Solutions, answers, and hints for selected problems

Complete solutions of some problems are given. Answers only are given for some other problems. For still others, only hints or partial solutions are given. Asterisks in "A Modern Approach to Probability Theory" by Fristedt and Gray identify the problems that are treated in this supplement.

For Chapter 29

29-5. Let $r = \sum_{j=1}^{m} r_j$.

$$P[X\{1\} = k] = \frac{\binom{r_1}{k}\binom{r-r_1}{n-k}}{\binom{r}{n}}$$

$$P[X\{1\} = k, X\{2\} = l] = \frac{\binom{r_1}{k}\binom{r_2}{l}\binom{r-r_1-r_2}{n-k-l}}{\binom{r}{n}}$$

$$P[X\{1\} = k, X\{2\} = l, X\{3\} = m] = \frac{\binom{r_1}{k}\binom{r_2}{l}\binom{r_3}{m}\binom{r-r_1-r_2-r_3}{n-k-l-m}}{\binom{r}{n}}$$

29-8. $P[X(B) = z] = \frac{(\sharp B)^z (n-\sharp B)^{r-z}}{n^r}\binom{r}{z}$, $0 \le z \le r$. Thus the distribution of $X(B)$ is binomial with parameters $\frac{\sharp B}{n}$ and r.

29-13. Let $(V_n : n \ge 0)$ be a renewal sequence. Define a random measure X on Z^+ by $X\{n\} = V_n$. Clearly X is a point process and its intensity measure equals the potential measure of the renewal sequence.

29-18. We use the formula for the probability that a Poisson random variable equals 0. For $v \ge 0$,

$$P[V \ge v] = P[X(\{0, 1, \dots, v-1\}) = 0] = e^{-v} .$$

Then

$$P[V = v] = P[V \ge v] - P[V \ge (v+1)] = e^{-v} - e^{-(v+1)} = (1 - e^{-1})e^{-v} .$$

29-23. Write

$$Y \cup \{0\} = \{0 = Y_0 < Y_1 < Y_2 < \dots\} ,$$

and let $(S_0 = 0, S_1, S_2, \dots)$ be a random walk having exponentially distributed steps with mean c^{-1}. For an arbitrary positive integer n we will show that $(Y_1, \dots, Y_n)$ and $(S_1, \dots, S_n)$ have the same distribution, thereby finishing the proof. We will verify that the distribution of each of these random vectors has the same density with respect to n-dimensional Lebesgue measure—namely,

$$(y_1,\dots,y_n) \rightsquigarrow \begin{cases} c^n e^{-cy_n} & \text{if } 0 < y_1 < \cdots < y_n \\ 0 & \text{otherwise}\,. \end{cases} \tag{0.1}$$

To check that this is the correct density for $(Y_1,\dots,Y_n)$ we integrate it over a set of the form $\prod_{i=1}^n [u_i, v_i)$, where

$$0 = v_0 < u_1 < v_1 < u_2 < \cdots < u_n < v_n = \infty\,.$$

We get

$$\begin{aligned} e^{-cu_n}\prod_{i=1}^{n-1} c(v_i-u_i) &= \Big(\prod_{i=1}^{n-1} c(v_i-u_i)e^{-c(v_i-u_i)}\Big)\Big(\prod_{i=1}^{n} e^{-(u_i-v_{i-1})}\Big) \\ &= \Big(\prod_{i=1}^{n-1} P[\sharp(Y\cap[u_i,v_i))=1]\Big)\Big(\prod_{i=1}^{n} P[\sharp(Y\cap[v_{i-1},u_i))=0]\Big) \\ &= P[Y_i\in[u_i,v_i) \text{ for } 1\le i\le n]\,, \end{aligned}$$

as desired.

We know that the density of $\big((S_1-S_0)\,,\,(S_2-S_1)\,,\,\dots\,,\,(S_n-S_1)\big)$ is

$$(x_1,\dots,x_n) \rightsquigarrow \begin{cases} \prod_{i=1}^n ce^{-cx_i} & \text{if each } x_i > 0 \\ 0 & \text{otherwise}\,. \end{cases}$$

We can get the density of $(S_1,\dots,S_n)$ by using the linear transformation $y_k = x_1+\cdots+x_k$, $1\le k\le n$, the Jacobian of which equals 1. The result is the desired density (7.15).

29-24. *Hint:* One approach is to start with sequences U and V having the desired properties and then use Problem 23 to show that $\{(U_n,V_n)\colon n=1,2,\dots\}$ is a Poisson point process with intensity measure $\lambda\times\mu$.

29-26. c^{-3}

29-29. π, $\frac{\pi}{2}$

29-34. $h \rightsquigarrow \frac{1}{n}\sum_{i=1}^n h(i)$ for $r=1$; $h\rightsquigarrow \frac{1}{n}\big[\sum_{i=1}^n [h(i)]^{-1}\big]\prod_{j=1}^n h(j)$ for $r=n-1$

29-39. $h\rightsquigarrow \exp\big(-\sum_{\psi\in\Psi}(1-h(\psi))\big)$, where Ψ is the countable set

29-43. The probability generating functional of $X+Y$ is

$$\begin{aligned} h \rightsquigarrow E\Big(\prod_{\psi\in\Psi}[h(\psi)]^{(X+Y)(\{\psi\})}\Big) &= E\Big(\prod_{\psi\in\Psi}[h(\psi)]^{X(\{\psi\})}[h(\psi)]^{Y(\{\psi\})}\Big) \\ &= E\Big(\Big[\prod_{\psi\in\Psi}[h(\psi)]^{X(\{\psi\})}\Big]\Big[\prod_{\psi\in\Psi}[h(\psi)]^{Y(\{\psi\})}\Big]\Big) \\ &= E\Big(\prod_{\psi\in\Psi}[h(\psi)]^{X(\{\psi\})}\Big)\,E\Big(\prod_{\psi\in\Psi}[h(\psi)]^{Y(\{\psi\})}\Big)\,, \end{aligned}$$

which is the product of the probability generating functionals of X and Y.

29-50. Suppose that $X_n \xrightarrow{\mathcal{D}} X$ as $n \to \infty$; that is, $Q_n \to Q$, where Q_n and Q denote the distributions of X_n and X, respectively. Let h be in the domain of the probability generating functional of Q (and thus of each Q_n). Assume first that h is bounded below by a positive constant. Then the function

$$\pi \rightsquigarrow \int \log(1/h)\, d\pi$$

is continuous, and thus the same is true for the function

$$\pi \rightsquigarrow e^{-\int \log(1/h)\, d\pi}\,. \tag{0.2}$$

For this latter function it is straightforward to remove the assumption that h be bounded below by a positive constant (of course, using the conventions $\infty \cdot 0 = 0$ and $e^{-\infty} = 0$). That

$$\int e^{-\int \log(1/h)\, d\pi}\, dQ_n \to \int e^{-\int \log(1/h)\, d\pi}\, dQ$$

follows from the continuity of the function (7.16). That the limiting probability generating functional has the property described in the theorem is a consequence of Proposition 16 which says that all probability generating functionals have a more general property.

For the converse suppose that F is the limit of a sequence of probability generating functionals corresponding to a sequence $(Q_n \colon n = 1, 2, \dots)$ of distributions of point processes in a locally compact Polish space Ψ, and that F satisfies the condition in the theorem. Let C be any compact subset of Ψ. By using Lemma 1 one can show that there exists a compact set B such that every point of C is an interior point of B and that therefore there exists a continuous $[(1 - \frac{1}{m}), 1]$-valued function h_m such that $h_m(\psi) = 1 - \frac{1}{m}$ for $\psi \in C$ and $h_m(\psi) = 1$ for $\psi \in B^c$.

Let $\varepsilon > 0$ Since $\mathrm{F}(\mathrm{h_m}) \to 1$ as $m \to \infty$, we can fix m so that for all n

$$\begin{aligned} Q_n\{\pi \colon \pi(C) \le z\} &\ge \int_{\{\pi \colon \pi(C) \le z\}} \prod_{\psi} [h_m(\psi)]^{\pi(\{\psi\})}\, Q_n(d\pi) \\ &> 1 - \tfrac{\varepsilon}{2} - \textstyle\int_{\{\pi \colon \pi(C) > z\}} \prod_\psi [h_m(\psi)]^{\pi(\{\psi\})}\, Q_n(d\pi) \\ &\ge 1 - \tfrac{\varepsilon}{2} - \big(1 - \tfrac{1}{m}\big)^{z+1}, \end{aligned}$$

which is larger than $1 - \varepsilon$ for sufficiently large z. By Theorem 19, every subsequence of (Q_n) has a convergent subsequence. By the first paragraph of this proof, F is the probability generating functional of every subsequential limit. By Theorem 14 all subsequential limits are identical. Therefore, the sequence (Q_n) itself converges to a limit whose probability generating functional is F.