Solutions, answers, and hints for selected problems

Complete solutions of some problems are given. Answers only are given for some other problems. For still others, only hints or partial solutions are given. Asterisks in "A Modern Approach to Probability Theory" by Fristedt and Gray identify the problems that are treated in this supplement.

For Chapter 22

22-10. If X_3 were to exist so that (X_1, X_2, X_3) is exchangeable, then, since $X_1 + X_2 = 0$ with probability 1, it would follow that $X_1 + X_3 = 0$ and $X_2 + X_3 = 0$ with probability 1. By solving three equations in three unknowns it would then follow that $X_1 = 0$ with probability 1, a contradiction.

22-11. *Hint:* Apply $E(P(A \mid \mathcal{G})) = P(A)$ for various choices for A.

22-14. uniform on the set of those $\binom{n}{[n+S_n(\omega)]/2}$ sequences of ± 1's that contain $[n + S_n(\omega)]/2 \quad 1$'s and $[n - S_n(\omega)]/2 \quad -1$'s. [Comment: The answer does not depend on p.]

22-16. first term equals 1 with probability $\frac{\alpha}{\alpha+\beta}$. conditional distribution of second term given first term: equals 1 with probability $\frac{\alpha+1}{\alpha+\beta+1}$ if first term equals 1 and equals 1 with probability $\frac{\alpha}{\alpha+\beta+1}$ if first term equals 0. distribution of first two terms: equals $(1, 1)$ with probability $\frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$ and equals $(0, 0)$ with probability $\frac{\beta(\beta+1)}{(\alpha+\beta)(\alpha+\beta+1)}$ and equals $(1, 0)$ and $(0, 1)$ each with probability $\frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)}$

22-21. By exchangeability, the correlation of I_m and I_n is the same as that of I_1 and I_2 if $n \neq m$; of course, it equals 1 if $n = m$.

The correlation of I_1 and I_2 equals $\frac{c}{x_0+y_0+c}$, which approaches 0 as $(x_0, y_0) \to (\infty, \infty)$ and approaches 1 as $c \to \infty$.

For large (x_0, y_0) the knowledge of the color of a fixed number c of balls in the urn hardly influences the probability that a blue ball will be drawn. For large c, the second ball drawn is very likely to be of the same color as the first ball since after the first ball is drawn almost all the balls in the urn will have the same color as the first ball.

22-22. Using the fact that $\prod_n \frac{a+cn}{b+cn} = 0$ if $0 \leq a < b$ and $0 \leq c$, we have

$$\begin{aligned}
P[I_n = 0 \text{ for } n > m] &= E(P[I_n = 0 \text{ for } n > m \mid \sigma(X_m, Y_m)]) \\
&= E\biggl(\prod_{n=m+1}^{\infty} \frac{Y_m + (n-m-1)c}{X_m + Y_m + (n-m-1)c} \biggr) \\
&= E(0) = 0
\end{aligned}$$

for each fixed m. Hence

$$\begin{aligned}
P(\liminf\{\omega\colon I_n(\omega) = 0\}) &= P\biggl(\bigcup_{m=1}^{\infty} \bigcap_{n>m} \{\omega\colon I_n(\omega) = 0\} \biggr) \\
&\leq \sum_{m=1}^{\infty} P\biggl(\bigcap_{n>m} \{\omega\colon I_n(\omega) = 0\} \biggr) \\
&= \sum_{m=1}^{\infty} 0 = 0\,,
\end{aligned}$$

from which it follows that the first event in the problem has probability 1. That the second event given there also has probability 1 follows by applying the result already proved to the sequence $((1 - I_n)\colon n = 1, 2, \dots)$, an application which is seen to be valid by interchanging the colors of the balls.

22-24. $\frac{(m-1)!}{m^{(n-1)}}\, S\bigl((n-1), (m-1)\bigr)$