has a subsequence $((a_{n_k}, b_{n_k})\colon k = 1, 2, \dots)$ that converges to a member (a, b) of $A \times B$, because $A \times B$ is compact. Since summation of coordinates is a continuous function on $A \times B$, the sequence $(a_{n_k} + b_{n_k})$ converges to the member $a + b$ of $A + B$. Hence, $A + B$ is compact. (By bringing the product space $A \times B$ into the argument we have avoided a proof involving a subsequence of a subsequence.)

10-52. For each φ: mean equals $\frac{4\sqrt{2}}{\pi}$ and variance equals $1 + \frac{2}{\pi} - \frac{16}{\pi^2}$

For Chapter 11

11-12. The one-point sets $\{0\}$ and $\{\pi\}$ each have probability $2^{n-1}3^{-n}$. The probability of any measurable B disjoint from each of these one-point sets is the product of $\frac{1}{2\pi}(1 - 2^n 3^{-n})$ and the Lebesgue measure of B.

11-13.

$$P\Big(\big\{\omega\colon \big(N(\omega) - 1, S_{N(\omega)-1}(\omega)\big) = (m, k)\big\}\Big) = r\binom{m}{k-m} q^{k-m} p^{2m-k}$$

for $m \leq k \leq 2m$ and 0 otherwise. $E(S_{N-1}) = \frac{p+2q}{r}$

11-14. for B a Borel subset of $\mathbb{R}^+$,

$$P\big(\{\omega\colon N(\omega) - 1 = m, S_{N(\omega)-1}(\omega) \in B\}\big) = Q(\{\infty\})Q^{*m}(B)\,;$$

$$E(S_{N-1}) = \frac{1}{Q(\{\infty\})} E(S_1\,;\,\{\omega\colon S_1(\omega) < \infty\})$$

11-17. Suppose that N is a stopping time. Then, for all $n \in \overline{\mathbb{Z}}^+$,

$$\{\omega\colon N(\omega) \leq n\} \in \mathcal{F}_n\,,$$

which for $n = 0$ is the desired conclusion $\{\omega\colon N(\omega) = 0\} \in \mathcal{F}_0$. Suppose $0 < n < \infty$. Then

$$\{\omega\colon N(\omega) < n\} \in \mathcal{F}_{n-1} \subseteq \mathcal{F}_n\,.$$

Therefore,

$$\{\omega\colon N(\omega) = n\} = \{\omega\colon N(\omega) \leq n\} \setminus \{\omega\colon N(\omega) < n\} \in \mathcal{F}_n\,.$$

We complete the proof in this direction by noting that

$$\{\omega\colon N(\omega) = \infty\} = \{\omega\colon N(\omega) \leq \infty\} \setminus \bigcup_{m=0}^{\infty} \{\omega\colon N(\omega) \leq m\}$$

and that all the events on the right side are members of $\mathcal{F}_\infty$.

For the converse we assume that $\{\omega\colon N(\omega) = n\} \in \mathcal{F}_n$ for all $n \in \overline{\mathbb{Z}}^+$. Then, whether $n < \infty$ or $n = \infty$,

$$\{\omega\colon N(\omega) \leq n\} = \bigcup_{m \leq n} \{\omega\colon N(\omega) = m\}\,.$$

All events on the right are members of $\mathcal{F}_n$ because filtrations are increasing. Therefore, the event on the left is a member of $\mathcal{F}_n$, as desired.

11-24. Let $A \in \mathcal{F}_M$. Then

$$\begin{aligned} A \cap \{\omega \colon N(\omega) \leq n\} &= A \cap \Big[\{\omega \colon M(\omega) \leq n\} \cap \{\omega \colon N(\omega) \leq n\}\Big] \\ &= \Big[A \cap \{\omega \colon M(\omega) \leq n\}\Big] \cap \{\omega \colon N(\omega) \leq n\}, \end{aligned}$$

which, being the intersection of two members of $\mathcal{F}_n$, is a member of $\mathcal{F}_n$. Hence $A \in \mathcal{F}_N$. Therefore $\mathcal{F}_M \subseteq \mathcal{F}_N$.

11-28.

$$\sigma_1(s) = \frac{1 - \sqrt{1 - 4p(1-p)s^2}}{2(1-p)s}, \quad 0 \leq s < 1$$

For n finite and even, the probability is 0 that n equals the hitting time of $\{1\}$. For $n = 2m - 1$, the hitting time of $\{1\}$ equals n with probability

$$\frac{1}{2m-1} \binom{2m-1}{m} p^m (1-p)^{m-1}.$$

The hitting time of $\{1\}$ equals ∞ with probability 0 or $(1-2p)/(1-p)$ according as $p \geq \frac{1}{2}$ or not.

If $p \geq \frac{1}{2}$, the global supremum equals ∞ with probability 1. If $p < \frac{1}{2}$, the global maximum exists a.s. and is geometrically distributed; the global maximum equals x with probability $\frac{1-2p}{1-p}(\frac{p}{1-p})^x$.

11-30. *Hint:* Use the Stirling Formula.

11-32. Let $(Z_j \colon j \geq 1)$ be a sequence of independent random variables with common distribution R (as used in the theorem). From the theorem we see that $(0, T_1, T_2, \dots)$ is distributed like a random walk with steps Z_j. Thus,

$$\begin{aligned} P(\{\omega \colon V(\omega) = k\}) &= P(\{\omega \colon Z_k(\omega) = \infty,\ Z_j(\omega) < \infty \text{ for } j < k\}) \\ &= P(\{\omega \colon T_1(\omega) = \infty\}) \Big[P(\{\omega \colon T_1(\omega) < \infty\})\Big]^{k-1}. \end{aligned}$$

Set $k = 1$ to obtain the first equality in (11.6). The above calculation also shows that V is geometrically distributed unless $P(\{\omega \colon V(\omega) = \infty\}) = 1$. Thus, it only remains to prove the second equality in (11.6).

Notice that

$$V = \sum_{n=0}^{\infty} I_{\{\omega \colon S_n(\omega) = 0\}}.$$

Take expected values of both sides to obtain

$$E(V) = \sum_{n=0}^{\infty} Q^{*n}(\{0\}).$$

If the right side equals ∞, then $V = \infty$ a.s., for otherwise it would be geometrically distributed and have finite mean. If the right side is finite, then $E(V) < \infty$, and, so, V is geometrically distributed and, as for all geometrically distributed random variables with smallest value 1, $\frac{1}{E(V)} = P(\{\omega \colon V(\omega) = 1\})$.

11-40. $m!/m^m$

11-41. For $m = 3$, let Q_n denote the distribution of S_n.

$$Q_n(\{\emptyset\}) = \begin{cases} \frac{1}{4}(1 + 3^{-(n-1)}) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

$$Q_n(\{\{1\}\}) = Q_n(\{\{2\}\}) = Q_n(\{\{3\}\}) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{1}{4}(1 + 3^{-n}) & \text{if } n \text{ is odd} \end{cases}$$

$$Q_n(\{\{1,2\}\}) = Q_n(\{\{1,3\}\}) = Q_n(\{\{2,3\}\}) = \begin{cases} \frac{1}{4}(1 - 3^{-n}) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

$$Q_n(\{\{1,2,3\}\}) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{1}{4}(1 - 3^{-(n-1)}) & \text{if } n \text{ is odd} \end{cases}$$

11-42. probability that $\{\emptyset\}$ is hit at time n or sooner: $[1 - (\frac{1}{2})^n]^m$; probability that $\{\{1, 2, \dots, k\}\}$ is hit at the positive time $n < \infty$:

$$(\tfrac{1}{2})^{nk}\Big[\big(1 - (\tfrac{1}{2})^n\big)^{m-k} - \big(1 - (\tfrac{1}{2})^{n-1}\big)^{m-k}\Big]\,;$$

probability that hitting time of $\{1, \dots, m-1\}$ equals ∞: $(2^m - 2)/(2^m - 1)$

11-45. For $n \geq 1$ the distribution of S_n assigns equal probability to each one-point event. The sequence S is an independent sequence of random variables. For $n \geq 1$, the probability that the first return time to 0 equals n is $(\frac{1}{m})(1 - \frac{1}{m})^{n-1}$, where m is the number of members of the group.

For Chapter 12

12-10. (ii) Let $Z_n = X_1 I_{\{\omega\colon |X_1(\omega)| \leq n\}}$. Then $|Z_n(\omega)| \leq |X_1(\omega)|$ for each n and ω. Since $E(|X_1|) < \infty$ and $Z_n(\omega) \to X_1(\omega)$ for every ω for which $X_1(\omega)$ is finite, the Dominated Convergence Theorem applies to give $E(Z_n) \to E(X_1)$. Since X_1 and X_n have identical distributions, Z_n and Y_n also have identical distributions and hence the same expected value. Therefore $E(Y_n) \to E(X_1)$.

12-16. Let G denote the distribution function of $|X_1|$. Then

$$\begin{aligned} \sum_{m=1}^{\infty} P(\{\omega\colon |X_{2m}(\omega)| > 2cm\}) &= \sum_{m=1}^{\infty} [1 - G(2cm)] \\ &\geq \frac{1}{2c} \sum_{m=1}^{\infty} \int_{2cm}^{2c(m+1)} [1 - G(2cx)]\, dx \\ &= \frac{1}{2c} \int_{2c}^{\infty} [1 - G(2cx)]\, dx \\ &= \frac{1}{4c^2} \int_{4c^2}^{\infty} [1 - G(y)]\, dy\,, \end{aligned}$$

which, by Corollary 20 of Chapter 4, equals ∞, since $E(|X_1|) = \infty$. By the Borel-Cantelli Lemma, (12.1) is true.

To prove (12.2) we note that if $|X_{2m}(\omega)| > 2cm$, then $|S_{2m}(\omega)| > cm$ or $|S_{2m-1}(\omega)| > cm$ from which it follows that

$$\left|\frac{S_{2m}(\omega)}{2m}\right| \vee \left|\frac{S_{2m-1}(\omega)}{2m-1}\right| > \frac{c}{2}\,.$$

From (12.1) we see that, for almost every ω, this inequality happens for infinitely many m. Hence, 0 is the probability of the event consisting of those ω for which $S_n(\omega)/n$ converges to a number having absolute value less than $\frac{c}{2}$. Now let $c \to \infty$ through a countable sequence to conclude that (12.2) is true.

12-17. $E(S_n) = \prod_{k=1}^n E(X_k) = 2^{-n}$. An application of the Strong Law of Large Numbers to the sequence defined by $\log S_n = \sum_{k=1}^n \log X_k$ gives

$$\lim_{n\to\infty} \frac{\log S_n}{n} = E(\log X_1) = \int_0^1 \log x\,dx = -1 \text{ a.s.}$$

Since almost sure convergence implies convergence in probability, we conclude that, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} P(\{\omega\colon e^{-(1+\varepsilon)n} < S_n < e^{-(1-\varepsilon)n}\}) = 1\,.$$

Thus, with high probability $E(S_n)/S_n$ is very large for large n. There is some small probability that S_n is not only much larger than e^{-n}, but even much larger than 2^{-n}, and it is the contribution of this small probability to the expected value that makes $E(S_n)$ much larger (in the sense of quotients, not differences) than the typical values of S_n. The random variable S_n represents the length of the stick that has been obtained by starting with a stick of length 1 and breaking off n pieces from the stick, the length of the piece kept (or the piece broken off) at the n^{th} stage being uniformly distributed on $(0, S_{n-1})$.

12-19. $(1+p)(1-p)$, $(1+p)(1-p)^2$,

$$\frac{(1-p)^2}{1-p+p^2}\,, \quad \frac{(1+p-p^2+p^3-p^4)(1-p)}{1-p^2+2p^3-p^4}$$

12-27. Let $A \in \bigotimes_{n=1}^\infty \mathcal{G}$ and $\varepsilon > 0$. (We are only interested in exchangeable A but the first part of the argument does not use exchangeability.) By Lemma 18 of Chapter 9, there exists an integer p and a measurable subset D of $\prod_{n=1}^p \Psi$ such that $P(A \triangle B) < \varepsilon$, where

$$B = D \times \left(\bigotimes_{n=p+1}^\infty \Psi\right).$$

Define a permutation π of $\mathbb{Z}^+ \setminus \{0\}$ by

$$\pi(n) = \begin{cases} n+p & \text{if } n \le p \\ n-p & \text{if } p < n \le 2p \\ n & \text{if } 2p < n\,. \end{cases}$$

Let $\hat\pi$ denote the corresponding permutation of Ω.

It is easy to check the following set-theoretic relation:

$$A \cap \hat\pi(A) \subseteq \Big[A \triangle B\Big] \cup \Big[B \cap \hat\pi(B)\Big] \cup \Big[\hat\pi(B) \triangle \hat\pi(A)\Big]\,.$$

Hence

$$P\big(A \cap \hat\pi(A)\big) \le P\big(A \triangle B\big) + P\big(B \cap \hat\pi(B)\big) + P\big(\hat\pi(B) \triangle \hat\pi(A)\big)\,. \tag{7.3}$$

The first term on the right side of (7.3) is less than ε. Since $P(C) = P(\hat\pi(C))$ for any $C \in \bigotimes_{n=1}^{\infty} \mathcal{G}$,

$$P\big(\hat\pi(B) \triangle \hat\pi(A)\big) = P\big(\hat\pi(B \triangle A)\big) = P\big(B \triangle A\big) < \varepsilon\,.$$

Thus the third term on the right side of (7.3) is also less than ε. Therefore

$$P\big(A \cap \hat\pi(A)\big) < P\big(B \cap \hat\pi(B)\big) + 2\varepsilon \tag{7.4}$$

Now assume that A is exchangeable. Then $A \cap \hat\pi(A) = A$. Also, it is clear that B and $\hat\pi(B)$ are independent, and so

$$P(B \cap \hat\pi(B)) = P(B)P(\hat\pi(B)) = [P(B)]^2\,.$$

Another easily obtained fact is that $P(B) < P(A) + \varepsilon$. From (7.4), we therefore obtain

$$P(A) < \big(P(A) + \varepsilon\big)^2 + 2\varepsilon \le [P(A)]^2 + 4\varepsilon + \varepsilon^2\,.$$

Algebraic manipulations give

$$P(A)[1 - P(A)] < 4\varepsilon + \varepsilon^2\,.$$

Let $\varepsilon \searrow 0$ to obtain $P(A)[1 - P(A)] = 0$, as desired.

12-30. (i) exchangeable but not tail, (ii) exchangeable and tail, (iii) neither exchangeable nor tail (but the Hewitt-Savage 0-1 Law can still be used to prove that the given event has probability 0 or 1) [Comment: the tail σ-field is a sub-σ-field of the exchangeable σ-field, so there is no random-walk example of an event that is tail but not exchangeable. This observation does not mean that the Kolmogorov 0-1 Law is a corollary of the Hewitt-Savage 0-1 Law, because there are settings where the Kolmogorov 0-1 Law applies and it is not even meaningful to speak of the exchangeable σ-field.]

12-35. $\sum P(\{\omega\colon |X_n(\omega)| > 1/n^2\}) \le \sum(1/n^2) < \infty$. By the Borel Lemma, for almost every ω, $|X_n(\omega)| \le (1/n^2)$ for all but finitely many n. By the comparison test for numerical series, $\sum X_n(\omega)$ converges (in fact, absolutely) for such ω.

12-40. by the Three-Series Theorem: Let b be any positive number, and define Y_n as in the theorem. By the Markov Inequality,

$$P(\{\omega\colon X_n(\omega) > b\}) \le \frac{E(X_n)}{b} = \frac{1}{bn^2}\,.$$

Thus the series (12.8) converges. Since $0 \le Y_n \le X_n$, $0 \le E(Y_n) \le \frac{1}{n^2}$. Hence, the series (12.9) converges. Also,

$$\mathrm{Var}(Y_n) \le E(Y_n^2) \le bE(Y_n) \le bE(X_n) = \frac{b}{n^2}\,.$$

Thus the series (12.10) converges. Therefore, $\sum X_n$ converges a.s. (Notice that this proof did not use the fact that the random variables are geometrically distributed.)

by Corollary 26: The distribution of X_n is geometric with parameter $(n^2+1)^{-1}$. Thus the variance is $(n^2+1)/n^4 < 2/n^2$. The series of these terms converges, as does the series of expectations. An appeal to Corollary 26 finishes the proof.

by Monotone Convergence Theorem: $E(\sum X_n) = \sum E(X_n) < \infty$. A random variable with finite expectation is finite a.s. Therefore, $\sum X_n$ is finite a.s. (Notice that for this proof, as for the proof by the Three-Series Theorem, the geometric nature of the distributions was not used.)

12-41. $\sum c_n^2 < \infty$

12-45. One place it breaks down is very early in the proof where the statement $\sum_{k=1}^n X_k(\omega) \neq \sum_{k=1}^m X_k(\omega)$ is replaced by the statement $\sum_{k=m+1}^n X_k(\omega) \neq 0$. These two statements are equivalent if the state space is $\mathbb{R}^d$, but if the state space is $\overline{\mathbb{R}}^+$ it is possible for the first of these two statements to be false, with both sums equal to ∞, and the second to be true.

For Chapter 13

13-15. if and only if the supports of the two uniform distributions have the same length

13-19. $k \to \frac{1-p}{1+p} p^{|k|}$; $v \rightsquigarrow \frac{(1-p)^2}{1+p^2-2p\cos v}$. [$p$ is the parameter of the (unsymmetrized) geometric distribution.]

13-30. mean equals $\sum_{k=1}^m k^{-1}$ and variance equals $\sum_{k=1}^m k^{-2}$

13-34. *Hint:* Let

$$f(\alpha) = \int_0^\infty \frac{1}{u^2+y^2}\, e^{-\alpha y}\, dy$$

and find a simple formula for $f'' + u^2 f$.

13-48.

$$\frac{2\pi}{\sqrt{a^2-b^2}} \left(\frac{a - \sqrt{a^2-b^2}}{b} \right)^{|n|}$$

13-72. yes

For Chapter 14

14-2. At any x where both F and G are continuous, $F(x) = G(x)$. The set of points where F is discontinuous is countable because F is monotone. The same is true for G. The set $\mathcal{D}$ of points where both F and G are continuous, and thus equal, is dense, because it has a countable complement. For any $y \in \mathbb{R}$, there exists a decreasing sequence $(x_1, x_2, \dots)$ in $\mathcal{D}$ such that $x_k \searrow y$ as $k \nearrow \infty$. The right continuity of F and G and the equality $F(x_k) = G(x_k)$ for each k then yield $F(y) = G(y)$.

14-4. We will first show that $Q_n\{x\} \to \frac{\lambda^x}{x!} e^{-\lambda}$ for each $x \in \mathbb{Z}^+$. The factor $e^{-\lambda}$ arises as the limit of $(1 - \frac{\lambda}{n})^n$. The factor λ^x already appears in the formula for $Q_n\{x\}$, and

$x!$ appears there implicitly as part of the binomial coefficient. To finish this part of the proof we need to show

$$\lim_{n\to\infty} \frac{n!}{(n-x)!n^x} \big(1-\frac{\lambda}{n}\big)^{-x} = 1\,.$$

The second factor obviously has the limit 1 and the first factor can be written as

$$\prod_{k=0}^{x-1} \big(1-\frac{k}{n}\big)$$

which also has limit 1.

We will finish the proof by showing that

$$\lim_{n\to\infty} \sum_{x\le y} Q_n(x) = \sum_{x\le y} \frac{\lambda^x}{x!}\, e^{-\lambda}$$

for every $y \in \mathbb{R}$. On the left side the limit and summation can be interchanged because the summation has only finitely many nonzero terms. The desired equality then follows from the preceding paragraph.

This problem could also be done by using Proposition 8 which appears later in Chapter 14.

14-6. standard gamma distributions. For $x > 0$,

$$\lim_{\gamma\searrow 0} \frac{1}{\Gamma(\gamma)} \int_0^x u^{\gamma-1}e^{-u}\, du = 1 - \big[\lim_{\gamma\searrow 0} \frac{1}{\Gamma(\gamma)}\big]\big[\lim_{\gamma\searrow 0} \int_x^\infty u^{\gamma-1}e^{-u}\, du\big]\,.$$

The first limit in the product of two limits equals 0 and by the Dominated Convergence Theorem, the second limit equals $\int_x^\infty u^{-1}e^{-u}\,du < \infty$, a dominating function being $(u^{-1}\vee 1)e^{-u}$. We conclude that

$$\lim_{\gamma\searrow 0} \frac{1}{\Gamma(\gamma)} \int_0^x u^{\gamma-1}e^{-u}\, du = 1$$

for $x > 0$ from which convergence to the delta distribution at 0 follows (despite the fact that we did not obtain convergence to 1 at $x = 0$).

14-10. Fix $x \ge 0$ and $r > 0$. We want to show

$$\lim_{m\to\infty} \sum_{k=0}^{\lfloor mx\rfloor} \big(\frac{1}{m}\big)^r \frac{(r)_k^{\uparrow}}{k!} \big(1-\frac{1}{m}\big)^k = \frac{1}{\Gamma(r)} \int_0^x u^{r-1}e^{-u}\, du\,,$$

which is equivalent to

$$\lim_{m\to\infty} \sum_{k=1}^{\lfloor mx\rfloor} \big(\frac{1}{m}\big)^r \frac{(r)_k^{\uparrow}}{k!} \big(1-\frac{1}{m}\big)^k = \frac{1}{\Gamma(r)} \int_0^x u^{r-1}e^{-u}\, du\,, \tag{7.5}$$

because the term $\frac{1}{m}$, obtained by setting $k = 0$, approaches 0 as $m \to \infty$.

The sum on the left side of (7.5) can be written as

$$\int_0^x g_m(u)\, du\,,$$

where

$$g_m(u) = \begin{cases} \left(\frac{k}{m}\right)^{r-1} \frac{(r)_k^{\uparrow}}{k^{r-1}\,k!} \left(1 - \frac{1}{m}\right)^k & \text{if } k-1 < mu \le k \text{ for } k = 1, 2, \dots, \lfloor mx \rfloor \\ 0 & \text{otherwise}; \end{cases}$$

and the right side can be written as

$$\int_0^x g(u)\,du\,,$$

where

$$g(u) = \frac{1}{\Gamma(r)}\,u^{r-1}e^{-u}\,.$$

The plan is to show that $g_m(u) \to g(u)$ as $m \to \infty$ for each u in the interval $(0, x)$ and to find a function h that has finite integral and dominates each g_m, for then the desired conclusion will follow immediately from the Dominated Convergence Theorem. We will consider the three factors in g_m separately. It is important to keep in mind that k depends on u and m and that in particular, $k \to \infty$ as $m \to \infty$ for each fixed $u \in (0, x)$, as this dependence is not explicit in the notation.

It is clear that $\left(\frac{k}{m}\right)^{r-1} \to u^{r-1}$ for $u \in (0, x)$. In case $r \le 1$, $\left(\frac{k}{m}\right)^{r-1} \le u^{r-1}$. In case $r > 1$, $\left(\frac{k}{m}\right)^{r-1} \le x^{r-1}$. Thus, we have constructed one factor of what we hope will be the dominating function h: u^{r-1} in case $r \le 1$ and the constant x^{r-1} in case $r > 1$.

The second factor in $g_m(u)$ equals

$$\frac{1}{\Gamma(r)}\,\frac{\Gamma(r+k)}{k^{r-1}\,\Gamma(k-1)}\,.$$

We use the Stirling Formula to obtain the limit:

$$\begin{aligned}
&\frac{1}{\Gamma(r)}\,\lim_{k\to\infty} \frac{\Gamma(r+k)}{k^{r-1}\,\Gamma(k+1)} \\
&= \frac{1}{\Gamma(r)}\,\lim_{k\to\infty} \frac{\sqrt{2\pi}(r+k)^{r+k-\frac{1}{2}}\,e^{-(r+k)}}{k^{r-1}\,\sqrt{2\pi}\,(k+1)^{k+\frac{1}{2}}\,e^{-(k+1)}} \\
&= \frac{e^{-(r-1)}}{\Gamma(r)}\,\lim_{k\to\infty} \left(1+\frac{r}{k}\right)^{r-1}\left(1+\frac{r-1}{k+1}\right)^{-1/2}\left(1+\frac{r-1}{k+1}\right)^{k+1} \\
&= \frac{1}{\Gamma(r)}\,.
\end{aligned}$$

The second factor in $g_m(u)$ is thus bounded as a function of k, the bound possibly depending on r. Such a constant bound will be the second factor we will use in constructing the dominating function h.

For the third factor in $g_m(u)$ we observe that

$$\left(1-\frac{1}{m}\right)^{mu+1} < \left(1-\frac{1}{m}\right)^{k} \le \left(1-\frac{1}{m}\right)^{mu}, \tag{7.6}$$

from which it follows that

$$\left(1-\frac{1}{m}\right)^{k} \to e^{-u}\,.$$

Moreover, (7.6) and the inequality $(1-\frac{1}{m})^m < e^{-1}$ imply that e^{-u} is a dominating function for the third factor in $g_m(u)$.

Our candidate for a dominating function $h(u)$ having finite integral is a constant multiple of $u^{r-1}e^{-u}$ in case $r \leq 1$ and a constant multiple of e^{-u} in case $r > 1$. Both these function have finite integral on the interval $[0, x]$, as desired.

For $r = 0$, each $Q_{p,r}$ is the delta distribution at 0, and, therefore, $\lim_{m\to\infty} R_m$ equals this delta distribution.

14-14. Let G denote the standard Gumbel distribution function defined in Problem 13. For $a > 0$ and $b \in \mathbb{R}$,

$$G(ax+b) = e^{-e^{-(ax+b)}} = e^{-ce^{-ax}},$$

where $c = e^{-b} > 0$.

14-16. For any real constant x,

$$\sum_{n=1}^{\infty} P[X_n > c] = \infty\,.$$

By the Borel-Cantelli Lemma, $M_n \to \infty$ a.s. as $n \to \infty$ and, hence,

$$\{\omega\colon \lim_{n\to\infty} [M_n(\omega) - \log n] \text{ exists and } > m\}$$

is a tail event of the sequence $(X_k\colon k = 1, 2, \dots)$ for every m. By the Kolmogorov 0-1 Law, the almost sure limit of $(M_n - \log n)$ must equal a constant if it exists. On the other hand, by the preceding problem the almost sure limit, if it exists, must have a Gumbel distribution. Therefore, the almost sure limit does not exist.

The sequence does not converge in probability, for if it did, there would be a subsequence that converges almost surely and the argument of the preceding paragraph would show that the distribution of the limit would have to be a delta distribution rather than a Gumbel distribution.

The preceding problem does imply that

$$\frac{M_n - \log n}{\log n} \xrightarrow{\mathcal{D}} 0 \quad \text{as } n \to \infty$$

and, therefore, that

$$\frac{M_n}{\log n} \to 1 \quad \text{in probability as } n \to \infty\,.$$

In Example 6 of Chapter 9 the stronger conclusion of almost sure convergence was obtained using calculations not needed for either this or the preceding problem.

14-22. Weibull: mean $= -\Gamma(1+\frac{1}{\alpha})$, variance $= \Gamma(1+\frac{2}{\alpha}) - [\Gamma(1+\frac{1}{\alpha})]^2$; Fréchet: mean is finite if and only if $\alpha > 1$ in which case it equals $\Gamma(1-\frac{1}{\alpha})$, variance is finite if and only if $\alpha > 2$ in which case it equals $\Gamma(1-\frac{2}{\alpha}) - [\Gamma(1-\frac{1}{\alpha})]^2$

14-35. $Q_n\{0\} = 1 - \frac{1}{n}$, $Q_n\{n^2\} = \frac{1}{n}$

14-37. We need to show

$$\lim_{z\searrow 1} Q_z(-\infty, x] = \frac{c-1}{c}$$

for all positive finite x. That is, we must show

$$\lim_{z \searrow 1} \frac{1}{\zeta(z)} \sum_{k=1}^{\lfloor c^{1/(z-1)} x \rfloor} \frac{1}{k^z} = \frac{c-1}{c}\,.$$

We may replace $\frac{1}{\zeta(z)}$ by $z-1$ because the ratio of these two functions approaches 1 as $z \searrow 1$ (as may be checked by bounding the sum that defines the Riemann zeta function by formulas involving integrals). We can bound the above sum by using:

$$\int_1^m \frac{1}{x^z}\, dz < \sum_{k=1}^m \frac{1}{k^z} < 1 + \int_1^m \frac{1}{x^z}\, dz\,;$$

that is,

$$\frac{1}{z-1}\left(1 - \frac{1}{m^{z-1}}\right) < \sum_{k=1}^m \frac{1}{k^z} < 1 + \frac{1}{z-1}\left(1 - \frac{1}{m^{z-1}}\right);$$

Replace m by $\lfloor c^{1/(z-1)} x \rfloor$, multiply by $z-1$, and let $z \searrow 1$ to obtain the desired limit $1 - \frac{1}{c}$.

14-44. Since $|\beta_n(u)| \leq 1$ for every u and n, we only need show that $1 - \Re(\beta_n(u)) \to 0$ for each u. This will follow from the hypothesis in the lemma and the inequality

$$1 - \Re\bigl(\beta(2u)\bigr) \leq 4\bigl[1 - \Re\bigl(\beta(u)\bigr)\bigr]\,,$$

which we will now prove to be valid for all characteristic functions β.

Using the positive definiteness of β we have

$$\begin{aligned} &+ \beta(0-0) z_1 \bar{z}_1 + \beta(u-0) z_1 \bar{z}_2 + \beta(2u-0) z_1 \bar{z}_3 \\ &\quad + \beta(0-u) z_2 \bar{z}_1 + \beta(u-u) z_2 \bar{z}_2 + \beta(2u-u) z_2 \bar{z}_3 \\ &\qquad + \beta(0-2u) z_3 \bar{z}_1 + \beta(u-2u) z_3 \bar{z}_2 + \beta(2u-2u) z_3 \bar{z}_3 \geq 0\,. \end{aligned}$$

Setting $z_1 = 1$, $z_2 = -2$, $z_3 = 1$, noting that $\beta(-v) = \overline{\beta(v)}$, and using $\beta(0) = 1$, we obtain

$$6 - 8\Re\bigl(\beta(u)\bigr) + 2\Re\bigl(\beta(2u)\bigr) \geq 0\,,$$

from which follows

$$8\bigl[1 - \Re\bigl(\beta(u)\bigr)\bigr] \geq 2\bigl[1 - \Re\bigl(\beta(2u)\bigr)\bigr]\,,$$

as desired. (Notice that the characteristic function of the standard normal distribution shows that 4 is the smallest possible constant for the inequality proved above, but it does not resolve the issue of whether $\leq$ can be replaced by $<$ for $u \neq 0$.)

14-48. The probability generating function $\rho_{p,r}$ of $Q_{p,r}$ is given by

$$\rho_{p,r}(s) = \sum_{x=0}^{\infty} (1-p)^r \binom{-r}{x} p^x s^x = (1-p)^r (1-ps)^{-r}\,.$$

Clearly, $(p,r) \rightsquigarrow \rho_{p,r}(s)$ is a continuous function on

$$\{(p,r)\colon 0 \leq p < 1\,,\; r \geq 0\}$$

for each fixed s, so the same is true of the function $(p,r) \rightsquigarrow Q_{p,r}$.

14-49. Example 1. The moment generating function of Q_n is

$$u \rightsquigarrow \frac{1}{n+1}\sum_{k=0}^{\infty}\Big(1+\frac{1}{n}\Big)^{-k}e^{-uk/n} = \frac{1}{n+1}\cdot\frac{1}{1-\frac{e^{-u/n}}{1+\frac{1}{n}}} = \frac{1}{n\big(1-e^{-u/n}+\frac{1}{n}\big)}\,,$$

which, as $n \to \infty$, approaches, pointwise, the function $u \rightsquigarrow \frac{1}{u+1}$, the moment generating function of the exponential distribution. An appeal to Theorem 19 finishes the proof.

14-52. Let V be the constant random variable 3 and let V_n be normally distributed with mean 3 and variance n^{-2}. Let $b_n = 3$ and $a_n = n^{-1}$. Then $(V_n - b_n)/a_n$ is normally distributed with mean 0 and variance 1 for every n even though $a_n \to 0$ as $n \to \infty$.

For Chapter 15

15-1. $\sum_{k=1}^{\infty}|X_k| \le 5\sum_{k=1}^{\infty}6^{-k} = 1$. Hence the series converges absolutely a.s. and therefore, it converges a.s., in probability, and in distribution; this is true without the independence assumption. The remainder of this solution, which concerns the limiting distribution and its characteristic function does use the independence assumption. The characteristic function of X_k is the function

$$v \rightsquigarrow \frac{1}{3}\Big(\cos\frac{v}{6^k} + \cos\frac{3v}{6^k} + \cos\frac{5v}{6^k}\Big)\,.$$

Therefore the characteristic function of $\sum_{k=1}^{\infty}X_k$ is the function

$$v \rightsquigarrow \prod_{k=1}^{\infty}\left[\frac{1}{3}\Big(\cos\frac{v}{6^k} + \cos\frac{3v}{6^k} + \cos\frac{5v}{6^k}\Big)\right]. \tag{7.7}$$

A direct simplification of this formula is not easy, so we will obtain the distribution by a method that does not rely on characteristic functions.

Calculations for $n = 1, 2, 3$ lead to the conjecture that the distribution Q_n of $\sum_{k=1}^{n}X_k$ is given by

$$Q_n\{m6^{-n}\} = 6^{-n} \quad \text{for } m \text{ odd, } |m| < 6^n\,.$$

This is easily proved by induction once it is noted that

$$\frac{m}{6^n} + \frac{5}{6^{n+1}} < \frac{m+2}{6^n} - \frac{5}{6^{n+1}}\,.$$

Then it is easy to let $n \to \infty$ to conclude that the distribution of $\sum_{k=1}^{\infty}X_k$ is the uniform distribution on $(-1, 1)$.

A sidelight: we have proved that the infinite product (7.7) equals the characteristic function of the uniform distribution on $(-1, 1)$ —namely $\frac{\sin v}{v}$.

15-6. 0.10

15-9. are not (except for the delta distribution at 0 in case one regards it as a degenerate Poisson distribution)

15-14. strict type consisting of positive constants (note: negative constants constitute another strict type)

15-16. *Hint:* The function g given by

$$g(u) = \int_0^\infty \frac{1}{x^{3/2}} e^{-\frac{b}{x} - ux}\, dx$$

can be evaluated by relating $g'(u)$ to the integral that can be obtained for $g(u)$ by using the substitution $y = \frac{c}{x}$ with an appropriate c. $a = \frac{1}{\sqrt{2}}$

15-20. (i) $\left(\frac{1-p}{1-p-\varepsilon}\right)^{1-p-\varepsilon}\left(\frac{p}{p+\varepsilon}\right)^{p+\varepsilon}$ (ii) $\left(1 + \frac{\varepsilon}{E(X_1)}\right)e^{-\varepsilon/E(X_1)}$

15-28. $\sup_{\{z\colon\, z-n \text{ even}\}}\Big|P[S_n = z] - \frac{1}{\sqrt{2\pi np(1-p)}}\exp\Big(-\frac{[z-n(2p-1)]^2}{8np(1-p)}\Big)\Big| = o\big(n^{-1/2}\big)$ as $n \to \infty$

For Chapter 16

16-1. *Hint:* Use Example 1.

16-6. 0.309 at 0; 0.215 at ± 1; 0.093 at ± 2; 0.029 at ± 3; 0.007 at ± 4; 0.001 at ± 5; 0.000 elsewhere (Comment: Using a certain table we found values that did not come close to summing to 1, so we concluded that either that table has errors or we were reading it incorrectly. We used another table.)

16-12. Suppose that $Q_k \to Q$ as $k \to \infty$. Fix n and suppose that there exist distributions R_k such that $R_k^{*n} = Q_k$. Let β_k, and γ_k denote the characteristic functions of Q_k and R_k, respectively. Because the family $\{Q_k \colon k = 1, 2, \dots\}$ is relatively sequentially compact, the family $\{\beta_k \colon k = 1, 2, \dots\}$ is equicontinuous at 0, by Theorem 13 of Chapter 14. Thus there exists some open interval B containing 0 such that $\beta_k(u) \neq 0$ for $u \in B$ and all k. So (Problem 7 of Appendix E), $\psi_k(u) = -\log \circ \beta_k(u)$ is well-defined for $u \in B$ and all k, and the family $\{\psi_k \colon k = 1, 2, \dots\}$ is equicontinuous at 0. For $u \in B$, $\gamma_k(u) = \exp\big(-\frac{1}{n}\psi_k(u)\big)$. Hence $\{\gamma_k \colon k = 1, 2, \dots\}$ is equicontinuous at 0. By Theorem 13 of Chapter 14 the family $\{R_k \colon k = 1, 2, \dots\}$ is relatively sequentially compact, and, therefore, the sequence (R_k) contains a convergent subsequence; let R denote the limit of such a subsequence. Since the convolution of convergent sequences converges to the convolution of the limit, $R^{*n} = Q$ as desired. [Comment: For fixed n we only used $R_k^{*n} = Q_k$ for each k, rather than the full strength of infinite divisibility. If Q is infinitely divisible we can strengthen the conclusion: From the forthcoming Proposition 3 it follows that β is never 0 and therefore that R is the unique distribution whose characteristic function is $\exp \circ (\frac{1}{n} \log \circ \beta)$ and moreover, it equals the limit of the sequence (R_k).]

16-13. By Proposition 1 the product of two infinitely divisible characteristic functions is infinitely divisible. The factors we use are the characteristic function of the compound Poisson distribution corresponding to ν, as in (16.1), and the function

$$u \rightsquigarrow \exp\bigg(i\Big[\eta - \int_{\mathbb{R}\setminus\{0\}} \chi\, d\nu\Big]u - \frac{\sigma^2 u^2}{2}\bigg),$$

known by Problem 9 to be infinitely divisible. The product equals $\exp \circ (-\psi)$, which is, therefore, an infinitely divisible characteristic function. For $\sigma = 0$ and $\eta = \int \chi\, d\nu$, the second factor is the function $u \rightsquigarrow 1$ and thus we obtain the compound Poisson