Solutions, answers, and hints for selected problems

Complete solutions of some problems are given. Answers only are given for some other problems. For still others, only hints or partial solutions are given. Asterisks in "A Modern Approach to Probability Theory" by Fristedt and Gray identify the problems that are treated in this supplement.

For Chapter 19

19-4. The function $t \rightsquigarrow t$ is monotone (and therefore of bounded variation) on $[0,1]$ and, for each ω, the function $W(\omega,\cdot)$ is continuous. Hence (see Appendix D), we may use integration by parts to rewrite the given functional as

$$x(1) - \int_0^1 t\,dx(t) = \int_0^1 (1-t)\,dx(t)\,,$$

which in turn is the limit of Riemann-Stieltjes sums:

$$\lim_{k\to\infty} \sum_{i=1}^{k} \bigl(1-\tfrac{i}{k}\bigr)\bigl(x(\tfrac{i}{k}) - x(\tfrac{i-1}{k})\bigr)\,.$$

Under Wiener measure, this sum is the sum of k independent normally distributed random variables each of which has mean 0 and the i^{th} of which has variance $(1-\frac{i}{k})^2\frac{1}{k}$. Therefore the Riemann-Stieltjes sum itself is normally distributed with mean 0 and variance

$$\sum_{i=1}^{k} (1-\tfrac{i}{k})^2\,\tfrac{1}{k}\,.$$

This variance is a Riemann sum for the Riemann integral

$$\int_0^1 (1-t)^2\,dt = \tfrac{1}{3}\,.$$

By Problem 8 of Chapter 14 we see that the answer to the problem is: Gaussian with mean 0 and variance $\frac{1}{3}$.

19-8. We treat the case $m = n$; the case $m = 0$ is similar. Following along the lines of the argument in the text, but using the fact that $K(x) = 1$ is possible if

$T(x) > 1$ and impossible if $x(\frac{1}{n}) < 0$, we obtain

$$\begin{aligned}
&Q_n(\{x\colon K(x) = 1\}) \\
&= \frac{1}{2} \sum_{\substack{j=2 \\ j\text{even}}}^{n} \frac{1}{j-1} \binom{j}{j/2} \frac{1}{2^j} \binom{(n-j)}{(n-j)/2} \frac{1}{2^{(n-j)}} + \frac{1}{2} \sum_{\substack{j=n+2 \\ j\text{even}}}^{\infty} \frac{1}{j-1} \binom{j}{j/2} \frac{1}{2^j}\,,
\end{aligned}$$

which, because of Lemma 12, equals

$$\frac{1}{2} \binom{n}{n/2} 2^{-n} + \frac{1}{2} \sum_{\substack{j=n+2 \\ j\text{even}}}^{\infty} \frac{1}{j-1} \binom{j}{j/2} \frac{1}{2^j}\,.$$

A straightforward induction proof that

$$\sum_{\substack{j=n+2 \\ j\text{even}}}^{\infty} \frac{1}{j-1} \binom{j}{j/2} \frac{1}{2^j} = \binom{n}{n/2} 2^{-n}$$

completes the proof. [For $n = 0$ (the starting value for the induction proof), the left side equals the probability—namely 1 —that the time of first return to 0 equals some finite value, and 1 is also the value of the right side when $n = 0$.]

19-11. $\frac{2+\pi}{2\pi} \approx 0.82$

19-27. We need to show that the value of the derivative of the moment generating function at 0 equals $-ab$. By definition, the derivative there equals

$$\begin{aligned}
&\lim_{u \searrow 0} \frac{\sinh(a\sqrt{2u}) + \sinh(b\sqrt{2u}) - \sinh((a+b)\sqrt{2u})}{u \sinh((a+b)\sqrt{2u})} \\
&= \lim_{w \searrow 0} \frac{2[\sinh(aw) + \sinh(bw) - \sinh((a+b)w)]}{w^2 \sinh((a+b)w)}\,.
\end{aligned}$$

Now three applications of the l'Hospital Rule yield the desired result.