Solutions, answers, and hints for selected problems

Complete solutions of some problems are given. Answers only are given for some other problems. For still others, only hints or partial solutions are given. Asterisks in "A Modern Approach to Probability Theory" by Fristedt and Gray identify the problems that are treated in this supplement.

For Chapter 20

20-5. $E(X)$

20-6. Proof of (iv): By the Cauchy-Schwarz Inequality

$$E(|X - X_n|) = E(|X - X_n|1) \le \sqrt{E(|X - X_n|^2)}\sqrt{E(1^2)} = \sqrt{E((X - X_n)^2)} \to 0\,.$$

Proof of (iii), using (iv):

$$\limsup E(|X_n|) \le E(|X|) + \limsup E(|X_n - X|) = E(|X|)$$

and

$$\begin{aligned} E(|X|) &\le \liminf \bigl[E(|X_n|) + E(|X - X_n|)\bigr] \\ &\le \liminf E(|X_n|) + \limsup E(|X - X_n|) = \liminf E(|X_n|)\,, \end{aligned}$$

from which the desired conclusion follows.

20-15. By the sentence preceding the problem, $E(V_i) = 0$ for each i and $E(Z) = E(X)$. Hence, $E(X-Z) = 0$. Our task has become that of showing $E((X-Z)Y_i) = 0$ for each i. In view of the fact that each Y_i is a linear combination of 1 and the various V_j and that we have already shown that $E((X - Z)1) = 0$, we can reformulate our task as that of showing that $E(XV_j) = E(ZV_j)$ for each j.

From the definition of Z we obtain

$$E(ZV_j) = \langle X, 1\rangle E(V_j) + \sum_{i=1}^{m} \langle X, V_i\rangle E(V_i V_j) = \langle X, V_j\rangle = E(XV_j)\,.$$