Chapter II
ℓ_p-spaces

6 Entropy numbers and eigenvalues

6.1 Preliminaries and notation

The main aim of Chapter II is to study entropy numbers in (weighted) ℓ_p-spaces. This will be done in the Sections 7–9. In the present section we describe briefly the necessary abstract background without proofs. We follow closely [ET96] where proofs, further details, explanations, and more references are given.

A *quasi-norm* on a complex linear space B is a map $\|\cdot|B\|$ from B to the non-negative reals $\mathbb{R}_+$ such that

$$\|x|B\| = 0 \text{ if, and only if, } x = 0, \tag{6.1}$$

$$\|\lambda x|B\| = |\lambda|\,\|x|B\| \quad \text{for all scalars } \lambda \in \mathbb{C} \text{ and all } x \in B, \tag{6.2}$$

there is a constant C such that for all $x \in B$ and $y \in B$

$$\|x+y|B\| \leq C(\|x|B\| + \|y|B\|). \tag{6.3}$$

Plainly $C \geq 1$; if $C = 1$ is allowed then $\|\cdot|B\|$ is a norm in B. As usual, B is called a *quasi-Banach space* if every Cauchy sequence with respect to $\|\cdot|B\|$ is a convergent sequence.

Given any $p \in (0,1]$, a *p-norm* on a complex linear space B is a map $\|\cdot|B\| \to \mathbb{R}_+$ which satisfies (6.1), (6.2), and instead of (6.3),

$$\|x+y|B\|^p \leq \|x|B\|^p + \|y|B\|^p \quad \text{for } x, y \in B. \tag{6.4}$$

Two quasi-norms or p-norms $\|\cdot|B\|_1$ and $\|\cdot|B\|_2$ are said to be *equivalent* if there is a constant $c \geq 1$ such that for all $x \in B$,

$$c^{-1}\|x|B\|_1 \leq \|x|B\|_2 \leq c\,\|x|B\|_1. \tag{6.5}$$

It can be shown (see [Kön86], p. 47 or [DeVL93], p. 20) that if $\|\cdot|B\|_1$ is a quasi-norm on B then there exists $p \in (0,1]$ and a p-norm $\|\cdot|B\|_2$ on B which is equivalent to $\|\cdot|B\|_1$.

Let A, B be quasi-Banach spaces and let $T : A \to B$ linear. Just as for the Banach space case, T will be called *bounded* or *continuous* if

$$\|T\| = \sup\{\|Ta|B\| : a \in A, \|a|A\| \leq 1\} < \infty. \tag{6.6}$$

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The family of all such T will be denoted by $L(A,B)$ or $L(A)$ if $A=B$. Otherwise terminology which is standard in the context of Banach spaces will be taken without further comment to quasi-Banach spaces. In particular if $T \in L(B)$ then $\sigma(T)$ stands for its spectrum.

In [ET96], pp. 3–7, we developed a Riesz theory for compact operators $T \in L(B)$ in quasi-Banach spaces B parallel to the well-known assertions in the Banach spaces case. Especially, if

$T \in L(B)$ is compact, then $\sigma(T) \setminus \{0\}$ consists of an at most countably infinite number of eigenvalues of finite algebraic multiplicity which may accumulate only at the origin.

If B is a quasi-Banach space then $U_B = \{b \in B : \|b|B\| \le 1\}$ stands for the unit ball in B.

6.2 Definition *Let A, B be quasi-Banach spaces and let $T \in L(A,B)$. Then for all $k \in \mathbb{N}$, the kth entropy number $e_k(T)$ of T is defined by*

$$e_k(T) = \inf\left\{ \varepsilon > 0 : T(U_A) \subset \bigcup_{j=1}^{2^{k-1}} (b_j + \varepsilon U_B) \text{ for some } b_1, \dots, b_{2^{k-1}} \in B \right\}. \tag{6.7}$$

6.3 Remark This formulation coincides with the definition given in [ET96], p. 7, which simply generalizes to quasi-Banach spaces what has been done before for Banach spaces. Further comments and some discussions may be found in [ET96], pp. 7–9, and, in greater detail, in [CaS90] and [EEv87].

6.4 Proposition *Let A, B, C be quasi-Banach spaces, let $S, T \in L(A,B)$ and suppose that $R \in L(B,C)$.*

(i) $\|T\| \ge e_1(T) \ge e_2(T) \ge \dots; e_1(T) = \|T\|$ *if B is a Banach space.*

(ii) *For all $k, l \in \mathbb{N}$*

$$e_{k+l-1}(R \circ S) \le e_k(R)\, e_l(S). \tag{6.8}$$

(iii) *If B is a p-Banach space, where $0 < p \le 1$, then for all $k, l \in \mathbb{N}$*

$$e^p_{k+l-1}(S+T) \le e^p_k(S) + e^p_l(T). \tag{6.9}$$

6.5 Remark This formulation coincides with Lemma 1 in [ET96], pp. 7,8, where also a simple proof may be found. In case of quasi-Banach spaces it may happen that $\|T\| > e_1(T)$, see [ET96], Remark 4 on p. 9.

6.6 Compact operators

Recall that $T \in L(B)$ is compact if, and only if, for every $\varepsilon > 0$ there is a finite ε-net covering $T(U_B)$. By (6.7) this is the same as

$$T \in L(B) \text{ is compact if, and only if, } e_k(T) \to 0 \text{ for } k \to \infty. \tag{6.10}$$

6.7 Interpolation properties

The entropy numbers behave very well with respect to real interpolation of quasi-Banach spaces. We gave in [ET96], pp. 13–15, a rather careful treatment of this subject, which in turn was based on [HaT94a]. Further properties in the context of Banach spaces and historical comments may be found in [Tri78], 1.16.2, and [Pie80], 12.1.

6.8 Eigenvalues

Again let B be a (complex) quasi-Banach space and let $T \in L(B)$ be compact. As we mentioned at the end of 6.1 the spectrum of T, apart from the point 0, consists solely of eigenvalues of finite algebraic multiplicity: let $\{\mu_k(T)\}_{k\in\mathbb{N}}$ be the sequence of all non-zero eigenvalues of T, repeated according to algebraic multiplicity and ordered so that

$$|\mu_1(T)| \geq |\mu_2(T)| \geq \ldots \to 0. \tag{6.11}$$

If T has only $m(< \infty)$ distinct eigenvalues and M is the sum of their algebraic multiplicities we put $\mu_n(T) = 0$ for all $n > M$.

6.9 Theorem *Let T and $\{\mu_k(T)\}_{k\in\mathbb{N}}$ be as in* 6.8. *Then*

$$\left(\prod_{m=1}^{k} |\mu_m(T)| \right)^{\frac{1}{k}} \leq \inf_{n\in\mathbb{N}} 2^{\frac{n}{2k}} \, e_n(T), \quad k \in \mathbb{N}. \tag{6.12}$$

6.10 Corollary *For all $k \in \mathbb{N}$*

$$|\mu_k(T)| \leq \sqrt{2}\, e_k(T). \tag{6.13}$$

6.11 Remark This is Carl's famous inequality which connects spectral properties of compact operators with the geometry of the map T described in terms of entropy numbers. (6.13) in the context of Banach spaces was proved by Carl in [Carl81]. In [ET96], pp. 18–20, we gave a proof of (6.12) which generalizes the proof given in [CaT80] from Banach spaces to quasi-Banach spaces. Plainly, (6.13) follows from (6.11) and (6.12) with $n = k$.

6.12 Remark Further results, comments, references and, in particular comparisons of entropy numbers with other geometric quantities, especially approximation numbers, may be found in [ET96], [CaS90], [EEv87], [Kön86], [Pie87], and [LGM96].

7 The spaces ℓ_p^M

7.1 Preliminaries and notation

We follow again [ET96], p. 97. Let $M \in \mathbb{N}$ and let $0 < p \leq \infty$. By ℓ_p^M we shall mean the linear space of all complex M-tuples $y = (y_j)$, endowed with the quasi-norm

$$\|y|\ell_p^M\| = \left(\sum_{j=1}^{M} |y_j|^p \right)^{\frac{1}{p}}, \quad \text{if} \quad 0 < p < \infty, \tag{7.1}$$

and

$$\|y|\ell_\infty^M\| = \sup_j |y_j|, \quad \text{if} \quad p = \infty. \tag{7.2}$$

Let

$$U_p^M = \{ y \in \ell_p^M : \ \|y|\ell_p^M\| \leq 1 \} \tag{7.3}$$

be the closed unit ball in ℓ_p^M. Since $\mathbb{C}^M$ may be identified with $\mathbb{R}^{2M}$, we shall understand by the volume of U_p^M the Lebesgue measure of

$$\left\{ (x_1, \ldots, x_{2M}) \in \mathbb{R}^{2M} : \sum_{j=1}^{M} (x_{2j-1}^2 + x_{2j}^2)^{\frac{p}{2}} \leq 1 \right\}. \tag{7.4}$$

Let $p \in (0, \infty]$ be given. There are two positive constants c_1 and c_2 (which may depend on p) such that for all $M \in \mathbb{N}$

$$c_1 M^{-\frac{1}{p}} \leq \left(vol\, U_p^M \right)^{\frac{1}{2M}} \leq c_2 M^{-\frac{1}{p}}. \tag{7.5}$$

This follows from the Proposition in [ET96], p. 97, and the end of the proof on p. 98.

Plainly the identity from $\ell_{p_1}^M$ in $\ell_{p_2}^M$ is a compact operator. Our aim is to estimate the corresponding entropy numbers according to Definition 6.2. In what follows we assume that $\log = \log_2$ is taken with respect to the base 2. First we complement the results in [ET96], p. 98.

7.2 Proposition *Let $0 < p_1 \leq \infty$, $0 < p_2 \leq \infty$ and for each $k \in \mathbb{N}$ let e_k be the entropy numbers of the embedding*

$$id : \ell_{p_1}^M \to \ell_{p_2}^M.$$

Then

$$e_k \geq c \quad if \quad 1 \leq k \leq \log(2M), \tag{7.6}$$

and

$$e_k \geq c 2^{-\frac{k}{2M}} (2M)^{\frac{1}{p_2} - \frac{1}{p_1}} \text{ if } k \in \mathbb{N}, \tag{7.7}$$

where c is a positive constant which is independent of M (and k) but may depend upon p_1 and p_2.

Proof.
Step 1. We prove (7.6). Let $y = (y_j) \in \ell_p^M$ for some p where all components y_j are zero with exception of one component which is either 1 or -1. There are 2M such elements belonging to $U_{p_1}^M$ and $U_{p_2}^M$. Let y^1 and y^2 be two such points and assume that they belong to the same $\ell_{p_2}^M$-ball of radius ε, hence

$$y^1 \in x + \varepsilon U_{p_2}^M \quad \text{and} \quad y^2 \in x + \varepsilon U_{p_2}^M \quad \text{for some} \quad x \in \ell_{p_2}^M. \tag{7.8}$$

For some c which is independent of M and $\overline{p_2} = \min(p_2, 1)$ we have

$$\begin{aligned} c \le \|y^1 - y^2|\ell_{p_2}^M\|^{\overline{p_2}} \le \\ \|y^1 - x|\ell_{p_2}^M\|^{\overline{p_2}} \ + \ \|y^2 - x|\ell_{p_2}^M\|^{\overline{p_2}} \ \le \ 2\varepsilon^{\overline{p_2}}. \end{aligned} \tag{7.9}$$

Now (7.6) follows from (7.9), $2^{k-1} \le M < 2M$ and (6.7).

Step 2. We prove (7.7). We cover $U_{p_1}^M$ with 2^{k-1} balls in $\ell_{p_2}^M$ of radius ε chosen in an appropriate way. Then we have by the interpretation (7.4)

$$\begin{aligned} vol\, U_{p_1}^M &\le 2^{k-1}\varepsilon^{2M}\, vol\, U_{p_2}^M \\ &\le 2^k e_k^{2M}\, vol\, U_{p_2}^M. \end{aligned} \tag{7.10}$$

Now (7.7) follows from (7.10) and (7.5).

7.3 Theorem *Let* $0 < p_1 \le p_2 \le \infty$ *and for each* $k \in \mathbb{N}$ *let* e_k *be the* k*th entropy number of the embedding*

$$id : \ell_{p_1}^M \to \ell_{p_2}^M.$$

Then

$$c_1 \le e_k \le c_2 \quad \textit{if} \quad 1 \le k \le \log(2M), \tag{7.11}$$

$$e_k \le c_2 \left(k^{-1} \log(1 + \frac{2M}{k}) \right)^{\frac{1}{p_1} - \frac{1}{p_2}} \quad \textit{if} \quad \log(2M) \le k \le 2M, \tag{7.12}$$

$$\begin{aligned} c_1 2^{-\frac{k}{2M}} (2M)^{\frac{1}{p_2} - \frac{1}{p_1}} &\le e_k \\ &\le c_2 2^{-\frac{k}{2M}} (2M)^{\frac{1}{p_2} - \frac{1}{p_1}} \quad \textit{if} \quad k \ge 2M, \end{aligned} \tag{7.13}$$

where c_1 *and* c_2 *are positive constants which are independent of* M *(and* k*) but may depend upon* p_1 *and* p_2*.*

7.4 Remark The estimate from below is covered by Proposition 7.2. A proof of the estimate from above may be found in [ET96], pp. 98–101.

7.5 Remark If $1 \le p_1 < p_2 \le \infty$, then also the estimate (7.12) is an equivalence as in the two other cases: see [Schü84] and [Kön86], 3.c.8, pp. 190–191.

8 Weighted ℓ_p-spaces

8.1 Preliminaries and notation

Let $d > 0$, $\delta \geq 0$ and $(M_j)_{j\in\mathbb{N}_0}$ be a sequence of natural numbers. We always assume that there are two positive numbers c_1 and c_2 with

$$c_1 \leq M_j\, 2^{-jd} \leq c_2 \quad \text{for every} \quad j \in \mathbb{N}_0. \tag{8.1}$$

Let $0 < p \leq \infty$ and $0 < q \leq \infty$. Then by $\ell_q(2^{j\delta}\ell_p^{M_j})$ we shall mean the linear space of all complex sequences $x = (x_{j,l}:\ j \in \mathbb{N}_0;\ l = 1,\ldots,M_j)$ endowed with the quasi-norm

$$\|x|\ell_q(2^{j\delta}\ell_p^{M_j})\| = \left(\sum_{j=0}^{\infty}(\sum_{l=1}^{M_j} 2^{j\delta p}|x_{j,l}|^p)^{\frac{q}{p}}\right)^{\frac{1}{q}} \tag{8.2}$$

with the obvious modifications according to (7.2) if $p = \infty$ and/or $q = \infty$. In case of $\delta = 0$ we write $\ell_q(\ell_p^{M_j})$ and if, in addition $p = q$, then we have the ℓ_p-spaces with the components ordered in the given way. Plainly, $\ell_q(2^{j\delta}\,\ell_p^{M_j})$ consists of dyadic blocks of spaces $\ell_p^{M_j}$ as introduced in 7.1 clipped together via the weights $2^{j\delta}$. We are interested in the counterpart of Theorem 7.3. Let $d > 0$, $\delta > 0$ and

$$0 < p_1 \leq p_2 \leq \infty, \quad 0 < q_1 \leq \infty, \quad 0 < q_2 \leq \infty. \tag{8.3}$$

Then the identity map

$$id: \quad \ell_{q_1}(2^{j\delta}\,\ell_{p_1}^{M_j}) \to \ell_{q_2}(\ell_{p_2}^{M_j}) \tag{8.4}$$

is compact, where M_j is restricted by (8.1). To prove this claim we use the decomposition

$$id = \sum_{j=0}^{\infty} id_j \tag{8.5}$$

where

$$\begin{aligned} id_j\, x &= (\delta_{jk}\, x_{k,l}:\ k \in \mathbb{N}_0;\ l = 1,\ldots,M_k) \\ &= (0,\ldots,0,x_{j,1},\ldots,x_{j,M_j},0,0,\ldots) \end{aligned} \tag{8.6}$$

selects the jth block. We have

$$\begin{aligned} \|id_j\,x|\ell_{q_2}(\ell_{p_2}^{M_k})\| &= \|(x_{j,l})|\,\ell_{p_2}^{M_j}\| \\ &\leq \|(x_{j,l})|\,\ell_{p_1}^{M_j}\| \\ &\leq 2^{-j\delta}\,\|x|\,\ell_{q_1}(2^{k\delta}\,\ell_{p_1}^{M_k})\|. \end{aligned} \tag{8.7}$$

Now by (8.5) and (8.7) it follows that id is compact.

8.2 Theorem *Let $d > 0$, $\delta > 0$, and $M_j \in \mathbb{N}$ with* (8.1)*. Let p_1, p_2, q_1, q_2 be given by* (8.3)*. Let e_k be the entropy numbers of the compact operator id in* (8.4) *according to Definition* 6.2*. There are two positive numbers c and C such that*

$$c\,k^{-\frac{\delta}{d}+\frac{1}{p_2}-\frac{1}{p_1}} \le e_k \le C\,k^{-\frac{\delta}{d}+\frac{1}{p_2}-\frac{1}{p_1}}, \quad k \in \mathbb{N}. \tag{8.8}$$

Proof.
Step 1. First we prove the left-hand side of (8.8). In the commutative diagram

$$\begin{tikzcd}
\ell_{p_1}^{M_j} \arrow[rr, "id^j"] \arrow[d, "id"'] & & \ell_{q_1}\Big(2^{j\delta}\ell_{p_1}^{M_j}\Big) \arrow[d, "id"] \\
\ell_{p_2}^{M_j} & & \ell_{q_2}\Big(\ell_{p_2}^{M_j}\Big) \arrow[ll, "id_j"']
\end{tikzcd}$$

the operator id_j is given as in (8.6), now acting in the indicated slightly modified way, whereas id^j maps $\ell_{p_1}^{M_j}$ identically onto $2^{j\delta}\,\ell_{p_1}^{M_j}$ interpreted as a dyadic block of $\ell_{q_1}(2^{j\delta}\,\ell_{p_1}^{M_j})$. In what follows we reserve id for the identity given by (8.4), otherwise we indicate the spaces involved. Hence

$$id\,\Big(\ell_{p_1}^{M_j} \to \ell_{p_2}^{M_j}\Big) = id_j \circ id \circ id^j, \quad j \in \mathbb{N}. \tag{8.9}$$

Plainly,

$$\|id^j\| = 2^{j\delta} \quad \text{and} \quad \|id_j\| = 1, \tag{8.10}$$

and consequently by (6.8)

$$e_k\Big(id:\ \ell_{p_1}^{M_j} \to \ell_{p_2}^{M_j}\Big) \le 2^{j\delta}\,e_k, \quad k \in \mathbb{N},\ j \in \mathbb{N}. \tag{8.11}$$

By (7.13) with $k = 2M_j$ we obtain

$$e_{2M_j} \ge c\,2^{-j\delta}\,2^{jd(\frac{1}{p_2}-\frac{1}{p_1})}, \quad j \in \mathbb{N}. \tag{8.12}$$

By (8.1) and the monotonicity properties of the entropy numbers described in Proposition 6.4(i) it follows that

$$e_k \ge c\,k^{-\frac{\delta}{d}+\frac{1}{p_2}-\frac{1}{p_1}}, \quad k \in \mathbb{N}, \tag{8.13}$$

for some $c > 0$.

Step 2. The estimate from above is more complicated. Let $J \in \mathbb{N}$ and

$$L\,\delta = J\,\delta + Jd(\frac{1}{p_1} - \frac{1}{p_2}); \tag{8.14}$$

in particular $L \geq J$. We put $e_t = e_{[t]}$ if $t \geq 1$ and assume $L \in \mathbb{N}$. We wish to prove

$$e_{2^{Jd}} \leq c\,2^{-J\delta + Jd(\frac{1}{p_2} - \frac{1}{p_1})}, \quad J \in \mathbb{N}. \tag{8.15}$$

This is equivalent to the estimate from above in (8.8). We split the sum in (8.5) in three parts,

$$id = \sum_{j=0}^{J} id_j + \sum_{j=J+1}^{L} id_j + \sum_{j=L+1}^{\infty} id_j. \tag{8.16}$$

Of course here id_j is considered as a map between the two spaces in (8.4) according to (8.6) in contrast to id_j in the above diagram in Step 1. There is no danger of confusion. In particular by (8.7) we have

$$\left\| \sum_{j=L+1}^{\infty} id_j \right\| \leq c\,2^{-L\delta}, \tag{8.17}$$

which by (8.14) coincides with the right-hand side of (8.15). Let $\varrho = \min(1, p_2, q_2)$. It is easy to see that $\ell_{q_2}(\ell_{p_2}^{M_j})$ is a ϱ-Banach space. Then we obtain by (6.9), (8.16), and (8.17)

$$e_k^\varrho \leq c\,2^{-L\delta\varrho} + \sum_{j=0}^{J} e_{k_j}^\varrho\,(id_j) + \sum_{j=J+1}^{L} e_{k_j}^\varrho\,(id_j) \tag{8.18}$$

where $k = \sum_{j=0}^{L} k_j$. By (8.7) we have

$$e_{k_j}(id_j) = 2^{-j\delta}\,e_{k_j}\left(id : \ell_{p_1}^{M_j} \to \ell_{p_2}^{M_j}\right), \tag{8.19}$$

and hence by Theorem 7.3

$$e_{k_j}(id_j) \leq$$

$$c\,2^{-j\delta}\left[k_j^{-1}\,\log(c\,2^{jd}\,k_j^{-1})\right]^{\frac{1}{p_1} - \frac{1}{p_2}} \quad \text{if} \quad k_j \leq 2M_j, \tag{8.20}$$

which covers also (7.11) and

$$e_{k_j}(id_j) \leq c\,2^{-j\delta}\,2^{-\frac{k_j}{2M_j}}\,(2M_j)^{\frac{1}{p_2} - \frac{1}{p_1}} \quad \text{if} \quad k_j > 2M_j. \tag{8.21}$$

Now we choose

$$k_j \;=\; 2^{Jd}\, 2^{-(J-j)\varepsilon} \quad \text{if} \quad j = 0, \dots, J \tag{8.22}$$

and

$$k_j \;=\; 2^{Jd}\, 2^{-(j-J)\varkappa} \quad \text{if} \quad j = J+1, \dots, L, \tag{8.23}$$

where ε and $\varkappa$ are positive numbers which will be chosen later on. We obtain

$$k \;\sim\; 2^{Jd}, \tag{8.24}$$

where «$\sim$» indicates equivalences (two-sided estimates up to unimportant positive constants which are independent of J). We deal with the two sums in (8.18) separately.

Step 3. Let $j = 0, \dots, J$. By (8.22) we have

$$k_j \;=\; 2^{jd}\, 2^{(J-j)(d-\varepsilon)} \;\geq\; 2^{jd}, \tag{8.25}$$

where we choose $0 < \varepsilon < d$. By (8.1) and (8.20), (8.21) it follows that we can always apply (8.21) in that case. Then we obtain

$$e_{k_j}\,(id_j) \;\leq\; c\, 2^{\lambda_{j,J}} \tag{8.26}$$

with

$$\lambda_{j,J} \;=\; -J\delta + Jd(\frac{1}{p_2} - \frac{1}{p_1}) + (J-j)\left(\delta - d(\frac{1}{p_2} - \frac{1}{p_1})\right) - c\, 2^{(J-j)(d-\varepsilon)} \tag{8.27}$$

and consequently

$$\sum_{j=0}^{J} e^{\varrho}_{k_j}\,(id_j) \;\leq\; c\, 2^{-J\delta\varrho + J\varrho d(\frac{1}{p_2} - \frac{1}{p_1})}. \tag{8.28}$$

Step 4. Let $j = J+1, \dots, L$. By (8.23) we have

$$k_j \;\leq\; c\, 2^{jd}. \tag{8.29}$$

Hence we can always apply (8.20) and obtain

$$\begin{aligned} e_{k_j}\,(id_j) &\leq c\, 2^{-j\delta} \left[2^{-Jd + (j-J)\varkappa} \log\left(c\, 2^{(j-J)d}\; 2^{(j-J)\varkappa}\right)\right]^{\frac{1}{p_1} - \frac{1}{p_2}} \\ &\leq c\, 2^{-J\delta + Jd(\frac{1}{p_2} - \frac{1}{p_1})}\; 2^{(J-j)[\delta + \varkappa(\frac{1}{p_2} - \frac{1}{p_1})]}\, [(d+\varkappa)(j-J)]^{\frac{1}{p_1} - \frac{1}{p_2}}. \end{aligned} \tag{8.30}$$

We choose $\varkappa > 0$ such that

$$\varkappa\,(\frac{1}{p_1} \;-\; \frac{1}{p_2}) \;<\; \delta \tag{8.31}$$

and obtain

$$\sum_{j=J+1}^{L} e^{\varrho}_{k_j}\,(id_j) \;\leq\; c\, 2^{-J\delta\varrho + Jd\varrho(\frac{1}{p_2} - \frac{1}{p_1})}. \tag{8.32}$$

Step 5. By (8.24), (8.14), (8.18), (8.28), and (8.32) we have

$$e_{c\, 2^{Jd}} \;\leq\; c'\, 2^{-J\delta + Jd(\frac{1}{p_2} - \frac{1}{p_1})}, \quad J \in \mathbb{N}, \tag{8.33}$$

where c and c' are appropriate positive constants. This coincides essentially with (8.15) and completes the proof of the right-hand side of (8.8).

8.3 Remark In the case of Banach spaces, which means that the numbers p_1, p_2, q_1 and q_2 in (8.3) are larger than or equal to 1, the above theorem is more or less known, see [Kühn84]. In that paper the proof is based on interpolation properties of entropy numbers and entropy numbers of diagonal operators in ℓ_p-spaces due to Carl, see [CaS90].

8.4 Estimates of constants

Theorem 8.2 is the basis for the study of entropy numbers of embedding operators in function spaces. Usually, in non-limiting situations, all parameters p_1, p_2, q_1, q_2, d and δ are fixed and there is no need to have additional information on the dependence of c and C in (8.8) on these parameters. However in some limiting cases we deal with a sequence of target spaces and we have to know how C in (8.8) depends on p_2, q_2 and δ, whereas the dependence of C on p_1 , q_1 and d is not so interesting for our later purposes. To facilitate the estimates we assume in addition

$$1 \le p_2 \le \infty, \quad 1 \le q_2 \le \infty, \quad \text{and} \quad 0 < \delta \le 1. \tag{8.34}$$

This is not really necessary, but sufficient for our later purposes.

8.5 Corollary *Under the hypotheses of Theorem* 8.2*, complemented by* (8.34)*, we have*

$$e_k \le c\,\delta^{-1-2(\frac{1}{p_1}-\frac{1}{p_2})}\, k^{-\frac{\delta}{d}+\frac{1}{p_2}-\frac{1}{p_1}}, \quad k \in \mathbb{N}, \tag{8.35}$$

for some positive constant c which is independent of p_2, q_2, and δ (but may depend on p_1, q_1, and d).

Proof. We follow the arguments in the Steps 2–5 of the proof of Theorem 8.2. By (8.34) we have $\varrho = 1$ in (8.18). We estimate the constant c in the first term on the right-hand side of (8.18). By (8.7) and (8.17) we have

$$\left\| \sum_{j=L+1}^{\infty} id_j \right\| \le \sum_{j=L+1}^{\infty} \|id_j\| \le \sum_{j=L+1}^{\infty} 2^{-j\delta} \le c\,\delta^{-1}\, 2^{-L\delta}, \tag{8.36}$$

where c is independent of δ. Next we remark that we may assume that the constant c_2 in (7.11)–(7.13) is independent of p_2. We refer to [ET96], Remark 2 on p. 101. But this is not a deep result. It follows immediately from Theorem 7.3 with $p_2 = \infty$ and the interpolation properties of the entropy numbers mentioned in 6.7. Having this in mind it follows that the constants c in (8.20) and (8.21) are independent of p_2 , q_2 and δ. We may choose $\varepsilon = \frac{d}{2}$ in (8.22) and (8.25). Hence ε is not of interest for us. As for $\varkappa$ in (8.23) and (8.31) we may choose $\varkappa = \frac{\delta p_1}{2}$. Then (8.24) must be substituted now by

$$2^{Jd} \le k \le \frac{c}{\delta}\, 2^{Jd}, \tag{8.37}$$

where c is independent of δ (and p_2 and q_2). Now by the above remarks about the constants in Theorem 7.3 and $\delta \le 1$ it follows from (8.26) and (8.27) that the constant c in (8.28) (now with $\varrho = 1$) is independent of p_2, q_2 and δ. We estimate the constant c in (8.32) (again with $\varrho = 1$). By the above choice of $\varkappa$ it follows from (8.30)

$$\sum_{j=J+1}^{L} e_{k_j}(id_j) \le c_1\, 2^{-J\delta + Jd(\frac{1}{p_2}-\frac{1}{p_1})}\, \delta^{\frac{1}{p_2}-\frac{1}{p_1}} \sum_{l=1}^{L-J} 2^{-c_2 l\delta}\, (\delta l)^{\frac{1}{p_1}-\frac{1}{p_2}} \tag{8.38}$$

where c_1 and c_2 are independent of δ, p_2 and q_2. The last factor can be estimated from above by

$$\int_0^\infty e^{-c_3\delta t}\, (\delta t)^{\frac{1}{p_1}-\frac{1}{p_2}}\, dt \le c_4\, \delta^{-1}. \tag{8.39}$$

Now by (8.18), (8.36), (8.28) and (8.38) with (8.39) we obtain

$$e_k \le c\, \delta^{-1-\frac{1}{p_1}+\frac{1}{p_2}}\, 2^{-J\delta + Jd(\frac{1}{p_2}-\frac{1}{p_1})}, \tag{8.40}$$

where c is independent of δ, p_2, and q_2, and k is given by (8.37). With $2^{Jd} \sim k\delta$ in (8.40) we have

$$e_k \le c\, \delta^{-1-2(\frac{1}{p_1}-\frac{1}{p_2})}\, k^{-\frac{\delta}{d}+\frac{1}{p_2}-\frac{1}{p_1}}, \quad k \in \mathbb{N}. \tag{8.41}$$

The proof of (8.35) is complete.

8.6 Remark The restrictions $p_2 \ge 1$ and $q_2 \ge 1$ are unimportant. Otherwise one has $\varrho < 1$ in (8.18). There is no problem to follow the above reasoning in this more general case.

8.7 Comparison The estimates for entropy numbers of compact embeddings between function spaces will be based in non-limiting cases on (8.8), whereas in some limiting cases we need the additional information given in the above corollary. Although the context is slightly different (so far) one can compare the exponent $1 + 2(\frac{1}{p_1} - \frac{1}{p_2})$ of δ in (8.35) with the exponents $1 + \frac{2}{p}$ in [ET96], p. 130, formula (8), and $-\frac{2s}{n} - \varepsilon$ in [ET96], p. 139, formula (3), where any $\varepsilon > 0$ is admitted. It comes out that $1 + \frac{2}{p}$ originates precisely from $1 + 2(\frac{1}{p_1} - \frac{1}{p_2})$ (restricted to the treated case), whereas $-\frac{2s}{n} - \varepsilon$ is somewhat better ($\varepsilon = 1$ would be the direct counterpart). On the other hand the situation considered in [ET96], p. 139, is more special. We return in 23.5 to these comparisons in greater detail and shed more light upon these admittedly somewhat cryptical remarks.

8.8 A digression: Matrix operators

It is not our aim to discuss the spectral theory of compact operators acting in ℓ_p-spaces. This has been done in great detail by A. Pietsch, B. Carl and other mathematicians. We refer to [Pie87], esp. pp. 230–231, [Kön86], esp. pp. 150–151, and [CaS90]. Our intention here is simply to demonstrate the power of Theorem 8.2. For that purpose we estimate the distribution of eigenvalues of some matrix operators in ℓ_p-spaces. We avoid any technical complications and we are far from the most general case which can be treated in that way. In this sense we leave it to the interested reader to compare the results obtained here with the more systematic treatments in the above-mentioned books. Let $0 < p \le \infty$; recall that ℓ_p is the linear space of all complex sequences $x = (x_k : k \in \mathbb{N})$ endowed with the quasi-norm

$$\|x|\ell_p\| = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} \tag{8.42}$$

with the obvious modification if $p = \infty$. Let

$$A = (a_{lk} : l \in \mathbb{N},\ k \in \mathbb{N}), \quad a_{lk} \in \mathbb{C}, \tag{8.43}$$

and as usual let

$$Ax = \left(\sum_{k=1}^{\infty} a_{lk}\, x_k : \quad l \in \mathbb{N} \right) \quad \text{for} \quad x = (x_k : k \in \mathbb{N}). \tag{8.44}$$

Let $d > 0$ and $\delta > 0$, and $M_j \sim 2^{jd}$ according to (8.1) with $j \in \mathbb{N}_0$. We put

$$M^j = \sum_{m=0}^{j-1} M_m \sim 2^{jd}, \quad j \in \mathbb{N}; \quad \text{and} \quad M^0 = 0, \tag{8.45}$$

and assume that the entries a_{lk} can be represented as

$$a_{lk} = 2^{-j\delta}\, b^j_{lm}, \quad l \in \mathbb{N}, \tag{8.46}$$

$$k = M^j + m \quad \text{for} \quad j \in \mathbb{N}_0 \quad \text{and} \quad m = 1, \ldots, M_j, \tag{8.47}$$

and

$$\sum_{l=0}^{\infty} \left(\sum_{j=0}^{\infty} \sum_{m=1}^{M_j} |b^j_{lm}| \right)^p < \infty, \quad 0 < p \le \infty, \tag{8.48}$$

with the obvious modification if $p = \infty$. In other words, for fixed l we sum first the absolute values of the entries in the lth row. Hence,

$$B = (b_{lk} = b^j_{lm} : l \in \mathbb{N},\ k \text{ given by (8.47)}) \tag{8.49}$$

is a so-called *Hille-Tamarkin matrix*, see the above references. In particular, A can be decomposed by

$$A = B \circ D, \tag{8.50}$$

where D is a diagonal matrix with the entries $d_k = 2^{-j\delta}$, where k is given by (8.47). As we shall see, A generates a compact operator in ℓ_p. Then we can apply the Riesz theory mentioned in 6.1, 6.6 and 6.8. In particular, the non-zero eigenvalues $\mu_k(A)$ of A, repeated according to algebraic multiplicity, can be ordered as in (6.11).

8.9 Proposition *Let $0 < p \le \infty$ and let A be the above operator. Then there is a positive constant c such that*

$$|\mu_k(A)| \le c\, k^{-\frac{\delta}{d}-\frac{1}{p}}, \quad k \in \mathbb{N}. \tag{8.51}$$

Proof. We decompose A as

$$A = B \circ id \circ D, \tag{8.52}$$

where D and B have the above meaning. We claim

$$\begin{aligned} D &: \ell_p \to \ell_p\,(2^{j\delta}\,\ell_p^{M_j}), \\ id &: \ell_p\,(2^{j\delta}\,\ell_p^{M_j}) \to \ell_\infty, \\ B &: \ell_\infty \to \ell_p. \end{aligned} \tag{8.53}$$

The first line is obvious where we used the notation introduced in (8.2). By Theorem 8.2 the operator id is compact and

$$e_k\,(id) \le C\, k^{-\frac{\delta}{d}-\frac{1}{p}}, \quad k \in \mathbb{N}. \tag{8.54}$$

Let $x = (x_k : k \text{ given by } (8.47)) \in \ell_\infty$. Then by (8.49)

$$\begin{aligned} |(Bx)_l| &= \left| \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} b^j_{lm}\, x_{M^j+m} \right| \\ &\le \|x|\ell_\infty\| \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} |b^j_{lm}|. \end{aligned} \tag{8.55}$$

Now the last line in (8.53) is a consequence of (8.48) and (8.55). Since D and B are bounded, (8.52), (8.54), and (6.8) prove

$$e_k(A) \le c\, k^{-\frac{\delta}{d}-\frac{1}{p}}, \quad k \in \mathbb{N}. \tag{8.56}$$

Finally, (8.51) is a consequence of Corollary 6.10.

8.10 Remark It can be easily seen that the exponent in (8.51) is sharp: Let A be a diagonal operator, $a_{kl} = 0$ if $k \neq l$ and

$$a_{kk} = k^{-\frac{\delta}{d}-\frac{1}{p}} (\log k)^{\alpha}. \tag{8.57}$$

By (8.47) and (8.48) with (8.45) we have

$$\sum_{l=0}^{\infty} \left(\sum_{j=0}^{\infty} \sum_{m=1}^{M_j} |b^j_{lm}| \right)^p \sim \sum_{j=0}^{\infty} 2^{-jd} j^{\alpha p} 2^{jd} < \infty \tag{8.58}$$

if $\alpha < -\frac{1}{p}$. In that case A has the required properties. On the other hand, a_{kk} are the eigenvalues of A. Hence the exponent in (8.51) is the best possible.

9 Weighted ℓ_p-spaces: a generalization

9.1 Preliminaries and notation

Unfortunately Theorem 8.2 and Corollary 8.5 are not completely sufficient for our later purposes. We need something like an ℓ_u-version of these two assertions. Fortunately it comes out that these generalizations are nothing more than a technical appendix to the results just mentioned. We use the same notation as in 8.1. In particular, let $d > 0$, $\delta \geq 0$ and $(M_j)_{j\in\mathbb{N}_0}$ be a sequence of natural numbers with (8.1) for some positive numbers c_1 and c_2. Let again $\ell_q(2^{j\delta}\,\ell_p^{M_j})$ with $0 < p \leq \infty$ and $0 < q \leq \infty$ be the quasi-Banach space introduced in 8.1 and quasi-normed by (8.2). Let, in addition, $\mu \geq 0$ and $0 < u \leq \infty$. Then by

$$\ell_u \left[2^{\mu m}\, \ell_q\, (2^{j\delta}\, \ell_p^{M_j}) \right]$$

we shall mean the linear space of all $\ell_q(2^{j\delta}\,\ell_p^{M_j})$-valued sequences $x = (x^m : m \in \mathbb{N}_0)$ endowed with the quasi-norm

$$\left\| x \,|\, \ell_u \left[2^{\mu m}\, \ell_q\, (2^{j\delta}\, \ell_p^{M_j}) \right] \right\| = \left(\sum_{m=0}^{\infty} 2^{\mu m u} \left\| x^m | \ell_q(2^{j\delta}\, \ell_p^{M_j}) \right\|^u \right)^{\frac{1}{u}} \tag{9.1}$$

with the obvious modification according to the vector-valued version of (7.2) if $u = \infty$. In case of $\mu = \delta = 0$ we write $\ell_u\,[\,\ell_q(l_p^{M_j})\,]$ in accordance with the notation introduced in 8.1. We are interested in an extension of Theorem 8.2. Let $d > 0$, $\delta > 0$, $\mu > 0$,

$$0 < p_1 \leq p_2 \leq \infty \tag{9.2}$$

and

$$0 < q_1 \leq \infty,\; 0 < q_2 \leq \infty,\; 0 < u_1 \leq \infty,\; 0 < u_2 \leq \infty. \tag{9.3}$$

Then the identity map

$$id: \quad \ell_{u_1}\left[2^{\mu m}\,\ell_{q_1}\,(2^{j\delta}\,\ell_{p_1}^{M_j})\right] \;\to\; \ell_{u_2}\left[\ell_{q_2}\,(\ell_{p_2}^{M_j})\right] \tag{9.4}$$

is compact. This is simply the extension of what had been said in 8.1, see (8.4)–(8.7), from the scalar case to the ℓ_u-valued case. However this generalization is an immediate consequence of $\mu > 0$. Now Theorem 8.2 can be rather easily extended to the vector-valued case.

9.2 Theorem *Let $d > 0$, $\delta > 0$, $\mu > 0$, and $M_j \in \mathbb{N}$ with (8.1). Let p_1, p_2, q_1, q_2, u_1, u_2 be given by* (9.2) *and* (9.3). *Let e_k be the entropy numbers of the compact operator id according to* (9.4). *There are two positive numbers c and C such that*

$$c\,k^{-\frac{\delta}{d}+\frac{1}{p_2}-\frac{1}{p_1}} \;\le\; e_k \;\le\; C\,k^{-\frac{\delta}{d}+\frac{1}{p_2}-\frac{1}{p_1}}, \quad k \in \mathbb{N}. \tag{9.5}$$

Proof.
Step 1. The estimate from below is covered by Step 1 of the proof of Theorem 8.2.

Step 2. We reduce the estimate from above to the corresponding scalar case in Theorem 8.2. Let

$$id_m: \quad x \;\mapsto\; x^m, \quad \text{where} \quad x = (x^l)_{l\in\mathbb{N}_0} \tag{9.6}$$

has the same meaning as in (9.1). Then we have

$$id \;=\; \sum_{m=0}^{\infty} id_m. \tag{9.7}$$

Let, for brevity, $a = \frac{\delta}{d} + \frac{1}{p_1} - \frac{1}{p_2}$, and let $J \in \mathbb{N}$ and

$$L = [\frac{a}{\mu}J] \in \mathbb{N}_0. \tag{9.8}$$

Of course $a > 0$. Then it follows that

$$\left\| \sum_{l=L+1}^{\infty} id_l \right\| \;\le\; c\,2^{-\mu L} \;\le\; c'\,2^{-aJ}. \tag{9.9}$$

Let

$$k_l = 2^J\,2^{-l\varepsilon} \quad \text{where } l = 0,\ldots,L \quad \text{and} \quad \varepsilon a < \mu. \tag{9.10}$$

Then we have

$$k = \sum_{l=0}^{L} k_l \;\sim\; 2^J \tag{9.11}$$

and by (8.8)

$$e_{k_l}\,(id_l) \;\le\; c\,2^{-\mu l}\,2^{-aJ}\,2^{la\varepsilon}; \quad l = 0,\ldots,L. \tag{9.12}$$

Now by (9.9)–(9.11) and (6.9) it follows that

$$e_{c_1 2^J}\,(id) \;\le\; c_2\,2^{-aJ}, \quad J \in \mathbb{N}, \tag{9.13}$$

for some $c_1 > 0$ and $c_2 > 0$. This proves the right-hand side of (9.5).

9.3 Remark We are also interested in an extension of Corollary 8.5 to the vector-valued case. In accordance with (8.34) we assume in addition

$$1 \leq p_2 \leq \infty,\ 1 \leq q_2 \leq \infty,\ 1 \leq u_2 \leq \infty, \text{ and } 0 < \delta \leq 1. \tag{9.14}$$

Again, these conditions are not really necessary, but sufficient for our later purposes.

9.4 Corollary *Under the hypotheses of Theorem* 9.2 *complemented by* (9.14) *we have*

$$e_k \leq c\,\delta^{-1-2(\frac{1}{p_1}-\frac{1}{p_2})}\,k^{-\frac{\delta}{d}+\frac{1}{p_2}-\frac{1}{p_1}}, \quad k \in \mathbb{N}, \tag{9.15}$$

for some positive constant c which is independent of p_2, q_2, u_2, and δ (but may depend on p_1, q_1, u_1, d and μ).

Proof. By slight modification we may assume that L in (9.8) and ε in (9.10) are chosen independently of the indicated numbers. Then we have the same situation in (9.9) and (9.11) by a similar argument as in (8.36). Replacing the constant c in (9.12) by the corresponding constant on the right-hand side of (8.35) we obtain (9.13) with the desired constant. This proves (9.15).

9.5 Remark As for comments we refer to 8.6 and 8.7.

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