Solutions, answers, and hints for selected problems

Complete solutions of some problems are given. Answers only are given for some other problems. For still others, only hints or partial solutions are given. Asterisks in "A Modern Approach to Probability Theory" by Fristedt and Gray identify the problems that are treated in this supplement.

For Chapter 14

14-2. At any x where both F and G are continuous, $F(x) = G(x)$. The set of points where F is discontinuous is countable because F is monotone. The same is true for G. The set $\mathcal{D}$ of points where both F and G are continuous, and thus equal, is dense, because it has a countable complement. For any $y \in \mathrm{R}$, there exists a decreasing sequence $(x_1, x_2, \dots)$ in $\mathcal{D}$ such that $x_k \searrow y$ as $k \nearrow \infty$. The right continuity of F and G and the equality $F(x_k) = G(x_k)$ for each k then yield $F(y) = G(y)$.

14-4. We will first show that $Q_n\{x\} \to \frac{\lambda^x}{x!} e^{-\lambda}$ for each $x \in \mathrm{Z}^+$. The factor $e^{-\lambda}$ arises as the limit of $(1 - \frac{\lambda}{n})^n$. The factor λ^x already appears in the formula for $Q_n\{x\}$, and $x!$ appears there implicitly as part of the binomial coefficient. To finish this part of the proof we need to show

$$\lim_{n\to\infty} \frac{n!}{(n-x)!n^x} \big(1 - \frac{\lambda}{n}\big)^{-x} = 1\,.$$

The second factor obviously has the limit 1 and the first factor can be written as

$$\prod_{k=0}^{x-1} (1 - \frac{k}{n})$$

which also has limit 1.

We will finish the proof by showing that

$$\lim_{n\to\infty} \sum_{x\le y} Q_n(x) = \sum_{x \le y} \frac{\lambda^x}{x!}\, e^{-\lambda}$$

for every $y \in \mathrm{R}$. On the left side the limit and summation can be interchanged because the summation has only finitely many nonzero terms. The desired equality then follows from the preceding paragraph.

This problem could also be done by using Proposition 8 which appears later in Chapter 14.

14-6. standard gamma distributions. For $x > 0$,

$$\lim_{\gamma \searrow 0} \frac{1}{\Gamma(\gamma)} \int_0^x u^{\gamma-1} e^{-u}\, du = 1 - \Big[\lim_{\gamma \searrow 0} \frac{1}{\Gamma(\gamma)}\Big] \Big[\lim_{\gamma \searrow 0} \int_x^\infty u^{\gamma-1} e^{-u}\, du\Big]\,.$$

The first limit in the product of two limits equals 0 and by the Dominated Convergence Theorem, the second limit equals $\int_x^\infty u^{-1} e^{-u}\, du < \infty$, a dominating function being $(u^{-1} \vee 1)e^{-u}$. We conclude that

$$\lim_{\gamma \searrow 0} \frac{1}{\Gamma(\gamma)} \int_0^x u^{\gamma-1} e^{-u}\, du = 1$$

for $x > 0$ from which convergence to the delta distribution at 0 follows (despite the fact that we did not obtain convergence to 1 at $x = 0$).

14-10. Fix $x \geq 0$ and $r > 0$. We want to show

$$\lim_{m \to \infty} \sum_{k=0}^{\lfloor mx \rfloor} \Big(\frac{1}{m}\Big)^r \frac{(r)_k^\uparrow}{k!} \Big(1 - \frac{1}{m}\Big)^k = \frac{1}{\Gamma(r)} \int_0^x u^{r-1} e^{-u}\, du\,,$$

which is equivalent to

$$\lim_{m \to \infty} \sum_{k=1}^{\lfloor mx \rfloor} \Big(\frac{1}{m}\Big)^r \frac{(r)_k^\uparrow}{k!} \Big(1 - \frac{1}{m}\Big)^k = \frac{1}{\Gamma(r)} \int_0^x u^{r-1} e^{-u}\, du\,, \tag{0.1}$$

because the term $\frac{1}{m}$, obtained by setting $k = 0$, approaches 0 as $m \to \infty$.

The sum on the left side of (7.5) can be written as

$$\int_0^x g_m(u)\, du\,,$$

where

$$g_m(u) = \begin{cases} \big(\frac{k}{m}\big)^{r-1} \frac{(r)_k^\uparrow}{k^{r-1}\, k!} \big(1 - \frac{1}{m}\big)^k & \text{if } k-1 < mu \leq k \text{ for } k = 1, 2, \ldots, \lfloor mx \rfloor \\ 0 & \text{otherwise}\,; \end{cases}$$

and the right side can be written as

$$\int_0^x g(u)\, du\,,$$

where

$$g(u) = \frac{1}{\Gamma(r)}\, u^{r-1} e^{-u}\,.$$

The plan is to show that $g_m(u) \to g(u)$ as $m \to \infty$ for each u in the interval $(0, x)$ and to find a function h that has finite integral and dominates each g_m, for then the desired conclusion will follow immediately from the Dominated Convergence Theorem. We will consider the three factors in g_m separately. It is important to keep in mind that k depends on u and m and that in particular, $k \to \infty$ as $m \to \infty$ for each fixed $u \in (0, x)$, as this dependence is not explicit in the notation.

It is clear that $\big(\frac{k}{m}\big)^{r-1} \to u^{r-1}$ for $u \in (0, x)$. In case $r \leq 1$, $\big(\frac{k}{m}\big)^{r-1} \leq u^{r-1}$. In case $r > 1$, $\big(\frac{k}{m}\big)^{r-1} \leq x^{r-1}$. Thus, we have constructed one factor of what we hope will be the dominating function h: u^{r-1} in case $r \leq 1$ and the constant x^{r-1} in case $r > 1$.

The second factor in $g_m(u)$ equals

$$\frac{1}{\Gamma(r)}\,\frac{\Gamma(r+k)}{k^{r-1}\,\Gamma(k-1)}\,.$$

We use the Stirling Formula to obtain the limit:

$$\begin{aligned}
&\frac{1}{\Gamma(r)}\,\lim_{k\to\infty}\frac{\Gamma(r+k)}{k^{r-1}\,\Gamma(k+1)}\\
&=\frac{1}{\Gamma(r)}\,\lim_{k\to\infty}\frac{\sqrt{2\pi}(r+k)^{r+k-\frac{1}{2}}\,e^{-(r+k)}}{k^{r-1}\sqrt{2\pi}\,(k+1)^{k+\frac{1}{2}}\,e^{-(k+1)}}\\
&=\frac{e^{-(r-1)}}{\Gamma(r)}\,\lim_{k\to\infty}\big(1+\frac{r}{k}\big)^{r-1}\,\big(1+\frac{r-1}{k+1}\big)^{-1/2}\,\big(1+\frac{r-1}{k+1}\big)^{k+1}\\
&=\frac{1}{\Gamma(r)}\,.
\end{aligned}$$

The second factor in $g_m(u)$ is thus bounded as a function of k, the bound possibly depending on r. Such a constant bound will be the second factor we will use in constructing the dominating function h.

For the third factor in $g_m(u)$ we observe that

$$\big(1-\frac{1}{m}\big)^{mu+1}<\big(1-\frac{1}{m}\big)^{k}\le\big(1-\frac{1}{m}\big)^{mu}\,, \tag{0.2}$$

from which it follows that

$$\big(1-\frac{1}{m}\big)^{k}\to e^{-u}\,.$$

Moreover, (7.6) and the inequality $(1-\frac{1}{m})^m<e^{-1}$ imply that e^{-u} is a dominating function for the third factor in $g_m(u)$.

Our candidate for a dominating function $h(u)$ having finite integral is a constant multiple of $u^{r-1}e^{-u}$ in case $r\le 1$ and a constant multiple of e^{-u} in case $r>1$. Both these function have finite integral on the interval $[0,x]$, as desired.

For $r=0$, each $Q_{p,r}$ is the delta distribution at 0, and, therefore, $\lim_{m\to\infty}R_m$ equals this delta distribution.

14-14. Let G denote the standard Gumbel distribution function defined in Problem 13. For $a>0$ and $b\in\mathrm{R}$,

$$G(ax+b)=e^{-e^{-(ax+b)}}=e^{-ce^{-ax}}\,,$$

where $c=e^{-b}>0$.

14-16. For any real constant x,

$$\sum_{n=1}^{\infty}P[X_n>c]=\infty\,.$$

By the Borel-Cantelli Lemma, $M_n\to\infty$ a.s. as $n\to\infty$ and, hence,

$$\{\omega\colon\ \lim_{n\to\infty}[M_n(\omega)-\log n]\text{ exists and }>m\}$$

is a tail event of the sequence $(X_k\colon k=1,2,\dots)$ for every m. By the Kolmogorov 0-1 Law, the almost sure limit of $(M_n-\log n)$ must equal a constant if it exists. On the other hand, by the preceding problem the almost sure limit, if it exists, must have a Gumbel distribution. Therefore, the almost sure limit does not exist.

The sequence does not converge in probability, for if it did, there would be a subsequence that converges almost surely and the argument of the preceding paragraph would show that the distribution of the limit would have to be a delta distribution rather than a Gumbel distribution.

The preceding problem does imply that

$$\frac{M_n - \log n}{\log n} \xrightarrow{\mathcal{D}} 0 \quad \text{as } n \to \infty$$

and, therefore, that

$$\frac{M_n}{\log n} \to 1 \quad \text{in probability as } n \to \infty\,.$$

In Example 6 of Chapter 9 the stronger conclusion of almost sure convergence was obtained using calculations not needed for either this or the preceding problem.

14-22. Weibull: mean $= -\Gamma(1+\frac{1}{\alpha})$, variance $= \Gamma(1+\frac{2}{\alpha}) - [\Gamma(1+\frac{1}{\alpha})]^2$; Fréchet: mean is finite if and only if $\alpha > 1$ in which case it equals $\Gamma(1-\frac{1}{\alpha})$, variance is finite if and only if $\alpha > 2$ in which case it equals $\Gamma(1-\frac{2}{\alpha}) - [\Gamma(1-\frac{1}{\alpha})]^2$

14-35. $Q_n\{0\} = 1 - \frac{1}{n}$, $Q_n\{n^2\} = \frac{1}{n}$

14-37. We need to show

$$\lim_{z \searrow 1} Q_z(-\infty, x] = \frac{c-1}{c}$$

for all positive finite x. That is, we must show

$$\lim_{z \searrow 1} \frac{1}{\zeta(z)} \sum_{k=1}^{\lfloor c^{1/(z-1)} x \rfloor} \frac{1}{k^z} = \frac{c-1}{c}\,.$$

We may replace $\frac{1}{\zeta(z)}$ by $z-1$ because the ratio of these two functions approaches 1 as $z \searrow 1$ (as may be checked by bounding the sum that defines the Riemann zeta function by formulas involving integrals). We can bound the above sum by using:

$$\int_1^m \frac{1}{x^z}\,dz < \sum_{k=1}^m \frac{1}{k^z} < 1 + \int_1^m \frac{1}{x^z}\,dz\,;$$

that is,

$$\frac{1}{z-1}\big(1 - \frac{1}{m^{z-1}}\big) < \sum_{k=1}^m \frac{1}{k^z} < 1 + \frac{1}{z-1}\big(1 - \frac{1}{m^{z-1}}\big)\,;$$

Replace m by $\lfloor c^{1/(z-1)} x \rfloor$, multiply by $z-1$, and let $z \searrow 1$ to obtain the desired limit $1 - \frac{1}{c}$.

14-44. Since $|\beta_n(u)| \le 1$ for every u and n, we only need show that $1 - \mathrm{R}(\beta_\mathrm{n}(\mathrm{u})) \to 0$ for each u. This will follow from the hypothesis in the lemma and the inequality

$$1 - \mathrm{R}\big(\beta(2\mathrm{u})\big) \le 4\big[1 - \mathrm{R}\big(\beta(\mathrm{u})\big)\big]\,,$$

which we will now prove to be valid for all characteristic functions β.

Using the positive definiteness of β we have

$$\begin{aligned}&+\beta(0-0)z_1\bar z_1+\beta(u-0)z_1\bar z_2+\beta(2u-0)z_1\bar z_3\\ &\quad+\beta(0-u)z_2\bar z_1+\beta(u-u)z_2\bar z_2+\beta(2u-u)z_2\bar z_3\\ &\qquad+\beta(0-2u)z_3\bar z_1+\beta(u-2u)z_3\bar z_2+\beta(2u-2u)z_3\bar z_3\ge 0\,.\end{aligned}$$

Setting $z_1=1$, $z_2=-2$, $z_3=1$, noting that $\beta(-v)=\overline{\beta(v)}$, and using $\beta(0)=1$, we obtain

$$6-8\mathrm{R}\big(\beta(\mathrm{u})\big)+2\mathrm{R}\big(\beta(2\mathrm{u})\big)\ge 0\,,$$

from which follows

$$8\big[1-\mathrm{R}\big(\beta(\mathrm{u})\big)\big]\ge 2\big[1-\mathrm{R}\big(\beta(2\mathrm{u})\big)\big]\,,$$

as desired. (Notice that the characteristic function of the standard normal distribution shows that 4 is the smallest possible constant for the inequality proved above, but it does not resolve the issue of whether $\le$ can be replaced by $<$ for $u\ne 0$.)

14-48. The probability generating function $\rho_{p,r}$ of $Q_{p,r}$ is given by

$$\rho_{p,r}(s)=\sum_{x=0}^{\infty}(1-p)^r\binom{-r}{x}p^x s^x=(1-p)^r(1-ps)^{-r}\,.$$

Clearly, $(p,r)\rightsquigarrow\rho_{p,r}(s)$ is a continuous function on

$$\{(p,r)\colon 0\le p<1\,,\ r\ge 0\}$$

for each fixed s, so the same is true of the function $(p,r)\rightsquigarrow Q_{p,r}$.

14-49. Example 1. The moment generating function of Q_n is

$$u\rightsquigarrow\frac{1}{n+1}\sum_{k=0}^{\infty}\big(1+\frac{1}{n}\big)^{-k}e^{-uk/n}=\frac{1}{n+1}\cdot\frac{1}{1-\frac{e^{-u/n}}{1+\frac{1}{n}}}=\frac{1}{n\big(1-e^{-u/n}+\frac{1}{n}\big)}\,,$$

which, as $n\to\infty$, approaches, pointwise, the function $u\rightsquigarrow\frac{1}{u+1}$, the moment generating function of the exponential distribution. An appeal to Theorem 19 finishes the proof.

14-52. Let V be the constant random variable 3 and let V_n be normally distributed with mean 3 and variance n^{-2}. Let $b_n=3$ and $a_n=n^{-1}$. Then $(V_n-b_n)/a_n$ is normally distributed with mean 0 and variance 1 for every n even though $a_n\to 0$ as $n\to\infty$.