PREFACE

This book is not a textbook, and does not attempt to be a definitive statement of theory. If this monograph is a compendium of anything, it is one of techniques, specifically those leading to the proof of Thomason's étale cohomological descent theorem for Bott periodic K-theory, and perhaps beyond.

The proof given here for Thomason's result exposes most of the major extant ideas of the homotopy theory of presheaves of spectra on étale sites. This theory specializes to both sheaf cohomology theories and generalized cohomology theories in the traditional topological sense: a generalized étale cohomology theory is a cohomology theory which is represented by a presheaf of spectra on an étale site, by analogy with the way an ordinary spectrum represents a cohomology theory on spaces. There are many potential examples of such, but only étale K-theory has so far been studied in depth.

The homotopy theory of presheaves of spectra is a language as well as a technical device, which the original proof of the Thomason theorem predates and partially anticipates. I have maintained for several years now that his result should be proved entirely within this language, to clarify the proof and its relation with the Lichtenbaum-Quillen conjecture. This has been done successfully here, but it's been a long time coming.

There are several reasons for this, the chief among them being the requirement for an adequate theory of smash products of presheaves of spectra. It doesn't suffice to take an arbitrary theory of smash products of spectra and try to make it functorial, because there is a constant need to manipulate the resulting objects with closed model axioms to solve homotopy coherence problems: the standard constructions of K-theory spectra are, after all, not quite functorial. A theory of smash products of spectra and smash products of diagrams of spectra is therefore required which is sensitive to closed model structures, particularly those descended from the work of Bousfield and Friedlander.

For now, Adams' theory of handicrafted smash products works best for such purposes. A handicrafted smash product, or just a smash product, is a special case of the output of a diagonal functor from bispectra, or spectrum objects in spectra, to the category of spectra. The category of bispectra carries a strict and a stable closed model structure, both of which are promoted from the corresponding Bousfield-Friedlander theories for ordinary spectra, in such a way that the diagonal functor preserves stable weak equivalences and fibrations, and also defines an equivalence of the associated stable homotopy categories. The category of bispectra is a different, but equivalent, model for the stable category. Corresponding theories exist more generally for n-fold spectra, and for presheaves of bispectra and n-fold spectra on arbitrary Grothendieck sites. The strict and stable closed model structures for presheaves of bispectra, in particular, are bootstrapped from the corresponding theories for presheaves of spectra, following the relation between bispectra and ordinary spectra. The diagonal functor from presheaves of bispectra to presheaves of spectra plays the same role as in the ordinary case, and a theory of smash products of presheaves of spectra results, which includes a sensible description of function spectrum objects.

These results about smash products are the subject of the first three chapters of this book. The first applications appear in the third chapter, and include a projection formula for the abstract transfer associated to finite Galois extensions for presheaves of spectra defined on étale sites of fields.

The applications of the theory of smash products of presheaves of spectra concern K-theory, but one has to first make sure that K-theory presheaves are acceptable input for the smash products machine. The K-theory presheaf is most generally constructed on an arbitrary diagram of schemes by feeding a pseudo-functor taking values in symmetric monoidal categories arising from vector bundles on schemes to a combination of Waldhausen's $S.$-construction and supercoherence theory. The cup product on such a diagram is induced by a pseudo-natural bilinear map given on each of the underlying schemes by tensor product. One has to make sense of this idea, and show how to use it — this is the subject of Chapter 5. Supercoherence plus $S.$ is verbose and perhaps a bit ugly, but the output in this case is worth the drudgery: in conjunction with the results from the early part of the book, the theory implies that the cup product in K-theory and étale K-theory is induced by a maps of presheaves of bispectra. Supercoherence is used to show that the K-theory transfer for a finite Galois extension is equivariant for the action of the Galois group up to canonical stable equivalence, and is natural with respect to induced field extensions; our freshly minted approach to cup products implies that the projection formula is equivariant as well. The chapter closes with a demonstration that the K-theory norm for a finite Galois extension is homotopic to an abstract norm which is defined by adding up the actions of the Galois group elements. This in itself is not new, but the homotopy preserves a rather subtle kind of equivariance which becomes crucial in the proof of Thomason's theorem.

Some attempt is made at the outset of Chapter 5 to bring the reader into the supercoherence frame of mind, but none of more technical definitions or proofs from [28] are repeated here. This is the approach taken to the literature in general: I can be mildly encouraging here and there (particularly in Chapters 2, 3 and 6) about the published theory of simplicial presheaves and presheaves of spectra ([22], [23], [24] and [27]), but I quote from it ruthlessly. Similar treatment is given to other sources that I consider to be standard, such as the book of Bousfield and Kan [9], Milne's book [42], anything written by either Serre or Quillen, and the Bousfield-Friedlander paper [8].

There are, however, no references in the text to unpublished material in the absence of specific workarounds, and any required folk theorems are actually proved. This is responsible, in particular, for the content of the Chapter 4, which either plugs little gaps in the literature, or presents certain existing results in a fashion that is more to my taste. I might have inadvertently done something new in places — the discussion of the derived category that appears in the sixth section may be an example.

The sixth chapter begins with an introduction to generalized étale cohomology theories in the large, including a formal description of the construction of the descent spectral sequence from Postnikov towers. The discussion quickly moves, in subsequent sections, to one of generalized Galois cohomology and its relationship with generalized Čech cohomology, which is the technical heart of the matter in all discussions of Galois descent for the K-theory of fields. A generalized Galois cohomology theory is generalized étale cohomology theory for a field; it is also a special case of a theory which is represented by a presheaf of spectra on the site which underlies the classifying topos of a profinite group. The latter observation is essential for the applications of the final chapter. The sixth chapter closes with a discussion of bigness of the big étale site, and its relatives. It turns out that it is often technically very convenient, and maybe even necessary, to work with sites which are bigger than the big site; the last section of Chapter 6 is supposed to alleviate any set-theoretic discomfort this may cause.

The final chapter is a bit of a giant. The first five sections are occupied with the proof of Thomason's descent theorem, and its corollaries. The key point is that Bott periodic K-theory and its associated étale (or Galois) invariants coincide for fields containing the right primitive roots of unity and having a finite Tate-Tsen filtration. This is what I call Thomason's descent theorem: everything else follows from this statement by means involving greatly reduced technical exertion. There is a parallel with the Lichtenbaum-Quillen conjecture which is drawn in the final section, in that it is explicitly shown that if you can prove Lichtenbaum-Quillen for the fields appearing in the hard case of Thomason's descent theorem, then it's all over. The slick method for going from statements about the K-theory of fields to the K-theory of more general schemes involves a separate section on the Nisnevich topology. This section contains a new and short proof of the Nisnevich descent theorem for the K-theory of regular separated Noetherian schemes. The brevity of this proof is achieved by replacing Godement resolutions with globally fibrant models.

In all, this book consists of seven chapters. Each of these has its own formal introduction.

This main part of the research and writing for the preliminary version of this monograph was conducted over a period of three years, beginning in 1991–92, while I was a visiting professor at the University of British Columbia. Work on the book continued while visiting Université Paris VII in May of 1993, and was completed while I held a Distinguished Research Professorship in the 1993–94 academic year at the University of Western Ontario. This work was supported throughout by grants from the Natural Sciences and Engineering

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I would like to affectionately thank my wife Catharine and children Courtney and Bryce for their patience and emotional support during the various gestation periods of this manuscript.

This book is dedicated to the memory of
Robert W. Thomason (1952–1995).

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