Solutions, answers, and hints for selected problems

Complete solutions of some problems are given. Answers only are given for some other problems. For still others, only hints or partial solutions are given. Asterisks in "A Modern Approach to Probability Theory" by Fristedt and Gray identify the problems that are treated in this supplement.

For Chapter 28

28-4. It suffices to prove that

$$\begin{aligned} &P[(X_m, X_{m+k}, \dots, X_{m+(d-1)k}) \in A] \\ &= P[(X_{m+k}, X_{m+2k}, \dots, X_{m+dk}) \in A] \end{aligned} \tag{0.1}$$

for every positive integer d and every Borel set $A \subseteq \Psi^d$, where Ψ denotes the common target of the X_j. Set

$$B = \{(x_0, x_1, \dots, x_{m+(d-1)k}) \in \Psi^{m+(d-1)k+1} \colon (x_m, x_{m+k}, \dots, x_{m+(d-1)k}) \in A\}\,.$$

Then the left side of (7.12) equals

$$P[(X_0, X_1, \dots, X_{m+(d-1)k}) \in B]$$

and the right side equals

$$P[(X_k, X_{k+1}, \dots, X_{m+dk}) \in B]\,.$$

These are equal by Problem 3.

28-6. *Hint:* From the given sequence obtain the desired joint distributions of every finite set of random variables. Use this information to construct a sequence $(Y_0, Y_{-1}, Y_{-2}, \dots)$ using Theorem 3 of Chapter 22. Then treat $(\dots, Y_{-2}, Y_{-1}, Y_0)$ as a single random object and use it as the first member of a random sequence to be constructed using Theorem 3 of Chapter 22 again, with the next members being $Y_1, Y_2, \dots$.

28-21. Let A be an set for which $R(A) \neq S(A)$. By Problem 18,

$$(I_A\,,\, I_A \circ \tau\,,\, I_A \circ \tau^2\,,\, \dots)$$

is ergodic. By the Birkhoff Ergodic Theorem the sequence

$$\Big(\frac{1}{n}\sum_{k=0}^{n-1} I_A \circ \tau^k \colon n = 1, 2, \dots\Big)$$

converges to $R(A)$ with R-probability 1 and also to $S(A)$ with S-probability 1. Since $S(A) \neq R(A)$, these two events are disjoint, and thus the mutual singularity is established.

28-23. Suppose first that a is rational, say p/q in lowest terms with q positive. Then the following set is easily seen to be shift-invariant and have Lebesgue measure $\frac{1}{2}$:

$$\{x \in [0,1) \colon x \in [\tfrac{p}{q}, \tfrac{2p+1}{2q}) \text{ for some } p\}\,.$$

Now suppose that a is irrational. Rotation through angle $2\pi a$ generates a shift transformation on $[0,1)^\infty$. It is clear that any shift-invariant distribution is determined by the initial distribution on $[0,1)$, but it may be that some choices for that distribution do not yield a shift-invariant measure on $[0,1)^\infty$. In fact, we will prove that the only initial distribution that does yield a shift-invariant measure on $[0,1)^\infty$ is Lebesgue measure.

For every $n \in \mathrm{Z}^+$ and 'left-closed, right-open subinterval' J of $[0,1)$, possibly with 'wrap-around', any shift-invariant measure assigns the same value to J and the interval J_{na} obtained by adding na to each endpoint of $J \mod n$. For any left-closed, right-open interval K having the same length as J, a sequence ($n_k \in \mathrm{Z}^+ \colon \mathrm{k} = 1, 2, \dots$) can be chosen so that

$$K = \lim_{k\to\infty} J_{n_k\, a}\,.$$

Hence all open intervals having the same length have the same measure, and therefore the only initial distribution that yields a shift-invariant distribution is Lebesgue measure.

Since there is only one shift-invariant distribution, that distribution is extremal and by Theorem 4, therefore ergodic. The Weyl Equidistribution Theorem is then an immediate consequence of the Birkhoff Ergodic Theorem.

28-25. $Q\{i\}T(i,j)$

28-28. Suppose that X is strongly mixing and consider any $A \in \mathcal{T}$. For each n there exists B_n such that $A = \tau^{-n}(B_n)$. As $n \to \infty$,

$$\begin{aligned}\bigl|Q(A) - [Q(A)]^2\bigr| &= \bigl|Q(A \cap \tau^{-n}(B_n)) - Q(A)Q(\tau^{-n}(B_n))\bigr| \\ &= \bigl|Q(A \cap \tau^{-n}(B_n)) - Q(A)Q(B_n)\bigr| \to 0\,.\end{aligned}$$

Therefore $Q(A)$, being a solution of $\bigl|Q(A) - [Q(A)]^2\bigr| = 0$, equals 0 or 1, as desired.

For the converse we assume that $\mathcal{T}$ is 0-1 trivial and fix a member A of $\mathcal{H}$. Then for all $B \in \mathcal{H}$ and all positive integers n,

$$\begin{aligned}\bigl|Q(A \cap \tau^{-n}(B)) - Q(A)Q(B)\bigr| &= \bigl|E_Q\bigl(I_A I_{\tau^{-n}(B)} - Q(A) I_{\tau^{-n}(B)}\bigr)\bigr| \\ &= \bigl|E_Q\bigl([Q(A \mid \mathcal{H}_n) - Q(A)]\, I_{\tau^{-n}(B)}\bigr)\bigr| \\ &\leq E_Q\bigl(\bigl|Q(A \mid \mathcal{H}_n) - Q(A)\bigr|\bigr)\,, \end{aligned} \tag{0.2}$$

where E_Q denotes expectation based on the distribution Q and

$$\mathcal{H}_n = \{\tau^{-n}(C) \colon C \in \mathcal{H}\}\,.$$

To finish the proof we only need show that (7.13) approaches 0 as $n \to \infty$, the uniformity in B resulting from the fact that (7.13) does not depend on B. By the Bounded Convergence Theorem, we only need show

$$\lim_{n\to\infty} \big[Q(A \mid \mathcal{H}_n) - Q(A)\big] = 0\,.$$

By the Reverse Martingale Convergence Theorem, this limit does exist and equals $Q(A \mid \mathcal{T}) - \mathcal{Q}(\mathcal{A})$, a random variable which has mean 0 and which, since $\mathcal{T}$ is 0-1 trivial, is a.s. constant. Therefore it must equal 0 a.s. as desired.

28-30. NEEDS TO BE DONE.

28-45. Let X denote a stationary Gaussian sequence with correlation function $(m,n) \rightsquigarrow \rho^{|m-n|}$. The result is obvious if $\rho = \pm 1$, so we assume $|\rho| < 1$. Following the hint, the conditional distribution of X_n given $(X_0, X_1, \dots, X_{n-1})$ is Gaussian with a constant variance and mean

$$\begin{pmatrix}\rho & \rho^2 & \dots & \rho^n\end{pmatrix} \begin{pmatrix} 1 & \rho & \dots & \rho^{n-1} \\ \rho & 1 & \dots & \rho^{n-2} \\ \vdots & \vdots & & \vdots \\ \rho^{n-1} & \rho^{n-2} & \dots & 1 \end{pmatrix}^{-1} \begin{pmatrix} X_{n-1} \\ X_{n-2} \\ \vdots \\ X_0 \end{pmatrix}. \tag{0.3}$$

Since the first matrix is a row matrix that is a multiple of the first column of the matrix whose inverse is in the formula, the matrix product (7.14) is some multiple of X_{n-1}, and this is all that is needed to show that X is Markov.