Preface

This book deals with several aspects of fractal geometry in $\mathbb{R}^n$ which are closely connected with Fourier analysis, function spaces, and appropriate (pseudo)differential operators. It emerged quite recently that some modern techniques in the theory of function spaces are intimately related to methods in fractal geometry. Special attention is paid to spectral properties of fractal (pseudo)differential operators; in particular we shall play the drum with a fractal layer.

In some sense this book may be considered as the fractal twin of [ET96], where we developed adequate methods to handle spectral problems of degenerate pseudodifferential operators in $\mathbb{R}^n$ and in bounded domains. Besides a few special properties of function spaces we relied there on sharp estimates of entropy numbers of compact embeddings between these spaces and their relations to the distribution of eigenvalues. Some of the main assertions of the present book are based on just these techniques but now in a fractal setting. Since virtually nothing of these new methods is available in literature, a substantial part of what we have to say deals with recent developments in the theory of function spaces, also for their own sake. In this respect the book might also be considered as a continuation of [Tri83] and [Tri92].

We give a brief description of the contents of the book. Chapter I deals with fractals in $\mathbb{R}^n$. We present the basic material without proofs following closely [Fal85] and [Mat95]. Isotropic, anisotropic and nonisotropic d-sets and related self-affine fractals are discussed in greater details. Chapter II is devoted to entropy numbers in weighted l_p-spaces with a dyadic block structure. In Chapter III we introduce the function spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ referring mainly to [Tri83, Tri92, ET96] for details as far as basic facts are concerned. On the other hand we give a new proof of atomic representations in these spaces. The method used allows us to atomize the atoms, which results in subatomic (or quarkonial) decompositions which prove to be useful in connection with entropy numbers. This method works even for vector-valued function spaces. One obtains a total decoupling and decomposition in elementary building blocks which resembles the Taylor expansion of analytic functions. The final section of this chapter deals with the Hausdorff dimension of the graph of a real function $f(x)$ belonging to the Hölder spaces $\mathcal{C}^s(\mathbb{R}^n)$ with $0 < s < 1$, demonstrating the close connection between atomic representations of functions and methods of fractal geometry. Chapter IV is devoted to function spaces on and of fractals. First we characterize the Hausdorff dimension $\dim_H \Gamma$ of a Borel set Γ in $\mathbb{R}^n$ with $\dim_H \Gamma < n$ in terms of distributions and function spaces. Furthermore spaces $L_p(\Gamma)$ and Besov spaces $\mathbb{B}^s_{pq}(\Gamma)$, $s > 0$, on compact fractals Γ are studied, especially their relations to the spaces $B^\sigma_{pq}(\mathbb{R}^n)$ and the asymptotics of the entropy numbers of compact embeddings between them. On that basis we introduce «regular» elliptic operators on compact d-sets via quadratic forms and study the distribution of their eigenvalues. The final Chapter V deals with spectra of fractal pseudodifferential operators of several types. Let

$$B = b\,a(\cdot,D), \quad a(\cdot,D) \in \Psi^{-\varkappa}_{1,\varrho}, \quad \varkappa > 0, \quad 0 \le \varrho \le 1,$$

(Hörmander classes) and

$$b \in \mathbb{B}^s_{pq}(\mathbb{R}^n), \quad s < 0, \quad supp\, b \quad \text{compact},$$

(pseudodifferential operators with fractal coefficients). These operators are compact in suitable spaces (under some restrictions on the parameters involved, in particular $\varkappa > 2|s|$) and one obtains for the eigenvalues μ_k (counted with respect to their algebraic multiplicities) the estimate

$$|\mu_k| \le c\, k^{-\frac{\varkappa - 2|s|}{n}}, \quad k \in \mathbb{N}.$$

For fractal pseudodifferential operators of type

$$B = b_1\, a(\cdot,D)\, b_2, \quad a(\cdot,D) \in \Psi^{-\varkappa}_{1,\varrho}, \quad \varkappa > 0, \quad 0 \le \varrho \le 1,$$

where $b_1 \in L_{r_1}(\Gamma)$, $b_2 \in L_{r_2}(\Gamma)$ and Γ is a compact d-set with $0 < d < n$, we obtain the estimate

$$|\mu_k| \le c\, k^{-\frac{\varkappa - n + d}{d}}, \quad k \in \mathbb{N},$$

for the distribution of the eigenvalues μ_k (counted with respect to their algebraic multiplicities) in suitable spaces (again under some restrictions on the parameters involved, in particular $\varkappa - n + d > d(\frac{1}{r_1} + \frac{1}{r_2})$). The exponent of k is sharp. In several (self-adjoint) cases one has even

$$c_1 k^{-\frac{\varkappa - n + d}{d}} \le |\mu_k| \le c_2 k^{-\frac{\varkappa - n + d}{d}}, \quad k \in \mathbb{N}, \tag{$*$}$$

for some c_1, c_2 with $0 < c_1 < c_2 < \infty$. Compared with the «Weyl exponent» $\frac{\varkappa}{n}$ one must replace n by d and one has the additional *fractal defect* $n - d$. Hence one obtains both sub-Weylian and super-Weylian behaviour. In particular one can play the (n-dimensional) drum, given by a smooth bounded domain Ω in $\mathbb{R}^n$, with a compact fractal layer Γ, where $\Gamma \subset \Omega$. Let Γ be an (isotropic, anisotropic, or nonisotropic) d-set; then the corresponding operator looks like

$$(-\Delta)^{-1} \circ tr^\Gamma$$

where tr^Γ is closely connected with the trace operator tr_Γ and $-\Delta$ stands for the Dirichlet Laplacian with respect to Ω. In case of isotropic d-sets one has $(*)$ with $\varkappa = 2$ provided that $n - 2 < d < n$. Even more interesting are anisotropic and nonisotropic d-sets in the plane (the music of the ferns). In that cases two-sided estimates for the μ_k's are given. At the end of this chapter we deal briefly with Schrödinger operators having fractal potentials and with some related nonlinear problems.

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