Solutions, answers, and hints for selected problems

Complete solutions of some problems are given. Answers only are given for some other problems. For still others, only hints or partial solutions are given. Asterisks in “A Modern Approach to Probability Theory” by Fristedt and Gray identify the problems that are treated in this supplement.

For Chapter 25

25-1. Define a random sequence T by $T_0 = 0$ and (25.1). Fix a finite sequence $(x_1, \dots, x_{r+s})$ such that $x_r = 1$ and let p denote the number of 1’s in this sequence. Define a finite sequence $(t_0, t_1, \dots, t_p)$ by $t_0 = 0$ and

$$t_k = \inf\{m > t_{k-1} \colon x_m = 1\}\,.$$

Then the probability on the left side of (25.2) equals

$$\begin{aligned} P[T_k - T_{k-1} = t_k - t_{k-1} \text{ for } 1 \le k \le p \\ \text{and } T_{p+1} - T_p > r + s - t_p]\,, \end{aligned} \tag{0.1}$$

and, since $t_k = r$ for some k, the probability on the right side of (25.2) equals

$$\begin{aligned} &P[T_k - T_{k-1} = t_k - t_{k-1} \text{ for } k \text{ for which } t_k \le r] \\ &\cdot P[T_k - T_{k-1} = t_k - t_{k-1} \text{ for } k \le p \text{ for which } t_k > r \\ &\qquad \text{and } T_{p+1} - T_p > r + s - t_p]\,. \end{aligned}$$

If T is a random walk, then this product equals (7.11), and so (25.2) holds.

For the converse assume that (25.2) holds. *Hint:* To prove that T is a random walk use Proposition 3 of Chapter 11.

25-5. Since the measure generating function of R^{*k} is φ^k we have

$$\begin{aligned} \sum_{n=0}^{\infty} U\{n\} s^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} R^{*k}\{n\} s^n = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} R^{*k}\{n\} s^n \\ &= \sum_{k=0}^{\infty} \varphi^k(s) = \frac{1}{1-\varphi(s)} \end{aligned}$$

for $0 \le s < 1$.

25-8. The function $s \rightsquigarrow 1 + s^2/4(1-s)$ is the measure generating function of the given sequence. Setting this function equal to $1/(1-\varphi)$ gives the formula $\varphi(s) = s^2(2-s)^{-2}$. To show that the given sequence is a potential sequence, we only need show that φ as just calculated is the measure generating function of some probability distribution on $\overline{\mathbb{Z}}^+ \setminus \{0\}$. We will do this by expanding in a power series and checking that all the coefficients are positive, that the coefficient of s^0 is 0, and that $\varphi(1-) \le 1$. Provided that all the checks are affirmative we will at the same time get a formula for the waiting time distribution R.

Clearly $\varphi(1-) = 1$, so if it develops that there is a corresponding waiting time distribution R, then $R\{\infty\} = 0$. By the Binomial Theorem,

$$s^2(2-s)^{-2} = \frac{s^2}{4}\big(1-\frac{s}{2}\big)^{-2} = \frac{s^2}{4}\sum_{n=0}^{\infty}\binom{-2}{n}\big(-\frac{s}{2}\big)^n$$
$$= \sum_{n=2}^{\infty}\binom{-2}{n-2}\big(-\frac{s}{2}\big)^n = \sum_{n=1}^{\infty}\frac{n-1}{2^n}s^n\,.$$

Therefore $R\{n\} = (n-1)2^{-n}$ for $n = 1, 2, 3, \dots$.

25-14. *Hint:* Problem 13 may be useful.

25-15. (ii). yes; $U\{0\} = 1$, $U\{1\} = p$, $U\{n\} = p^2$ for $n \ge 2$; $R\{\infty\} = 0$,

$$R\{n\} = p\,\frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-} - p(1-p)\,\frac{\lambda_+^{n-1} - \lambda_-^{n-1}}{\lambda_+ - \lambda_-}\,,$$

where $\lambda_\pm = \frac{1}{2}\big[1 - p \pm \sqrt{(1-p)(1+3p)}\big]$ (It may be of some interest that each $R\{n\}$ is a polynomial function of p.)

(v) no, unless $p = \frac{1}{2}$

(vii) yes; $U\{0\} = 1$, $U\{n\} = 0$ for n odd, $U\{n\} = \binom{n-1}{n/2}[p(1-p)]^{n/2}$ for $n \ge 2$ and even; measure generating function of U:

$$s \rightsquigarrow 1 + \sum_{k=1}^{\infty}\binom{2k-1}{k}[p(1-p)]^k s^{2k} = \frac{1}{2} + \frac{1}{2}\sum_{k=0}^{\infty}\binom{-1/2}{k}[-4p(1-p)s^2]^k$$
$$= \frac{1}{2}\big[1 + \big(1 - 4p(1-p)s^2\big)^{-1/2}\big]\,;$$

measure generating function of R:

$$s \rightsquigarrow \frac{1 - 2p(1-p)s^2 - \big(1 - 4p(1-p)s^2\big)^{1/2}}{2p(1-p)s^2} = 2\sum_{k=1}^{\infty}\binom{1/2}{k+1}[-4p(1-p)s^2]^k$$
$$= \sum_{k=1}^{\infty}\frac{1}{k+1}\binom{2k}{k}[p(1-p)]^k s^{2k}\,;$$

$R\{n\} = 0$ for n odd, $R\{n\} = \frac{2}{n+2}\binom{n}{n/2}[p(1-p)]^{n/2}$ for n even, $R\{\infty\} = \frac{|2p-1|}{p\vee(1-p)}$ [Notice that the coefficient $\frac{2}{n+2}\binom{n}{n/2}$ in the formula for $R\{n\}$, n even, is the $(n/2)^{\text{th}}$ Catalan number.]

25-20. for B a set of consecutive integers, $P(N(B) > 0) = 1 - p^{\sharp B}$, in notation of Problem 12

25-29. $\frac{\sigma^2 + \mu(\mu-1)}{2\mu}$, where μ is mean and σ^2 (possibly ∞) is variance

25-36. $R\{n\} = \frac{1}{2n-1}\binom{2n}{n}4^{-n}$, $U\{n\} = \binom{2n}{n}4^{-n}$

25-39. The solution of Problem 28 of Chapter 11 gives the measure generating function of the waiting time distribution for strict ascending ladder times:

$$\varphi^{++}(s) = \frac{1 - \sqrt{1 - 4p(1-p)s^2}}{2(1-p)s} \,.$$

The measure generating function of the waiting time distribution for weak descending ladder times can then be obtained from Theorem 22:

$$\varphi^{-}(s) = \frac{1 + 2(1-p)s - \sqrt{1 - 4p(1-p)s^2}}{2} \,.$$

It is straightforward to use the Binomial Theorem to obtain the waiting time distributions and potential measures corresponding to these two measure generating functions. The other two types of ladder times can be treated by interchanging p and $1-p$.