Solutions, answers, and hints for selected problems

Complete solutions of some problems are given. Answers only are given for some other problems. For still others, only hints or partial solutions are given. Asterisks in "A Modern Approach to Probability Theory" by Fristedt and Gray identify the problems that are treated in this supplement.

For Chapter 9

9-1. Ω_1 and Ω_2 each have six members, Ω has 36 members. Each of $\mathcal{F}_1$, $\mathcal{G}_1$, $\mathcal{F}_2$ and $\mathcal{G}_2$ has $2^6 = 64$ members. $\mathcal{F}$ has 2^{36} members and $\mathcal{R}$ has 64^2 members.

9-6. $x \rightsquigarrow 1 - \lim_{\varepsilon \searrow 0} \prod_n [1 - F_n(x+\varepsilon)]$ and $\prod_n F_n$. The example $F_n = I_{[(1/n),\infty)}$ shows that one may not just set $\varepsilon = 0$ in the first of the two answers.

9-7. exponential with mean $\lambda_1\lambda_2/(\lambda_1+\lambda_2)$

9-10. Fix $B_k \in \sigma(\mathcal{E}_k)$ for $k \in K$. For each such k there are disjoint members $A_{k,i}$, $1 \le i \le r_k$, of $\mathcal{E}_k$ such that

$$B_k = \bigcup_{i=1}^{r_k} A_{k,i}\,.$$

Hence,

$$\begin{aligned}
P\Big(\bigcap_{k\in K} B_k\Big) &= P\Big(\bigcap_{k\in K}\bigcup_{i=1}^{r_k} A_{k,i}\Big) = P\Big(\bigcup_{(i_k \le r_k\colon k\in K)}\bigcap_{k\in K} A_{k,i_k}\Big)\\
&= \sum_{(i_k \le r_k\colon k\in K)} P\Big(\bigcap_{k\in K} A_{k,i_k}\Big) = \sum_{(i_k \le r_k\colon k\in K)}\prod_{k\in K} P(A_{k,i_k})\\
&= \prod_{k\in K}\sum_{i=1}^{r_k} P(A_{k,i_k}) = \prod_{k\in K} P(B_k)\,.
\end{aligned}$$

(Contrast this proof with the proof of Proposition 3.)

9-14. For each event B, let

$$\mathcal{D}_B = \{D\colon P(D\cap B) = P(D)\,P(B)\}\,.$$

Clearly each $\mathcal{D}_B$ is closed under proper differences. By continuity of measure it is also closed under monotone limits and, hence, it is a Sierpiński class.

Denote the two members of L by 1 and 2. By hypothesis, $\mathcal{E}_1 \subseteq \mathcal{D}_B$ for each $B \in \mathcal{E}_2$. By the Sierpiński Class Theorem, $\sigma(\mathcal{E}_1) \subseteq \mathcal{D}_B$ for each $B \in \mathcal{E}_2$. Therefore

$\mathcal{E}_2 \subseteq \mathcal{D}_A$ for each $A \in \sigma(\mathcal{E}_1)$. Another application of the Sierpiński Class Theorem gives $\sigma(\mathcal{E}_2) \subseteq \mathcal{D}_A$ for every $A \in \sigma(\mathcal{E}_1)$, which is the desired conclusion.

9-15. The criterion is that for each finite subsequence $(A_{k_1}, \dots, A_{k_n})$,

$$P(A_{k_1} \cap \dots \cap A_{k_n}) = P(A_{k_1}) \dots P(A_{k_n})\,.$$

9-23. Let us first confirm the appropriateness of the hint. Because the proposition treats x and y symmetrically, we only need prove the first of the two assertions in the proposition. To do that we need to show that $\{x\colon f(x,y) \in B\} \in \mathcal{G}$ for every measurable B in the target of f and every y. Suppose that we show that the R-valued function $x \rightsquigarrow (I_B \circ f)(x,y)$ is measurable. Then it will follow that the inverse image of $\{1\}$ of this function is measurable. Since this inverse image equals $\{x\colon f(x,y) \in B\}$, the assertion in the hint is correct.

Since f is measurable, any function of the form $I_B \circ f$, where B is a measurable subset of the target of f, is the indicator function of some measurable set $A \in \mathcal{G} \times \mathcal{H}$. Thus, our task has become that of showing that $x \rightsquigarrow I_A(x,y)$ is measurable for each such A.

Let $\mathcal{C}$ denote the collection of sets $A \subseteq \Psi \times \Theta$ such that $x \rightsquigarrow I_A(x,y)$ is measurable for each fixed y. This class $\mathcal{C}$ contains all measurable rectangles, and the class of all measurable rectangles is closed under finite intersections. Since differences and monotone limits of measurable functions are measurable, the Sierpiński Class Theorem implies that $\mathcal{C}$ contains the indicator functions of all sets in $\mathcal{G} \times \mathcal{H}$, as desired.

9-27. The independence of X and Y is equivalent to the distribution of (X,Y) being a product measure $Q_1 \times Q_2$. By the Fubini Theorem,

$$\begin{aligned} E(|XY|) &= \int \left(\int |x|\,|y|\, Q_2(dy) \right) Q_1(dx) \\ &= \int |x| E(|Y|)\, Q_2(dx) = E(|X|)\, E(|Y|) < \infty\,. \end{aligned}$$

Thus we may apply the Fubini Theorem again:

$$\begin{aligned} E(XY) &= \int \left(\int xy\, Q_2(dy) \right) Q_1(dx) \\ &= \int x E(Y)\, Q_2(dx) = E(X)\, E(Y)\,. \end{aligned}$$

9-29. *Hint:* The crux of the matter is to show that, in the presence of independence, the existence of $E(X+Y)$ implies the existence of both $E(X)$ and $E(Y)$ and, moreover, it is not the case that one of $E(X)$ and $E(Y)$ equals ∞ and the other equals $-\infty$.

9-33. $\frac{2}{3}$

9-41. Method 1: The left side divided by the right side equals

$$\frac{\int_x^\infty e^{-u^2/2\sigma^2}\,du}{\sigma^2 x^{-1} e^{-x^2/2\sigma^2}}\,.$$

Both numerator and denominator approach 0 as $x \to \infty$; so we use the l'Hospital Rule. After differentiating we multiply throughout by $e^{x^2/2\sigma^2}$. The result is that we need to calculate the limit of

$$\frac{-1}{-\sigma^2 x^{-2} - 1}\,.$$

The limit equals 1, as desired.

Method 2: Let $\delta > 0$. For $x > \sigma/\sqrt{\delta}$,

$$\begin{aligned}\frac{1}{\sqrt{2\pi\sigma^2}} \int_x^\infty & e^{-u^2/2\sigma^2}\,du \\ &< \frac{1}{\sqrt{2\pi\sigma^2}} \int_x^\infty (1 + \frac{\sigma^2}{u^2}) e^{-u^2/2\sigma^2}\,du \\ &\qquad < \frac{1+\delta}{\sqrt{2\pi\sigma^2}} \int_x^\infty e^{-u^2/2\sigma^2}\,du\,.\end{aligned}$$

The expression between the two inequality signs is equal to the right side of (9.12). (The motivation behind these calculations is to replace the integrand by a slightly different integrand that has a simple antiderivative. One way to discover such an integrand is to try integration by parts along a few different paths, and, then, if, for one of these paths, the new integral is small compared with the original integral, combine it with the original integral. Of course, Method 1 is simple and straightforward, but it depends on being given the asymptotic formula in advance.)

9-42. $a_n = \sqrt{2\sigma^2 \log n}$

9-45. 0

9-47. If $x_i \le v + \delta$ for every positive δ, then $x_i \le v$; hence, the infimum that one would naturally place in (9.13), where the minimum appears, is attained and, therefore, the minimum exists. As j in the right side of (9.13) is increased, the set described there becomes smaller or stays constant and, therefore, its minimum becomes larger or stays constant. So (9.14) is true. The function $v \rightsquigarrow \sharp\{i\colon x_i \le v\}$ has a jump of size $\sharp\{i\colon x_i = v\}$, possibly 0, at each v. But the size of this jump equals the number of different values for the integer j that yield this value of v for the minimum in the right side of (9.13). Thus, (9.15) is true. The image of $\chi^{(d)}$ consists of all $y \in \mathrm{R^d}$ for which $y_1 \le y_2 \le \cdots \le y_d$. For such a y the cardinality of its inverse image equals

$$\frac{d!}{\prod_{j=1}^d (d_j!)^{1/d_j}}\,,$$

where d_j denotes the number of coordinates of y which equal y_j, including y_j itself.

To prove $\chi^{(d)}$ continuous it suffices to prove that each of its coordinate functions is uniformly continuous. Let $\varepsilon > 0$. Suppose that x and w are members of $\mathrm{R^d}$ for which $|x - w| < \varepsilon$. Then

$$\{i\colon x_i \le v\} \subseteq \{i\colon w_i \le v + \varepsilon\}\,.$$

Hence

$$\sharp\{i\colon x_i \le v\} \ge j \Longrightarrow \sharp\{i\colon w_i \le v + \varepsilon\} \ge j\,.$$

Since $[\chi^{(d)}(x)]_j$ is the smallest v for which the left side is true, we have

$$\sharp\{i\colon w_i \le [\chi^{(d)}(x)]_j + \varepsilon\} \ge j\,.$$

Therefore, $[\chi^{(d)}(w)]_j \le [\chi^{(d)}(x)]_j + \varepsilon$. The roles of x and w may be interchanged to complete the proof.

9-49. The density is $d!$ on the set of points in $[0,1]^d$ whose coordinates are in increasing order, and 0 elsewhere.

9-51. For $n = 1, 2, \ldots,$

$$P(\{\omega\colon N(\omega) = n\}) = \frac{n}{(n+1)!}\,.$$

Also, $E(N) = e - 1$. The support of the distribution of Z is $[0,1]$ and its density there is $z \rightsquigarrow (1-z)e^{1-z}$.

9-52. 1/16

9-53. $E(X) = \infty$ if $z \le 2$; $E(X) = \frac{\zeta(z-1)}{\zeta(z)}$ if $z > 2$. $\mathrm{Var}(X) = \infty$ if $2 < z \le 3$;

$$\mathrm{Var}(X) = \frac{\zeta(z-2)\zeta(z) - [\zeta(z-1)]^2}{\zeta(z)^2} \qquad \text{if } z > 3\,.$$

The probability that X is divisible by m equals $1/m^z$ which approaches $\frac{1}{m}$ as $z \searrow 1$.

9-57. The distribution of the polar angle has density

$$\theta \rightsquigarrow \frac{\Gamma(2\gamma)}{4^\gamma[\Gamma(\gamma)]^2}|\sin 2\theta|^{2\gamma-1}\,.$$

The norm is a nonnegative random variable the square of which has a gamma distribution with parameter 2γ.