Solutions, answers, and hints for selected problems

Complete solutions of some problems are given. Answers only are given for some other problems. For still others, only hints or partial solutions are given. Asterisks in "A Modern Approach to Probability Theory" by Fristedt and Gray identify the problems that are treated in this supplement.

For Chapter 8

8-8. Application of the Fatou Lemma to the sequence $(g-f_n \colon n \geq 1)$ of nonnegative measurable functions gives

$$\liminf \int (g - f_n)\, d\mu \geq \int \liminf (g - f_n)\, d\mu = \int (g - \limsup f_n)\, d\mu \geq 0\,.$$

Since $\int g d\mu < \infty$, we may use linearity to obtain

$$\int g d\mu - \limsup \int f_n\, d\mu \geq \int g\, d\mu - \int \limsup f_n\, d\mu \geq 0\,.$$

Subtraction of $\int g\, d\mu$ followed by multiplication by -1 gives the last two inequalities in (8.2). The first two inequalities in (8.2) can be obtained in a similar manner using $g + f_n$, and the middle inequality in (8.2) is obvious.

Under the additional hypothesis that $\lim f_n = f$, the first and last finite quantities in (8.2) are equal, and therefore all four finite quantities are equal. Thus $\int |f|\, d\mu < \infty$ and $\int f_n\, d\mu \to \int f\, d\mu$. Applying what we have already proved to the sequence $(|f - f_n| \colon n \geq 1)$, each member of which is bounded by $2g$, we obtain

$$\lim \int |f - f_n|\, d\mu = \int \bigl(\lim |f - f_n|\bigr)\, d\mu = \int 0\, d\mu = 0\,.$$

8-12. Let $I_{t,c}$ denote the indicator function of $\{\omega \colon |X_t(\omega)| \geq c\}$.

$$E(|X_t| I_{t,c}) = E(|X_t|^{1-p} I_{t,c} |X_t|^p) \leq c^{1-p} E(|X_t|^p) \leq c^{1-p} k \to 0 \quad \text{as } c \to \infty\,.$$

8-22. By Theorem 14 the assertion to be proved can be stated as:

$$\lim_{\gamma \to \infty} \int \theta_\gamma\, d\lambda = \int \theta\, d\lambda\,,$$

where λ denotes Lebesgue measure on R and

$$\theta(v) = \begin{cases} e^{-v^2/2} & \text{if } v \geq 0 \\ 0 & \text{otherwise}\,. \end{cases}$$

The plan is to use the Dominated Convergence Theorem. Thus we may restrict our attention to $v \geq 0$ throughout.

We take logarithms of the integrands:

$$(\log \circ \theta_\gamma)(v) = (\gamma - 1)\log(1 + v\gamma^{-1/2}) - v\gamma^{1/2}\,.$$

The Taylor Formula with remainder (or an argument based on the Mean-Value Theorem) shows that $(\log \circ \theta_\gamma)(v)$ lies between

$$(\gamma - 1)(v\gamma^{-1/2} - \tfrac{1}{2}v^2\gamma^{-1}) - v\gamma^{1/2}$$

and

$$(\gamma - 1)(v\gamma^{-1/2} - \tfrac{1}{2}v^2\gamma^{-1} + \tfrac{1}{3}v^3\gamma^{-3/2}) - v\gamma^{1/2}\,,$$

both of which approach $-v^2/2$ as $\gamma \to \infty$. Thus, to complete the proof we only need find a dominating function having finite integral.

The integrands θ_γ are nonnegative. It is enough to show, for $\gamma \geq 1$, that $\theta_\gamma(x) \leq (1+v)e^{-v}$, since this last function of v has finite integral on $[0, \infty)$. Clearly, $\theta_\gamma(v) \leq (1 + v\gamma^{-1/2})\theta_\gamma(v)$, the logarithm of which equals

$$\gamma \log(1 + v\gamma^{-1/2}) - v\gamma^{1/2}\,. \tag{0.1}$$

Differentiation with respect to γ and writing x for $v\gamma^{-1/2}$ gives

$$\log(1 + x) - \frac{x(2 + x)}{2(1 + x)}\,, \tag{0.2}$$

a function which equals 0 when $x = 0$ and is, by Problem 21, a decreasing function of x. Thus, (7.2) is nonpositive when $x \geq 0$. For $\gamma \geq 1$ [which we may assume without loss of generality], (7.1) is no larger than the value $\log(1 + v) - v$ it attains when $\gamma = 1$. The exponential of this value is the desired function $(1 + v)e^{-v}$. [Comment: The introduction of the factor $(1 + v\gamma^{-1/2})$ in the sentence containing (7.1) was for the purpose of obtaining a decreasing function of γ.]

8-26. *Hint:* The absolute value of the integral is bounded by

$$2\sqrt[3]{n^2}\max\Big|\log\Big(1 + \frac{x - n}{n}\Big)\Big|\max\big(x^n e^{-x}\big)\,,$$

where each maximum is over those x for which $|x - n| \leq \sqrt[3]{n^2}$. Apply the Mean-Value Theorem to the logarithmic function, standard methods of differential calculus to the function $x \rightsquigarrow x^n e^{-x}$, and the Stirling Formula to $n!$. (Note: If one works with the product of the maximum of the function $x \rightsquigarrow x^n$ and the maximum of the function $x \rightsquigarrow e^{-x}$ one does not get an inequality that is sharp enough to give the desired conclusion.)

8-35. Define a σ-finite measure ν by

$$\nu(A) = \int_A f\, d\lambda\,,$$

where λ denotes Lebesgue measure on R, so that f is the density of ν with respect to Lebesgue measure. In particular,

$$\nu((a, b]) = \int_a^b f(x)\, dx$$

for all $a < b$. By an appropriate version of the Fundamental Theorem of Calculus,

$$\mu((a,b]) = F(b) - F(a) = \int_a^b f(x)\,dx$$

for all $a < b$. Thus, μ and ν agree on intervals of the form $(a,b]$. By the Uniqueness Theorem, they are the same measure.