Solutions, answers, and hints for selected problems

Complete solutions of some problems are given. Answers only are given for some other problems. For still others, only hints or partial solutions are given. Asterisks in "A Modern Approach to Probability Theory" by Fristedt and Gray identify the problems that are treated in this supplement.

For Chapter 27

27-2. μ denotes De Finetti measure; for $i = 1,2,3$, $\mu\{y_i\} = \frac{1}{3}$, where $y_i\{1\} = y_i\{6-i\} = \frac{1}{2}$

27-4. De Finetti measure equals delta measure at uniform distribution on $\{x \in$ Z: 1 $\leq$ x $\leq$ 12$\}$

27-6. Yes. By letting p_i equal the value assigned to the one-point set $\{i\}$ by a probability measure on $\{1,2,3,4\}$, the probability measure itself is represented by an ordered 4-tuple (p_1,p_2,p_3,p_4) The De Finetti measure assigns probability

$\frac{1}{512}$ to $(1,0,0,0)$ and to each of the other 3 permutations thereof;
$\frac{1}{128}$ to $(\frac{3}{4},\frac{1}{4},0,0)$ and to each of the other 11 permutations thereof;
$\frac{3}{128}$ to $(\frac{1}{2},\frac{1}{4},\frac{1}{4},0)$ and to each of the other 11 permutations thereof;
$\frac{3}{256}$ to $(\frac{1}{2},\frac{1}{2},0,0)$ and to each of the other 5 permutations thereof;
$\frac{35}{64}$ to $(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$.

27-10. μ denotes De Finetti measure; $\mu\{m/n\} = P[Z_1 + \cdots + Z_n = m]$.

27-15. $P[X_1 = X_2 = 1] = P[X_1 = X_2 = 0] = \frac{5n-6}{12(n-1)}$
$P[X_1 = -X_2 = 1] = P[X_1 = -X_2 = -1] = \frac{n}{12(n-1)}$

27-31. $\alpha +$ the numbers of 1's, $\beta +$ the number of 0's

27-32. conditional distribution of (Y, X_{m+1}, X_{m+2}) has density with respect to $\mu \times \gamma \times \gamma$, where γ denotes counting measure on $\{0,1\}$; density is

$$(p, z_1, z_2) \rightsquigarrow \frac{p^{(X_1+\cdots+X_m+z_1+z_2)}\,(1-p)^{(m+2)-(X_1+\cdots+X_m+z_1+z_2)}}{\int_{[0,1]} x^{(X_1+\cdots+X_m)}\,(1-x)^{m-(X_1+\cdots+X_m)}\,\mu(dx)}\,;$$

integration in p and z_2 gives conditional density with respect to γ of X_{m+1}:

$$z_1 \rightsquigarrow \frac{\int_{[0,1]} p^{(X_1+\cdots+X_m+z_1)}\,(1-p)^{(m+1)-(X_1+\cdots+X_m+z_1)}\,\mu(dp)}{\int_{[0,1]} x^{(X_1+\cdots+X_m)}\,(1-x)^{m-(X_1+\cdots+X_m)}\,\mu(dx)}\,;$$

multiplication by z_1 and integration in z_1 give the conditional expectation of X_{m+1}:

$$\frac{\int_{[0,1]} p^{(X_1+\cdots+X_m+1)}\,(1-p)^{m-(X_1+\cdots+X_m)}\,\mu(dp)}{\int_{[0,1]} x^{(X_1+\cdots+X_m)}\,(1-x)^{m-(X_1+\cdots+X_m)}\,\mu(dx)}\,,$$

which equals

$$\frac{X_1+\cdots+X_m+1}{m+2}$$

in case μ is the standard uniform distribution.

27-39. density of each of X_1 and X_2: $x \rightsquigarrow \frac{1}{2}e^{-x} + \frac{1}{4}e^{-x/2}$; density of (X_1, X_2): $(x_1, x_2) \rightsquigarrow \frac{1}{4}\big(e^{-x_1-x_2/2} + e^{-x_2-x_1/2}\big)$; De Finetti measure assigns probability 1 to the set of uniform two-point distributions, the density of the two points being $\{y_1, y_2\} \rightsquigarrow \frac{1}{2}\big(e^{-y_1-y_2/2} + e^{-y_2-y_1/2}\big)$, $0 < y_1 < y_2$.

27-47. conditional distribution of reciprocal of mean of Y given $(X_1, \dots, X_m)$ is gamma with main parameter $m+1$ and scaling parameter $1 + \sum_{j=1}^{m} X_j$

27-52. The stick-breaking random walk breaks a stick into random pieces in such a way that, say, the sizes of the first three pieces determines how much of the stick is left for pieces 4, 5, ..., to share but gives no information about the relative sizes of these pieces. Certain information about $(X_1, \dots, X_m)$ might, for example, give information about the sizes of pieces 1, 2, and 3, without giving information about the relative sizes of the remaining pieces. (Comment: The authors of this book find this explanation to be neither complete nor satisfactory, but it is the best that they could do.)

27-55. The formula is trivial when $k = 0$; it is $1 = 1/1$. Assume it is true for k and multiply both sides by

$$P[X_{k+1} = x_{k+1} \mid X_1 = x_1, \dots, X_k = x_k] = \frac{c + \gamma_{x_{k+1}}}{k + \sum_{i=1}^{d} \gamma_i}\,,$$

where c equals the number of x_j, $j \le k$, for which $x_j = x_{k+1}$. The result follows.