Solutions, answers, and hints for selected problems

Complete solutions of some problems are given. Answers only are given for some other problems. For still others, only hints or partial solutions are given. Asterisks in "A Modern Approach to Probability Theory" by Fristedt and Gray identify the problems that are treated in this supplement.

For Chapter 21

21-3. By Definition 1: Clearly, $P(B \mid \mathcal{G})I_A$ is a member of $\mathbf{L_2}(\Omega, \mathcal{G}, \mathbf{P})$. Let $Y \in \mathbf{L_2}(\Omega, \mathcal{G}, \mathbf{P})$. To finish the proof we must show

$$E\big([I_{A\cap B} - P(B \mid \mathcal{G})I_A]\, Y\big) = 0\,.$$

That is we must show that

$$E\big([I_B - P(B \mid \mathcal{G})]\,[I_A Y]\big) = 0\,.$$

In view of the fact that $I_A Y$ is $\mathcal{G}$-measurable, this statement follows from the definition of $P(B \mid \mathcal{G})$.

By Proposition 2: Let $X = P(B \mid \mathcal{G})I_A$. Condition (i) of Proposition 2 is clearly satisfied by X. To check condition (ii), let $C \in \mathcal{G}$. Then we must show that

$$E(XI_C) = P((A \cap B) \cap C)\,.$$

That is, we must show that

$$E(P(B \mid \mathcal{G})I_{A\cap C}) = P(B \cap (A \cap C))\,.$$

In view of the fact that $A \cap C \in \mathcal{G}$, this last statement follows from Proposition 2 applied to $P(B \mid \mathcal{G})$.

[Comment: Notice the similarity between the two proofs. Proposition 2 says that the orthogonality condition entailed in Definition 1 need only be checked for indicator functions of members of $\mathcal{G}$ rather than for every member of $\mathbf{L_2}(\Omega, \mathcal{G}, \mathbf{P})$.]

21-5. The right side X of (21.1) is obviously $\sigma(C)$-measurable. To check the second condition in Proposition 2 we only have to consider the four members of $\sigma(C)$. Obviously $E(XI_\emptyset) = 0 = P(A \cap \emptyset)$. Also,

$$E(XI_C) = \frac{P(A \cap C)}{P(C)} E(I_C I_C) = P(A \cap C)$$

and similarly,

$$E(XI_{C^c}) = \frac{P(A \cap C^c)}{P(C^c)} E(I_{C^c} I_{C^c}) = P(A \cap C^c)\,.$$

Finally,

$$E(XI_\Omega) = E(XI_C) + E(XI_{C^c}) = P(A \cap C) + P(A \cap C^c) = P(A \cap \Omega)\,.$$

21-8. (ii)

$$\omega \rightsquigarrow \begin{cases} 1 & \text{if } \omega_1 + \omega_2 + \omega_3 + \omega_4 = 4 \\ \frac{1}{2} & \text{if } \omega_1 + \omega_2 + \omega_3 + \omega_4 = 2 \\ \frac{1}{6} & \text{if } \omega_1 + \omega_2 + \omega_3 + \omega_4 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$\frac{1}{6}$ for the particular given ω

(v)

$$\omega \rightsquigarrow \begin{cases} \frac{1}{4} & \text{if } \omega_1 + \omega_2 + \omega_3 + \omega_4 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$\frac{1}{4}$ for the particular given ω

21-9. The general formula is

$$16 \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{1} \sum_{l=0}^{1} P(A \cap B_{i,j,k,l}) I_{B_{i,j,k,l}}\,,$$

where

$$B_{i,j,k,l} = \{\psi \colon \psi_1 = 2i-1, \psi_2 = 2j-1, \psi_3 = 2k-1, \psi_4 = 2l-1\}\,.$$

(ii)

$$\omega \rightsquigarrow \begin{cases} 1 & \text{if } \omega_1 + \omega_2 = 2 \\ 0 & \text{otherwise} \end{cases}$$

0 for the particular given ω

(v) same answer as problem 8

21-10. For each positive integer m and almost every ω,

$$P(\limsup A_n \mid \mathcal{G})(\omega) \le P\Big(\bigcup_{n=m}^{\infty} A_n \mid \mathcal{G}\Big)(\omega) \le \sum_{n=m}^{\infty} P(A_n \mid \mathcal{G})(\omega)\,.$$

For those ω for which the sum on the right is finite, that sum can be made arbitrarily close to 0 by choosing m sufficiently large (depending on ω). For such an ω the probability on the far left must equal 0 since it does not depend on m. This completes the proof of the first of the two assertions in the problem.

21-12. $\psi \rightsquigarrow \psi_1$

21-13. It is possible that the image of V is not a measurable subset of Ψ.

21-17. $v \rightsquigarrow v$

21-24. With Q denoting the distribution of Y and δ_x the delta distribution at x, a conditional distribution is the function

$$(\omega, B) \rightsquigarrow Q\big([X(\omega),\infty)\big)\delta_{X(\omega)}(B) + Q\big(B \cap (-\infty, X(\omega)\big)\,.$$

(Various functions are presented via this notation: one function of two variables, functions of B for various fixed values of ω, and functions of ω for various fixed values of B.)

21-25. With Q denoting any fixed distribution [for instance, the (unconditional) distribution of X and δ_c denoting the delta distribution at c, a conditional distribution is $g \circ |X|$, where

$$g(w) = \begin{cases} \frac{f(-w)}{f(-w)+f(w)}\,\delta_{-w} + \frac{f(w)}{f(-w)+f(w)}\,\delta_w & \text{if } f(-w)+f(w) \neq 0 \\ \quad Q & \text{if } f(-w)+f(w) = 0\,. \end{cases}$$

21-30.

$$(\omega, x) \rightsquigarrow \begin{cases} \frac{1}{\lambda}\,e^{-(x-t)/\lambda} & \text{if } X(\omega) \geq t\,,\ x \geq t \\ \frac{1}{\lambda(1-e^{-t/\lambda})}\,e^{-x/\lambda} & \text{if } X(\omega) < t\,,\ 0 \leq x \leq t \\ 0 & \text{otherwise} \end{cases}$$

21-34. The density is

$$(x_1, \dots, x_{d-1}, y) \rightsquigarrow \frac{(y - x_1 - \cdots - x_{d-1})^{\gamma_d - 1} e^{-y}}{\Gamma(\gamma_d)} \prod_{i=1}^{d-1} \frac{x_i^{\gamma_i - 1}}{\Gamma(\gamma_i)}$$

for $x_i \geq 0, y \geq x_1 + \cdots + x_{d-1}$.

Let $Y = X_1 + \cdots + X_d$. A conditional density of $(X_1, \dots, X_{d-1})$ given $\sigma(Y)$ is

$$\begin{aligned} &(\omega, (x_1, \dots, x_{d-1})) \\ &\rightsquigarrow \frac{\big(1 - \frac{x_1}{Y(\omega)} - \cdots - \frac{x_{d-1}}{Y(\omega)}\big)^{\gamma_d - 1}\Gamma(\gamma_1 + \cdots + \gamma_d)}{\Gamma(\gamma_d)\,[Y(\omega)]^{d-1}} \prod_{i=1}^{d-1} \frac{\big(\frac{x_i}{Y(\omega)}\big)^{\gamma_i - 1}}{\Gamma(\gamma_i)} \end{aligned}$$

for $x_j \geq 0, x_1 + \cdots + x_{d-1} \leq Y(\omega)$ if $Y(\omega) > 0$ and $\rightsquigarrow$ the unconditional density of $(X_1, \dots, X_{d-1})$ if $Y(\omega) \leq 0$. [Note the relationship to the Dirichlet distribution which is described in an optional section of Chapter 10.]

21-44. Let Ω consist of the four points corresponding to two independent fair coins. Let $\mathcal{G}$ denote the σ-field generated by the first coin and $\mathcal{H}$ the σ-field generated by the second coin. By definition, $(\mathcal{G}, \mathcal{H})$ is an independent pair and it is clear that $\sigma(\mathcal{G}, \mathcal{H})$ consists of all subsets of Ω. Thus, any σ-field consisting of subsets of Ω is a sub-σ-field of $\sigma(\mathcal{G}, \mathcal{H})$. Let $\mathcal{K}$ be the σ-field generated by the event that exactly 1 head is flipped. Given $\mathcal{K}$ the conditional probability of any member of $\mathcal{G}$ different from $\emptyset$ and Ω equals $\frac{1}{2}$ as does the conditional of any such member of $\mathcal{H}$. But, there is no event that has conditional probability given $\mathcal{K}$ equal to $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.