A straightforward induction proof that

$$\sum_{\substack{j=n+2 \\ j \text{even}}}^{\infty} \frac{1}{j-1}\binom{j}{j/2}\frac{1}{2^j} = \binom{n}{n/2}2^{-n}$$

completes the proof. [For $n = 0$ (the starting value for the induction proof), the left side equals the probability—namely 1 —that the time of first return to 0 equals some finite value, and 1 is also the value of the right side when $n = 0$.]

19-11. $\frac{2+\pi}{2\pi} \approx 0.82$

19-27. We need to show that the value of the derivative of the moment generating function at 0 equals $-ab$. By definition, the derivative there equals

$$\begin{aligned}
&\lim_{u\searrow 0} \frac{\sinh(a\sqrt{2u}) + \sinh(b\sqrt{2u}) - \sinh((a+b)\sqrt{2u})}{u\sinh((a+b)\sqrt{2u})} \\
&= \lim_{w\searrow 0} \frac{2[\sinh(aw) + \sinh(bw) - \sinh((a+b)w)]}{w^2\sinh((a+b)w)}\,.
\end{aligned}$$

Now three applications of the l'Hospital Rule yield the desired result.

For Chapter 20

20-5. $E(X)$

20-6. Proof of (iv): By the Cauchy-Schwarz Inequality

$$E(|X-X_n|) = E(|X-X_n|1) \le \sqrt{E(|X-X_n|^2)}\sqrt{E(1^2)} = \sqrt{E((X-X_n)^2)} \to 0\,.$$

Proof of (iii), using (iv):

$$\limsup E(|X_n|) \le E(|X|) + \limsup E(|X_n - X|) = E(|X|)$$

and

$$\begin{aligned}
E(|X|) &\le \liminf\big[E(|X_n|) + E(|X-X_n|)\big] \\
&\le \liminf E(|X_n|) + \limsup E(|X-X_n|) = \liminf E(|X_n|)\,,
\end{aligned}$$

from which the desired conclusion follows.

20-15. By the sentence preceding the problem, $E(V_i) = 0$ for each i and $E(Z) = E(X)$. Hence, $E(X - Z) = 0$. Our task has become that of showing $E((X - Z)Y_i) = 0$ for each i. In view of the fact that each Y_i is a linear combination of 1 and the various V_j and that we have already shown that $E((X - Z)1) = 0$, we can reformulate our task as that of showing that $E(XV_j) = E(ZV_j)$ for each j.

From the definition of Z we obtain

$$E(ZV_j) = \langle X, 1\rangle E(V_j) + \sum_{i=1}^{m}\langle X, V_i\rangle E(V_iV_j) = \langle X, V_j\rangle = E(XV_j)\,.$$

For Chapter 21

21-3. By Definition 1: Clearly, $P(B \mid \mathcal{G})I_A$ is a member of $\mathbf{L}_2(\Omega, \mathcal{G}, P)$. Let $Y \in \mathbf{L}_2(\Omega, \mathcal{G}, P)$. To finish the proof we must show

$$E\Big([I_{A\cap B} - P(B \mid \mathcal{G})I_A]\,Y\Big) = 0\,.$$

That is we must show that

$$E\Big([I_B - P(B \mid \mathcal{G})]\,[I_A Y]\Big) = 0\,.$$

In view of the fact that $I_A Y$ is $\mathcal{G}$-measurable, this statement follows from the definition of $P(B \mid \mathcal{G})$.

By Proposition 2: Let $X = P(B \mid \mathcal{G})I_A$. Condition (i) of Proposition 2 is clearly satisfied by X. To check condition (ii), let $C \in \mathcal{G}$. Then we must show that

$$E(XI_C) = P((A \cap B) \cap C)\,.$$

That is, we must show that

$$E(P(B \mid \mathcal{G})I_{A\cap C}) = P(B \cap (A \cap C))\,.$$

In view of the fact that $A \cap C \in \mathcal{G}$, this last statement follows from Proposition 2 applied to $P(B \mid \mathcal{G})$.

[Comment: Notice the similarity between the two proofs. Proposition 2 says that the orthogonality condition entailed in Definition 1 need only be checked for indicator functions of members of $\mathcal{G}$ rather than for every member of $\mathbf{L}_2(\Omega, \mathcal{G}, P)$.]

21-5. The right side X of (21.1) is obviously $\sigma(C)$-measurable. To check the second condition in Proposition 2 we only have to consider the four members of $\sigma(C)$. Obviously $E(XI_\emptyset) = 0 = P(A \cap \emptyset)$. Also,

$$E(XI_C) = \frac{P(A \cap C)}{P(C)} E(I_C I_C) = P(A \cap C)$$

and similarly,

$$E(XI_{C^c}) = \frac{P(A \cap C^c)}{P(C^c)} E(I_{C^c} I_{C^c}) = P(A \cap C^c)\,.$$

Finally,

$$E(XI_\Omega) = E(XI_C) + E(XI_{C^c}) = P(A \cap C) + P(A \cap C^c) = P(A \cap \Omega)\,.$$

21-8. (ii)

$$\omega \rightsquigarrow \begin{cases} 1 & \text{if } \omega_1 + \omega_2 + \omega_3 + \omega_4 = 4 \\ \frac{1}{2} & \text{if } \omega_1 + \omega_2 + \omega_3 + \omega_4 = 2 \\ \frac{1}{6} & \text{if } \omega_1 + \omega_2 + \omega_3 + \omega_4 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$\frac{1}{6}$ for the particular given ω

(v)

$$\omega \rightsquigarrow \begin{cases} \frac{1}{4} & \text{if } \omega_1 + \omega_2 + \omega_3 + \omega_4 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$\frac{1}{4}$ for the particular given ω

21-9. The general formula is

$$16 \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{1} \sum_{l=0}^{1} P(A \cap B_{i,j,k,l}) I_{B_{i,j,k,l}} \,,$$

where

$$B_{i,j,k,l} = \{\psi \colon \psi_1 = 2i-1, \psi_2 = 2j-1, \psi_3 = 2k-1, \psi_4 = 2l-1\}\,.$$

(ii)

$$\omega \rightsquigarrow \begin{cases} 1 & \text{if } \omega_1 + \omega_2 = 2 \\ 0 & \text{otherwise} \end{cases}$$

0 for the particular given ω

(v) same answer as problem 8

21-10. For each positive integer m and almost every ω,

$$P(\limsup A_n \mid \mathcal{G})(\omega) \leq P\Big(\bigcup_{n=m}^{\infty} A_n \mid \mathcal{G}\Big)(\omega) \leq \sum_{n=m}^{\infty} P(A_n \mid \mathcal{G})(\omega)\,.$$

For those ω for which the sum on the right is finite, that sum can be made arbitrarily close to 0 by choosing m sufficiently large (depending on ω). For such an ω the probability on the far left must equal 0 since it does not depend on m. This completes the proof of the first of the two assertions in the problem.

21-12. $\psi \rightsquigarrow \psi_1$

21-13. It is possible that the image of V is not a measurable subset of Ψ.

21-17. $v \rightsquigarrow v$

21-24. With Q denoting the distribution of Y and δ_x the delta distribution at x, a conditional distribution is the function

$$(\omega, B) \rightsquigarrow Q\big([X(\omega), \infty)\big)\delta_{X(\omega)}(B) + Q\big(B \cap (-\infty, X(\omega)\big)\,.$$

(Various functions are presented via this notation: one function of two variables, functions of B for various fixed values of ω, and functions of ω for various fixed values of B.)

21-25. With Q denoting any fixed distribution [for instance, the (unconditional) distribution of X and δ_c denoting the delta distribution at c, a conditional distribution is $g \circ |X|$, where

$$g(w) = \begin{cases} \frac{f(-w)}{f(-w)+f(w)}\,\delta_{-w} + \frac{f(w)}{f(-w)+f(w)}\,\delta_w & \text{if } f(-w)+f(w) \neq 0 \\ \quad Q & \text{if } f(-w)+f(w) = 0\,. \end{cases}$$

21-30.

$$(\omega, x) \rightsquigarrow \begin{cases} \frac{1}{\lambda}\, e^{-(x-t)/\lambda} & \text{if } X(\omega) \geq t\,,\ x \geq t \\ \frac{1}{\lambda(1-e^{-t/\lambda})}\, e^{-x/\lambda} & \text{if } X(\omega) < t\,,\ 0 \leq x \leq t \\ 0 & \text{otherwise} \end{cases}$$

21-34. The density is

$$(x_1,\dots,x_{d-1},y) \rightsquigarrow \frac{(y-x_1-\cdots-x_{d-1})^{\gamma_d-1}e^{-y}}{\Gamma(\gamma_d)}\prod_{i=1}^{d-1}\frac{x_i^{\gamma_i-1}}{\Gamma(\gamma_i)}$$

for $x_i \geq 0, y \geq x_1+\cdots+x_{d-1}$.

Let $Y = X_1+\cdots+X_d$. A conditional density of $(X_1,\dots,X_{d-1})$ given $\sigma(Y)$ is

$$\begin{aligned}&(\omega,(x_1,\dots,x_{d-1}))\\ &\quad\rightsquigarrow \frac{\left(1-\frac{x_1}{Y(\omega)}-\cdots-\frac{x_{d-1}}{Y(\omega)}\right)^{\gamma_d-1}\Gamma(\gamma_1+\cdots+\gamma_d)}{\Gamma(\gamma_d)\,[Y(\omega)]^{d-1}}\prod_{i=1}^{d-1}\frac{\left(\frac{x_i}{Y(\omega)}\right)^{\gamma_i-1}}{\Gamma(\gamma_i)}\end{aligned}$$

for $x_j \geq 0, x_1+\cdots+x_{d-1} \leq Y(\omega)$ if $Y(\omega) > 0$ and $\rightsquigarrow$ the unconditional density of $(X_1,\dots,X_{d-1})$ if $Y(\omega) \leq 0$. [Note the relationship to the Dirichlet distribution which is described in an optional section of Chapter 10.]

21-44. Let Ω consist of the four points corresponding to two independent fair coins. Let $\mathcal{G}$ denote the σ-field generated by the first coin and $\mathcal{H}$ the σ-field generated by the second coin. By definition, $(\mathcal{G},\mathcal{H})$ is an independent pair and it is clear that $\sigma(\mathcal{G},\mathcal{H})$ consists of all subsets of Ω. Thus, any σ-field consisting of subsets of Ω is a sub-σ-field of $\sigma(\mathcal{G},\mathcal{H})$. Let $\mathcal{K}$ be the σ-field generated by the event that exactly 1 head is flipped. Given $\mathcal{K}$ the conditional probability of any member of $\mathcal{G}$ different from $\emptyset$ and Ω equals $\frac{1}{2}$ as does the conditional of any such member of $\mathcal{H}$. But, there is no event that has conditional probability given $\mathcal{K}$ equal to $\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$.

For Chapter 22

22-10. If X_3 were to exist so that (X_1,X_2,X_3) is exchangeable, then, since $X_1+X_2=0$ with probability 1, it would follow that $X_1+X_3=0$ and $X_2+X_3=0$ with probability 1. By solving three equations in three unknowns it would then follow that $X_1=0$ with probability 1, a contradiction.

22-11. *Hint:* Apply $E(P(A \mid \mathcal{G})) = P(A)$ for various choices for A.

22-14. uniform on the set of those $\binom{n}{[n+S_n(\omega)]/2}$ sequences of ±1's that contain $[n+S_n(\omega)]/2$ 1's and $[n-S_n(\omega)]/2$ -1's. [Comment: The answer does not depend on p.]

22-16. first term equals 1 with probability $\frac{\alpha}{\alpha+\beta}$. conditional distribution of second term given first term: equals 1 with probability $\frac{\alpha+1}{\alpha+\beta+1}$ if first term equals 1 and equals 1 with probability $\frac{\alpha}{\alpha+\beta+1}$ if first term equals 0. distribution of first two terms: equals $(1,1)$ with probability $\frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$ and equals $(0,0)$ with probability $\frac{\beta(\beta+1)}{(\alpha+\beta)(\alpha+\beta+1)}$ and equals $(1,0)$ and $(0,1)$ each with probability $\frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)}$

22-21. By exchangeability, the correlation of I_m and I_n is the same as that of I_1 and I_2 if $n \neq m$; of course, it equals 1 if $n = m$.

The correlation of I_1 and I_2 equals $\frac{c}{x_0+y_0+c}$, which approaches 0 as $(x_0,y_0) \to (\infty,\infty)$ and approaches 1 as $c \to \infty$.

For large (x_0,y_0) the knowledge of the color of a fixed number c of balls in the urn hardly influences the probability that a blue ball will be drawn. For large c, the second ball drawn is very likely to be of the same color as the first ball since after the first ball is drawn almost all the balls in the urn will have the same color as the first ball.

22-22. Using the fact that $\prod_n \frac{a+cn}{b+cn} = 0$ if $0 \le a < b$ and $0 \le c$, we have

$$\begin{aligned} P[I_n = 0 \text{ for } n > m] &= E(P[I_n = 0 \text{ for } n > m \mid \sigma(X_m, Y_m)]) \\ &= E\biggl(\prod_{n=m+1}^{\infty} \frac{Y_m + (n-m-1)c}{X_m + Y_m + (n-m-1)c}\biggr) \\ &= E(0) = 0 \end{aligned}$$

for each fixed m. Hence

$$\begin{aligned} P(\liminf\{\omega\colon I_n(\omega) = 0\}) &= P\biggl(\bigcup_{m=1}^{\infty} \bigcap_{n>m} \{\omega\colon I_n(\omega) = 0\}\biggr) \\ &\le \sum_{m=1}^{\infty} P\biggl(\bigcap_{n>m} \{\omega\colon I_n(\omega) = 0\}\biggr) \\ &= \sum_{m=1}^{\infty} 0 = 0\,, \end{aligned}$$

from which it follows that the first event in the problem has probability 1. That the second event given there also has probability 1 follows by applying the result already proved to the sequence $((1-I_n)\colon n = 1, 2, \dots)$, an application which is seen to be valid by interchanging the colors of the balls.

22-24. $\frac{(m-1)!}{m^{(n-1)}}\, S\bigl((n-1), (m-1)\bigr)$

For Chapter 23

23-11. *Hint:* Use Problem 14 of Chapter 5.

23-17. Let $\omega = (0, 1]$, $\mathcal{F}$ the Borel σ-field, and P Lebesgue measure. Let $X_n = nI_{(0,1/n)}$. Then $X_n(\omega) \to 0$ for every ω and $E(X_n) = 1$, so the (unconditional) Dominated Convergence Theorem must not apply. Let

$$\mathcal{G} = \sigma\bigl((2^{-m}, 2^{-(m-1)}]\colon m = 1, 2, \dots\bigr)\,.$$

The random variable $Y(\omega) = \frac{1}{\omega}$ dominates every X_n and satisfies $E(Y \mid \mathcal{G})(\omega) = 2^m \log 2$ for $2^{-m} < \omega \le 2^{-(m-1)}$. In particular $E(Y \mid \mathcal{G})(\omega) < \infty$ for every ω. Hence the Conditional Dominated Convergence Theorem applies. We conclude that $E(X_n \mid \mathcal{G})(\omega) \to 0$ for almost every ω, a fact that we could have also obtained by directly by observing that $E(X_n \mid \mathcal{G})(\omega) = 0$ for $n > \frac{2}{\omega}$.

23-23. Problem 21 of Chapter 21

23-30. $\frac{1}{2}$ (for all b), which is larger than $\frac{1}{3}$, the (unconditional) expectation. The following paragraphs present various ways of looking at the situation.

Fix b. If, before the random experiment begins, it is understood that one will be told whether or not b is between X and Y, one will clearly want to assign a larger value to the expectation of $Y - X$ in case b is between X and Y and a smaller value otherwise.

An appropriate weighted average of these two numbers equals $\frac{1}{3}$, so, as expected, the first of these two numbers is larger than $\frac{1}{3}$.

Knowing that exactly one of two order statistics from the uniform distribution on $(0,1)$ is larger than b gives no reason for biasing one's estimate for it among the various values larger than b. Thus, the conditional mean of its excess over b is half the distance from b to 1 —namely, $\frac{1-b}{2}$. Similarly the conditional mean of the difference between b and the smaller of the two order statistics is $\frac{b}{2}$. The sum of these two conditional expectations is $\frac{1}{2}$, independently of b.

Here is a second method of getting an intuitive feel for the value $\frac{1}{2}$. Fix the number b. Pick three iid points Z_1, Z_2, and Z_3 on a circle of circumference 1. Cut the circle at Z_1 in order to straighten it into a unit interval with the counterclockwise direction on the circle corresponding locally to the direction of increase on the unit interval. Then set the smaller of Z_2 and Z_3 equal to X and the larger equal to Y. The condition that b be between X and Y is the condition that as one traverses the circle counterclockwise the contacts with either Z_2 or Z_3 alternate with the contacts with either Z_1 or b. Among such possible arrangements, there is probability $\frac{1}{2}$ that b lies in the long interval and Z_1 in the short interval determined by Z_2 and Z_3 and probability $\frac{1}{2}$ that the opposite relations hold. So the average length of the interval in which b lies is $\frac{1}{2}$.

23-33. By Problem 27 and Proposition 6, there exist choices of $E(X^+ I_B \mid \mathcal{H})$ and $E(X^- I_B \mid \mathcal{H})$ such that

$$E(E(X^+ \mid \mathcal{G})I_B \mid \mathcal{H})(\omega) = E(E(X^+ I_B \mid \mathcal{G}) \mid \mathcal{H})(\omega) = E(X^+ I_B \mid \mathcal{H})(\omega)$$

and

$$E(E(X^- \mid \mathcal{G})I_B \mid \mathcal{H})(\omega) = E(E(X^- I_B \mid \mathcal{G}) \mid \mathcal{H})(\omega) = E(X^- I_B \mid \mathcal{H})(\omega)$$

for every sample point ω. Subtraction gives

$$\begin{aligned} &E(E(X^+ \mid \mathcal{G})I_B \mid \mathcal{H})(\omega) - E(E(X^- \mid \mathcal{G})I_B \mid \mathcal{H})(\omega) \\ &= E(XI_B \mid \mathcal{H})(\omega) \end{aligned} \tag{7.9}$$

for every ω for which the right side of (7.9) [that is, the right side of (23.9)] exists. At such an ω at least one of the two terms on the left side is finite.

We will focus on

$$A \stackrel{\text{def}}{=} \{\omega\colon E(E(X^+ \mid \mathcal{G})I_B \mid \mathcal{H})(\omega) < \infty\}\,.$$

For each $\omega \in A$,

$$\int_{[0,\infty]} x\, Z(\omega, dx) < \infty\,,$$

where Z is the conditional distribution of $E(X^+ \mid \mathcal{G})I_B$. So $E(Z(\cdot, \{\infty\})I_A) = 0$. From the definition of conditional probability we then obtain

$$P\Big(\{\omega\colon [E(X^+ \mid \mathcal{G})I_B](\omega) = \infty\} \cap A\Big) = 0\,.$$

Therefore the left side of (7.9) can be rewritten as

$$E\Big([E(X^+ \mid \mathcal{G}) - E(X^- \mid \mathcal{G})]I_B \;\Big|\; \mathcal{H}\Big)(\omega) \tag{7.10}$$

for almost every ω for which the right side of (7.9) is less than ∞. Similarly, this can be done for almost every ω for which the right side of (7.10) is greater than $-\infty$, in particular for almost every ω for which the right side of (7.9) equals ∞.

The upshot is that for almost every ω for which the right side of (7.9) exists, the left side of (7.9) can be rewritten as (7.10) in which the inside difference between two conditional expectations is not of the form $\infty - \infty$. Therefore linearity of conditional expectation may be used to complete the proof.

23-42. *Hint:* Apply the Conditional Chebyshev Inequality and then take (unconditional) expectations of both sides.

For Chapter 24

24-2. The 'if' part is obvious. For the proof of 'only if' fix n. The inequality in the problem is obviously true with equality in case $m = 0$ and it is true by definition if $m = 1$. To complete an inductive proof, let $m > 1$ and assume that

$$E(X_{n+(m-1)} \mid \mathcal{F}_n) \geq X_n \text{ a.s.}$$

Since $\mathcal{F}_n \subseteq \mathcal{F}_{n+(m-1)}$,

$$\begin{aligned} E\big(X_{n+m} \mid \mathcal{F}_n\big) &= E\big(E(X_{n+m} \mid \mathcal{F}_{n+(m-1)}) \mid \mathcal{F}_n\big) \\ &\geq E\big(X_{n+(m-1)} \mid \mathcal{F}_n\big) \geq X_n \text{ a.s.} \end{aligned}$$

24-8. We treat the real and imaginary parts simultaneously. Let $\mathcal{F}_n = \sigma(S_0, \dots, S_n)$ and denote the steps of the random walk by $X_1, X_2, \dots$. Then

$$\begin{aligned} E(Y_{n+1} \mid \mathcal{F}_n) &= \frac{1}{(\varphi(u))^{n+1}} E\big(e^{iuS_n} e^{iuX_{n+1}} \mid \mathcal{F}_n\big) \\ &= \frac{1}{(\varphi(u))^{n+1}} e^{iuS_n} E\big(e^{iuX_{n+1}} \mid \mathcal{F}_n\big) \\ &= \frac{1}{(\varphi(u))^{n}} e^{iuS_n} = Y_n\,. \end{aligned}$$

[Remark: We have proved that the real and imaginary parts of $(Y_n \colon n = 0, 1, \dots)$ are martingales with respect to the minimal filtration for the random walk, which may possibly contain larger σ-fields than the corresponding σ-fields in the minimal filtration for the sequence (Y_n).]

24-10. Proof of uniqueness: Suppose that conditions (i)-(iv) of the proposition hold as stated and that they also hold with some sequences Z and U in place of Y and V, respectively. By subtraction

$$Z_n - Y_n = V_n - U_n\,.$$

Thus $Z_n - Y_n$ is $\mathcal{F}_{n-1}$-measurable, and, hence,

$$Z_n - Y_n = E\big((Z_n - Y_n) \mid \mathcal{F}_{n-1}\big) = Z_{n-1} - Y_{n-1}\,.$$

This fact combined with $Z_0 - Y_0 = 0$, a consequence of $U_0 = V_0 = 0$, gives $Z_n = Y_n$, and therefore $U_n = V_n$ for every n.

24-11. Let $\mathcal{F}_n = \sigma(S_0, \dots, S_n)$. Then

$$\begin{aligned} E\big((S_{n+1}^2 - S_n^2) \mid \mathcal{F}_n\big) &= E\big((S_{n+1} - S_n)^2 \mid \mathcal{F}_n\big) + 2E\big((S_{n+1} - S_n)S_n \mid \mathcal{F}_n\big) \\ &\geq 0 + 2S_n E\big((S_{n+1} - S_n) \mid \mathcal{F}_n\big) = 0\,, \end{aligned}$$

as desired. [Remark: See the remark in the solution of Problem 8.] $V_n = n\,\mathrm{Var}(S_1)$.

24-20. *Hint:* $|X_n| I_{[T>n]} = X_n I_{[T>n]} \leq X_T I_{[T>n]} \leq X_T$

24-23. *Hint:* Use two relevant previous results; do not do any hard work.

24-26. The sequence $(X_n : n \geq 0)$, being uniformly bounded, is uniformly integrable. By Theorem 12 and the Optional Sampling Theorem, $E(X_T) \leq E(X_0) = f_0$; Clearly $E(X_T) \geq g\,P[X_T = g]$. Hence $f_0 \geq g\,P[X_T = g]$, as desired.

24-33.

$$\begin{aligned} E([S_{T_n} - \tfrac{1}{2}T_n]^2) = \mathrm{Var}(S_1)E(T_n) = 2^{-1}[1 - 2^{-n}] &\nearrow 2^{-1} = E([S_T - \tfrac{1}{2}T]^2) \\ \mathrm{Var}(S_{T_n}) = 2^{-n}[1 - 2^{-n}] &\searrow 0 = \mathrm{Var}(S_T) \\ E\big(\mathrm{Var}(S_{T_n} \mid T_n)\big) = 2^{-(n+1)} &\searrow 0 = E\big(\mathrm{Var}(S_T \mid T)\big) \end{aligned}$$

For $n > 1$, $\mathrm{Var}(S_{T_n}) < E(S_1)E(T_n)$, thus highlighting the importance of the assumption in Theorem 15 of mean 0 for the steps.

24-41. Suppose that X is a uniformly integrable martingale. By the theorem it has an almost sure limit $Y = X_\infty$ such that $(X_n : n \in \overline{\mathbb{Z}}+)$ is both a submartingale and a supermartingale—that is, a martingale. Hence $E(Y \mid \mathcal{F}_n) = X_n$. Moreover, Y is $\mathcal{F}_\infty$-measurable, so $E(Y \mid \mathcal{F}_\infty) = Y$.

For the converse, suppose that Y has finite expectation and

$$X_n = E(Y \mid \mathcal{F}_n)$$

for each $n \in \mathbb{Z}^+$. Take expectations of both sides to obtain $E(X_n) = E(Y)$, which is finite. For $k < n$,

$$E\big(X_n \mid \mathcal{F}_k\big) = E\big(E(Y \mid \mathcal{F}_n) \mid \mathcal{F}_k\big) = E\big(Y \mid \mathcal{F}_k\big) = X_k\,.$$

Therefore with $X_\infty = Y$, $(X_n : n \in \overline{\mathbb{Z}}^+)$ is a martingale with respect to the filtration $(\mathcal{G}_n : n \in \overline{\mathbb{Z}}^+)$, where $\mathcal{G}_n = \mathcal{F}_n$ for $n < \infty$ and

$$\mathcal{G}_\infty = \sigma(Y, \mathcal{F}_\infty)\,.$$

To prove that $\{X_n : n \in \mathbb{Z}^+\}$ is uniformly integrable we let $A_{n,r} = [|X_n| > r]$ and note that, for any $m > 0$,

$$\begin{aligned} E\big(|X_n|\,;\,A_{n,r}\big) &= E\big(|E(Y \bigm| \mathcal{F}_n)|\,;\,A_{n,r}\big) \leq E\big(E(|Y| \bigm| \mathcal{F}_n)\,;\,A_{n,r}\big) \\ &= E\big(|Y|\,;\,A_{n,r}\big) \leq mP(A_{n,r}) + E(|Y|\,;\,[|Y| > m])\,. \end{aligned}$$

Since, by dominated convergence, the second term approaches 0 as $m \to \infty$, we can finish the proof of uniform integrability by showing that $P(B_{n,r}) + P(C_{n,r}) \to 0$ as

$r \to \infty$ uniformly in n, where $B_{n,r} = [X_n > r]$ and $C_{n,r} = [X_n < -r]$. That this is so follows from

$$P(B_{n,r}) \leq \frac{1}{r} E(X_n\,;\, B_{n,r}) = \frac{1}{r} E(Y\,;\, B_{n,r}) \leq \frac{1}{r} E(|Y|)\,,$$

$$P(C_{n,r}) \leq -\frac{1}{r} E(X_n\,;\, C_{n,r}) = -\frac{1}{r} E(Y\,;\, C_{n,r}) \leq \frac{1}{r} E(|Y|)\,,$$

and the observation that $E(|Y|)$ is a finite number independent of r and n. From the theorem $(X_1, X_2, \dots)$ has an $\mathbf{L}_1$ and a.s. limit Z that is $\mathcal{F}_\infty$ measurable.

To prove that $Z = E(Y \mid \mathcal{F}_\infty)$ we only need show that $E((Z - Y)\,;\, D) = 0$ for every $D \in \mathcal{F}_\infty$. For $D \in \mathcal{F}_n$ we have

$$\begin{aligned} E((Z - Y)\,;\, D) &= E\Big(E((Z - Y)\,;\, D \mid \mathcal{F}_n)\Big) \\ &= E\Big(I_D E((Z - Y) \mid \mathcal{F}_n)\Big) = E(\big(I_D(X_n - X_n)\big) = 0\,, \end{aligned}$$

where I_D denotes the indicator function of D. Thus the desired equality is true for all $D \in \cup_{n=0}^{\infty} \mathcal{F}_n$, a collection that is closed under finite intersections, contains the entire probability space Ω, and generates $\mathcal{F}_\infty$. By linearity of expectation the set of D for which $E((Y - Z)\,;\, D) = 0$ is closed under proper differences, and, since Y and Z both have means, dominated convergence shows that it is closed under monotone limits. An appeal to the Sierpiński Class Theorem completes the proof.

24-42. The martingale $(V_n : n \in \mathbb{Z}^+)$, being bounded, is obviously uniformly integrable. Hence, $\lim V_n$ exists; call this limiting proportion of blue balls V_∞. From the fact that the martingale property is preserved when V_∞ is adjoined to the sequence $(V_n : n \in \mathbb{Z}^+)$, we conclude that the expected limiting proportion of blue balls conditioned on the contents of the urn at any particular time is the proportion of blue balls in the urn at that time.

24-45. Let Y be a $(-\infty, 0]$-valued random variable for which $E(Y) = -\infty$. Let $X_n = Y \vee (-n)$. Then $X_n(\omega) \to Y(\omega)$ for every ω. For $n = 0, 1, 2, \dots$, let $\mathcal{G}_n = \sigma(Y)$. Then $(\mathcal{G}_n : n = 0, 1, 2, \dots)$ is a reverse filtration to which $(X_n : n = 0, 1, 2, \dots)$ is adapted. Clearly $E(X_n) > -\infty$ for every n. The inequality

$$E(X_n \mid \mathcal{G}_{n+1}) = X_n \geq X_{n+1}$$

shows that $(X_0, X_1, \dots)$ is a reverse submartingale.

For Chapter 25

25-1. Define a random sequence T by $T_0 = 0$ and (25.1). Fix a finite sequence $(x_1, \dots, x_{r+s})$ such that $x_r = 1$ and let p denote the number of 1's in this sequence. Define a finite sequence $(t_0, t_1, \dots, t_p)$ by $t_0 = 0$ and

$$t_k = \inf\{m > t_{k-1} : x_m = 1\}\,.$$

Then the probability on the left side of (25.2) equals

$$\begin{aligned} P[T_k - T_{k-1} = t_k - t_{k-1} \text{ for } 1 \leq k \leq p \\ \text{and } T_{p+1} - T_p > r + s - t_p]\,, \end{aligned} \tag{7.11}$$

and, since $t_k = r$ for some k, the probability on the right side of (25.2) equals

$$\begin{aligned} &P[T_k - T_{k-1} = t_k - t_{k-1} \text{ for } k \text{ for which } t_k \leq r] \\ &\cdot P[T_k - T_{k-1} = t_k - t_{k-1} \text{ for } k \leq p \text{ for which } t_k > r \\ &\qquad \text{and } T_{p+1} - T_p > r + s - t_p]\,. \end{aligned}$$

If T is a random walk, then this product equals (7.11), and so (25.2) holds.

For the converse assume that (25.2) holds. *Hint:* To prove that T is a random walk use Proposition 3 of Chapter 11.

25-5. Since the measure generating function of R^{*k} is φ^k we have

$$\begin{aligned} \sum_{n=0}^{\infty} U\{n\} s^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} R^{*k}\{n\} s^n = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} R^{*k}\{n\} s^n \\ &= \sum_{k=0}^{\infty} \varphi^k(s) = \frac{1}{1-\varphi(s)} \end{aligned}$$

for $0 \leq s < 1$.

25-8. The function $s \rightsquigarrow 1 + s^2/4(1-s)$ is the measure generating function of the given sequence. Setting this function equal to $1/(1-\varphi)$ gives the formula $\varphi(s) = s^2(2-s)^{-2}$. To show that the given sequence is a potential sequence, we only need show that φ as just calculated is the measure generating function of some probability distribution on $\overline{\mathbb{Z}}^+ \setminus \{0\}$. We will do this by expanding in a power series and checking that all the coefficients are positive, that the coefficient of s^0 is 0, and that $\varphi(1-) \leq 1$. Provided that all the checks are affirmative we will at the same time get a formula for the waiting time distribution R.

Clearly $\varphi(1-) = 1$, so if it develops that there is a corresponding waiting time distribution R, then $R\{\infty\} = 0$. By the Binomial Theorem,

$$\begin{aligned} s^2(2-s)^{-2} = \frac{s^2}{4}\bigl(1-\tfrac{s}{2}\bigr)^{-2} &= \frac{s^2}{4}\sum_{n=0}^{\infty}\binom{-2}{n}\bigl(-\tfrac{s}{2}\bigr)^n \\ &= \sum_{n=2}^{\infty}\binom{-2}{n-2}\bigl(-\tfrac{s}{2}\bigr)^n = \sum_{n=1}^{\infty}\frac{n-1}{2^n}s^n\,. \end{aligned}$$

Therefore $R\{n\} = (n-1)2^{-n}$ for $n = 1, 2, 3, \dots$.

25-14. *Hint:* Problem 13 may be useful.

25-15. (ii). yes; $U\{0\} = 1$, $U\{1\} = p$, $U\{n\} = p^2$ for $n \geq 2$; $R\{\infty\} = 0$,

$$R\{n\} = p\,\frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-} - p(1-p)\,\frac{\lambda_+^{n-1} - \lambda_-^{n-1}}{\lambda_+ - \lambda_-}\,,$$

where $\lambda_\pm = \frac{1}{2}\bigl[1 - p \pm \sqrt{(1-p)(1+3p)}\bigr]$ (It may be of some interest that each $R\{n\}$ is a polynomial function of p.)

(v) no, unless $p = \frac{1}{2}$

(vii) yes; $U\{0\}=1$, $U\{n\}=0$ for n odd, $U\{n\}=\binom{n-1}{n/2}[p(1-p)]^{n/2}$ for $n\geq 2$ and even; measure generating function of U:

$$s \rightsquigarrow 1+\sum_{k=1}^{\infty}\binom{2k-1}{k}[p(1-p)]^k s^{2k} = \frac{1}{2}+\frac{1}{2}\sum_{k=0}^{\infty}\binom{-1/2}{k}[-4p(1-p)s^2]^k$$
$$= \frac{1}{2}\Big[1+\big(1-4p(1-p)s^2\big)^{-1/2}\Big]\,;$$

measure generating function of R:

$$s \rightsquigarrow \frac{1-2p(1-p)s^2-\big(1-4p(1-p)s^2\big)^{1/2}}{2p(1-p)s^2} = 2\sum_{k=1}^{\infty}\binom{1/2}{k+1}[-4p(1-p)s^2]^k$$
$$= \sum_{k=1}^{\infty}\frac{1}{k+1}\binom{2k}{k}[p(1-p)]^k s^{2k}\,;$$

$R\{n\}=0$ for n odd, $R\{n\}=\frac{2}{n+2}\binom{n}{n/2}[p(1-p)]^{n/2}$ for n even, $R\{\infty\}=\frac{|2p-1|}{p\vee(1-p)}$ [Notice that the coefficient $\frac{2}{n+2}\binom{n}{n/2}$ in the formula for $R\{n\}$, n even, is the $(n/2)^{\text{th}}$ Catalan number.]

25-20. for B a set of consecutive integers, $P(N(B)>0)=1-p^{\#B}$, in notation of Problem 12

25-29. $\frac{\sigma^2+\mu(\mu-1)}{2\mu}$, where μ is mean and σ^2 (possibly ∞) is variance

25-36. $R\{n\}=\frac{1}{2n-1}\binom{2n}{n}4^{-n}$, $U\{n\}=\binom{2n}{n}4^{-n}$

25-39. The solution of Problem 28 of Chapter 11 gives the measure generating function of the waiting time distribution for strict ascending ladder times:

$$\varphi^{++}(s)=\frac{1-\sqrt{1-4p(1-p)s^2}}{2(1-p)s}\,.$$

The measure generating function of the waiting time distribution for weak descending ladder times can then be obtained from Theorem 22:

$$\varphi^{-}(s)=\frac{1+2(1-p)s-\sqrt{1-4p(1-p)s^2}}{2}\,.$$

It is straightforward to use the Binomial Theorem to obtain the waiting time distributions and potential measures corresponding to these two measure generating functions. The other two types of ladder times can be treated by interchanging p and $1-p$.

For Chapter 26

26-5.

$$Q_{n+1}(B)=P[X_{n+1}\in B]=E\big(P([X_{n+1}\in B]\mid\mathcal{F}_n)\big)$$
$$=E\big(\mu_{X_n}(B)\big)=\int\mu_x(B)Q_n(dx)=(Q_nT)(B)$$
$$E(f\circ X_{n+1}\mid\mathcal{F}_n)=\int f(y)\mu_{X_n}(dy)=(Tf)\circ X_n$$