for the probability that a Poisson random variable equals 0. For $v \ge 0$,

$$P[V \ge v] = P[X(\{0, 1, \dots, v-1\}) = 0] = e^{-v} .$$

Then

$$P[V = v] = P[V \ge v] - P[V \ge (v+1)] = e^{-v} - e^{-(v+1)} = (1 - e^{-1})e^{-v} .$$

29-23. Write

$$Y \cup \{0\} = \{0 = Y_0 < Y_1 < Y_2 < \dots\} ,$$

and let $(S_0 = 0, S_1, S_2, \dots)$ be a random walk having exponentially distributed steps with mean c^{-1}. For an arbitrary positive integer n we will show that $(Y_1, \dots, Y_n)$ and $(S_1, \dots, S_n)$ have the same distribution, thereby finishing the proof. We will verify that the distribution of each of these random vectors has the same density with respect to n-dimensional Lebesgue measure—namely,

$$(y_1, \dots, y_n) \rightsquigarrow \begin{cases} c^n e^{-cy_n} & \text{if } 0 < y_1 < \cdots < y_n \\ 0 & \text{otherwise}\,. \end{cases} \tag{7.15}$$

To check that this is the correct density for $(Y_1, \dots, Y_n)$ we integrate it over a set of the form $\prod_{i=1}^n [u_i, v_i)$, where

$$0 = v_0 < u_1 < v_1 < u_2 < \cdots < u_n < v_n = \infty\,.$$

We get

$$\begin{aligned} e^{-cu_n} \prod_{i=1}^{n-1} c(v_i - u_i) &= \left(\prod_{i=1}^{n-1} c(v_i - u_i) e^{-c(v_i-u_i)}\right)\left(\prod_{i=1}^{n} e^{-(u_i - v_{i-1})}\right) \\ &= \left(\prod_{i=1}^{n-1} P[\#(Y \cap [u_i, v_i)) = 1]\right)\left(\prod_{i=1}^{n} P[\#(Y \cap [v_{i-1}, u_i)) = 0]\right) \\ &= P[Y_i \in [u_i, v_i) \text{ for } 1 \le i \le n]\,, \end{aligned}$$

as desired.

We know that the density of $\big((S_1 - S_0)\,,\, (S_2 - S_1)\,,\, \dots\,,\, (S_n - S_1)\big)$ is

$$(x_1, \dots, x_n) \rightsquigarrow \begin{cases} \prod_{i=1}^n ce^{-cx_i} & \text{if each } x_i > 0 \\ 0 & \text{otherwise}\,. \end{cases}$$

We can get the density of $(S_1, \dots, S_n)$ by using the linear transformation $y_k = x_1 + \cdots + x_k$, $1 \le k \le n$, the Jacobian of which equals 1. The result is the desired density (7.15).

29-24. *Hint:* One approach is to start with sequences U and V having the desired properties and then use Problem 23 to show that $\{(U_n, V_n)\colon n = 1, 2, \dots\}$ is a Poisson point process with intensity measure $\lambda \times \mu$.

29-26. c^{-3}

29-29. π, $\frac{\pi}{2}$

29-34. $h \rightsquigarrow \frac{1}{n}\sum_{i=1}^n h(i)$ for $r = 1$; $h \rightsquigarrow \frac{1}{n}\left[\sum_{i=1}^n [h(i)]^{-1}\right]\prod_{j=1}^n h(j)$ for $r = n-1$

29-39. $h \rightsquigarrow \exp\big(-\sum_{\psi\in\Psi}(1 - h(\psi))\big)$, where Ψ is the countable set

29-43. The probability generating functional of $X + Y$ is

$$\begin{aligned} h \rightsquigarrow E\Big(\prod_{\psi\in\Psi}[h(\psi)]^{(X+Y)(\{\psi\})}\Big) &= E\Big(\prod_{\psi\in\Psi}[h(\psi)]^{X(\{\psi\})}[h(\psi)]^{Y(\{\psi\})}\Big) \\ &= E\Big(\Big[\prod_{\psi\in\Psi}[h(\psi)]^{X(\{\psi\})}\Big]\Big[\prod_{\psi\in\Psi}[h(\psi)]^{Y(\{\psi\})}\Big]\Big) \\ &= E\Big(\prod_{\psi\in\Psi}[h(\psi)]^{X(\{\psi\})}\Big)\,E\Big(\prod_{\psi\in\Psi}[h(\psi)]^{Y(\{\psi\})}\Big)\,, \end{aligned}$$

which is the product of the probability generating functionals of X and Y.

29-50. Suppose that $X_n \xrightarrow{\mathcal{D}} X$ as $n \to \infty$; that is, $Q_n \to Q$, where Q_n and Q denote the distributions of X_n and X, respectively. Let h be in the domain of the probability generating functional of Q (and thus of each Q_n). Assume first that h is bounded below by a positive constant. Then the function

$$\pi \rightsquigarrow \int \log(1/h)\, d\pi$$

is continuous, and thus the same is true for the function

$$\pi \rightsquigarrow e^{-\int \log(1/h)\, d\pi}\,. \tag{7.16}$$

For this latter function it is straightforward to remove the assumption that h be bounded below by a positive constant (of course, using the conventions $\infty \cdot 0 = 0$ and $e^{-\infty} = 0$). That

$$\int e^{-\int \log(1/h)\, d\pi}\, dQ_n \to \int e^{-\int \log(1/h)\, d\pi}\, dQ$$

follows from the continuity of the function (7.16). That the limiting probability generating functional has the property described in the theorem is a consequence of Proposition 16 which says that all probability generating functionals have a more general property.

For the converse suppose that $\mathfrak{F}$ is the limit of a sequence of probability generating functionals corresponding to a sequence $(Q_n \colon n = 1, 2, \dots)$ of distributions of point processes in a locally compact Polish space Ψ, and that $\mathfrak{F}$ satisfies the condition in the theorem. Let C be any compact subset of Ψ. By using Lemma 1 one can show that there exists a compact set B such that every point of C is an interior point of B and that therefore there exists a continuous $[(1-\frac{1}{m}), 1]$-valued function h_m such that $h_m(\psi) = 1 - \frac{1}{m}$ for $\psi \in C$ and $h_m(\psi) = 1$ for $\psi \in B^c$.

Let $\varepsilon > 0$ Since $\mathfrak{F}(h_m) \to 1$ as $m \to \infty$, we can fix m so that for all n

$$\begin{aligned}
Q_n\{\pi \colon \pi(C) \le z\} &\ge \int_{\{\pi \colon \pi(C) \le z\}} \prod_{\psi} [h_m(\psi)]^{\pi(\{\psi\})}\, Q_n(d\pi) \\
&> 1 - \tfrac{\varepsilon}{2} - \int_{\{\pi \colon \pi(C) > z\}} \prod_{\psi} [h_m(\psi)]^{\pi(\{\psi\})}\, Q_n(d\pi) \\
&\ge 1 - \tfrac{\varepsilon}{2} - \left(1 - \tfrac{1}{m}\right)^{z+1},
\end{aligned}$$

which is larger than $1 - \varepsilon$ for sufficiently large z. By Theorem 19, every subsequence of (Q_n) has a convergent subsequence. By the first paragraph of this proof, $\mathfrak{F}$ is the probability generating functional of every subsequential limit. By Theorem 14 all subsequential limits are identical. Therefore, the sequence (Q_n) itself converges to a limit whose probability generating functional is $\mathfrak{F}$.

For Chapter 30

30-1. $I_{[2+(1/n),\infty)}$ right-continuous; pointwise limit $I_{(2,\infty)}$ not right-continuous at 2.

30-10. The moment generating function is

$$u \rightsquigarrow E\Big(e^{-uY_t}\Big) = E\Big(\exp\Big(-u \sum_{x\in[0,\infty]} xX\big((0,t]\times\{x\}\big)\Big)\Big)$$

$$= E\left(\prod_{\substack{s\in(0,t]\\ x\in[0,\infty]}} \left[e^{-ux}\right]^{X\{(s,x)\}}\right). \tag{7.17}$$

For calculating (7.17), we may replace $(0,t]$ by $[0,t]$. The function $(s,x) \rightsquigarrow e^{-ux}$ is a continuous function on the compact set $[0,t]\times[0,\infty]$, taking the value 0 at (s,∞) if $u>0$ and the value 1 there if $u=0$. Therefore we may apply Proposition 15 of Chapter 29 to conclude that (7.17) equals

$$\exp\Big(-\int_{[0,t]\times[0,\infty]} (1-e^{-ux})\,\kappa(\lambda\times Q)(d(s,x))\Big)$$
$$= \exp\Big(-\kappa t\int_{[0,\infty]} (1-e^{-ux})\,Q(dx)\Big)\,.$$

We could have treated the problem as a single-variable problem by working with the Poisson point process X_t, the restriction of X to $(0,t]\times[0,\infty]$.

In view of Remark 1, Q might be a probability measure on $(0,\infty]$, which is not compact. We could handle this setting, by adjoining 0 to $(0,\infty]$ and specifying $Q\{0\} = 0$, or by approximating $x \rightsquigarrow e^{-ux}$ by continuous functions that equal 1 for small x.

It is not possible to treat characteristic functions by adjoining $\pm\infty$ to $\mathbb{R}$ in order to obtain compactness, because one will then lose continuity. Approximation of the functions $x \rightsquigarrow e^{ivx}$ by functions that are continuous everywhere and constant for large x is a method that works. By then going to the limit one obtains the characteristic function of Y_t:

$$v \rightsquigarrow \exp\Big(-\kappa t\int_{[0,\infty]} (1-e^{ivx})\,Q(dx)\Big)\,.$$

30-13. $1-e^{-t\nu[y,\infty]}$

30-16. Set

$$\widetilde{R}_y(B) = R(\{v\in\mathbf{D}^+[0,1]\colon yv\in B\})\,,$$

and let $0=t_0\le t_1<t_2<\cdots<t_d=1$. The proof relies on showing that

$$P[Z_{t_i}-Z_{t_{i-1}}\le b_i \text{ for } 1\le i\le d]$$
$$= \int_{(0,\infty)} \widetilde{R}_y(\{z\in\mathbf{D}^+[0,1]\colon z_{t_i}-z_{t_{i-1}}\le b_i \text{ for } 1\le i\le d\})\,ae^{-ay}\,dy \tag{7.18}$$

for positive numbers b_i.

The left side of (7.18) equals

$$\prod_{i=1}^{d}\int_0^{b_i} \frac{a^{(t_i-t_{i-1})}\,\theta_i^{(t_i-t_{i-1})-1}\,e^{-a\theta_i}}{\Gamma(t_i-t_{i-1})}\,d\theta_i$$

$$= a\prod_{i=1}^{d}\int_0^{b_i} \frac{\theta_i^{(t_i-t_{i-1})-1}\,e^{-a\theta_i}}{\Gamma(t_i-t_{i-1})}\,d\theta_i\,. \tag{7.19}$$

The right side of (7.18) equals

$$\int_0^\infty R(\{v \in \mathbf{D}^+[0,1] \colon v_{t_i} - v_{t_{i-1}} \le \frac{b_i}{y} \text{ for } 1 \le i \le d\})\, ae^{-ay}\, dy\,.$$

From Problem 15, we can rewrite this expression in terms of a Dirichlet distribution:

$$a \int_{\substack{0<\rho_i \le b_i/y\,,\, i\le(d-1)\\ 0<1-\rho_1-\cdots-\rho_{d-1}\le b_d/y}} \int_0^\infty e^{-ay} \frac{(1-\rho_1-\cdots-\rho_{d-1})^{(t_d-t_{d-1})-1}}{\Gamma(t_d-t_{d-1})} \cdot \prod_{i=1}^{d-1} \frac{\rho_i^{(t_i-t_{i-1})-1}}{\Gamma(t_i-t_{i-1})}\, dy\, d\rho_{d-1}\, \ldots\, d\rho_1\,. \tag{7.20}$$

For $1 \le i \le d-1$, let $\theta_i = y\rho_i$, and also let $\theta_d = y(1-\rho_1-\cdots-\rho_{d-1})$. The Jacobian of this transformation is $\frac{d(\theta_1,\ldots,\theta_{d-1},\theta_d)}{d(\rho_1,\ldots,\rho_{d-1},y)} = y^{d-1}$; hence this change of variables turns (7.20) into (7.19), as desired.

30-25. negative binomial with parameters $1/\big(1+E(Y_1)\big)$ and $\tau E(Z_1)$

30-31. (iii): Let $\varepsilon > 0$, and denote the distribution of Z_t by Q_t. Then for $\varepsilon t < 1$,

$$\begin{aligned}
\big(1-e^{-1}\big)P[Z_t > \varepsilon t] &\le \int_{(\varepsilon t,\infty)} \big(1-e^{-x/(\varepsilon t)}\big)\, Q_t(dx)\\
&\le 1-\exp\Big(-t\int_{(0,1]}\big(1-e^{-y/(\varepsilon t)}\big)\,\nu(dy)\Big)\\
&\le t\int_{(0,1]}\big(1-e^{-y/(\varepsilon t)}\big)\,\nu(dy)\Big)\\
&\le \varepsilon^{-1}\int_{(0,\varepsilon t]} y\,\nu(dy) + t\int_{(\varepsilon t,1]} \nu(dy)\,. \qquad (7.21)
\end{aligned}$$

The first term in (7.21) goes to 0 as $t \searrow 0$. To treat the second term, let $\delta > 0$ and choose $r \in (0,1)$ so that $\int_{(0,r]} s\,\nu(ds) < \delta$. Then as $t \searrow 0$,

$$t\int_{(\varepsilon t,1]} \nu(dy) \le \int_{(\varepsilon t,r]} s\,\nu(ds) + t\nu(r,1] \to \int_{(0,r]} s\,\nu(ds) < \delta\,.$$

Since δ is an arbitrary positive number, it follows that

$$\lim_{t\searrow 0}\; t\int_{(\varepsilon t,1]} \nu(dy) = 0\,,$$

as desired. *Hint:* for (vi): For any $t \in (0,1]$ there exists a nonnegative integer n such that $t > 2^{-n-1}$ and

$$\frac{Z_t}{t} \le 2^{n+1} Z_{2^{-n}}\,.$$

30-32. The carelessness might be ignoring the term ‘almost’ in the phrase ‘ a.s.’.

30-35. in case $\alpha < 1$, 0 or ∞ according as $\beta < \frac{1}{\alpha}$ or $\beta \ge \frac{1}{\alpha}$; in case $\alpha = 1$, 0 if and only if $\beta < 1$, and ∞ if and only if $\beta > 1$