Solutions, answers, and hints for selected problems

Complete solutions of some problems are given. Answers only are given for some other problems. For still others, only hints or partial solutions are given. Asterisks in "A Modern Approach to Probability Theory" by Fristedt and Gray identify the problems that are treated in this supplement.

For Chapter 30

30-1. $I_{[2+(1/n),\infty)}$ right-continuous; pointwise limit $I_{(2,\infty)}$ not right-continuous at 2.

30-10. The moment generating function is

$$\begin{aligned} u \rightsquigarrow E\big(e^{-uY_t}\big) &= E\Big(\exp\Big(-u \sum_{x\in[0,\infty]} xX\big((0,t]\times\{x\}\big)\Big)\Big) \\ &= E\left(\prod_{\substack{s\in(0,t] \\ x\in[0,\infty]}} \big[e^{-ux}\big]^{X\{(s,x)\}}\right). \end{aligned} \tag{0.1}$$

For calculating (7.17), we may replace $(0,t]$ by $[0,t]$. The function $(s,x) \rightsquigarrow e^{-ux}$ is a continuous function on the compact set $[0,t]\times[0,\infty]$, taking the value 0 at (s,∞) if $u>0$ and the value 1 there if $u=0$. Therefore we may apply Proposition 15 of Chapter 29 to conclude that (7.17) equals

$$\begin{aligned} &\exp\Big(-\int_{[0,t]\times[0,\infty]} (1-e^{-ux})\,\kappa(\lambda\times Q)(d(s,x))\Big) \\ &= \exp\Big(-\kappa t \int_{[0,\infty]} (1-e^{-ux})\,Q(dx)\Big)\,. \end{aligned}$$

We could have treated the problem as a single-variable problem by working with the Poisson point process X_t, the restriction of X to $(0,t]\times[0,\infty]$.

In view of Remark 1, Q might be a probability measure on $(0,\infty]$, which is not compact. We could handle this setting, by adjoining 0 to $(0,\infty]$ and specifying $Q\{0\}=0$, or by approximating $x \rightsquigarrow e^{-ux}$ by continuous functions that equal 1 for small x.

It is not possible to treat characteristic functions by adjoining $\pm\infty$ to R in order to obtain compactness, because one will then lose continuity. Approximation of the functions $x \rightsquigarrow e^{ivx}$ by functions that are continuous everywhere and constant for large x is a method that works. By then going to the limit one obtains the characteristic function of Y_t:

$$v \rightsquigarrow \exp\Big(-\kappa t \int_{[0,\infty]} (1 - e^{ivx})\, Q(dx)\Big)\,.$$

30-13. $1 - e^{-t\nu[y,\infty]}$

30-16. Set

$$\widetilde{R}_y(B) = R(\{v \in \mathbf{D}^+[0,1] \colon yv \in B\})\,,$$

and let $0 = t_0 \le t_1 < t_2 < \cdots < t_d = 1$. The proof relies on showing that

$$\begin{aligned} &P[Z_{t_i} - Z_{t_{i-1}} \le b_i \text{ for } 1 \le i \le d] \\ &= \int_{(0,\infty)} \widetilde{R}_y(\{z \in \mathbf{D}^+[0,1] \colon z_{t_i} - z_{t_{i-1}} \le b_i \text{ for } 1 \le i \le d\})\, ae^{-ay}\, dy \end{aligned} \tag{0.2}$$

for positive numbers b_i.

The left side of (7.18) equals

$$\begin{aligned} &\prod_{i=1}^{d} \int_0^{b_i} \frac{a^{(t_i - t_{i-1})}\, \theta_i^{(t_i - t_{i-1})-1}\, e^{-a\theta_i}}{\Gamma(t_i - t_{i-1})}\, d\theta_i \\ &= a \prod_{i=1}^{d} \int_0^{b_i} \frac{\theta_i^{(t_i - t_{i-1})-1}\, e^{-a\theta_i}}{\Gamma(t_i - t_{i-1})}\, d\theta_i\,. \end{aligned} \tag{0.3}$$

The right side of (7.18) equals

$$\int_0^\infty R(\{v \in \mathbf{D}^+[0,1] \colon v_{t_i} - v_{t_{i-1}} \le \frac{b_i}{y} \text{ for } 1 \le i \le d\})\, ae^{-ay}\, dy\,.$$

From Problem 15, we can rewrite this expression in terms of a Dirichlet distribution:

$$\begin{aligned} a \int_{\substack{0<\rho_i\le b_i/y\,,\, i\le(d-1) \\ 0<1-\rho_1-\cdots-\rho_{d-1}\le b_d/y}} \int_0^\infty e^{-ay} \frac{(1-\rho_1 - \cdots - \rho_{d-1})^{(t_d - t_{d-1})-1}}{\Gamma(t_d - t_{d-1})} \\ \cdot \prod_{i=1}^{d-1} \frac{\rho_i^{(t_i - t_{i-1})-1}}{\Gamma(t_i - t_{i-1})}\, dy\, d\rho_{d-1}\, \ldots\, d\rho_1\,. \end{aligned} \tag{0.4}$$

For $1 \le i \le d-1$, let $\theta_i = y\rho_i$, and also let $\theta_d = y(1-\rho_1-\cdots-\rho_{d-1})$. The Jacobian of this transformation is $\frac{d(\theta_1,\ldots,\theta_{d-1},\theta_d)}{d(\rho_1,\ldots,\rho_{d-1},y)} = y^{d-1}$; hence this change of variables turns (7.20) into (7.19), as desired.

30-25. negative binomial with parameters $1/\big(1 + E(Y_1)\big)$ and $\tau E(Z_1)$

30-31. (iii): Let $\varepsilon > 0$, and denote the distribution of Z_t by Q_t. Then for $\varepsilon t < 1$,

$$\begin{aligned}\big(1-e^{-1}\big)P[Z_t > \varepsilon t] &\leq \int_{(\varepsilon t,\infty)} \big(1-e^{-x/(\varepsilon t)}\big)\, Q_t(dx) \\ &\leq 1-\exp\Big(-t\int_{(0,1]} \big(1-e^{-y/(\varepsilon t)}\big)\,\nu(dy)\Big) \\ &\leq t\int_{(0,1]} \big(1-e^{-y/(\varepsilon t)}\big)\,\nu(dy)\Big) \\ &\leq \varepsilon^{-1}\int_{(0,\varepsilon t]} y\,\nu(dy) + t\int_{(\varepsilon t,1]} \nu(dy)\,. \end{aligned} \tag{0.5}$$

The first term in (7.21) goes to 0 as $t \searrow 0$. To treat the second term, let $\delta > 0$ and choose $r \in (0,1)$ so that $\int_{(0,r]} s\,\nu(ds) < \delta$. Then as $t \searrow 0$,

$$t\int_{(\varepsilon t,1]} \nu(dy) \leq \int_{(\varepsilon t,r]} s\,\nu(ds) + t\nu(r,1] \to \int_{(0,r]} s\,\nu(ds) < \delta\,.$$

Since δ is an arbitrary positive number, it follows that

$$\lim_{t\searrow 0}\, t\int_{(\varepsilon t,1]} \nu(dy) = 0\,,$$

as desired. *Hint:* for (vi): For any $t \in (0,1]$ there exists a nonnegative integer n such that $t > 2^{-n-1}$ and

$$\frac{Z_t}{t} \leq 2^{n+1} Z_{2^{-n}}\,.$$

30-32. The carelessness might be ignoring the term 'almost' in the phrase ' a.s.'.

30-35. in case $\alpha < 1$, 0 or ∞ according as $\beta < \frac{1}{\alpha}$ or $\beta \geq \frac{1}{\alpha}$; in case $\alpha = 1$, 0 if and only if $\beta < 1$, and ∞ if and only if $\beta > 1$