Solutions, answers, and hints for selected problems

Complete solutions of some problems are given. Answers only are given for some other problems. For still others, only hints or partial solutions are given. Asterisks in "A Modern Approach to Probability Theory" by Fristedt and Gray identify the problems that are treated in this supplement.

For Chapter 16

16-1. *Hint:* Use Example 1.

16-6. 0.309 at 0; 0.215 at ± 1; 0.093 at ± 2; 0.029 at ± 3; 0.007 at ± 4; 0.001 at ± 5; 0.000 elsewhere (Comment: Using a certain table we found values that did not come close to summing to 1, so we concluded that either that table has errors or we were reading it incorrectly. We used another table.)

16-12. Suppose that $Q_k \to Q$ as $k \to \infty$. Fix n and suppose that there exist distributions R_k such that $R_k^{*n} = Q_k$. Let β_k, and γ_k denote the characteristic functions of Q_k and R_k, respectively. Because the family $\{Q_k \colon k = 1, 2, \dots\}$ is relatively sequentially compact, the family $\{\beta_k \colon k = 1, 2, \dots\}$ is equicontinuous at 0, by Theorem 13 of Chapter 14. Thus there exists some open interval B containing 0 such that $\beta_k(u) \neq 0$ for $u \in B$ and all k. So (Problem 7 of Appendix E), $\psi_k(u) = -\log \circ \beta_k(u)$ is well-defined for $u \in B$ and all k, and the family $\{\psi_k \colon k = 1, 2, \dots\}$ is equicontinuous at 0. For $u \in B$, $\gamma_k(u) = \exp\bigl(-\frac{1}{n}\psi_k(u)\bigr)$. Hence $\{\gamma_k \colon k = 1, 2, \dots\}$ is equicontinuous at 0. By Theorem 13 of Chapter 14 the family $\{R_k \colon k = 1, 2, \dots\}$ is relatively sequentially compact, and, therefore, the sequence (R_k) contains a convergent subsequence; let R denote the limit of such a subsequence. Since the convolution of convergent sequences converges to the convolution of the limit, $R^{*n} = Q$ as desired. [Comment: For fixed n we only used $R_k^{*n} = Q_k$ for each k, rather than the full strength of infinite divisibility. If Q is infinitely divisible we can strengthen the conclusion: From the forthcoming Proposition 3 it follows that β is never 0 and therefore that R is the unique distribution whose characteristic function is $\exp \circ (\frac{1}{n} \log \circ \beta)$ and moreover, it equals the limit of the sequence (R_k).]

16-13. By Proposition 1 the product of two infinitely divisible characteristic functions is infinitely divisible. The factors we use are the characteristic function of the compound Poisson distribution corresponding to ν, as in (16.1), and the function

$$u \rightsquigarrow \exp\Big(i\Big[\eta - \int_{\mathrm{R}\setminus\{0\}} \chi\,d\nu\Big]u - \frac{\sigma^2 u^2}{2}\Big)\,,$$

known by Problem 9 to be infinitely divisible. The product equals $\exp\circ(-\psi)$, which is, therefore, an infinitely divisible characteristic function. For $\sigma = 0$ and $\eta = \int \chi\,d\nu$, the second factor is the function $u \rightsquigarrow 1$ and thus we obtain the compound Poisson characteristic function corresponding to an arbitrary finite measure ν.

16-14. Define ν_j, $1 \le j \le 3$, by

$$\begin{aligned} \nu_1(B) &= \nu\big(B \cap [-1,1]\big)\,;\\ \nu_2(B) &= \nu\big(B \cap (-\infty,-1)\big)\,;\\ \nu_3(B) &= \nu\big(B \cap (1,\infty)\big)\,. \end{aligned}$$

Write $\psi = \sum_{j=1}^4 \psi_j$, where

$$\begin{aligned} \psi_1(u) &= \int_{\mathrm{R}\setminus\{0\}} \big(1 - e^{iuy} + iuy\big)\,\nu_1(dy)\,;\\ \psi_2(u) &= \int_{\mathrm{R}\setminus\{0\}} \big(1 - e^{iuy}\big)\,\nu_2(dy)\,;\\ \psi_3(u) &= \int_{\mathrm{R}\setminus\{0\}} \big(1 - e^{iuy}\big)\,\nu_3(dy)\,;\\ \psi_4(u) &= iu\big(-\eta - \nu(-\infty,-1) + \nu(1,\infty)\big) + \frac{\sigma^2 u^2}{2}\,. \end{aligned}$$

Then X has the same distribution as $\sum_{j=1}^4 X_j$, where $(X_j \colon 1 \le j \le 4)$ is an independent quadruple and, for $1 \le j \le 4$, X_j is infinitely divisible with characteristic function $\exp\circ(-\psi_j)$. In view of the linearity of expectation, strengthened as in Problem 29 of Chapter 9 for independent random variables, and the linearity of variance for independent random variables, we have thus replaced the original problem by four subsidiary problems—to show:

$$\begin{aligned} &E(X_1) = 0\,, && \mathrm{Var}(X_1) = \int_{[-1,1]\setminus\{0\}} y^2\,\nu(dy)\,;\\ &E(X_2) = \int_{(-\infty,-1)} y\,\nu(dy)\,, && \mathrm{Var}(X_2) = \int_{(-\infty,-1)} y^2\,\nu(dy)\,;\\ &E(X_3) = \int_{(1,\infty)} y\,\nu(dy)\,, && \mathrm{Var}(X_3) = \int_{(1,\infty)} y^2\,\nu(dy)\,;\\ &E(X_4) = \eta + \nu(-\infty,-1) - \nu(1,\infty)\,, && \mathrm{Var}(X_4) = \sigma^2\,. \end{aligned}$$

(Comments: In defining ψ_2 and ψ_3, but not ψ_1 we were able to split off the term involving χ. It is important that no assumptions about existence of expectations or about finiteness of either expectations or variances are being made.)

The formulas involving X_4 are the known formulas for the mean and variance of a Gaussian random variable. Standard applications of the Dominated Convergence

Theorem, based on bounds from Appendix E, show that ψ_1 has derivatives of all orders, in particular orders 1 and 2, which may be calculated by differentiating under the integral sign. Thus,

$$\psi_1'(u) = \int_{[-1,1]\setminus\{0\}} (-iye^{iuy} + iy)\,\nu(dy)$$

and

$$\psi_1''(u) = \int_{[-1,1]\setminus\{0\}} y^2 e^{iuy}\,\nu(dy)\,.$$

The first and second derivatives of $\exp\circ(-\psi_1)$ exist (because those of ψ_1 do) and equal the functions $-\psi_1'\cdot(\exp\circ(-\psi_1))$ and $(-\psi_1''+(\psi_1')^2)\cdot(\exp\circ(-\psi))$. Inserting $u=0$ gives 0 for the first derivative and $\int_{[-1,1]\setminus\{0\}} y^2\,\nu(dy)$ for the second, as desired.

Turning to X_3, with the intention of skipping X_2 because its treatment is so similar to that of X_3, we note that the desired formulas are obvious in case ν_3 is the zero measure and recognize that for other ν_3 we may use Example 2. In this latter case we replace ν_3 by λR where R is a probability measure on $(1,\infty)$. In terms of the notation of Example 2 we see that X_3 has the same distribution as

$$\sum_{k=1}^{\infty} Y_k I_{[M\geq k]}\,,$$

Using the independence of each pair (Y_k, M) and monotone convergence we obtain

$$\begin{aligned} E(X_3) &= \Big(\int_{(1,\infty)} y\,R(dy)\Big)\sum_{k=1}^{\infty} P[M\geq k] \\ &= \Big(\int_{(1,\infty)} y\,R(dy)\Big)E(M) = \int_{(1,\infty)} y\,\nu(dy)\,. \end{aligned}$$

We go for the second moment rather than directly for the variance (a useful strategy when monotone convergence is being used):

$$\begin{aligned} E(X_3^2) &= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} E(Y_k Y_l I_{[M\geq k\vee l]}) \\ &= 2\sum_{k=2}^{\infty}\sum_{l=1}^{k-1} E(Y_k)E(Y_l)P[M\geq k] + \sum_{k=1}^{\infty} E(Y_k^2)P[M\geq k] \\ &= 2\Big(\int_{(1,\infty)} y\,R(dy)\Big)^2 \sum_{k=2}^{\infty}(k-1)P[M\geq k] \\ &\quad + \Big(\int_{(1,\infty)} y^2\,R(dy)\Big)\sum_{k=1}^{\infty} P[M\geq k]\,. \end{aligned} \tag{0.1}$$

The second term in (7.8) is what we want to prove the variance to be, so we only need prove that the first term equals $(E(X_3))^2$. To do this we only need show that

$2\sum_{k=2}^{\infty}(k-1)P[M \geq k] = \lambda^2$, which is a consequence of the following calculation:

$$\begin{aligned}
2\sum_{k=2}^{\infty}(k-1)P[M \geq k] &= 2\sum_{k=2}^{\infty}\sum_{m=k}^{\infty}(k-1)P[M=m] \\
&= 2\sum_{m=2}^{\infty}\sum_{k=2}^{m}(k-1)P[M=m] \\
&= \sum_{m=0}^{\infty} m(m-1)P[M=m] \\
&= E(M^2) - E(M) = (\lambda^2+\lambda) - \lambda = \lambda^2 .
\end{aligned}$$

16-17. If $\eta = 0$ and ν is symmetric about 0, the characteristic exponent is real because the function

$$y \rightsquigarrow -\sin uy + u\chi(y)$$

is an odd function for each u. Therefore the corresponding distribution is symmetric about 0 and its characteristic exponent has the form shown.

For the converse suppose that the characteristic function is real. It follows that the characteristic exponent is real since it is continuous and equals the real number 0 at 0. Then

$$-\eta u + \int_{\mathrm{R}\setminus\{0\}} \big(-\sin uy + u\chi(y)\big)\,\nu(dy) = 0$$

for every u. Another way to get 0 is to replace η by $\eta_0 = 0$ and ν by ν_0 defined by $\nu_0(B) = \frac{1}{2}(\nu(B) + \nu(-B))$. This change, together with no change in σ also leaves the real part of the characteristic exponent unchanged. By the uniqueness of the triples in Lévy-Khinchin representations (Lemma 11) it follows that $\eta = 0$ and $\nu = \nu_0$. We are done since it is obvious that ν_0 is symmetric about 0. (Comment: Another approach is to use the measure ζ defined in Lemma 7.)

16-20. Let X have a compound Poisson distribution with corresponding Lévy measure ν. Write $\nu = \nu_- + \nu_+$, where $\nu_-(0,\infty) = 0$ and $\nu_+(-\infty,0) = 0$. Then X has the same distribution as $X_- + X_+$, where (X_-, X_+) is an independent pair of compound Poisson random variables with corresponding Lévy measures ν_- and ν_+, the independence being a consequence of the factorization of (16.1) induced by $\nu = \nu_- + \nu_+$. If ν_- is not the zero measure, then by Problem 19 there is positive probability that $X_- < 0$ and $X_+ = 0$ and thus positive probability that $X < 0$. Therefore, ν_- must be the zero measure if $P[X \geq 0] = 1$.

16-25. The moment generating functions of a gamma distribution has the form $v \rightsquigarrow (1+\frac{v}{a})^{-\gamma}$. Accordingly, we want to find (ξ,ν) (with $\nu\{\infty\} = 0$) such that

$$\gamma\log\big(1+\frac{v}{a}\big) = \xi v + \int_{(0,\infty)} (1-e^{-vy})\,\nu(dy)\,.$$

By letting $v \to \infty$ we see that the shift $\xi = 0$. Then differentiation of both sides, with differentiation inside the integral being justified by the Monotone Convergence

Theorem (or in some other manner), gives

$$\frac{\gamma}{a+v} = \int_{(0,\infty)} e^{-vy} y\, \nu(dy)\,.$$

It is now easy to see that the Lévy measure ν has the density $y \rightsquigarrow \gamma y^{-1} e^{-ay}$ with respect to Lebesgue measure on $(0,\infty)$.

16-33. Statement: Let $((\xi_n, \nu_n),\ n = 1,2,\dots)$, satisfy: every $\xi_n \in \mathrm{R}^+$ and every ν_n is a Lévy measure for $\overline{\mathrm{R}}^+$. For each n, let Q_n be the infinitely divisible distribution on $\overline{\mathrm{R}}^+$ corresponding to (ξ_n, ν_n) via the relation

$$\int_{[0,\infty]} e^{-vx}\, Q_n(dx) = \exp\Big(-\xi_n v - \int_{(0,\infty]} (1 - e^{-vy})\, \nu_n(dy)\Big)\,.$$

Then the sequence $(Q_n \colon n = 1,2,\dots)$ converges to a distribution on $\overline{\mathrm{R}}^+$ different from the delta distribution at ∞ if and only if there exist $\xi \in \mathrm{R}^+$ and a Lévy measure ν for $\overline{\mathrm{R}}^+$ for which the following two conditions both hold:

$$\nu[x,\infty] = \lim_{n\to\infty} \nu_n[x,\infty] \quad \text{if } 0 < x \text{ and } \nu\{x\} = 0\,;$$

$$\begin{aligned}\xi &= \lim_{\varepsilon\searrow 0} \limsup_{n\to\infty} \Big(\xi_n + \int_{(0,\varepsilon]} y\, \nu_n(dy)\Big)\\ &= \lim_{\varepsilon\searrow 0} \liminf_{n\to\infty} \Big(\xi_n + \int_{(0,\varepsilon]} y\, \nu_n(dy)\Big)\,.\end{aligned}$$

In case these conditions are satisfied the limit of the sequence $(Q_n \colon n \geq 1)$ is the infinitely divisible distribution with moment generating function

$$v \rightsquigarrow \exp\Big(-\xi v - \int_{(0,\infty]} (1 - e^{-vy})\, \nu(dy)\Big)\,.$$

16-41. limiting distribution: two-sided Poisson supported by set of integral multiples of c; characteristic exponent: $u \rightsquigarrow 1 - \cos cu$.

16-42. limit exists; corresponding triple: $(0,1,\nu)$, where ν has support $\{-1,1\}$ and $\nu\{-1\} = \nu\{1\} = \frac{1}{2}$; characteristic exponent of the limit (not requested in the problem) is

$$u \rightsquigarrow \frac{u^2}{2} + 1 - \cos u\,.$$

16-50. *Hint:* Fix u and let $\varepsilon > 0$. By (E.2) and (E.3) of Appendix E and Lemma 20, the characteristic functions $\beta_{k,n}$ and corresponding characteristic exponents $\psi_{k,n}$ satisfy

$$\big(1 - \beta_{k,n}(u)\big) \leq \psi_{k,n}(u) \leq (1+\varepsilon)\big(1 - \beta_{k,n}(u)\big)$$

for all sufficiently large n (depending on u) and all $k \leq n$.

16-54. uan condition satisfied so Theorem 25 applicable; $u \rightsquigarrow e^{-(\log 2)u}$

16-59. Q exists and characterized by triple $(0,0,\nu)$, where $\frac{d\nu}{d\lambda}(y) = \frac{(1-|y|)^2}{2} \vee 0$; $Q\{0\} = e^{-1/3}$

16-68. limit exists; $(0, \frac{1}{2\sqrt{3}}\sqrt{\log 2}, 0)$ is corresponding triple for its Lévy-Khinchin representation