Solutions, answers, and hints for selected problems

Complete solutions of some problems are given. Answers only are given for some other problems. For still others, only hints or partial solutions are given. Asterisks in "A Modern Approach to Probability Theory" by Fristedt and Gray identify the problems that are treated in this supplement.

For Chapter 12

12-10. (ii) Let $Z_n = X_1 I_{\{\omega\colon |X_1(\omega)| \leq n\}}$. Then $|Z_n(\omega)| \leq |X_1(\omega)|$ for each n and ω. Since $E(|X_1|) < \infty$ and $Z_n(\omega) \to X_1(\omega)$ for every ω for which $X_1(\omega)$ is finite, the Dominated Convergence Theorem applies to give $E(Z_n) \to E(X_1)$. Since X_1 and X_n have identical distributions, Z_n and Y_n also have identical distributions and hence the same expected value. Therefore $E(Y_n) \to E(X_1)$.

12-16. Let G denote the distribution function of $|X_1|$. Then

$$\begin{aligned}\sum_{m=1}^{\infty} P(\{\omega\colon |X_{2m}(\omega)| > 2cm\}) &= \sum_{m=1}^{\infty} [1 - G(2cm)] \\ &\geq \frac{1}{2c} \sum_{m=1}^{\infty} \int_{2cm}^{2c(m+1)} [1 - G(2cx)]\, dx \\ &= \frac{1}{2c} \int_{2c}^{\infty} [1 - G(2cx)]\, dx \\ &= \frac{1}{4c^2} \int_{4c^2}^{\infty} [1 - G(y)]\, dy\,,\end{aligned}$$

which, by Corollary 20 of Chapter 4, equals ∞, since $E(|X_1|) = \infty$. By the Borel-Cantelli Lemma, (12.1) is true.

To prove (12.2) we note that if $|X_{2m}(\omega)| > 2cm$, then $|S_{2m}(\omega)| > cm$ or $|S_{2m-1}(\omega)| > cm$ from which it follows that

$$\Big|\frac{S_{2m}(\omega)}{2m}\Big| \vee \Big|\frac{S_{2m-1}(\omega)}{2m-1}\Big| > \frac{c}{2}\,.$$

From (12.1) we see that, for almost every ω, this inequality happens for infinitely many m. Hence, 0 is the probability of the event consisting of those ω for which $S_n(\omega)/n$ converges to a number having absolute value less than $\frac{c}{2}$. Now let $c \to \infty$ through a countable sequence to conclude that (12.2) is true.

12-17. $E(S_n) = \prod_{k=1}^{n} E(X_k) = 2^{-n}$. An application of the Strong Law of Large

Numbers to the sequence defined by $\log S_n = \sum_{k=1}^{n} \log X_k$ gives

$$\lim_{n\to\infty} \frac{\log S_n}{n} = E(\log X_1) = \int_0^1 \log x\, dx = -1 \text{ a.s.}$$

Since almost sure convergence implies convergence in probability, we conclude that, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} P(\{\omega\colon e^{-(1+\varepsilon)n} < S_n < e^{-(1-\varepsilon)n}\}) = 1\,.$$

Thus, with high probability $E(S_n)/S_n$ is very large for large n. There is some small probability that S_n is not only much larger than e^{-n}, but even much larger than 2^{-n}, and it is the contribution of this small probability to the expected value that makes $E(S_n)$ much larger (in the sense of quotients, not differences) than the typical values of S_n. The random variable S_n represents the length of the stick that has been obtained by starting with a stick of length 1 and breaking off n pieces from the stick, the length of the piece kept (or the piece broken off) at the n^{th} stage being uniformly distributed on $(0, S_{n-1})$.

12-19. $(1+p)(1-p)$, $(1+p)(1-p)^2$,

$$\frac{(1-p)^2}{1-p+p^2}\,, \quad \frac{(1+p-p^2+p^3-p^4)(1-p)}{1-p^2+2p^3-p^4}$$

12-27. Let $A \in \bigotimes_{n=1}^{\infty} \mathcal{G}$ and $\varepsilon > 0$. (We are only interested in exchangeable A but the first part of the argument does not use exchangeability.) By Lemma 18 of Chapter 9, there exists an integer p and a measurable subset D of $\prod_{n=1}^{p} \Psi$ such that $P(A \triangle B) < \varepsilon$, where

$$B = D \times \left(\bigotimes_{n=p+1}^{\infty} \Psi \right).$$

Define a permutation π of $\mathrm{Z}^+ \setminus \{0\}$ by

$$\pi(n) = \begin{cases} n+p & \text{if } n \le p \\ n-p & \text{if } p < n \le 2p \\ n & \text{if } 2p < n\,. \end{cases}$$

Let $\hat{\pi}$ denote the corresponding permutation of Ω.

It is easy to check the following set-theoretic relation:

$$A \cap \hat{\pi}(A) \subseteq \big[A \triangle B\big] \cup \big[B \cap \hat{\pi}(B)\big] \cup \big[\hat{\pi}(B) \triangle \hat{\pi}(A)\big]\,.$$

Hence

$$P\big(A \cap \hat{\pi}(A)\big) \le P\big(A \triangle B\big) + P\big(B \cap \hat{\pi}(B)\big) + P\big(\hat{\pi}(B) \triangle \hat{\pi}(A)\big)\,. \tag{0.1}$$

The first term on the right side of (7.3) is less than ε. Since $P(C) = P(\hat{\pi}(C))$ for any $C \in \bigotimes_{n=1}^{\infty} \mathcal{G}$,

$$P\big(\hat{\pi}(B) \triangle \hat{\pi}(A)\big) = P\big(\hat{\pi}(B \triangle A)\big) = P\big(B \triangle A\big) < \varepsilon\,.$$

Thus the third term on the right side of (7.3) is also less than ε. Therefore

$$P\big(A \cap \hat{\pi}(A)\big) < P\big(B \cap \hat{\pi}(B)\big) + 2\varepsilon \tag{0.2}$$

Now assume that A is exchangeable. Then $A \cap \hat{\pi}(A) = A$. Also, it is clear that B and $\hat{\pi}(B)$ are independent, and so

$$P(B \cap \hat{\pi}(B)) = P(B)P(\hat{\pi}(B)) = [P(B)]^2 .$$

Another easily obtained fact is that $P(B) < P(A) + \varepsilon$. From (7.4), we therefore obtain

$$P(A) < \big(P(A) + \varepsilon\big)^2 + 2\varepsilon \le [P(A)]^2 + 4\varepsilon + \varepsilon^2 .$$

Algebraic manipulations give

$$P(A)[1 - P(A)] < 4\varepsilon + \varepsilon^2 .$$

Let $\varepsilon \searrow 0$ to obtain $P(A)[1 - P(A)] = 0$, as desired.

12-30. (i) exchangeable but not tail, (ii) exchangeable and tail, (iii) neither exchangeable nor tail (but the Hewitt-Savage 0-1 Law can still be used to prove that the given event has probability 0 or 1) [Comment: the tail σ-field is a sub-σ-field of the exchangeable σ-field, so there is no random-walk example of an event that is tail but not exchangeable. This observation does not mean that the Kolmogorov 0-1 Law is a corollary of the Hewitt-Savage 0-1 Law, because there are settings where the Kolmogorov 0-1 Law applies and it is not even meaningful to speak of the exchangeable σ-field.]

12-35. $\sum P(\{\omega\colon |X_n(\omega)| > 1/n^2\}) \le \sum(1/n^2) < \infty$. By the Borel Lemma, for almost every ω, $|X_n(\omega)| \le (1/n^2)$ for all but finitely many n. By the comparison test for numerical series, $\sum X_n(\omega)$ converges (in fact, absolutely) for such ω.

12-40. by the Three-Series Theorem: Let b be any positive number, and define Y_n as in the theorem. By the Markov Inequality,

$$P(\{\omega\colon X_n(\omega) > b\}) \le \frac{E(X_n)}{b} = \frac{1}{bn^2} .$$

Thus the series (12.8) converges. Since $0 \le Y_n \le X_n$, $0 \le E(Y_n) \le \frac{1}{n^2}$. Hence, the series (12.9) converges. Also,

$$\operatorname{Var}(Y_n) \le E(Y_n^2) \le bE(Y_n) \le bE(X_n) = \frac{b}{n^2} .$$

Thus the series (12.10) converges. Therefore, $\sum X_n$ converges a.s. (Notice that this proof did not use the fact that the random variables are geometrically distributed.)

by Corollary 26: The distribution of X_n is geometric with parameter $(n^2+1)^{-1}$. Thus the variance is $(n^2 + 1)/n^4 < 2/n^2$. The series of these terms converges, as does the series of expectations. An appeal to Corollary 26 finishes the proof.

by Monotone Convergence Theorem: $E(\sum X_n) = \sum E(X_n) < \infty$. A random variable with finite expectation is finite a.s. Therefore, $\sum X_n$ is finite a.s. (Notice that for this proof, as for the proof by the Three-Series Theorem, the geometric nature of the distributions was not used.)

12-41. $\sum c_n^2 < \infty$

12-45. One place it breaks down is very early in the proof where the statement $\sum_{k=1}^{n} X_k(\omega) \neq \sum_{k=1}^{m} X_k(\omega)$ is replaced by the statement $\sum_{k=m+1}^{n} X_k(\omega) \neq 0$.

These two statements are equivalent if the state space is R^{d}, but if the state space is $\overline{\mathrm{R}}^{+}$ it is possible for the first of these two statements to be false, with both sums equal to ∞, and the second to be true.