Solutions, answers, and hints for selected problems

Complete solutions of some problems are given. Answers only are given for some other problems. For still others, only hints or partial solutions are given. Asterisks in "A Modern Approach to Probability Theory" by Fristedt and Gray identify the problems that are treated in this supplement.

For Chapter 23

23-11. *Hint:* Use Problem 14 of Chapter 5.

23-17. Let $\omega = (0,1]$, $\mathcal{F}$ the Borel σ-field, and P Lebesgue measure. Let $X_n = nI_{(0,1/n)}$. Then $X_n(\omega) \to 0$ for every ω and $E(X_n) = 1$, so the (unconditional) Dominated Convergence Theorem must not apply. Let

$$\mathcal{G} = \sigma\big((2^{-m}, 2^{-(m-1)}]\colon m = 1, 2, \dots\big)\,.$$

The random variable $Y(\omega) = \frac{1}{\omega}$ dominates every X_n and satisfies $E(Y \mid \mathcal{G})(\omega) = 2^m \log 2$ for $2^{-m} < \omega \le 2^{-(m-1)}$. In particular $E(Y \mid \mathcal{G})(\omega) < \infty$ for every ω. Hence the Conditional Dominated Convergence Theorem applies. We conclude that $E(X_n \mid \mathcal{G})(\omega) \to 0$ for almost every ω, a fact that we could have also obtained by directly by observing that $E(X_n \mid \mathcal{G})(\omega) = 0$ for $n > \frac{2}{\omega}$.

23-23. Problem 21 of Chapter 21

23-30. $\frac{1}{2}$ (for all b), which is larger than $\frac{1}{3}$, the (unconditional) expectation. The following paragraphs present various ways of looking at the situation.

Fix b. If, before the random experiment begins, it is understood that one will be told whether or not b is between X and Y, one will clearly want to assign a larger value to the expectation of $Y - X$ in case b is between X and Y and a smaller value otherwise. An appropriate weighted average of these two numbers equals $\frac{1}{3}$, so, as expected, the first of these two numbers is larger than $\frac{1}{3}$.

Knowing that exactly one of two order statistics from the uniform distribution on $(0,1)$ is larger than b gives no reason for biasing one's estimate for it among the various values larger than b. Thus, the conditional mean of its excess over b is half the distance from b to 1 —namely, $\frac{1-b}{2}$. Similarly the conditional mean of the difference between b and the smaller of the two order statistics is $\frac{b}{2}$. The sum of these two conditional expectations is $\frac{1}{2}$, independently of b.

Here is a second method of getting an intuitive feel for the value $\frac{1}{2}$. Fix the number b. Pick three iid points Z_1, Z_2, and Z_3 on a circle of circumference 1. Cut the circle at Z_1 in order to straighten it into a unit interval with the counterclockwise direction on the circle corresponding locally to the direction of increase on the unit interval. Then set the smaller of Z_2 and Z_3 equal to X and the larger equal to Y. The condition that b be between X and Y is the condition that as one traverses the circle counterclockwise the contacts with either Z_2 or Z_3 alternate with the contacts with either Z_1 or b. Among such possible arrangements, there is probability $\frac{1}{2}$ that b lies in the long interval and Z_1 in the short interval determined by Z_2 and Z_3 and probability $\frac{1}{2}$ that the opposite relations hold. So the average length of the interval in which b lies is $\frac{1}{2}$.

23-33. By Problem 27 and Proposition 6, there exist choices of $E(X^+I_B \mid \mathcal{H})$ and $E(X^-I_B \mid \mathcal{H})$ such that

$$E(E(X^+ \mid \mathcal{G})I_B \mid \mathcal{H})(\omega) = E(E(X^+I_B \mid \mathcal{G}) \mid \mathcal{H})(\omega) = E(X^+I_B \mid \mathcal{H})(\omega)$$

and

$$E(E(X^- \mid \mathcal{G})I_B \mid \mathcal{H})(\omega) = E(E(X^-I_B \mid \mathcal{G}) \mid \mathcal{H})(\omega) = E(X^-I_B \mid \mathcal{H})(\omega)$$

for every sample point ω. Subtraction gives

$$\begin{aligned} &E(E(X^+ \mid \mathcal{G})I_B \mid \mathcal{H})(\omega) - E(E(X^- \mid \mathcal{G})I_B \mid \mathcal{H})(\omega) \\ &= E(XI_B \mid \mathcal{H})(\omega) \end{aligned} \tag{0.1}$$

for every ω for which the right side of (7.9) [that is, the right side of (23.9)] exists. At such an ω at least one of the two terms on the left side is finite.

We will focus on

$$A \stackrel{\text{def}}{=} \{\omega\colon E(E(X^+ \mid \mathcal{G})I_B \mid \mathcal{H})(\omega) < \infty\}\,.$$

For each $\omega \in A$,

$$\int_{[0,\infty]} x\, Z(\omega, dx) < \infty\,,$$

where Z is the conditional distribution of $E(X^+ \mid \mathcal{G})I_B$. So $E(Z(\cdot, \{\infty\})I_A) = 0$. From the definition of conditional probability we then obtain

$$P\big(\{\omega\colon [E(X^+ \mid \mathcal{G})I_B](\omega) = \infty\} \cap A\big) = 0\,.$$

Therefore the left side of (7.9) can be rewritten as

$$E\big([E(X^+ \mid \mathcal{G}) - E(X^- \mid \mathcal{G})]I_B \bigm| \mathcal{H}\big)(\omega) \tag{0.2}$$

for almost every ω for which the right side of (7.9) is less than ∞. Similarly, this can be done for almost every ω for which the right side of (7.10) is greater than $-\infty$, in particular for almost every ω for which the right side of (7.9) equals ∞.

The upshot is that for almost every ω for which the right side of (7.9) exists, the left side of (7.9) can be rewritten as (7.10) in which the inside difference between two conditional expectations is not of the form $\infty - \infty$. Therefore linearity of conditional expectation may be used to complete the proof.

23-42. *Hint:* Apply the Conditional Chebyshev Inequality and then take (unconditional) expectations of both sides.