Solutions, answers, and hints for selected problems

Complete solutions of some problems are given. Answers only are given for some other problems. For still others, only hints or partial solutions are given. Asterisks in "A Modern Approach to Probability Theory" by Fristedt and Gray identify the problems that are treated in this supplement.

For Chapter 10

10-5. normal with mean $\mu_1 + \mu_2$ and standard deviation $\sqrt{\sigma_1^2 + \sigma_2^2}$

10-7. $x \rightsquigarrow (1 - |x - 1|) \vee 0$

10-11. probability $\frac{1}{12}$ at each of the points $\frac{k\pi}{6}$ for $-5 \le k \le 6$

10-17. For $0 \le k \le n$,

$$\begin{aligned} &P(\{\omega\colon X_{N(\omega)}(\omega) = k \text{ and } N(\omega) = n\}) = P(\{\omega\colon X_n(\omega) = k \text{ and } N(\omega) = n\}) \\ &= P(\{\omega\colon X_n(\omega) = k\})\, P(\{\omega\colon N(\omega) = n\}) = \left[\binom{n}{k} p^k (1-p)^{n-k}\right] \left[\frac{\lambda^n e^{-\lambda}}{n!}\right]. \end{aligned}$$

We sum on n:

$$\frac{(p\lambda)^k e^{-\lambda}}{k!} \sum_{n=k}^{\infty} \frac{(\lambda(1-p))^{n-k}}{(n-k)!} = \frac{(p\lambda)^k e^{-p\lambda}}{k!}\,,$$

as desired.

10-21. The distribution of a single fair-coin flip is the square convolution root. If there were a cube convolution root Q, it would, by Problem 19, be supported by $\overline{\mathbb{Z}}^+$. If $Q(\{m\})$ were positive for some positive $m \in \overline{\mathbb{Z}}^+$, then $P(\{3m\})$ would also be positive, a contradiction. Thus, it would necessarily be that Q is the delta distribution δ_0, which is certainly not a cube root of P. Therefore P has no cube root.

10-30.

$$\begin{aligned} E(Y) &= \frac{1}{\gamma}(\gamma_1, \dots, \gamma_d) \\ \operatorname{Var}(Y_i) &= \frac{\gamma_i(\gamma - \gamma_i)}{\gamma^2(\gamma + 1)} \\ \operatorname{Cov}(Y_i Y_j) &= -\frac{\gamma_i \gamma_j}{\gamma^2(\gamma + 1)}\,, \quad i \ne j \end{aligned}$$

For the calculations of the above formulas one must avoid the error of treating the Dirichlet density in (10.4) as a d-dimensional density on the d-dimensional hypercube.

Here are the details of the calculation of $E(Y_1Y_2)$ under the assumption that $d \geq 4$. We replace y_d by $1 - y_1 - \cdots - y_{d-1}$ and discard the denominator $\sqrt{d}$ in (10.4) in order to obtain a density on a $(d-1)$-dimensional hypercube. (In fact, this replacement is done so often that the result of this displacement is often called the Dirichlet density.) Implicitly assuming that all variables are positive, setting

$$D = \{(y_3, \dots, y_{d-1})\colon y_3 + \cdots + y_{d-1} \leq 1\}\,,$$

and using the abbreviation $w = 1 - (y_3 + \dots y_{d-1})$, we obtain

$$\begin{aligned} E(Y_1Y_2) &= \frac{\Gamma(\gamma)}{\Gamma(\gamma_1)\,\Gamma(\gamma_2)\,\Gamma(\gamma_d)} \int_D \prod_{i=3}^{d-1} \frac{y_i^{\gamma_i - 1}}{\Gamma(\gamma_i)} \\ &\quad \cdot \int_0^w y_2^{\gamma_2} \int_0^{w-y_2} y_1^{\gamma_1}(w - y_2 - y_1)^{\gamma_d - 1}\, dy_1\, dy_2\, d(y_3, \dots, y_{d-1})\,. \end{aligned}$$

We substitute $(w - y_2)z_1$ for y_1 and then use Problem 34 of Chapter 3 for the evaluation of the innermost integral to obtain

$$\begin{aligned} E(Y_1Y_2) &= \frac{\gamma_1\Gamma(\gamma)}{\Gamma(\gamma_2)\,\Gamma(\gamma_1 + \gamma_d + 1)} \\ &\quad \cdot \int_D \prod_{i=3}^{d-1} \frac{y_i^{\gamma_i - 1}}{\Gamma(\gamma_i)} \int_0^w y_2^{\gamma_2}(w - y_2)^{\gamma_1 + \gamma_d}\, dy_2\, d(y_3, \dots, y_{d-1})\,. \end{aligned}$$

For the evaluation of the inner integral we substitute wz_2 for y_2; we get

$$E(Y_1Y_2) = \frac{\gamma_1\gamma_2\Gamma(\gamma)}{\Gamma(\gamma_2 + \gamma_1 + \gamma_d + 2)} \int_D \prod_{i=3}^{d-1} \frac{y_i^{\gamma_i - 1}}{\Gamma(\gamma_i)} w^{\gamma_2 + \gamma_1 + \gamma_d + 1}\, d(y_3, \dots, y_{d-1})\,.$$

By rearranging the constants appropriately we have come to the position of needing to calculate the integral of a Dirichlet density with parameters $\gamma_3, \dots, \gamma_{d-1}$, and $\gamma_2 + \gamma_1 + \gamma_d + 2$. Since the integral of the density of any probability distribution equals 1 we obtain

$$E(Y_1Y_2) = \frac{\gamma_1\gamma_2}{\gamma(\gamma + 1)}\,.$$

Since $Y_1 + \cdots + Y_d$ is a constant its variance equals 0. On the other hand, from the formula

$$\mathrm{Var}(Y_1 + \cdots + Y_d) = \sum_{j=1}^{d} \sum_{i=1}^{d} \mathrm{Cov}(Y_iY_j)$$

we see that the variance equals the sum of the entries of the covariance matrix. So, in this case, that sum is 0. But the determinant of any square matrix whose entries sum to 0 is 0, since a zero row is obtained by subtracting all the other rows from it.

10-33. Let F denote the desired distribution function. Clearly, $F(z) = 0$ for $z \leq 0$ and $F(z) = 1$ for $z \geq \frac{1}{3}$. Let $z \in (0, \frac{1}{3})$. From (10.4), $1 - F(z)$ equals $2/\sqrt{3}$ times the area of those ordered triples (z_1, z_2, z_3) satisfying $z_i > z$ for $i = 1, 2, 3$ and

$z_1 + z_2 + z_3 = 1$. This is the same as twice the area of those ordered pairs (z_1, z_2) such that $z_1 > z$, $z_2 > z$, and $1 - z_1 - z_2 > z$. Thus

$$1 - F(z) = 2 \int_z^{1-2z} \int_z^{1-z-z_1} dz_2 \, dz_1 = 1 - 6z + 9z^2 \,.$$

Therefore $F(z) = 6z - 9z^2$ for $0 < z < \frac{1}{3}$.

10-36. beta with parameters $d - 1$ and 2

10-37. The distribution has support $[0, \frac{1}{2}]$ and there the distribution function is given by

$$w \rightsquigarrow \tfrac{1}{4} + 3w^2 + 3w \log \tfrac{1}{2w} \,.$$

10-40. *Hint:* For C_1, C_2, and C_3 convex compact sets, show that

$$\{r_1 x_1 + r_2 x_2 + r_3 x_3 \colon x_i \in C_i, r_i \geq 0, r_1 + r_2 + r_3 = 1\}$$

is convex, closed, and a subset of both $(C_1 \vee C_2) \vee C_3$ and $C_1 \vee (C_2 \vee C_3)$.

10-43. $|\sin \varphi|$, $|\cos \varphi|$, $|\sin \varphi| \vee |\cos \varphi|$

10-47. For all φ and $-1 \leq w \leq 1$, the distribution function is

$$w \rightsquigarrow \left(\frac{\pi + w\sqrt{1 - w^2} - \arccos w}{\pi} \right)^3 .$$

10-48. Let A and B be two compact convex sets. Consider two arbitrary members $a_1 + b_1$ and $a_2 + b_2$ of $A + B$, where $a_i \in A$ and $b_i \in B$. Let $\kappa \in [0, 1]$. Then

$$\kappa(a_1 + b_1) + (1 - \kappa)(a_2 + b_2) = [\kappa a_1 + (1 - \kappa)a_2] + [\kappa b_1 + (1 - \kappa)b_2] \,,$$

which, in view of the fact that A and B are convex, is the sum of a member $\kappa a_1 + (1 - \kappa)a_2$ of A and a member $\kappa b_1 + (1 - \kappa)b_2$ of B, and thus is itself a member of $A + B$. Thus, convexity is proved.

It remains to prove that $A + B$ is compact. Consider a sequence $(a_n + b_n \colon n = 1, 2, \dots)$, where each $a_n \in A$ and each $b_n \in B$. The sequence $((a_n, b_n) \colon n = 1, 2, \dots)$ has a subsequence $((a_{n_k}, b_{n_k}) \colon k = 1, 2, \dots)$ that converges to a member (a, b) of $A \times B$, because $A \times B$ is compact. Since summation of coordinates is a continuous function on $A \times B$, the sequence $(a_{n_k} + b_{n_k})$ converges to the member $a + b$ of $A + B$. Hence, $A + B$ is compact. (By bringing the product space $A \times B$ into the argument we have avoided a proof involving a subsequence of a subsequence.)

10-52. For each φ: mean equals $\frac{4\sqrt{2}}{\pi}$ and variance equals $1 + \frac{2}{\pi} - \frac{16}{\pi^2}$