Solutions, answers, and hints for selected problems

Complete solutions of some problems are given. Answers only are given for some other problems. For still others, only hints or partial solutions are given. Asterisks in "A Modern Approach to Probability Theory" by Fristedt and Gray identify the problems that are treated in this supplement.

For Chapter 24

24-2. The 'if' part is obvious. For the proof of 'only if' fix n. The inequality in the problem is obviously true with equality in case $m = 0$ and it is true by definition if $m = 1$. To complete an inductive proof, let $m > 1$ and assume that

$$E(X_{n+(m-1)} \mid \mathcal{F}_n) \geq X_n \text{ a.s.}$$

Since $\mathcal{F}_n \subseteq \mathcal{F}_{n+(m-1)}$,

$$\begin{aligned} E\big(X_{n+m} \mid \mathcal{F}_n\big) &= E\big(E(X_{n+m} \mid \mathcal{F}_{n+(m-1)}) \mid \mathcal{F}_n\big) \\ &\geq E\big(X_{n+(m-1)} \mid \mathcal{F}_n\big) \geq X_n \text{ a.s.} \end{aligned}$$

24-8. We treat the real and imaginary parts simultaneously. Let $\mathcal{F}_n = \sigma(S_0, \dots, S_n)$ and denote the steps of the random walk by $X_1, X_2, \dots$. Then

$$\begin{aligned} E(Y_{n+1} \mid \mathcal{F}_n) &= \frac{1}{(\varphi(u))^{n+1}} E\big(e^{iuS_n} e^{iuX_{n+1}} \mid \mathcal{F}_n\big) \\ &= \frac{1}{(\varphi(u))^{n+1}} e^{iuS_n} E\big(e^{iuX_{n+1}} \mid \mathcal{F}_n\big) \\ &= \frac{1}{(\varphi(u))^{n}} e^{iuS_n} = Y_n \,. \end{aligned}$$

[Remark: We have proved that the real and imaginary parts of $(Y_n \colon n = 0, 1, \dots)$ are martingales with respect to the minimal filtration for the random walk, which may possibly contain larger σ-fields than the corresponding σ-fields in the minimal filtration for the sequence (Y_n).]

24-10. Proof of uniqueness: Suppose that conditions (i)-(iv) of the proposition hold as stated and that they also hold with some sequences Z and U in place of Y and V, respectively. By subtraction

$$Z_n - Y_n = V_n - U_n \,.$$

Thus $Z_n - Y_n$ is $\mathcal{F}_{n-1}$-measurable, and, hence,

$$Z_n - Y_n = E\big((Z_n - Y_n) \mid \mathcal{F}_{n-1}\big) = Z_{n-1} - Y_{n-1}\,.$$

This fact combined with $Z_0 - Y_0 = 0$, a consequence of $U_0 = V_0 = 0$, gives $Z_n = Y_n$, and therefore $U_n = V_n$ for every n.

24-11. Let $\mathcal{F}_n = \sigma(S_0, \dots, S_n)$. Then

$$\begin{aligned} E\big((S_{n+1}^2 - S_n^2) \mid \mathcal{F}_n\big) &= E\big((S_{n+1} - S_n)^2 \mid \mathcal{F}_n\big) + 2E\big((S_{n+1} - S_n)S_n \mid \mathcal{F}_n\big) \\ &\geq 0 + 2S_n E\big((S_{n+1} - S_n) \mid \mathcal{F}_n\big) = 0\,, \end{aligned}$$

as desired. [Remark: See the remark in the solution of Problem 8.] $V_n = n\operatorname{Var}(S_1)$.

24-20. *Hint:* $|X_n| I_{[T>n]} = X_n I_{[T>n]} \leq X_T I_{[T>n]} \leq X_T$

24-23. *Hint:* Use two relevant previous results; do not do any hard work.

24-26. The sequence $(X_n \colon n \geq 0)$, being uniformly bounded, is uniformly integrable. By Theorem 12 and the Optional Sampling Theorem, $E(X_T) \leq E(X_0) = f_0$; Clearly $E(X_T) \geq g\,P[X_T = g]$. Hence $f_0 \geq g\,P[X_T = g]$, as desired.

24-33.

$$\begin{aligned} E([S_{T_n} - \tfrac{1}{2}T_n]^2) = \operatorname{Var}(S_1)E(T_n) = 2^{-1}[1 - 2^{-n}] &\nearrow 2^{-1} = E([S_T - \tfrac{1}{2}T]^2) \\ \operatorname{Var}(S_{T_n}) = 2^{-n}[1 - 2^{-n}] &\searrow 0 = \operatorname{Var}(S_T) \\ E\big(\operatorname{Var}(S_{T_n} \mid T_n)\big) = 2^{-(n+1)} &\searrow 0 = E\big(\operatorname{Var}(S_T \mid T)\big) \end{aligned}$$

For $n > 1$, $\operatorname{Var}(S_{T_n}) < E(S_1)E(T_n)$, thus highlighting the importance of the assumption in Theorem 15 of mean 0 for the steps.

24-41. Suppose that X is a uniformly integrable martingale. By the theorem it has an almost sure limit $Y = X_\infty$ such that $(X_n \colon n \in \overline{\mathrm{Z}}+)$ is both a submartingale and a supermartingale—that is, a martingale. Hence $E(Y \mid \mathcal{F}_n) = X_n$. Moreover, Y is $\mathcal{F}_\infty$-measurable, so $E(Y \mid \mathcal{F}_\infty) = Y$.

For the converse, suppose that Y has finite expectation and

$$X_n = E(Y \mid \mathcal{F}_n)$$

for each $n \in \mathrm{Z}^+$. Take expectations of both sides to obtain $E(X_n) = E(Y)$, which is finite. For $k < n$,

$$E\big(X_n \mid \mathcal{F}_k\big) = E\big(E(Y \mid \mathcal{F}_n) \mid \mathcal{F}_k\big) = E\big(Y \mid \mathcal{F}_k\big) = X_k\,.$$

Therefore with $X_\infty = Y$, $(X_n \colon n \in \overline{\mathrm{Z}}^+)$ is a martingale with respect to the filtration $(\mathcal{G}_n \colon n \in \overline{\mathrm{Z}}^+)$, where $\mathcal{G}_n = \mathcal{F}_n$ for $n < \infty$ and

$$\mathcal{G}_\infty = \sigma(Y, \mathcal{F}_\infty)\,.$$

To prove that $\{X_n \colon n \in \mathrm{Z}^+\}$ is uniformly integrable we let $A_{n,r} = [|X_n| > r]$ and note that, for any $m > 0$,

$$\begin{aligned} E\big(|X_n|\,;\, A_{n,r}\big) &= E\big(|E(Y \bigm| \mathcal{F}_n)|\,;\, A_{n,r}\big) \leq E\big(E(|Y| \bigm| \mathcal{F}_n)\,;\, A_{n,r}\big) \\ &= E\big(|Y|\,;\, A_{n,r}\big) \leq mP(A_{n,r}) + E(|Y|\,;\, [|Y| > m])\,. \end{aligned}$$

Since, by dominated convergence, the second term approaches 0 as $m \to \infty$, we can finish the proof of uniform integrability by showing that $P(B_{n,r}) + P(C_{n,r}) \to 0$ as $r \to \infty$ uniformly in n, where $B_{n,r} = [X_n > r]$ and $C_{n,r} = [X_n < -r]$. That this is so follows from

$$P(B_{n,r}) \le \frac{1}{r} E(X_n\,;\, B_{n,r}) = \frac{1}{r} E(Y\,;\, B_{n,r}) \le \frac{1}{r} E(|Y|)\,,$$

$$P(C_{n,r}) \le -\frac{1}{r} E(X_n\,;\, C_{n,r}) = -\frac{1}{r} E(Y\,;\, C_{n,r}) \le \frac{1}{r} E(|Y|)\,,$$

and the observation that $E(|Y|)$ is a finite number independent of r and n. From the theorem $(X_1, X_2, \dots)$ has an $\mathbf{L_1}$ and a.s. limit Z that is $\mathcal{F}_\infty$ measurable.

To prove that $Z = E(Y \mid \mathcal{F}_\infty)$ we only need show that $E((Z - Y)\,;\, D) = 0$ for every $D \in \mathcal{F}_\infty$. For $D \in \mathcal{F}_n$ we have

$$\begin{aligned} E((Z-Y)\,;\, D) &= E\big(E((Z-Y)\,;\, D \mid \mathcal{F}_n)\big) \\ &= E\big(I_D E((Z-Y) \mid \mathcal{F}_n)\big) = E(\big(I_D(X_n - X_n)\big) = 0\,, \end{aligned}$$

where I_D denotes the indicator function of D. Thus the desired equality is true for all $D \in \cup_{n=0}^{\infty} \mathcal{F}_n$, a collection that is closed under finite intersections, contains the entire probability space Ω, and generates $\mathcal{F}_\infty$. By linearity of expectation the set of D for which $E((Y - Z)\,;\, D) = 0$ is closed under proper differences, and, since Y and Z both have means, dominated convergence shows that it is closed under monotone limits. An appeal to the Sierpiński Class Theorem completes the proof.

24-42. The martingale $(V_n \colon n \in \mathrm{Z}^+)$, being bounded, is obviously uniformly integrable. Hence, $\lim V_n$ exists; call this limiting proportion of blue balls V_∞. From the fact that the martingale property is preserved when V_∞ is adjoined to the sequence $(V_n \colon n \in \mathrm{Z}^+)$, we conclude that the expected limiting proportion of blue balls conditioned on the contents of the urn at any particular time is the proportion of blue balls in the urn at that time.

24-45. Let Y be a $(-\infty, 0]$-valued random variable for which $E(Y) = -\infty$. Let $X_n = Y \vee (-n)$. Then $X_n(\omega) \to Y(\omega)$ for every ω. For $n = 0, 1, 2, \dots$, let $\mathcal{G}_n = \sigma(Y)$. Then $(\mathcal{G}_n \colon n = 0, 1, 2, \dots)$ is a reverse filtration to which $(X_n \colon n = 0, 1, 2, \dots)$ is adapted. Clearly $E(X_n) > -\infty$ for every n. The inequality

$$E(X_n \mid \mathcal{G}_{n+1}) = X_n \ge X_{n+1}$$

shows that $(X_0, X_1, \dots)$ is a reverse submartingale.