Solutions, answers, and hints for selected problems

Complete solutions of some problems are given. Answers only are given for some other problems. For still others, only hints or partial solutions are given. Asterisks in "A Modern Approach to Probability Theory" by Fristedt and Gray identify the problems that are treated in this supplement.

For Chapter 26

26-5.

$$\begin{aligned} Q_{n+1}(B) = P[X_{n+1} \in B] &= E\big(P([X_{n+1} \in B] \mid \mathcal{F}_n)\big) \\ &= E\big(\mu_{X_n}(B)\big) = \int \mu_x(B) Q_n(dx) = (Q_n T)(B) \end{aligned}$$

$$E(f \circ X_{n+1} \mid \mathcal{F}_n) = \int f(y) \mu_{X_n}(dy) = (Tf) \circ X_n$$

26-19. Let f be the identity function on $[0,1]$. Clearly f is bounded and measurable. By Theorem 6, Y is a martingale where

$$Y_n = X_n - \sum_{k=0}^{n-1} (Gf) \circ X_k \,.$$

Solving for X_n gives a representation for X in terms of the martingale Y and a previsible sequence having the value 0 when $n = 0$. To show that this sequence is increasing, as required for a Doob decomposition, we only need show that Gf is a nonnegative function when f is the identity function. The following calculation does this:

$$Gf(x) = Tf(x) - x = E^x(X_1) - x \geq 0 \,,$$

the last equality using the fact that X is a submartingale.

26-28. $x \rightsquigarrow \frac{2}{x+1}$

26-29. *Hint:* Reminder: There is one value of x that is not required to satisfy the difference equation.

26-31. $x \rightsquigarrow e^{-1} \sum_{k=x}^{\infty} \frac{1}{k!}$

26-39. Denote the two states by x and y. By the last part of Problem 38, if one of the two states is transient so is the other. Now suppose that y is null recurrent; our goal is to show that x is not positive recurrent.

By the Renewal Theorem the sequence of entries of T^n in position y along the main diagonal converges to 0 as $n \to \infty$. We will complete the proof by finding an integer k and a positive constant c such that the entry in position y along the main diagonal in T^n is larger than c times the entry in position x along the main diagonal in T^{n-k} for all $n \geq k$, for then it will follow that the sequence of entries in T^{n-k} in position x along the main diagonal will converge to 0 as $n \to \infty$, implying that x is not positive recurrent.

One way to start at y and to then be there again at time n is to first be at state x at some time r, then be at x again at some time $n-k+r$, and then be at state y at time n. By first choosing r and then k appropriately one can make the product of the probabilities of the first and third of these three tasks a positive constant c.

We omit the part of the solution treating the periodicity issue.

26-43. Suppose that, for some k, all entries of T^k are positive. For any states x and y there is positive probability of being at y at time k if the starting state is x. Hence, $\pi_{xy} > 0$. Therefore, T is irreducible. Clearly, $T^m T^k = T^{m+k}$ has only positive entries for all nonnegative integers m, and thus 1 is the greatest common divisor of the powers of T for which the upper left entry (or any other diagonal entry) is positive. Aperiodicity follows.

For the converse suppose that T is irreducible and aperiodic. The sequence of numbers in a fixed diagonal position of $T^0, T^1, T^2, \dots$ is an aperiodic potential sequence, which, by Lemma 18 of Chapter 25, contains only finitely many zero terms. Thus, there exists an integer m such that all diagonal entries of T^m are positive. Hence, all diagonal entries of T^n are positive for $n \geq m$. Since T is irreducible, there is, for each x and y, an integer k_{xy} such that the entry in row x and column y of $T^{k_{xy}}$ is positive. Let $k = m + \max\{k_{xy}\}$. Since T^k can be obtained by multiplying $T^{k_{xy}}$ by a power of T at least as large as m, the entry in row x and column y of T^k is positive. Thus, all entries of T^k are positive, as desired.

26-52. starting state of interest denoted by 0; probabilities of absorption at the absorbing states -2 and -1, respectively:

$$\sum_{k=0}^{\infty} \frac{2^{2k-1}}{(3 \cdot 2^{2k-1} - 1)(3 \cdot 2^{2k} - 1)} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{2^{2k}}{(3 \cdot 2^{2k} - 1)(3 \cdot 2^{2k+1} - 1)}$$

probability of no absorption: 1/3

26-55. We can introduce infinitely many extra transient states in order to obtain a birth-death sequence. The transition distributions μ_x are given by

$$\mu_x\{x-1\} = \frac{x}{b} \wedge 1$$
$$\mu_x\{x+1\} = \frac{b-x}{b} \vee 0\,.$$

From Problem 54 we see the relevance of the following product:

$$\prod_{z=1}^{x} \frac{\frac{b-z+1}{b} \vee 0}{\frac{z}{b} \wedge 1} = \binom{b}{x}\,.$$

The number r as defined in Problem 54 can now be calculated:

$$r = \sum_{x=0}^{\infty} \binom{b}{x} = \sum_{x=0}^{b} \binom{b}{x} = 2^b \,.$$

The equilibrium distribution Q for the Ehrenfest urn sequence is given by

$$Q\{x\} = \frac{1}{2^b} \binom{b}{x}, \quad 0 \le x \le b \,.$$