Chapter 31. Geometric Properties

This chapter contains several results of geometric type for the cut polytope $\mathrm{CUT}_n^\square$. One of our objectives here is to study the geometric shape of $\mathrm{CUT}_n^\square$, in particular, in connection with its linear relaxation by the semimetric polytope $\mathrm{MET}_n^\square$ and with its convex (nonpolyhedral) relaxation by the elliptope $\mathcal{E}_n$.

We have already seen (in Section 26.3.3) that the polytope $\mathrm{CUT}_n^\square$ has a lot of symmetries. We are interested, for instance, in the following further questions: What are the edges of the polytope $\mathrm{CUT}_n^\square$? More generally, what is the structure of its faces of small dimension ? We can, in some sense, give an answer to this question up to dimension $\log_2 n$. Indeed, it turns out that $\mathrm{CUT}_n^\square$ has a lot of faces of dimension up to $\log_2 n$ in common with its relaxations $\mathrm{MET}_n^\square$ and $\mathcal{E}_n$ that arise by taking sets of cuts in general position (see Theorems 31.5.9 and 31.6.4).

As we have seen in the rest of Part V, $\mathrm{CUT}_n^\square$ has a great variety of facets, most of them having a very complicated structure. A legitimate question to ask is which ones are the most important among them ? Giving a precise definition of the word "important" in this question is not an easy task. However, it is intuitively clear that some facets are more essential than others; some facets have indeed a "big area" while some others contribute only to rounding off some little corners of the polytope. One way of measuring the importance of a facet is by computing the Euclidean distance of the hyperplane containing the facet to the barycentrum of $\mathrm{CUT}_n^\square$. It seems intuitively clear that facets that are close to the barycentrum are more important than facets that are far apart. It is conjectured that the triangle facets are the closest facets to the barycentrum; see Section 31.7 for results related to this conjecture. We remind from Chapter 27 that triangle inequalities share several other interesting properties.

The cut polytope is not a simplicial polytope (if $n \geq 5$) as some of its facets are not simplices. However, it seems that the great majority of its facets are simplices. This has been verified for $n \leq 7$, where it has been computed that about 97% of the facets are simplices. We group in Section 31.8 results on the simplex facets of $\mathrm{CUT}_n^\square$.

Section 31.5 presents several geometric properties of the elliptope $\mathcal{E}_n$, which was defined in Section 28.4.1 as the set of $n \times n$ symmetric positive semidefinite matrices with an all-ones diagonal. Up to a simple transformation, $\mathcal{E}_n$ is a (nonpolyhedral) relaxation of the cut polytope $\mathrm{CUT}_n^\square$.

One more interesting interpretation of the cut polytope is mentioned in Section 31.2; namely, the fact that the valid inequalities for $\mathrm{CUT}_n^\square$ yield inequalities

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for the pairwise angles among a set of n unit vectors in $\mathbb{R}^n$. (This is essentially a reformulation of the fact, stated in Section 6.4, that spherical distance spaces are ℓ_1-embeddable.) We describe in Section 31.3 some further implications of this result in connection with the completion problem for partial positive semidefinite matrices. In fact, this problem amounts to the description of projections of the elliptope $\mathcal{E}_n$. In general, the projected elliptope $\mathcal{E}(G)$ is contained in the image of $\mathrm{CUT}^{\square}(G)$ under the mapping $x \mapsto \cos(\pi x)$. It turns out that both bodies coincide when the graph G has no K_4-minor (see Theorem 31.3.7). Further results are given for larger classes of graphs in Section 31.3.

In Section 31.4 we consider the analogue completion problem for Euclidean distance matrices. In fact, this problem is nothing but the problem of describing projections of the negative type cone NEG_n. It turns out that there are several results for this problem, which are in perfect analogy with the known results for the positive semidefinite completion problem. We mention in Section 31.4.2 how the two completion problems can be linked (using, in particular, one of the metric transforms which was exposed in Chapter 9, namely, the Schoenberg transform).

In Section 31.1 we describe how cuts have been used for disproving a long standing conjecture of Borsuk.

31.1 Disproval of a Conjecture of Borsuk Using Cuts

The following question was asked by Borsuk [1933] more than sixty years ago:

> *Given a set X of points in $\mathbb{R}^d$, is it always possible to partition X into $d+1$ subsets, each having a smaller diameter than X ?*

We recall that the *diameter*[1] of a set $X \subseteq \mathbb{R}^d$ is defined as

$$\mathrm{diam}(X) := \max_{x,y \in X} \| x - y \|_2,$$

the maximum Euclidean distance between any two points of X. Borsuk's question has been answered in the negative by Kahn and Kalai [1993], who constructed a counterexample using cut vectors. We present here a variation of their counterexample, which is due to Nilli [1994].

Let $n = 4p$ where p is an odd prime integer, and $d := \binom{n}{2}$. As set of points $X \subseteq \mathbb{R}^d$, we take the set

$$X := \{\delta(S) \mid S \subseteq V_n,\ |S| \text{ is even and } 1 \in S\}$$

of all even cut vectors in K_n; hence, $|X| = 2^{n-2}$. Then, X provides a counterexample to Borsuk's question in the case when

[1] For a polytope P, there is another notion of diameter besides the geometric notion considered here. Namely, the diameter of P is also sometimes defined as the diameter of its 1-skeleton graph; for instance, the diameter (of the 1-skeleton graph of) the cut polytope is 1 (see Section 31.6).

$$\frac{2^{n-2}}{\sum_{i=0}^{p-1}\binom{n-1}{i}} > \binom{n}{2} + 1. \tag{31.1.1}$$

The smallest counterexample occurs in dimension $d = \binom{44}{2} = 946$ for $n = 44$, $p = 11$. The proof is based on the following result of Nilli [1994].

Lemma 31.1.2. *Let $n = 4p$ with p odd prime and let $\mathcal{E}$ denote the set of vectors $x \in \{\pm 1\}^n$ such that $x_1 = 1$ and x has an even number of positive components. If $\mathcal{F} \subseteq \mathcal{E}$ contains no two orthogonal vectors, then $|\mathcal{F}| \leq \sum_{p=0}^{p-1}\binom{n-1}{i}$.*

Proof. Observe that the scalar product of two elements $a, b \in \mathcal{E}$ is divisible by 4. Hence, by the assumption, $a^T b \not\equiv 0 \pmod p$ for any $a \neq b \in \mathcal{F}$. For each $a \in \mathcal{F}$, we consider the polynomial P_a in the variables $X_1, \ldots, X_n$ defined by

$$P_a(X) := \prod_{i=1}^{p-1}\Big(\sum_{j=1}^{n} a_j X_j - i\Big).$$

Then,

(i) $P_a(b) \equiv 0 \pmod p$ for all $a \neq b \in \mathcal{F}$,

(ii) $P_a(a) \not\equiv 0 \pmod p$ for all $a \in \mathcal{F}$.

Let Q_a denote the polynomial obtained from P_a by developing it and repeatedly replacing the product X_i^2 by 1 for each $i = 1, \ldots, n$. Hence, $Q_a(x) = P_a(x)$ for all $x \in \{\pm 1\}^n$. Therefore, Q_a also satisfies the relations (i),(ii) above. These relations permit to check that the set $\{Q_a \mid a \in \mathcal{F}\}$ is linearly independent over the field $GF(p)$. Hence, $|\mathcal{F}|$ is less than or equal to the dimension of the space of polynomials in $n-1$ variables (as $x_1 = 1$) of degree at most $p-1$ over $GF(p)$, which is precisely $\sum_{i=0}^{p-1}\binom{n-1}{i}$. ∎

We now show that the set X of all even cut vectors cannot be partitioned into $d+1$ subsets of smaller diameter. It turns out to be more convenient to work with ± 1-valued vectors rather than with the $(0,1)$-valued cut vectors. In other words, we show that the set

$$X_1 := \{xx^T \mid x \in \mathcal{E}\}$$

cannot be partitioned into $d+1$ subsets of smaller diameter if the condition (31.1.1) holds ($\mathcal{E}$ is defined as in Lemma 31.1.2). (Note that xx^T is the $n \times n$ symmetric matrix with entries $x_i x_j$ and, thus, all its diagonal entries are equal to 1. Hence, the vectors xx^T ($x \in \{\pm 1\}^n$) lie, in fact, in the space of dimension d.) Given $x, y \in \mathcal{E}$, we have

$$(\| xx^T - yy^T \|_2)^2 = 2n^2 - 2(x^T y)^2 \leq 2n^2$$

with equality if $x^T y = 0$. Hence, the diameter of X_1 is equal to $n\sqrt{2}$. Suppose that X_1 is partitioned into s subsets $Y^1 \cup \ldots \cup Y^s$, where each Y^i has diameter $< n\sqrt{2}$. Then, no two vectors in Y^i are orthogonal. We deduce from Lemma 31.1.2

that $|Y^i| \leq \sum_{j=0}^{p-1} \binom{n-1}{j}$ for all i. This implies that $2^{n-2} \leq s \sum_{j=0}^{p-1} \binom{n-1}{j}$. Therefore, the condition (31.1.1) implies that $s > \binom{n}{2} + 1 = d + 1$. This shows that, under the condition (31.1.1), the set X_1 (or X) cannot be partitioned into $d+1$ subsets of smaller diameter.

31.2 Inequalities for Angles of Vectors

Let $v_1, \ldots, v_n$ be n unit vectors in $\mathbb{R}^m$ ($m \geq 1$). Set

$$\theta_{ij} := \arccos(v_i^T v_j) \quad \text{for } 1 \leq i < j \leq n.$$

We consider the question of determining valid inequalities that are satisfied by the angles θ_{ij}. A classical result in 3-dimensional geometry asserts that

$$\theta_{12} \leq \theta_{13} + \theta_{23}, \;\; \theta_{13} \leq \theta_{12} + \theta_{23}, \;\; \theta_{23} \leq \theta_{12} + \theta_{13}, \;\; \theta_{12} + \theta_{13} + \theta_{23} \leq 2\pi$$

for the pairwise angles among three vectors in $\mathbb{R}^3$ (see Theorem 31.2.2 below). Observe that the above inequalities are nothing but the triangle inequalities (for the variable $\frac{\theta}{\pi}$). An analogue result holds in any dimension $m \geq 3$, as was shown in Theorem 6.4.5. We repeat the result here for convenience.

Theorem 31.2.1. *Let $v_1, \ldots, v_n$ be n unit vectors in $\mathbb{R}^m$ ($n \geq 3$, $m \geq 1$). Let $a \in \mathbb{R}^{E_n}$ and $a_0 \in \mathbb{R}$ such that the inequality $a^T x \leq a_0$ is valid for the cut polytope $\mathrm{CUT}_n^{\square}$. Then,*

$$\sum_{1 \leq i < j \leq n} a_{ij} \arccos(v_i^T v_j) \leq \pi a_0.$$

∎

Therefore, the valid inequalities for the cut polytope $\mathrm{CUT}_n^{\square}$ have the following nice interpretation: They yield valid inequalities for the pairwise angles among a set of n unit vectors. A whole wealth of such inequalities have been presented in the preceding paragraphs. As an example,

$$\sum_{1 \leq i < j \leq n} \arccos(v_i^T v_j) \leq \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \pi$$

for any n unit vectors $v_1, \ldots, v_n$. The question of determining the maximum value for the sum of pairwise angles among a set of vectors was first asked by Fejes Tóth [1959]; he conjectured that the above inequality holds and proved that this is the case for $n \leq 6$. The even case $n = 2p$ was settled by Sperling [1960] and the general case by Kelly [1970b].

In case $n = 3$ the statement from Theorem 31.2.1 can, in fact, be formulated as an equivalence[2].

Theorem 31.2.2. *The following assertions are equivalent for $\alpha, \beta, \gamma \in [0, \pi]$.*

[2]This fact has been known since long; see, e.g., Blumenthal [1953] (Lemma 43.1), or Berger [1987] (Corollary 18.6.10) or, more recently, Barrett, Johnson and Tarazaga [1993].

(i) *The matrix*

$$A := \begin{pmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{pmatrix}$$

is positive semidefinite.

(ii) *There exist three unit vectors* $v_1, v_2, v_3 \in \mathbb{R}^3$ *such that* $\alpha = \arccos(v_1^T v_2)$, $\beta = \arccos(v_1^T v_3)$ *and* $\gamma = \arccos(v_2^T v_3)$.

(iii) $\alpha \leq \beta + \gamma$, $\beta \leq \alpha + \gamma$, $\gamma \leq \alpha + \beta$ *and* $\alpha + \beta + \gamma \leq 2\pi$.

Proof. Clearly, (i) $\iff$ (ii). Now, $\det A$ can be expressed as:

$$\begin{aligned} \det A &= 1 + 2\cos\alpha \cdot \cos\beta \cdot \cos\gamma - \cos^2\alpha - \cos^2\beta - \cos^2\gamma \\ &= (1 - \cos^2\beta - \cos^2\gamma + \cos^2\beta \cdot \cos^2\gamma) \\ &\quad - (\cos^2\alpha + \cos^2\beta \cdot \cos^2\gamma - 2\cos\alpha \cdot \cos\beta \cdot \cos\gamma) \\ &= (1 - \cos^2\beta)(1 - \cos^2\gamma) - (\cos\alpha - \cos\beta \cdot \cos\gamma)^2 \\ &= \sin^2\beta \cdot \sin^2\gamma - (\cos\alpha - \cos\beta \cdot \cos\gamma)^2 \\ &= (\cos(\beta - \gamma) - \cos\alpha) \cdot (\cos\alpha - \cos(\beta + \gamma)) \end{aligned}$$

Hence, $A \succeq 0 \iff \det A \geq 0 \iff |\beta - \gamma| \leq \alpha \leq \beta + \gamma \leq 2\pi - \alpha$, which is equivalent to (iii). ∎

Some generalizations of this result will be presented in the next subsection; see, in particular, Theorem 31.3.7.

31.3 The Positive Semidefinite Completion Problem

We consider here the elliptope $\mathcal{E}_n$ and its projections on subsets of the entries. We recall from Section 28.4 that

$$\mathcal{E}_n = \{Y \ n \times n \text{ symmetric matrix} \mid Y \succeq 0,\ y_{ii} = 1 \ \forall i = 1, \ldots, n\}.$$

Given a subset E of $E_n := \{ij \mid 1 \leq i < j \leq n\}$, consider the graph $G := (V_n, E)$ and the projection $\mathcal{E}(G)$ of $\mathcal{E}_n$ on the subspace $\mathbb{R}^E$, i.e.,

$$\mathcal{E}(G) := \{x \in \mathbb{R}^E \mid \exists Y = (y_{ij}) \in \mathcal{E}_n \text{ such that } x_{ij} = y_{ij} \ \forall ij \in E\}.$$

Hence, $\mathcal{E}_n$ and $\mathcal{E}(K_n)$ are in one-to-one correspondence as the elements of $\mathcal{E}(K_n)$ are precisely the upper triangular parts of the matrices in $\mathcal{E}_n$.

Given a graph $G = (V_n, E)$ and $x \in \mathbb{R}^E$, denote by X the partial symmetric $n \times n$ matrix whose off-diagonal entries are specified only on the positions corresponding to edges in G (and the symmetric ones); the ijth-entry of X is x_{ij} for $ij \in E$ and the diagonal entries of X are all equal to 1. Then, $x \in \mathcal{E}(G)$ if and only if the partial matrix X can be completed to a positive semidefinite matrix. Hence, the positive semidefinite completion problem, which was introduced in Section 28.4, is the problem of testing membership in the elliptope $\mathcal{E}(G)$.

This problem has received a lot of attention in the literature, especially within the community of linear algebra. This is due, in particular, to its many applications (e.g., to probability and statistics, engineering, etc.) and to its close connection with other important matrix properties such as Euclidean distance matrices. (See, e.g., the survey of Johnson [1990] for a broad survey on completion problems.) We present here some results about the positive semidefinite completion problem that are most relevant to the topic of this book, namely, to cut and semimetric polyhedra. Indeed, it turns out that, for some graphs, the elliptope $\mathcal{E}(G)$ has a closed form description involving the cut and semimetric polytopes of G. We give here a compact presentation covering results obtained by several authors. The exposition in this section as well as in the next Section 31.4 follows essentially the survey paper by Laurent [1997d].

31.3.1 Results

Let $G = (V_n, E)$ be a graph and let $x \in \mathbb{R}^E$ with corresponding partial matrix X. Clearly, if $x \in \mathcal{E}(G)$ then every principal submatrix of X whose entries are all specified is positive semidefinite. In other words, if $K \subseteq V_n$ induces a clique in G then the projection x_K of x on the edge set of $G[K]$ belongs to the elliptope $\mathcal{E}(K)$ of the clique K. (Here, we use the same letter K for denoting the clique as a node set or as a graph.) Hence,

(31.3.1) $\quad x_K \in \mathcal{E}(K)$ for each clique K in G

is a necessary condition for $x \in \mathcal{E}(G)$, called *clique condition.* Another necessary condition for membership in $\mathcal{E}(G)$ can be deduced from the result in Section 31.2. Clearly, all the components of $x \in \mathcal{E}(G)$ belong to the interval $[-1, 1]$; hence, x can be parametrized as

$$x = \cos(\pi a), \quad \text{i.e., } x_e = \cos(\pi a_e) \text{ for all } e \in E,$$

where $0 \leq a_e \leq 1$ for all $e \in E$. Then, Theorem 31.2.1 can be reformulated as

$$\mathcal{E}(K_n) \subseteq \cos(\pi \mathrm{CUT}_n^\square) := \{\cos(\pi a) \mid a \in \mathrm{CUT}_n^\square\}.$$

By taking the projections of both sides on the subspace $\mathbb{R}^E$ indexed by the edge set of G, we obtain

$$\mathcal{E}(G) \subseteq \cos(\pi \mathrm{CUT}^\square(G)) := \{\cos(\pi a) \mid a \in \mathrm{CUT}^\square(G)\}.$$

In other words,

(31.3.2) $\quad a \in \mathrm{CUT}^\square(G)$

is a necessary condition for $x = \cos(\pi a) \in \mathcal{E}(G)$, called *cut condition.* As $\mathrm{CUT}^\square(G) \subseteq \mathrm{MET}^\square(G)$ (by (27.3.1)) we deduce that

(31.3.3) $\quad a \in \mathrm{MET}^\square(G)$

is also a necessary condition for $x = \cos(\pi a) \in \mathcal{E}(G)$, called *metric condition.*

None of the conditions (31.3.1), (31.3.2), or (31.3.3) suffices for characterizing $\mathcal{E}(G)$ in general. For instance, let $C = (V_n, E)$ be a circuit on $n \geq 4$ nodes and let $x \in \mathbb{R}^E$ be defined by $x_e := 1$ for all edges except $x_e := -1$ for one edge of C. Then, x satisfies (31.3.1) but $x \notin \mathcal{E}(C)$. As another example, consider the 4×4 matrix X with diagonal entries 1 and with off-diagonal entries $-\frac{1}{2}$. Then, $X \notin \mathcal{E}_4$ (as X is not positive semidefinite because $Xe = -\frac{1}{2}e$, where e denotes the all ones vector). Hence, the vector $x := (-\frac{1}{2}, \ldots, -\frac{1}{2}) \in \mathbb{R}^{E(K_4)}$ does not belong to $\mathcal{E}(K_4)$, while $\frac{1}{\pi}\arccos x = (\frac{2}{3}, \ldots, \frac{2}{3})$ belongs to $\mathrm{MET}^{\square}(K_4) = \mathrm{CUT}^{\square}(K_4)$.

Hence arises the question of characterizing the graphs G for which the conditions (31.3.1), (31.3.2), (31.3.3) (taken together or separately) suffice for the description of $\mathcal{E}(G)$. Let $\mathcal{P}_K$ (resp. $\mathcal{P}_M$, $\mathcal{P}_C$) denote the class of graphs G for which the clique condition (31.3.1) (resp. the metric condition (31.3.3), the cut condition (31.3.2)) is sufficient for the description of $\mathcal{E}(G)$.

We start with the description of the class $\mathcal{P}_K$. Recall that a graph is said to be *chordal* if every circuit of length ≥ 4 has a chord. We will also use the following characterization from Dirac [Di61]: A graph is chordal if and only if it can be obtained from cliques by means of clique sums.

Clearly, every graph $G \in \mathcal{P}_K$ must be chordal. (For, suppose that C is a chordless circuit in G of length ≥ 4; define $x \in \mathbb{R}^E$ by setting $x_e := 1$ for all edges e in C except $x_{e_0} := -1$ for one edge e_0 in C, and $x_e := 0$ for all remaining edges in G. Then, x satisfies (31.3.1) but $x \notin \mathcal{E}(G)$.) Grone, Johnson, Sá, and Wolkowicz [1984] show that $\mathcal{P}_K$ consists precisely of the chordal graphs. Namely,

Theorem 31.3.4. *For a graph $G = (V, E)$, we have*

$$\mathcal{E}(G) = \{x \in \mathbb{R}^E \mid x_K \in \mathcal{E}(K) \ \ \forall K \text{ clique in } G\}$$

if and only if G is chordal.

The proof relies upon Lemma 31.3.5 below, since cliques belong trivially to $\mathcal{P}_K$ and every chordal graph can be build from cliques by taking clique sums.

Lemma 31.3.5. *The class $\mathcal{P}_K$ is closed under taking clique sums.*

Proof. Let $G = (V, E)$ be the clique sum of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Suppose that $G_1, G_2 \in \mathcal{P}_K$; we show that $G \in \mathcal{P}_K$. For this, let $x \in \mathbb{R}^E$ such that $x_K \in \mathcal{E}(K)$ for every clique K in G. Then, for $i = 1, 2$, the projection of x on the subspace $\mathbb{R}^{E_i}$ belongs to $\mathcal{E}(G_i)$ and, thus, can be completed to a positive semidefinite matrix of order $|V_i|$. Hence, we can find vectors $u_j \in \mathbb{R}^k$ ($j \in V_1$) and $v_j \in \mathbb{R}^k$ ($j \in V_2$) such that $x_{ij} = u_i^T u_j$ for all $i, j \in V_1$ and $x_{ij} = v_i^T v_j$ for all $i, j \in V_2$. Now, by looking at the values on the common clique $V_1 \cap V_2$, we have that $u_i^T u_j = v_i^T v_j$ for all $i, j \in V_1 \cap V_2$. Hence, there exists an orthogonal $k \times k$ matrix A such that $Au_i = v_i$ for all $i \in V_1 \cap V_2$.

Now, the Gram matrix of the system of vectors: Au_i ($i \in V_1$), v_i ($i \in V_2 \setminus V_1$) provides a positive semidefinite completion of x, which shows that $x \in \mathcal{E}(G)$. ∎

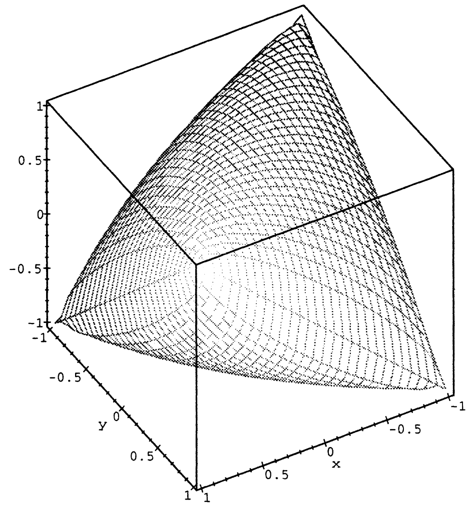


Figure 31.3.6: The elliptope $\mathcal{E}(K_3)$ of the complete graph on 3 nodes

We now turn to the description of the classes $\mathcal{P}_M$ and $\mathcal{P}_C$. Obviously,

$$\mathcal{P}_M \subseteq \mathcal{P}_C.$$

By Theorem 31.2.2 the graph K_3 belongs to $\mathcal{P}_M$. In other words, $\mathcal{E}(K_3) = \cos(\pi \mathrm{MET}_3^\square)$. Thus, $\mathcal{E}(K_3)$ is a 'deformation' via the cosine mapping of the 3-dimensional simplex $\mathrm{MET}_3^\square$; see Figure 31.3.6 for a picture of the elliptope $\mathcal{E}(K_3)$. As was observed earlier, the graph K_4 does not belong to $\mathcal{P}_C$. Laurent [1997b] shows that the classes $\mathcal{P}_M$ and $\mathcal{P}_C$ are identical and consist precisely of the graphs with no K_4-minor.

Theorem 31.3.7. *The following assertions are equivalent for a graph G:*

(i) $\mathcal{E}(G) = \{x = \cos(\pi a) \mid a \in \mathrm{CUT}^\square(G)\}$.

(ii) $\mathcal{E}(G) = \{x = \cos(\pi a) \mid a \in \mathrm{MET}^\square(G)\}$.

(iii) *G has no K_4-minor.*

The proof relies essentially upon the following decomposition result for graphs with no K_4-minor[3] [4] (see Duffin [1965]): A graph G has no K_4-minor if and

[3]A graph with no K_4-minor is also known under the name of (simple) *series-parallel graph.* We stress 'simple' as series-parallel graphs are allowed in general to contain loops and multiple edges. But, here, we consider only simple graphs.

[4]From this follows that every graph with no K_4-minor is a subgraph of a chordal graph (on the same node set) containing no clique of size 4.

only if $G = K_3$, or G is a subgraph of a clique k-sum ($k = 0, 1, 2$) of two smaller graphs (i.e., with less nodes than G), each having no K_4-minor. We state two intermediary results.

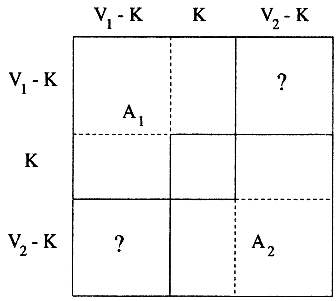


Figure 31.3.8

Lemma 31.3.9. *Each of the classes $\mathcal{P}_M$ and $\mathcal{P}_C$ is closed under taking minors.*

Proof. Let $G = (V, E)$ be a graph on $n = |V|$ nodes, let $e = uv$ be an edge in G and let G' be the graph obtained from G by deleting or contracting the edge e. We show that $G' \in \mathcal{P}_M$ (resp. $G' \in \mathcal{P}_C$) whenever $G \in \mathcal{P}_M$ (resp. $G \in \mathcal{P}_C$).

We first consider the case when $G' = G \backslash e$ is obtained by deleting e. We suppose first that $G \in \mathcal{P}_M$; we show that $G' \in \mathcal{P}_M$. For this, let $a \in \mathrm{MET}^{\square}(G')$; we show that $\cos(\pi a) \in \mathcal{E}(G')$. Let $b \in \mathrm{MET}^{\square}(G)$ whose projection on the edge set of G' is a. Then, $\cos(\pi b) \in \mathcal{E}(G)$ as $G \in \mathcal{P}_M$, which implies that its projection $\cos(\pi a)$ on the edge set of G' belongs to $\mathcal{E}(G')$.

Suppose now that $G \in \mathcal{P}_C$; we show that $G' \in \mathcal{P}_C$. The reasoning is similar. Indeed, if $a \in \mathrm{CUT}^{\square}(G')$, let $b \in \mathrm{CUT}^{\square}(G)$ whose projection on the edge set of G' is a; then, $\cos(\pi b) \in \mathcal{E}(G)$ which implies that $\cos(\pi a) \in \mathcal{E}(G')$.

We consider now the case when $G' = G/e$ is obtained by contracting edge e. Let w denote the node of G' obtained by contraction of edge $e = uv$. The proof is based on the following simple observation: Given $a \in \mathbb{R}^{E'}$ define $b \in \mathbb{R}^{E}$ by setting $b_{uv} := 0$, $b_{iu} := a_{iw}$ if i is adjacent to u in G, $b_{iv} := a_{iw}$ if i is adjacent to v in G, and $b_f := a_f$ for all remaining edges f of G. Then, $b \in \mathrm{MET}^{\square}(G)$ (resp. $b \in \mathrm{CUT}^{\square}(G)$) whenever $a \in \mathrm{MET}^{\square}(G')$ (resp. $a \in \mathrm{CUT}^{\square}(G')$). Suppose that $G \in \mathcal{P}_M$, let $a \in \mathrm{MET}^{\square}(G')$ and let $b \in \mathrm{MET}^{\square}(G')$ be defined as above. Then, $\cos(\pi b) \in \mathcal{E}(G)$. Hence, there exists a matrix $B \in \mathcal{E}_n$ extending $\cos(\pi b)$. If A denotes the matrix obtained from B by deleting the row and column indexed by u and renaming v as w, then $A \in \mathcal{E}_{n-1}$ and A extends $\cos(\pi a)$, which shows that $\cos(\pi a) \in \mathcal{E}(G')$. The proof is identical in the case of $\mathcal{P}_C$. ∎

Lemma 31.3.10. *The class $\mathcal{P}_M$ is closed under taking clique sums.*

Proof. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs in $\mathcal{P}_M$ such that $K := V_1 \cap V_2$ induces a clique in both G_1 and G_2 and there are no edges between

$V_1 \setminus V_2$ and $V_2 \setminus V_1$. Let $G = (V_1 \cup V_2, E_1 \cup E_2)$ denote their clique sum. We show that $G \in \mathcal{P}_M$. For this, let $a \in \mathrm{MET}^\square(G)$. The projection a_i of a on $\mathbb{R}^{E_i}$ belongs to $\mathrm{MET}^\square(G_i)$, which implies that $\cos(\pi a_i) \in \mathcal{E}(G_i)$ for $i = 1, 2$. Hence, there exists a matrix $A_i \in \mathcal{E}_{n_i}$ ($n_i := |V_i|$) extending $\cos(\pi a_i)$. Consider the partial symmetric matrix M shown in Figure 31.3.8, whose entries m_{uv} ($u \in V_1 \setminus V_2$, $v \in V_2 \setminus V_1$) remain to be specified. Hence, the entries of M are specified on the graph H defined as the clique sum (along K) of two complete graphs with respective node sets V_1 and V_2. As H is chordal, we deduce from Theorem 31.3.4 that M can be completed to a positive semidefinite matrix. This shows that $\cos(\pi a) \in \mathcal{E}(G)$ as M extends $\cos(\pi a)$. ∎

Proof of Theorem 31.3.7. As $\mathcal{P}_M \subseteq \mathcal{P}_C$, it suffices to verify that a graph in $\mathcal{P}_C$ has no K_4-minor and that a graph with no K_4-minor belongs to $\mathcal{P}_M$. The statement that a graph in $\mathcal{P}_C$ has no K_4-minor follows from Lemma 31.3.9 and the fact that $K_4 \notin \mathcal{P}_C$. Conversely, suppose that G has no K_4-minor. We show that $G \in \mathcal{P}_M$ by induction on the number of nodes. If $G = K_3$ then $G \in \mathcal{P}_M$ by Theorem 31.2.2. Otherwise, G is a subgraph of a clique sum of two smaller graphs G_1 and G_2 with no K_4-minors. Now, G_1 and G_2 belong to $\mathcal{P}_M$ by the induction assumption. This implies that $G \in \mathcal{P}_M$, using Lemmas 31.3.9 and 31.3.10. ∎

Let us now consider the class $\mathcal{P}_{KM}$ (resp. $\mathcal{P}_{KC}$) consisting of the graphs G for which the clique and metric conditions (31.3.1), (31.3.3) (resp. the clique and cut conditions (31.3.1), (31.3.2)) taken together suffice for the description of $\mathcal{E}(G)$. In view of the above results, it suffices here to assume that the clique condition (31.3.1) holds for all cliques of size ≥ 4. Obviously,

$$\mathcal{P}_{KM} \subseteq \mathcal{P}_{KC}.$$

In fact, the two classes $\mathcal{P}_{KM}$ and $\mathcal{P}_{KC}$ coincide. Several equivalent characterizations for the graphs in this class are known; they are presented below. First, we need some definitions.

Call *splitting* the converse operation to that of contracting an edge; hence, splitting a node u in a graph means replacing u by two adjacent nodes u' and u'' and replacing every edge uv in an arbitrary manner, either by $u'v$, or by $u''v$ (but in such a way that each of u' and u'' is adjacent to at least one node). (This operation can be seen as a special case of the splitting operation defined in Section 26.5.) See Figure 31.3.12 for an example. *Subdividing* an edge $e = uv$ means inserting a new node w and replacing edge e by the two edges uw and wv. Hence, this is a special case of splitting. A graph that can be constructed from a given graph G by subdividing its edges is called a *homeomorph* of G. Note that splitting a node of degree 2 or 3 amounts to subdividing one of the edges incident to that node. (Therefore, homeomorphs of K_4 and splittings of K_4 are the same notions; in particular, a graph has no K_4-minor if and only if it contains no homeomorph of K_4 as a subgraph.) Figure 31.3.11 shows a homeomorph of K_4; the dotted lines indicate paths.

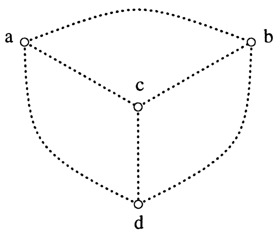
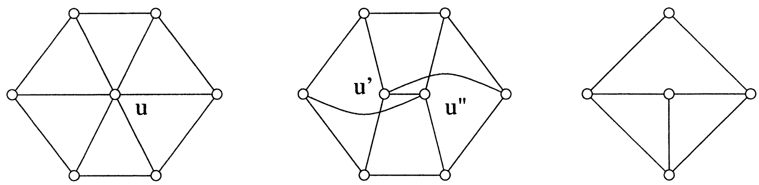


Figure 31.3.11: A homeomorph of K_4

Let $W_n := \nabla C_{n-1}$ denote the *wheel* on n nodes, obtained by adding a new node adjacent to all nodes of a circuit of length $n-1$. Hence, $W_4 = K_4$. Figure 31.3.12 (a) shows the wheel W_7 and (c) shows the graph $\widehat{W_4}$ obtained from K_4 by splitting one node. Clearly, W_n ($n \geq 5$) and any splitting of W_n ($n \geq 4$) do not belong to $\mathcal{P}_{KC}$ (by Theorem 31.3.7, since these graphs do not contain cliques of size 4 while having a K_4-minor).



(a) (b) (c)

Figure 31.3.12: (a) The wheel W_7; (b) Splitting node u in W_7; (c) The graph $\widehat{W_4}$

Several equivalent characterizations for the graphs in $\mathcal{P}_{KM}$ have been discovered by Barrett, Johnson and Loewy [1996]; more precisely, they show the equivalence of assertions (i), (iii), (iv), (v) in Theorem 31.3.13 below. Building upon their result, Johnson and McKee [1996] show the equivalence of (i) and (vi); in other words, the graphs in $\mathcal{P}_{KM}$ arise from the graphs in $\mathcal{P}_K$ and $\mathcal{P}_M$ by taking clique sums. Laurent [1997c] observes moreover the equivalence of (i) and (ii); hence, the two classes $\mathcal{P}_{KM}$ and $\mathcal{P}_{KC}$ coincide even though the cut condition (31.3.2) is stronger than the metric condition (31.3.3). We delay the proof of the next result till Section 31.3.2.

Theorem 31.3.13. *The following assertions are equivalent for a graph G:*

(i) *$G \in \mathcal{P}_{KM}$, i.e., $\mathcal{E}(G)$ consists of the vectors $x = \cos(\pi a)$ such that $a \in \mathrm{MET}^{\square}(G)$ and $x_K \in \mathcal{E}(K)$ for every clique K in G.*

(ii) *$G \in \mathcal{P}_{KC}$, i.e., $\mathcal{E}(G)$ consists of the vectors $x = \cos(\pi a)$ such that $a \in \mathrm{CUT}^{\square}(G)$ and $x_K \in \mathcal{E}(K)$ for every clique K in G.*

(iii) *No induced subgraph of G is W_n ($n \geq 5$) or a splitting of W_n ($n \geq 4$).*

(iv) *Every induced subgraph of G that contains a homeomorph of K_4 contains a clique of size* 4.

(v) *There exists a chordal graph G' containing G as a subgraph and having no new clique of size* 4.

(vi) *G can be obtained by means of clique sums from chordal graphs and graphs with no K_4-minor.*

We close the section with a result concerning the graphs whose elliptope is a polytope. It turns out that this occurs only in the most trivial case, when $\mathcal{E}(G) = [-1,1]^E$. Set

$$Q_n := \mathrm{Conv}(xx^T \mid x \in \{\pm 1\}^n)$$

and, for a graph $G = (V_n, E)$, let $Q(G)$ denote the projection of Q_n on the subspace $\mathbb{R}^E$ indexed by the edge set of G. Hence, Q_n (resp. $Q(G)$) is nothing but the image of the cut polytope $\mathrm{CUT}_n^\square$ (resp. $\mathrm{CUT}^\square(G)$) under the mapping $x \mapsto 1 - 2x$. Clearly,

$$Q_n \subseteq \mathcal{E}_n, \quad Q(G) \subseteq \mathcal{E}(G).$$

The following result of Laurent [1997b] characterizes the graphs for which equality $Q(G) = \mathcal{E}(G)$ holds; its proof is along the same lines as that of Theorem 31.3.7.

Theorem 31.3.14. *For a graph G, equality $Q(G) = \mathcal{E}(G)$ holds if and only if G has no K_3-minor, i.e., if G is a forest. Then, $\mathcal{E}(G) = [-1,1]^E$.* ∎

As the class of graphs G for which $\mathcal{E}(G)$ is a polytope is closed under taking minors, we deduce:

Corollary 31.3.15. *The elliptope $\mathcal{E}(G)$ of a graph G is a polytope if and only if G is a forest; then, $\mathcal{E}(G) = [-1,1]^E$.* ∎

31.3.2 Characterizing Graphs with Excluded Induced Wheels

We give here the full proof[5] of Theorem 31.3.13, which states several equivalent characterizations for the graphs containing no splittings of wheels as induced subgraphs. We show the following implications:

$$\text{(ii)} \Longrightarrow \text{(iii)} \Longrightarrow \text{(iv)} \Longrightarrow \text{(v)} \Longrightarrow \text{(i)} \ \text{and} \ \text{(i)} \Longleftrightarrow \text{(vi)},$$

the implication (i) $\Longrightarrow$ (ii) being obvious.

[5] The proof given here follows the exposition in Laurent [1997d]. It is based essentially on the original proofs of Barrett, Johnson and Loewy [1996] and Johnson and McKee [1996] . However, several parts have been simplified and shortened; in particular, the implications (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (i).

The following notion of 'path avoiding a clique' will be useful in the proof. Let $G = (V, E)$ be a graph, let K be a clique in G and let $a \in K$, $x \in V \setminus K$. A path P joining the two nodes a and x is said to *avoid the clique* K if P contains no other node of K besides a.

We start with some preliminary results.

Lemma 31.3.16. *The class $\mathcal{P}_{KC}$ is closed under taking induced subgraphs.*

Proof. Suppose $G = (V, E)$ belongs to $\mathcal{P}_{KC}$ and let $H = G[U]$ be an induced subgraph of G, where $U \subseteq V$. We show that $H \in \mathcal{P}_{KC}$. Let x be a vector indexed by the edge set of H satisfying (31.3.1) and (31.3.2); we show that $x \in \mathcal{E}(H)$. For this we extend x to a vector y indexed by the edge set of G by setting $y_{uv} := 0$ for an edge $uv \in E$ with $u \in U$, $v \in V \setminus U$ and $y_{uv} := 1$ for an edge $uv \in E$ contained in $V \setminus U$. It is clear that y satisfies (31.3.1). By assumption, $a := \frac{1}{\pi} \arccos x \in \mathrm{CUT}^{\square}(H)$; we verify that $b := \frac{1}{\pi} \arccos y \in \mathrm{CUT}^{\square}(G)$. Indeed, say

$$a = \sum_{S \subseteq U} \lambda_S \delta_H(S)$$

where $\lambda_S \geq 0$, $\sum_S \lambda_S = 1$. Then,

$$b = \frac{1}{2} \sum_{S \subseteq U} \lambda_S \left(\delta_G(S) + \delta_G(U \setminus S) \right),$$

which shows that $b \in \mathrm{CUT}^{\square}(G)$. Hence, y satisfies (31.3.2). Therefore, $y \in \mathcal{E}(G)$ which implies that $x \in \mathcal{E}(H)$. ∎

Lemma 31.3.17. *The class $\mathcal{P}_{KM}$ is closed under taking clique sums.* ∎

We omit the proof which is analogue to that of Lemma 31.3.10.

Lemma 31.3.18. *Let $G = (V, E)$ be a graph in which every induced subgraph containing a homeomorph of K_4 also contains a clique of size 4. Let K be a clique in G with $|K| \geq 4$, let $a, b, c \in K$, $v \in V \setminus K$, and let P_a (resp. P_b, P_c) be a path from a (resp. from b, c) to v avoiding the clique K. Then, there exists a node $w \in V \setminus K$ lying on one of the paths P_a, P_b or P_c which is adjacent to all three nodes a, b and c.*

Proof. Let W denote the set of nodes lying on the paths P_a, P_b or P_c. Clearly, there is a path avoiding K from every node $w \in W$ to each node in $\{a, b, c\}$. For $w \in W$, define $d(w)$ as the smallest sum $|Q_a| + |Q_b| + |Q_c|$, where Q_a, Q_b, Q_c are paths avoiding K that join w to a, b, c, respectively, in the graph $G[W]$. Suppose w is a node in W for which $d(w)$ is minimum and let Q_a, Q_b, Q_c be the corresponding paths, as defined above. Let $W_0 \subseteq W$ denote the set of nodes lying on Q_a, Q_b or Q_c. Then,

$$V(Q_a) \cap V(Q_b) = V(Q_a) \cap V(Q_c) = V(Q_b) \cap V(Q_c) = \{w\}.$$

Indeed, if z is a node in $V(Q_a) \cap V(Q_b)$ distinct from w, then it is easy to see that $d(z) < d(w)$. Hence, the three paths Q_a, Q_b, Q_c together with the edges ab, ac and bc form a homeomorph of K_4 contained in $G[W_0]$. By the assumption, $G[W_0]$ must contain a clique S of size 4. We show that

$$S = \{w, a, b, c\}.$$

Suppose that $w \notin S$. Then, S contains two nodes r, s that lie on a common path, say, on Q_a; say, w, r, s, a lie in that order along Q_a. Let $t \in S \setminus \{r, s\}$. We can suppose that t lies on Q_b (as t does not lie on Q_a, by minimality of $d(w)$). Then, $t = b$ (else, we would have $d(t) < d(w)$). Hence, S is of the form $\{r, s, b, c\}$ which implies that $d(r) < d(w)$, a contra diction. Therefore, the set S contains w; so, $S = \{w, r, s, t\}$ where r, s, t lie on Q_a, Q_b, Q_c, respectively. Now, $r = a$ (else, $d(r) < d(w)$); similarly, $s = b$ and $t = c$. This shows that $S = \{w, a, b, c\}$. ∎

Proposition 31.3.19. *Let $G = (V, E)$ be a graph satisfying the following conditions:*

(i) *Every induced subgraph of G containing a homeomorph of K_4 contains a clique of size* 4.

(ii) *G contains a clique of size* 4.

(iii) *For every maximal clique K in G, $a \in K$ and $v \in V \setminus K$, there exists a path avoiding K from a to v.*

Then, G is chordal.

Proof. We show the result by induction on the number n of nodes in G. The result holds trivially if $n = 4$ (as $G = K_4$). Let $n \geq 5$ and let K be a maximal clique in G of size ≥ 4. We can assume that the subgraph $G[V \setminus K]$ induced by $V \setminus K$ is connected. (Else, letting $W_1, \ldots, W_p$ denote the connected components of $G[V \setminus K]$, then $G_i := G[K \cup W_i]$ is chordal for each $i = 1, \ldots, p$, by the induction assumption. Hence, G is chordal as it is a clique sum of chordal graphs.) We show that $K = V$, i.e., that G is a complete graph. For this, suppose $K \neq V$. For each $x \in V \setminus K$, let $N(x)$ denote the set of nodes in K that are adjacent to x. We claim:

(a) If $x, y \in V \setminus K$ are adjacent and if $N(x) \not\subseteq N(y), N(y) \not\subseteq N(x)$, then $N(x) \cap N(y) = \emptyset, |N(x)| = |N(y)| = 1$.

Indeed, let $a \in N(x) \setminus N(y)$ and $b \in N(y) \setminus N(x)$. Suppose first that there exists $c \in N(x) \cap N(y)$. Then, the subgraph of G induced by $\{a, b, c, x, y\}$ contains a homeomorph of K_4 but no clique of size 4, contradicting (i). If $|N(x)| \geq 2$, we obtain again a contradiction with (i) by choosing now c in $N(x) \setminus \{a\}$. This shows (a). Next, we have:

(b) If $x \in V \setminus K$ and $|N(x)| = 1$, then $N(x) \subset N(y)$ for some $y \in V \setminus K$.

Say, $N(x) = \{a\}$. Let $b, c \in K \setminus \{a\}$ and let P_a, P_b, P_c be paths from x to a, b, c, respectively, that avoid K. By Lemma 31.3.18, there exists a node $y \in V \setminus K$ lying on one of these paths which is adjacent to a, b and c. Hence, $N(x) \subset N(y)$.

Call a set $N(x)$ ($x \in V \setminus K$) *maximal* if $N(x) = N(y)$ whenever $N(x) \subseteq N(y)$ for $y \in V \setminus K$. We show:

(c) Let $x \neq y \in V \setminus K$ for which $N(x)$ and $N(y)$ are both maximal. Then, $N(x) = N(y)$.

Suppose that $N(x) \neq N(y)$. Then, by (a) and (b), x and y are not adjacent. Let $(x, z_1, \ldots, z_p, y)$ be a path of shortest length joining x and y in $G[V \setminus K]$. Then, $N(z_1) \subseteq N(x)$ and $N(z_p) \subseteq N(y)$. Let us first assume that $N(z_i) \not\subseteq N(x)$ for some $i = 1, \ldots, p$. Let i be the smallest such index. Then, $N(z_1) \cup \ldots \cup N(z_{i-1}) \subseteq N(x)$ and $N(z_i) \not\subseteq N(x)$. Let $a \in N(x) \setminus N(z_i)$ and $b \in N(z_i) \setminus N(x)$. We claim that $N(z_1) \cup \ldots \cup N(z_{i-1}) \subseteq \{a\}$. For, suppose that there exists an element $a' \in N(z_1) \cup \ldots \cup N(z_{i-1})$ with $a' \neq a$. Then, applying Lemma 31.3.18, we find a node $w \in \{x, z_1, \ldots, z_{i-1}, z_i\}$ which is adjacent to all three nodes a, b and a'. This implies that $w = z_j$ ($j < i$) and, thus, $b \in N(z_j) \subseteq N(x)$, a contradiction. Therefore, $N(z_1) \cup \ldots \cup N(z_{i-1}) \subseteq \{a\}$. Let $c \in N(x) \setminus \{a\}$; then the subgraph of G induced by $\{a, b, c, x, z_1, \ldots, z_{i-1}, z_i\}$ contains a homeomorph of K_4 but no clique of size 4, contradicting (i). When $N(z_i) \not\subseteq N(y)$ for some $i = 1, \ldots, p$, we obtain a contradiction in the same manner as above. Hence, we have that $N(z_1) \cup \ldots \cup N(z_p) \subseteq N(x) \cap N(y)$. Taking $a \in N(x) \setminus N(y)$, $b \in N(y) \setminus N(x)$, $c \in N(x) \setminus \{a\}$, the subgraph of G induced by $\{a, b, c, x, z_1, \ldots, z_p, y\}$ contains a homeomorph of K_4 but no clique of size 4, yielding again a contradiction. Hence, (c) holds.

We can now conclude the proof. Let $N(x_0)$ denote the unique maximal set of the form $N(x)$ ($x \in V \setminus K$). Then, $N(x_0) = K$ (by (iii)). Hence, $K \cup \{x_0\}$ is a clique, which contradicts the maximality of K. ∎

Proof of Theorem 31.3.13.

The implication (ii) $\Longrightarrow$ (iii) follows from Lemma 31.3.16 since W_n ($n \geq 5$) and a splitting of W_n ($n \geq 4$) do not belong to $\mathcal{P}_{KC}$. The implication (vi) $\Longrightarrow$ (i) follows from Lemma 31.3.17 and the fact that chordal graphs and graphs with no K_4-minor belong to $\mathcal{P}_{KM}$.

(v) $\Longrightarrow$ (i) Suppose $G = (V, E)$ is a graph satisfying (v). Let $G' = (V, E')$ be a chordal graph such that $E \subseteq E'$ and every clique of size 4 in G' is, in fact, a clique in G. We show that $G \in \mathcal{P}_{KM}$. For this, let $x = \cos(\pi a) \in \mathbb{R}^E$ such that $a \in \mathrm{MET}^\square(G)$ and $x_K \in \mathcal{E}(K)$ for all cliques K in G. Let $b \in \mathrm{MET}^\square(G')$ extending a and set $y := \cos(\pi b)$. Then, y satisfies the clique condition (31.3.1) (as $y_K = x_K \in \mathcal{E}(K)$ for each clique K of size ≥ 4 in G'). As G' is chordal, we deduce that $y \in \mathcal{E}(G')$ and, thus, $x \in \mathcal{E}(G)$.

(iii) $\Longrightarrow$ (iv) Suppose $G = (V, E)$ is a graph for which there exists a subset $U \subseteq V$ such that $G[U]$ contains a homeomorph of K_4 and contains no clique of size 4. Choose such U of minimum cardinality; set $G' := G[U] := (U, E')$. Moreover, let

$H = (W, F)$ be a homeomorph of K_4 contained in G' having minimum number of edges. Then, $W = U$ (by minimality of $|U|$) and $H \neq K_4$ (by assumption). To fix ideas, suppose H is the graph shown in Figure 31.3.11; so, H consists of the six paths P_{ab}, P_{ac}, P_{bc}, P_{ad}, P_{bd} and P_{cd} (where P_{ab} denotes the path joining the nodes a and b, etc.); let us refer to the nodes a, b, c, d as the 'corners' of H.

We show that G' is a wheel or a splitting of a wheel. This is obvious if $|E' \setminus F| \leq 1$. So, we can suppose that $|E' \setminus F| \geq 2$. A first observation is:

(a) The end nodes of an edge $e \in E' \setminus F$ do not lie on a common path in H.

Indeed, suppose that the end nodes x and y of e lie, say, on the path P_{ab}. Let $P_{ab}(x, y)$ denote the subpath of P_{ab} joining x and y. Then, the graph obtained from H by deleting $P_{ab}(x, y)$ and adding the edge e is again a homeomorph of K_4 contained in G' but having less edges than H. This contradicts the minimality of H. Hence, (a) holds.

There are two possibilities for an edge $e = xy \in E' \setminus F$: Either, (I) e lies within a face of H (i.e., x and y lie on two paths in H sharing a common end node) or, (II) e connects two disjoint paths in H. We make two observations:

(b) Let $e = xy \in E' \setminus F$ where x, y are internal nodes in P_{ab}, P_{cd}, respectively. Then, $|P_{ac}| = |P_{bc}| = |P_{ad}| = |P_{bd}| = 1$.

Indeed, suppose $|P_{ac}| > 1$. Then, the graph obtained from H by adding e and deleting P_{ac} is a homeomorph of K_4 (with corners x, y, b, d) contained in G' with less edges than H. Similarly,

(c) Let $e = xy \in E' \setminus F$ lying in a face of H. Say, x, y lie on P_{ab}, P_{ac}, respectively. Then, (ci) $xa, ya \in E, |P_{bc}| = |P_{bd}| = |P_{cd}| = 1$, or (cii) $y = c, |P_{ac}| = |P_{bc}| == |P_{cd}| = 1$, or (ciii) $x = b, |P_{ab}| = |P_{bc}| = |P_{bd}| = 1$.

Suppose first that there exists an edge $e \in E' \setminus F$ of type (II). Say, $e = xy$ where x, y are internal nodes on P_{ab}, P_{cd}, respectively. Let $e' = x'y'$ be another edge in $E' \setminus F$. Then, e' is of type (I). (Indeed, if e' is of type (II) then e' connects the same paths P_{ab} and P_{cd} - this follows from (b) and the fact that $H \neq K_4$. Say, $x \neq x'$ and d, y', y, c lie in that order along P_{cd}. Then, adding e, e' to H and deleting P_{ad}, P_{bd} and the subpath $P_{cd}(d, y')$ creates a homeomorph of K_4 with less edges than H.) We can suppose without loss of generality that e' lies within the face of H containing a, b, c. By (c), e' is of the form cz where z lies on P_{ab}. Say, z lies between a and x. Then, adding e, e' to H and deleting P_{ad}, P_{ac} and $P_{ab}(a, z)$ creates a smaller homeomorph of K_4 than H.

Hence, we can now suppose that every edge in $E' \setminus F$ is of type (I), i.e., lies within a face of H. If $E' \setminus F$ contains an edge as in (ci), then it is easy to see that one can always find a smaller homeomorph of K_4 in G'. Hence, we can suppose that all edges in $E' \setminus F$ are as in (cii) or (ciii). Let $e = cx \in E' \setminus F$, where x is an internal node of P_{ab}. This implies easily that every other edge $e' \in E' \setminus F$

is of the form cz, where z lies on P_{ab}, P_{bd} or P_{ad}. Therefore, G' is a wheel or a splitting of a wheel.

(iv) $\Longrightarrow$ (v) Suppose that G satisfies the assumption (iv). We show that (v) holds by induction on the number of nodes in G. We can suppose that G contains a homeomorph of K_4; else, the result holds. By (iv), G has a clique of size 4. We can suppose, moreover, that there exist a maximal clique K in G, $a_0 \in K$, and $x_0 \in V \setminus K$ such that no path avoiding K from a to x exists; for, if not, G is chordal by Proposition 31.3.19 and we are done. Let S denote the set of nodes $b \in K$ for which there exists a path from x_0 to b avoiding K. Moreover, let T denote the set of nodes $x \in V \setminus K$ that can be joined to all nodes of S by some path avoiding K, and that cannot be joined to any other point of $K \setminus S$ by a path avoiding K. Then, $S \neq K$ (as $a_0 \notin S$) and $T \neq \emptyset$ (as $x_0 \in T$). Moreover, there is no edge between T and $(V \setminus K) \setminus T$, or $K \setminus S$. Consider the induced subgraphs $G[S \cup T]$ and $G[V \setminus T]$; both are proper subgraphs of G. By the induction assumption, there exists a chordal graph H_1 (resp. H_2) containing $G[S \cup T]$ (resp. $G[V \setminus T]$) as a subgraph and having no new clique of size 4. Let $H := H_1 \cup H_2$ denote the graph with edge set $E(H_1) \cup E(H_2)$. Then, H contains G as a subgraph. Moreover, H is chordal and H contains no new clique of size 4. This follows from the fact that H is, in fact, the clique sum of the two graphs H_1 and H_2 (along the clique S). Hence, G satisfies (v).

(i) $\Longrightarrow$ (vi). Let G be a graph in $\mathcal{P}_{KM}$. We show that G satisfies (vi) by induction on the number of nodes. We can suppose that G is connected (else, the result follows by induction) and that G contains a homeomorph of K_4. It suffices now to show that G contains a clique cutset, i.e., a clique K such that $G[V \setminus K]$ is disconnected. If G contains a simplicial[6] node v, then the set of neighbors of v forms a clique cutset. Suppose now that G contains no simplicial node. Using the implication (i) $\Longrightarrow$ (iv) (already shown above), we know that G contains a clique of size 4. Let K be a maximal clique in G of size ≥ 4 such that $G[V \setminus K]$ is connected (else, we are done). Observe that, for every $a \in K$ and $x \in V \setminus K$, there exists a path from a to x avoiding K. (Indeed, as a is not a simplicial node, a is adjacent to some node $w \in V \setminus K$. Now, v and w can be joined by some path in $G[V \setminus K]$, which yields a path from v to a avoiding K.) Hence, the graph G satisfies the conditions (i)-(iii) from Proposition 31.3.19. Therefore, G is chordal. This yields a contradiction as every chordal graph contains a simplicial vertex. This concludes the proof for (i) $\Longrightarrow$ (vi). ∎

31.4 The Euclidean Distance Matrix Completion Problem

Let (V_n, d) be a distance space with associated distance matrix D. We remind that D is said to be a Euclidean distance matrix when $(V_n, \sqrt{d})$ is isometrically ℓ_2-embeddable; that is, when d belongs to the negative type cone NEG_n.

[6]A node v in graph G is said to be *simplicial* if its set of neighbors induces a clique in G.

Given a subset E of $E_n = \{ij \mid 1 \leq i < j \leq n\}$, consider the graph $G = (V_n, E)$. Denote by NEG(G) the projection of the negative type cone NEG$_n$ on the subspace $\mathbb{R}^E$ indexed by the edge set E of G. Hence, a vector $d = (d_{ij})_{ij \in E}$ belongs to NEG(G) if and only if there exist vectors $u_1, \ldots, u_n \in \mathbb{R}^m$ (for some $m \geq 1$) such that

$$(31.4.1) \qquad \sqrt{d_{ij}} = \| u_i - u_j \|_2 \quad \text{for all } ij \in E.$$

To $d \in \mathbb{R}^E$ corresponds a partial symmetric $n \times n$ matrix $M = (m_{ij})$ whose entries are specified only on the diagonal positions and on the positions corresponding to edges in E; namely, $m_{ii} := 0$ for all $i \in V_n$ and $m_{ij} = m_{ji} := d_{ij}$ for all $ij \in E$. Then, $d \in$ NEG(G) if the unspecified entries of M can be chosen in such a way that one obtains a Euclidean distance matrix; that is, if M can be completed to a Euclidean distance matrix. Therefore, the completion problem for partial Euclidean distance matrices is that of characterizing membership in projections of the negative type cone.

Barvinok [1995] shows that, for $d \in$ NEG(G), there exists a system of vectors $u_1, \ldots, u_n \in \mathbb{R}^m$ satisfying (31.4.1) in dimension m bounded by

$$(31.4.2) \qquad m \leq \left\lfloor \frac{\sqrt{8|E|+1} - 1}{2} \right\rfloor .$$

A short proof for this fact can be given using Theorem 31.5.3 from the next section.

Proof of relation (31.4.2). For $d \in \mathbb{R}^E$ we have:

$$\begin{gathered} \exists\, u_1, \ldots, u_n \in \mathbb{R}^m \text{ such that } d_{ij} = (\| u_i - u_j \|_2)^2 \text{ for all } ij \in E \\ \Updownarrow \\ \exists \text{ symmetric } n \times n \text{ matrix } A \succeq 0 \text{ with rank} \leq m \text{ such that} \\ d_{ij} = a_{ii} + a_{jj} - 2a_{ij} \text{ for all } ij \in E. \end{gathered}$$

Consider the convex set $K := \{X \mid X \succeq 0,\ x_{ii} + x_{jj} - 2x_{ij} = d_{ij} \text{ for } ij \in E\}$. If $K \neq \emptyset$ (that is, if $d \in$ NEG(G)) and if $d \neq 0$ (then, K has extreme points), then any matrix $A \in K$ which is an extreme point of K has rank r satisfying $\binom{r+1}{2} \leq |E|$ (by Theorem 31.5.3). This condition is equivalent to the inequality in (31.4.2). ∎

We present in this section a closed form description of the projected negative type cone NEG(G) for several classes of graphs. In fact, one can formulate necessary conditions for membership in NEG(G) that are similar to the conditions (31.3.1), (31.3.2) and (31.3.3) considered in Section 31.3 for the positive semidefinite completion problem. Moreover, these conditions are sufficient for precisely the same classes of graphs as those coming up in Section 31.3.

In a first step, we formulate the results concerning the Euclidean distance matrix completion problem. Then, we show how they can be derived from the corresponding results for the positive semidefinite completion problem; here are used essentially the techniques on metric transforms developed in Chapter 9.

The exposition in this section follows again essentially the survey paper of Laurent [1997d].

31.4.1 Results

We formulate here some results for the Euclidean distance matrix completion problem; proofs are delayed till Section 31.4.2.

Let $K \subseteq V_n$ be a subset of nodes that induces a clique in G. For $d \in \mathbb{R}^E$ denote by d_K its projection on the edge set of $G[K]$. Clearly, if $d \in \mathrm{NEG}(G)$ then $d_K \in \mathrm{NEG}(K)$. Therefore, the condition

(31.4.3) $\qquad d_K \in \mathrm{NEG}(K)$ for every clique K in G

is a necessary condition for $d \in \mathrm{NEG}(G)$, again called *clique condition.* Bakonyi and Johnson [1995] characterize the graphs G for which the condition (31.4.3) is sufficient for the description of $\mathrm{NEG}(G)$. They show:

Theorem 31.4.4. *For a graph $G = (V_n, E)$, we have*

$$\mathrm{NEG}(G) = \{d \in \mathbb{R}^E \mid d_K \in \mathrm{NEG}(K) \;\; \forall K \textit{ clique in } G\}$$

if and only if G is chordal.

The condition (31.4.3) is not sufficient for the description of $\mathrm{NEG}(G)$ when G is not chordal. Indeed, suppose that G has a chordless circuit C of length ≥ 4. Let $x \in \mathbb{R}^E$ be defined by $x_e := 0$ for all edges e in C except $x_{e_0} := 1$ for one edge e_0 in C, $x_{ij} := 1$ for all edges ij with $i \in V(C)$, $j \in V \setminus V(C)$, and $x_{ij} := 0$ for all edges ij with $i, j \in V \setminus V(C)$. Then, x satisfies (31.4.3) but $x \notin \mathrm{NEG}(G)$. Another necessary condition can be easily formulated in terms of the cut cone. Namely, the condition

(31.4.5) $\qquad \sqrt{d} \in \mathrm{CUT}(G)$

is a necessary condition for $d \in \mathrm{NEG}(G)$, called *cut condition*; this follows from the fact that "$\ell_2 \Longrightarrow \ell_1$" (recall Proposition 6.4.12) and taking projections. Therefore,

(31.4.6) $\qquad \sqrt{d} \in \mathrm{MET}(G)$

is also a necessary condition for $d \in \mathrm{NEG}(G)$, called *metric condition.* The condition (31.4.6) characterizes $\mathrm{NEG}(G)$ in the case when $G = K_3$. This result has, in fact, already been mentioned in Remark 6.2.12; we repeat the proof for clarity.

Lemma 31.4.7. $\mathrm{NEG}_3 = \{d \in \mathbb{R}^3_+ \mid \sqrt{d} \in \mathrm{MET}_3\}$.

Proof. Let d be a distance on V_3 and set $d_{12} := a$, $d_{13} := b$, $d_{23} := c$. Let us consider the image of d under the covariance mapping (pointed at position 3) and the corresponding symmetric matrix

$$P := \begin{pmatrix} b & \frac{b+c-a}{2} \\ \frac{b+c-a}{2} & c \end{pmatrix}.$$

We use the fact that $d \in \mathrm{NEG}_3$ if and only if $P \succeq 0$ (recall Figure 6.2.3). Now, $P \succeq 0$ if and only if $\det P \geq 0$, i.e., if $4bc - (b+c-a)^2 \geq 0$. The latter condition can be rewritten as: $a^2 - 2a(b+c) + (b-c)^2 \leq 0$, which is equivalent to $b + c - 2\sqrt{bc} = (\sqrt{b} - \sqrt{c})^2 \leq a \leq b + c + 2\sqrt{bc} = (\sqrt{b} + \sqrt{c})^2$. Hence, we find the condition that $\sqrt{d} \in \mathrm{MET}_3$. ∎

More generally, Bakonyi and Johnson [1995] observe that the condition (31.4.6) suffices for the description of $\mathrm{NEG}(G)$ if G is a circuit. In fact, the following result holds, which is an analogue of Theorem 31.3.7 (Laurent [1997c]).

Theorem 31.4.8. *The following assertions are equivalent for a graph G:*

(i) $\mathrm{NEG}(G) = \{d \in \mathbb{R}^E_+ \mid \sqrt{d} \in \mathrm{CUT}(G)\}$.

(ii) $\mathrm{NEG}(G) = \{d \in \mathbb{R}^E_+ \mid \sqrt{d} \in \mathrm{MET}(G)\}$.

(iii) *G has no K_4-minor.*

The next result identifies the graphs for which the clique and metric conditions (resp. clique and cut conditions) suffice for the description of the cone $\mathrm{NEG}(G)$. The equivalence of (i) and (iii) is due to Johnson, Jones and Kroschel [1995] and that of (ii) and (iii) to Laurent [1997c].

Theorem 31.4.9. *The following assertions are equivalent for a graph G:*

(i) $\mathrm{NEG}(G) = \{d \in \mathbb{R}^E_+ \mid \sqrt{d} \in \mathrm{MET}(G)$ *and* $d_K \in \mathrm{NEG}(K)\ \forall K$ *clique in* $G\}$.

(ii) $\mathrm{NEG}(G) = \{d \in \mathbb{R}^E_+ \mid \sqrt{d} \in \mathrm{CUT}(G)$ *and* $d_K \in \mathrm{NEG}(K)\ \forall K$ *clique in* $G\}$.

(iii) *No induced subgraph of G is a wheel W_n ($n \geq 5$) or a splitting of a wheel W_n ($n \geq 4$).*

We conclude this section with a result of geometric flavor given in Bakonyi and Johnson [1995]; it follows as a direct application of Theorem 31.4.4.

Proposition 31.4.10. *Let $G = (V_n, E)$ be a chordal graph, let $K_1, \ldots, K_s$ denote its maximal cliques and let $d \in \mathbb{R}^E$, $R > 0$. Suppose that there exist vectors $u_1, \ldots, u_n \in \mathbb{R}^n$ satisfying* (i) *and* (ii)*:*

(i) $\| u_i - u_j \|_2 = d_{ij}$ *for all* $ij \in E$,

(ii) *for every* $r = 1, \dots, s$, *the vectors* u_i $(i \in K_r)$ *lie on a sphere of radius* R.

Then there exist vectors $v_1, \dots, v_n \in \mathbb{R}^n$ *satisfying* (i) *and all of them lying on a sphere of radius* R. ∎

31.4.2 Links Between the Two Completion Problems

There is an obvious analogy between the above results for the Euclidean distance matrix completion problem and the results from Section 31.3 for the positive semidefinite completion problem. Compare, in particular, Theorems 31.3.4 and 31.4.4, as well as Theorems 31.3.7 and 31.4.8, and Theorems 31.3.13 and 31.4.9. Following Laurent [1997c], we indicate here how to derive the results for the Euclidean distance matrix completion problem from those for the positive semidefinite completion problem.

For convenience let us introduce the following classes of graphs: $\mathcal{D}_K$ (resp. $\mathcal{D}_M, \mathcal{D}_C$) denotes the class of graphs for which the clique condition (31.4.3) (resp. metric condition (31.4.6), cut condition (31.4.5)) suffices for the description of NEG(G); and $\mathcal{D}_{KM}$ (resp. $\mathcal{D}_{KC}$) denotes the class of graphs for which the clique and metric (resp. clique and cut) conditions taken together suffice for the description of NEG(G).

It is also convenient to introduce a notation for the following classes of graphs, already encountered in the previous section. The class $\mathcal{G}_{ch}$ consists of all chordal graphs; the class $\mathcal{G}_{K4}$ consists of the graphs that do not contain K_4 as a minor; and the class $\mathcal{G}_{wh}$ consists of the graphs that do not contain a wheel W_n $(n \geq 5)$ or a splitting of a wheel W_n $(n \geq 4)$ as an induced subgraph.

Proving Theorems 31.4.4, 31.4.8 and 31.4.9 amounts to showing the equalities: $\mathcal{D}_K = \mathcal{G}_{ch}$, $\mathcal{D}_M = \mathcal{D}_C = \mathcal{G}_{K4}$, and $\mathcal{D}_{KM} = \mathcal{D}_{KC} = \mathcal{G}_{wh}$. For this, it suffices to verify the inclusions: $\mathcal{D}_K \subseteq \mathcal{G}_{ch}$, $\mathcal{P}_K \subseteq \mathcal{D}_K$; $\mathcal{D}_C \subseteq \mathcal{G}_{K4}$, $\mathcal{P}_M \subseteq \mathcal{D}_M$; and $\mathcal{D}_{KC} \subseteq \mathcal{G}_{wh}$, $\mathcal{P}_{KM} \subseteq \mathcal{D}_{KM}$. We do so in Lemmas 31.4.16 and 31.4.17 below.

Crucial for the proof are some links between the negative type cone and the elliptope. A first obvious link between the cone NEG(∇G) and the elliptope $\mathcal{E}(G)$ is provided by the covariance mapping (as defined in (27.3.8)). Namely, given vectors $x \in \mathbb{R}^E$ and $d \in \mathbb{R}^{E(\nabla G)}$ satisfying: $d_{i,n+1} = 1$ for all $i \in V_n$ and $d_{ij} = 2 - 2x_{ij}$ for all $ij \in E$, then

$$x \in \mathcal{E}(G) \Longleftrightarrow d \in \mathrm{NEG}(\nabla G). \tag{31.4.11}$$

Another essential tool is the following property of the Schoenberg transform from Theorem 9.1.1: For $d \in \mathbb{R}^{E_n}$,

$$d \in \mathrm{NEG}(K_n) \Longleftrightarrow \exp(-\lambda d) \in \mathcal{E}(K_n) \text{ for all } \lambda > 0. \tag{31.4.12}$$

(We remind that the notation $\exp(-\lambda d)$ means applying the exponential function componentwise, i.e., $\exp(-\lambda d) = (\exp(-\lambda d_{ij}))_{ij}$.) This relation remains valid at the level of arbitrary graphs. Namely,

Proposition 31.4.13. *Let $G = (V_n, E)$ be a graph and $d \in \mathbb{R}^E$. The following assertions are equivalent.*

(i) $d \in \mathrm{NEG}(G)$.

(ii) $\exp(-\lambda d) \in \mathcal{E}(G)$ *for all* $\lambda > 0$.

(iii) $1 - \exp(-\lambda d) \in \mathrm{NEG}(G)$ *for all* $\lambda > 0$.

Proof. (i) $\Longrightarrow$ (ii) follows from (31.4.12) and taking projections.
(ii) $\Longrightarrow$ (iii) Given $\lambda > 0$, define the vector $D \in \mathbb{R}^{E(\nabla G)}$ by $D_{i,n+1} = 1$ for $i \in V_n$ and $D_{ij} = 2 - 2\exp(-\lambda d_{ij})$ for $ij \in E$. Then, $D \in \mathrm{NEG}(\nabla G)$ (by relation (31.4.11)) which implies that $1 - \exp(-\lambda d) \in \mathrm{NEG}(G)$.
(iii) $\Longrightarrow$ (i) Let $v^T x \leq 0$ be a valid inequality for the cone $\mathrm{NEG}(G)$. We show that $v^T d \leq 0$. By assumption, $v^T(1 - \exp(-\lambda d)) \leq 0$. Expanding in series the exponential function, we obtain:

$$\begin{aligned} v^T(1 - \exp(-\lambda d)) &= \textstyle\sum_{ij \in E} v_{ij} (\sum_{p \geq 1} \frac{(-1)^{p-1}}{p!} \lambda^p d_{ij}^p) \\ &= \textstyle\sum_{p \geq 1} \frac{(-1)^{p-1} \lambda^p}{p!} \sum_{ij \in E} v_{ij} d_{ij}^p \leq 0. \end{aligned}$$

Dividing by λ and, then, letting $\lambda \longrightarrow 0$ yields: $\sum_{ij \in E} v_{ij} d_{ij} \leq 0$. This shows that $d \in \mathrm{NEG}(G)$, as d satisfies all the valid inequalities for $\mathrm{NEG}(G)$. ∎

From this we can derive the following result[7] permitting to link the two metric conditions (31.3.3) and (31.4.6).

Lemma 31.4.14. *Let $G = (V_n, E)$ be a graph and $d \in \mathbb{R}^E_+$. Then,*

$$\sqrt{d} \in \mathrm{MET}(G) \Longrightarrow \frac{1}{\pi} \arccos(e^{-\lambda d}) \in \mathrm{MET}^{\square}(G) \quad \textit{for all } \lambda > 0.$$

Proof. Note first that it suffices to show the result in the case when $G = K_n$ (as the general result will then follow by taking projections). Next, observe that it suffices to show the result in the case $n = 3$ (as $\mathrm{MET}(K_n)$ and $\mathrm{MET}^{\square}(K_n)$ are defined by inequalities that involve only three points). Now, we have: $\sqrt{d} \in \mathrm{MET}(K_3) \Longleftrightarrow d \in \mathrm{NEG}(K_3)$ (by Lemma 31.4.7); $d \in \mathrm{NEG}(K_3) \Longleftrightarrow \exp(-\lambda d) \in \mathcal{E}(K_3)$ for all $\lambda > 0$ (by Proposition 31.4.13); finally, $\exp(-\lambda d) \in \mathcal{E}(K_3) \Longleftrightarrow \frac{1}{\pi} \arccos(e^{-\lambda d}) \in \mathrm{MET}^{\square}(K_3)$ (by Theorem 31.2.2). ∎

One more useful preliminary result is the following.

[7] The implication in Lemma 31.4.14 holds, in fact, as an equivalence. The converse implication can be shown using the mean value theorem applied to the function $f(t) = \arccos(e^{-t^2})$ and letting λ tend to zero.

Lemma 31.4.15. *Let $W_n := \nabla C$ be a wheel on n nodes, with center u_0 and circuit C. Consider the vector d indexed by the edge set of W_n and defined by $d(u_0, u) := 1$ for each node u of C, $d(u, v) := 4$ for each edge uv of C. Then, $d \in \text{NEG}(W_n) \iff n$ is odd.*

Proof. Let x be the vector indexed by the edge set of C and taking value -1 on every edge. By (31.4.11), $d \in \text{NEG}(W_n)$ if and only if $x \in \mathcal{E}(C)$. The latter holds if and only if $\frac{1}{\pi} \arccos x \in \text{MET}^{\square}(C)$, that is, if and only if C has an even length. ∎

Lemma 31.4.16. *We have: $\mathcal{D}_K \subseteq \mathcal{G}_{ch}$, $\mathcal{D}_C \subseteq \mathcal{G}_{K4}$, and $\mathcal{D}_{KC} \subseteq \mathcal{G}_{wh}$.*

Proof. We show the inclusion: $\mathcal{D}_K \subseteq \mathcal{G}_{ch}$. For this, let $G = (V, E)$ be a non-chordal graph and let $C = (V(C), E(C))$ be a chordless circuit of length ≥ 4 in G. We define a vector $d \in \mathbb{R}^E$ satisfying (31.4.3) and such that $d \notin \text{NEG}(G)$ by setting $d_e := 0$ for all edges $e \in E(C)$ except $d_{e_0} := 1$ for one edge e_0 in C; $d_e := 1$ for every edge e joining a node of C to a node of $V \setminus V(C)$; and $d_e := 0$ for every edge e joining two nodes of $V \setminus V(C)$.
The example from Lemma 31.4.15 above shows that $K_4 = W_4$ does not belong to $\mathcal{D}_C$. The inclusion: $\mathcal{D}_C \subseteq \mathcal{G}_{K4}$ now follows after noting that $\mathcal{D}_C$ is closed under taking minors.
We finally check the inclusion: $\mathcal{D}_{KC} \subseteq \mathcal{G}_{wh}$. For this, let $G = (V, E)$ be a graph in $\mathcal{D}_{KC}$ and let $H := G[U]$ be an induced subgraph of G where $U \subseteq V$. Suppose in a first step that H is a wheel $W_n := \nabla C$ ($n \geq 5$) with center u_0. Consider the vector d indexed by the edge set of G and defined in the following manner: d takes value 4 on every edge of the circuit C excepet value 0 on one edge if n is odd; d takes value 1 on every edge joining the center u_0 of the wheel to a node of C; d takes value 1 on an edge between a node of C and a node outside the wheel; d takes value 0 on every remaining edge (i.e., an edge joining u_0 to a node outside the wheel or an edge joining two nodes outside the wheel). Then d satisfies (31.4.3) and $d \notin \text{NEG}(G)$ (by Lemma 31.4.15). Moreover d satisfies (31.4.5), i.e., $\sqrt{d} \in \text{CUT}(G)$. Indeed, say C is the circuit $(u_1, \ldots, u_{n-1})$. Then, $\sqrt{d} = \sum_{i=1}^{n-1} \delta_G(u_i)$ if n is even and $\sqrt{d} = \delta_G(\{u_1, u_{n-1}\}) + \sum_{i=2}^{n-2} \delta_G(u_i)$ if n is odd and (u_1, u_{n-1}) is the edge of C on which d takes value 0. Finally, if H is a splitting of a wheel W_n ($n \geq 4$), extend the above vector d by assigning value 0 to every new edge created during the splitting process. ∎

Lemma 31.4.17. *$\mathcal{P}_K \subseteq \mathcal{D}_K$, $\mathcal{P}_M \subseteq \mathcal{D}_M$, and $\mathcal{P}_{PM} \subseteq \mathcal{D}_{KM}$.*

Proof. We first verify the inclusion: $\mathcal{P}_K \subseteq \mathcal{D}_K$. Let G be a graph in $\mathcal{P}_K$; we show that $G \in \mathcal{D}_K$. For this, let $d \in \mathbb{R}^E$ satisfying (31.4.3); we show that $d \in \text{NEG}(G)$. By Proposition 31.4.13, $\exp(-\lambda d_K) \in \mathcal{E}(K)$ for every clique K in G and every $\lambda > 0$. As $G \in \mathcal{P}_K$, this implies that $\exp(-\lambda d) \in \mathcal{E}(G)$ for all $\lambda > 0$. Using again Proposition 31.4.13, we obtain that $d \in \text{NEG}(G)$.
Suppose now that $G \in \mathcal{P}_M$; we show that $G \in \mathcal{D}_M$. Let $d \in \mathbb{R}^E$ satisfy-

ing (31.4.6), i.e., $\sqrt{d} \in \mathrm{MET}(G)$. Then, by Lemma 31.4.14, $\frac{1}{\pi}\arccos(e^{-\lambda d}) \in \mathrm{MET}^{\square}(G)$ for all $\lambda > 0$. As $G \in \mathcal{P}_M$, this implies that $\exp(-\lambda d) \in \mathcal{E}(G)$ for all $\lambda > 0$. By Proposition 31.4.13, we obtain that $d \in \mathrm{NEG}(G)$.
The inclusion $\mathcal{P}_{KM} \subseteq \mathcal{D}_{KM}$ follows by combining the above arguments. ∎

31.5 Geometry of the Elliptope

In Section 28.4.1 was introduced the convex body $\mathcal{J}_n$ as a (nonpolyhedral) relaxation of the cut polytope $\mathrm{CUT}_n^{\square}$. We remind that

$$\begin{aligned}\mathcal{J}_n &= \{x \in \mathbb{R}^{E_n} \mid \sum_{1 \le i < j \le n} b_i b_j x_{ij} \le \frac{1}{4}(\sum_{i=1}^{n} b_i)^2 \text{ for all } b \in \mathbb{Z}^n\} \\ &= \{x \in \mathbb{R}^{E_n} \mid J - 2X \succeq 0\}\end{aligned}$$

where J is the all-ones matrix and, for $x \in \mathbb{R}^{E_n}$, X is the symmetric $n \times n$ matrix with zero diagonal and off-diagonal entries x_{ij}. We also remind that the elliptope $\mathcal{E}_n$ is defined as the set of $n \times n$ symmetric positive semidefinite matrices with an all-ones diagonal. Therefore,

$$x \in \mathcal{J}_n \Longleftrightarrow J - 2X \in \mathcal{E}_n.$$

Hence, the two convex sets $\mathcal{J}_n$ and $\mathcal{E}_n$ are essentially identical (up to the transformation $x \mapsto 1 - 2x$). The convex body $\mathcal{J}_n$ is a relaxation of $\mathrm{CUT}_n^{\square}$, i.e.,

$$\mathrm{CUT}_n^{\square} \subseteq \mathcal{J}_n.$$

Moreover, $\mathcal{J}_n$ provides a good approximation for $\mathrm{CUT}_n^{\square}$ in the sense of optimization (recall Theorem 28.4.7). In fact, the convex body $\mathcal{J}_n$ presents several geometric features, which may explain and provide further insight for its good behaviour in optimization. One such property is, for instance, the fact that the only vertices of $\mathcal{J}_n$ are the cut vectors. This result is given below as well as several other geometric properties. For convenience we will work with the elliptope $\mathcal{E}_n$ rather than with $\mathcal{J}_n$ itself.

We start with recalling some definitions. Let K be a convex set in $\mathbb{R}^d$. Given a boundary point x_0 of K, its *normal cone* $N(K, x_0)$ is defined as

$$N(K, x_0) := \{c \in \mathbb{R}^d \mid c^T x \le c^T x_0 \text{ for all } x \in K\}.$$

Hence, $N(K, x_0)$ consists of the normal vectors to the supporting hyperplanes of K at x_0. Then, the *supporting cone* at x_0 is defined by

$$C(K, x_0) := \{x \in \mathbb{R}^d \mid c^T x \le 0 \ \text{ for all } c \in N(K, x_0)\}.$$

The dimension of the normal cone permits to classify the boundary points. In particular, a boundary point x_0 is called a *vertex* of K if its normal cone $N(K, x_0)$ is full-dimensional. A subset $F \subseteq K$ is a face of K if, for all $x \in F$, $y, z \in K$ and

$0 \leq \alpha \leq 1$, $x = \alpha y + (1-\alpha)z$ implies that $y, z \in F$. In particular, an element $x_0 \in K$ is called *an extreme point* of K if the set $\{x_0\}$ is a face of K. In what follows we consider the two convex sets $\mathcal{E}_n$ and $\mathcal{J}_n$. When dealing with $\mathcal{E}_n$ we take the space of symmetric $n \times n$ matrices as ambient space, equipped with the inner product:

$$\langle A, B\rangle := \sum_{i,j=1}^{n} a_{ij} b_{ij} \text{ for two symmetric } n \times n \text{ matrices } A, B$$

and, when dealing with $\mathcal{J}_n$, the ambient space is the usual Euclidean space $\mathbb{R}^{\binom{n+1}{2}}$. We remind that $\text{Tr } A := \sum_{i=1}^{n} a_{ii}$ for an $n \times n$ matrix A.

We begin with the description of the polar of $\mathcal{E}_n$ and of its normal cones. These results are established by Laurent and Poljak [1995b, 1996a]; proofs can be found there.

Theorem 31.5.1. *The polar of $\mathcal{E}_n$ is given by*

$$(\mathcal{E}_n)^\circ = \{D - M \mid M \succeq 0, D \textit{ diagonal matrix with } \text{Tr } D = 1\}.$$

For $A \in \mathcal{E}_n$, its normal cone is defined by

$$N(\mathcal{E}_n, A) = \{D - M \mid M \succeq 0, \langle M, A\rangle = 0, D \textit{ diagonal matrix}\}.$$

Moreover, dim $N(\mathcal{E}_n, A) = n + \binom{n-r+1}{2}$, *where r is the rank of A.* ∎

Corollary 31.5.2. *The only vertices of $\mathcal{E}_n$ are the 'cut matrices' xx^T, for $x \in \{\pm 1\}^n$. In other words, the convex body $\mathcal{J}_n$ has 2^{n-1} vertices, namely, the cut vectors $\delta(S)$ for $S \subseteq V_n$.* ∎

We remind that, given $c \in \mathbb{R}^{E_n}$, $\max(c^T x \mid x \in \mathcal{J}_n)$ is an upper bound for the max-cut problem: $\max(c^T x \mid x \in \text{CUT}_n^\square)$. Equality holds between the bound and the max-cut precisely when c belongs to the normal cone of one of the cut vectors. That the cut vectors are the only boundary points having a full dimensional normal cone supports the idea that $\mathcal{J}_n$ approximates well $\text{CUT}_n^\square$. From Theorem 31.5.1 one obtains that the supporting cone $C(\mathcal{E}_n, A)$ at $A \in \mathcal{E}_n$ is the set

$$\{X \text{ symmetric } n \times n \mid x_{ii} = 0 \ \forall i = 1, \ldots, n, \ b^T X b \geq 0 \text{ for all } b \in \text{Ker} A\}.$$

In particular, at $A = J$ (the all-ones matrix), the supporting cone is $-\text{NEG}_n$. At every other vertex of $\mathcal{E}_n$, the supporting cone is an affine image of the negative type cone NEG_n (under the switching mapping). So, this makes one more connection between the elliptope and the negative type cone.

We now turn to the description of the faces of $\mathcal{E}_n$. We remind that $\mathcal{E}_n$ is obtained by taking the intersection of the cone PSD_n of positive semidefinite matrices with the linear space $W := \{X \mid x_{ii} = 1 \ \ \forall i = 1, \ldots, n\}$. The facial structure of the cone PSD_n is well understood (see Hill and Waters [1987]). It is, in some sense, rather simple. Indeed, given a matrix $A \in \mathrm{PSD}_n$ with rank r the smallest face $F_{\mathrm{PSD}}(A)$ of PSD_n that contains A is given by

$$F_{\mathrm{PSD}}(A) = \{X \in \mathrm{PSD}_n \mid \mathrm{Ker}X \supseteq \mathrm{Ker}A\}.$$

Hence, $F_{\mathrm{PSD}}(A)$ is isomorphic to the cone PSD_r and, thus, has dimension $\binom{r+1}{2}$. From this follows the description of the faces of $\mathcal{E}_n$. For $A \in \mathcal{E}_n$, the smallest face $F_{\mathcal{E}}(A)$ of $\mathcal{E}_n$ that contains A is equal to $F_{\mathrm{PSD}}(A) \cap W$ (as W is the only face of W). In other words,

$$F_{\mathcal{E}}(A) = \{X \in \mathcal{E}_n \mid \mathrm{Ker}X \supseteq \mathrm{Ker}A\}.$$

However, computing the dimension of $F_{\mathcal{E}}(A)$ requires more care[8]. This has been done by Li and Tam [1994]. For convenience, we state their result in a more general setting.

Theorem 31.5.3. *Let $A_1, \ldots, A_m$ be $n \times n$ symmetric matrices and $b_1, \ldots, b_m \in \mathbb{R}$. Consider the convex set*

$$K := \{X \in \mathrm{PSD}_n \mid \langle X, A_j \rangle = b_j \quad \forall j = 1, \ldots, m\}.$$

Let $A \in K$ and let $F_K(A)$ be the smallest face of K that contains A. Suppose that A has rank r and that $A = QQ^T$, where Q is an $n \times r$ matrix of rank r. Then,

$$\dim\ F_K(A) = \binom{r+1}{2} - \mathrm{rank}\ \{Q^T A_j Q \mid j = 1, \ldots, m\}.$$

Proof. Call a symmetric matrix B a *perturbation* of A if $A \pm \lambda B \in K$ for some $\lambda > 0$. Then, dim $F_K(A)$ is equal to the rank of the set of perturbations of A. We claim:

(a) B is a perturbation of $A \iff B = QRQ^T$ for some $r \times r$ symmetric matrix R and $\langle B, A_j \rangle = 0$ for all $j = 1, \ldots, m$.

If $B = QRQ^T$ then $A \pm \lambda B = Q(I \pm \lambda R)Q^T$ is clearly positive semidefinite if $\lambda > 0$ is small enough. Moreover, the condition: $\langle B, A_j \rangle = 0$ for all j ensures that $A \pm \lambda B \in K$. Conversely, suppose that B is a perturbation of A. So, $A \pm \lambda B \in K$ for some $\lambda > 0$. This implies that $\langle B, A_j \rangle = 0$ for all j. Complete Q to an $n \times n$ nonsingular matrix P. Set $C := P^{-1}B(P^{-1})^T$; that is, $B = PCP^T$. Then,

$$A \pm \lambda B = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} P^T \pm \lambda PCP^T = P\left(\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \pm \lambda \begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix}\right) P^T,$$

[8] Here arises also the question of characterizing the linear subspaces V of $\mathbb{R}^n$ such that $V \subseteq \mathrm{Ker}A$ for some $A \in \mathcal{E}_n$. Delorme and Poljak [1993b] show that a vector $b \in \mathbb{R}^n$ belongs to the kernel of some matrix $A \in \mathcal{E}_n$ if and only if b satisfies: $|b_i| \leq \sum_{1 \leq j \leq n,\ j \neq i} |b_j|$ for all $i = 1, \ldots, n$. An analogue combinatorial characterization for higher dimensional spaces is not known.

setting $C := \begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix}$. Hence, $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \pm \lambda \begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix} \succeq 0$. This implies that $C_{12} = C_{22} = 0$. Therefore, $B = QC_{11}Q^T$, where C_{11} is a symmetric $r \times r$ matrix. Hence, (a) holds.

Now, every perturbation of A is of the form $B = QRQ^T$ with $\langle B, A_j \rangle = 0$ for all j; that is, $\langle R, Q^T A_i Q \rangle = 0$ for all j. Hence, the dimension of $F_K(A)$ is equal to the dimension of the orthogonal complement of $\{Q^T A_j Q \mid j = 1, \ldots, m\}$ in the space of symmetric $r \times r$ matrices. Hence, we have the desired formula for $\dim\ F_K(A)$. ▮

Corollary 31.5.4. *Let $A \in \mathcal{E}_n$ with rank r, let $F_{\mathcal{E}}(A)$ denote the smallest face of $\mathcal{E}_n$ containing A, and suppose that A is the Gram matrix of the vectors $u_1, \ldots, u_n \in \mathbb{R}^r$. Then,*

$$\dim\ F_{\mathcal{E}}(A) = \binom{r+1}{2} - \text{rank } \{u_i u_i^T \mid i = 1, \ldots, n\}. \tag{31.5.5}$$

▮

In particular, one obtains bounds for the rank of extreme matrices[9] of $\mathcal{E}_n$.

Corollary 31.5.6. *Let $A \in \mathcal{E}_n$ with rank r. If A is an extreme point of $\mathcal{E}_n$ then $\binom{r+1}{2} \leq n$.* ▮

Moreover, as we see below, for every r such that $\binom{r+1}{2} \leq n$ there exists an extreme matrix in $\mathcal{E}_n$ having rank r. The formula (31.5.5) can be used for finding the possible dimensions for the faces of $\mathcal{E}_n$, as observed in Laurent and Poljak [1996a]. Namely,

Proposition 31.5.7. *Let $A \in \mathcal{E}_n$ with rank r and set $k :=$ dim $F_{\mathcal{E}}(A)$. Then,*

$$\max(0, \binom{r+1}{2} - n) \leq k \leq \binom{r}{2}.$$

Moreover, for every integers $r, k \geq 0$ satisfying the above inequality, there exists a matrix $A \in \mathcal{E}_n$ with rank r and with dim $F_{\mathcal{E}}(A) = k$.

Proof. The inequality from Proposition 31.5.7 follows from (31.5.5), after noting that $r \leq \text{rank } \{u_1 u_1^T, \ldots, u_n u_n^T\} \leq n$. The existence part relies essentially on a construction proposed in Grone, Pierce and Watkins [1990], which goes as follows. Let $e_1, \ldots, e_r$ denote the coordinate vectors in $\mathbb{R}^r$ and set $w_{ij} := \frac{1}{\sqrt{2}}(e_i + e_j)$ for $1 \leq i < j \leq r$. Then, the $\binom{r+1}{2}$ matrices: $e_i e_i^T$ $(i = 1, \ldots, r)$ and $w_{ij} w_{ij}^T$ $(1 \leq i < j \leq r)$ are linearly independent. Suppose first that $n = \binom{r+1}{2} - k$ where

[9] Solving this question has been the subject of several papers in the linear algebra literature; for example, by Christensen and Vesterstrøm [1979], Loewy [1980], Grone, Pierce and Watkins [1990].

$k \le \binom{r}{2}$. Hence, $r \le n \le \binom{r+1}{2}$. Define A as the Gram matrix of the following n vectors: $e_1, \ldots, e_r$ together with $n - r$ of the vectors w_{ij}. By construction, A has rank r and dim $F_{\mathcal{E}}(A) = \binom{r+1}{2} - n = k$. When $n > \binom{r+1}{2} - k$, we can take as matrix A the Gram matrix of the following n vectors: e_1 repeated $n - \binom{r+1}{2} + k + 1$ times, $e_2 \ldots e_r$, and $\binom{r}{2} - k$ of the w_{ij}'s. ∎

Therefore, the range $\mathcal{D}_n$ of the possible values for the dimension of the faces of $\mathcal{E}_n$ is given by:

$$\mathcal{D}_n = [0, \binom{k_n}{2}] \cup \bigcup_{r=k_n+1}^{n} [\binom{r+1}{2} - n, \binom{r}{2}],$$

where k_n is the smallest integer k such that $\binom{k+2}{2} - n > \binom{k_n}{2} + 1$, i.e., $2k_n > n$; that is, $k_n = \lfloor \frac{n}{2} \rfloor + 1$. For instance,

$$k_3 = 2,\ \mathcal{D}_3 = [0,1] \cup \{3\},$$

$$k_4 = 3,\ \mathcal{D}_4 = [0,3] \cup \{6\},$$

$$k_5 = 3,\ \mathcal{D}_5 = [0,3] \cup [5,6] \cup \{10\},$$

$$k_6 = 4,\ \mathcal{D}_6 = [0,6] \cup [9,10] \cup \{15\},$$

$$k_7 = 4,\ \mathcal{D}_7 = [0,6] \cup [8,10] \cup [14,15] \cup \{21\}.$$

One can verify on Figure 31.3.6 that the proper faces of $\mathcal{E}_3$ have dimension 0 (extreme points) or 1 (an edge between two cut vectors; there are six such faces). A detailed description of the faces of $\mathcal{E}_n$ can be found in Laurent and Poljak [1995b, 1996a] for $n = 3$ and $n = 4$, respectively.

Finally, the possible dimensions for the polyhedral faces of $\mathcal{E}_n$ are as follows; they were computed by Laurent and Poljak [1996a].

Theorem 31.5.8. *If F is a polyhedral face of $\mathcal{E}_n$ with dimension k, then $\binom{k+1}{2} \le n - 1$. Moreover, if all the vertices of F are cut matrices then F is a simplex. Conversely, for every integer $k \ge 1$ such that $\binom{k+1}{2} \le n - 1$, $\mathcal{E}_n$ has a polyhedral face of dimension k (which can be chosen to be a simplex with cut matrices as vertices).* ∎

Every polyhedral face of $\mathcal{E}_n$ with cut matrices as vertices yields clearly a face of the cut polytope. We describe below a construction for such polyhedral faces, due to Laurent and Poljak [1996a]. We need a definition in order to state the result.

Let $S_1, \ldots, S_k$ be k subsets of V_n. The cut vectors $\delta(S_1), \ldots, \delta(S_k)$ are said to be in *general position* if the set

$$\bigcap_{i \in I} S_i \cap \bigcap_{i \notin I} (V_n \setminus S_i)$$

is nonempty, for every subset $I \subseteq \{1, \ldots, k\}$. This implies that $2^k \leq n$, i.e., $k \leq \log_2 n$. Moreover, the cut vectors $\delta(S_1), \ldots, \delta(S_k)$ are linearly independent.

Theorem 31.5.9. *Let $\delta(S_1), \ldots, \delta(S_k)$ be k cuts in general position. Then, the set $F := \mathrm{Conv}(\delta(S_1), \ldots, \delta(S_k))$ is a face of the convex body $\mathcal{J}_n$. (Equivalently, the set $\mathrm{Conv}(x_1x_1^T, \ldots, x_kx_k^T)$ is a face of $\mathcal{E}_n$, where $x_h \in \mathbb{R}^n$ is defined by $x_h(i) := 1$ if $i \in S_h$ and $x_h(i) := -1$ if $i \in V_n \setminus S_h$, for $h = 1, \ldots, k$.) Therefore, F is also a face of the cut polytope $\mathrm{CUT}_n^{\square}$.* ∎

This result shows that $\mathcal{J}_n$ and $\mathrm{CUT}_n^{\square}$ share fairly many common faces, up to dimension $\lfloor \log_2 n \rfloor$. This supports again the idea that $\mathcal{J}_n$ approximates well the cut polytope $\mathrm{CUT}_n^{\square}$. In fact, the faces considered in Theorem 31.5.9 are also faces in common with the semimetric polytope $\mathrm{MET}_n^{\square}$; see Theorem 31.6.4.

31.6 Adjacency Properties

We now return to the study of the geometry of the cut polytope itself, as well as with respect to its linear relaxation by the semimetric polytope. We mention first some results on the faces of low dimension and, then, facts and questions about the small cut and semimetric polytopes.

31.6.1 Low Dimension Faces

A striking property of the cut polytope $\mathrm{CUT}_n^{\square}$ is that any two of its vertices form an edge of $\mathrm{CUT}_n^{\square}$. In fact, much more is true. In order the formulate the results, we need some definitions.

Let P be a polytope with set of vertices V. Given an integer $k \geq 1$, the polytope P is said to be *k-neighborly* if, for any subset $W \subseteq V$ of vertices such that $|W| \leq k$, the set $\mathrm{Conv}(W)$ is a a face of P. This implies, in particular, that every k vertices of P are affinely independent. Hence, every polytope is 1-neighborly and a polytope is 2-neighborly precisely when its 1-skeleton graph is a complete graph.

Given an integer d and a polyhedron P, we let $\phi_d(P)$ denote the set of d-dimensional faces of P.

Barahona and Mahjoub [1986] show that $\mathrm{CUT}_n^{\square}$ is 2-neighborly, i.e., that any two cut vectors are adjacent on $\mathrm{CUT}_n^{\square}$. In other words, the 1-skeleton graph of $\mathrm{CUT}_n^{\square}$ is a complete graph. Padberg [1989] shows the following stronger result: Any two cut vectors are adjacent on the rooted semimetric polytope $\mathrm{RMET}_n^{\square}$ (defined by the triangle inequalities going through a given node; recall Section 27.2). More generally, Deza, Laurent and Poljak [1992] show the following result.

Theorem 31.6.1. *Let W be a set of cut vectors such that $|W| \leq 3$. Then, the set $\mathrm{Conv}(W)$ is a simplex face of the semimetric polytope $\mathrm{MET}_n^{\square}$.*

Proof. Due to switching, we can suppose that the set W contains the zero cut vector $\delta(\emptyset)$. Let us first consider the case when $|W| = 2$; say, $W = \{\delta(\emptyset), \delta(S)\}$, where $S \neq \emptyset, V_n$. In order to show that the set $\text{Conv}(W)$ is a face of $\text{MET}_n^\square$, it suffices to find a vector $w \in \mathbb{R}^{E_n}$ satisfying the following property:

(a) $w^T x \leq 0$ for all $x \in \text{MET}_n^\square$, with equality if and only if $x \in \text{Conv}(W)$.

For this, set $w_{ij} := 0$ if $\delta(S)_{ij} = 1$ and $w_{ij} := -1$ otherwise. It is immediate to verify that w satisfies the desired property.

We now consider the case when $|W| = 3$; say, $W = \{\delta(\emptyset), \delta(S), \delta(T)\}$, where $\delta(S)$ and $\delta(T)$ are distinct and nonzero. Set $A := S \cap T$, $B := \overline{S} \cap T$, $C := S \cap \overline{T}$, and $D := \overline{S} \cap \overline{T}$. Again, we should find $w \in \mathbb{R}^{E_n}$ satisfying (a). Let us first suppose that the sets A, B, C, D are nonempty. Let $a \in A$, $b \in B$, $c \in C$, and $d \in D$. Define $w \in \mathbb{R}^{E_n}$ by setting $w_{ab} = w_{ac} = w_{bd} = w_{cd} := -1$, $w_{ad} = w_{bc} := 1$, $w_{ij} := -1$ if $i \neq j$ both belong to A, or B, or C, or D (denote by E the set of these pairs ij), and $w_{ij} := 0$ otherwise. Then, $w^T\delta(S) = w^T\delta(T) = 0$. Let $x \in \text{MET}_n^\square$. Then,

$$w^T x = -\sum_{ij \in E} x_{ij} + \sigma,$$

where

$$\sigma := x_{ad} + x_{bc} - x_{ab} - x_{bd} - x_{cd} - x_{ac}.$$

We have the relations:

(i) $\sigma = (x_{ad} - x_{ac} - x_{cd}) + (x_{bc} - x_{cd} - x_{bd}) + x_{cd} - x_{ab} \leq x_{cd} - x_{ab}$,

(ii) $\sigma = (x_{ad} - x_{ab} - x_{bd}) + (x_{bc} - x_{ab} - x_{ac}) + x_{ab} - x_{cd} \leq x_{ab} - x_{cd}$,

(iii) $\sigma = (x_{ad} - x_{ac} - x_{cd}) + (x_{bc} - x_{ab} - x_{ac}) + x_{ac} - b_{bd} \leq x_{ac} - b_{bd}$,

(iv) $\sigma = (x_{ad} - x_{ab} - x_{bd}) + (x_{bc} - x_{bd} - x_{cd}) + x_{bd} - x_{ac} \leq x_{bd} - x_{ac}$.

From (i)-(iv) we deduce that $\sigma \leq 0$. Therefore, $w^T x \leq 0$. Moreover, if $w^T x = 0$, then $x_{ij} = 0$ for all $ij \in E$ and $\sigma = 0$. Hence, using (i)-(iv), $x_{ab} = x_{cd} := \alpha$, $x_{ac} = x_{bd} := \beta$ for some $\alpha, \beta \geq 0$, $\alpha + \beta \leq 1$, and $x_{ad} = x_{bc} = \alpha + \beta$. From this follows easily that $x = \alpha\delta(S) + \beta\delta(T)$, which shows that $x \in \text{Conv}(W)$.

Finally, let us suppose that one of the sets A, B, C, D is empty. Say, $D = \emptyset$. Then, $A, B, C \neq \emptyset$; let $a \in A$, $b \in B$ and $c \in C$. We now define $w \in \mathbb{R}^{E_n}$ by $w_{ab} = w_{ac} := -1$, $w_{bc} := 1$, $w_{ij} := -1$ if $i \neq j$ both belong to A, or B, or C, and $w_{ij} := 0$ otherwise. It can be verified as above that w satisfies (a). ∎

Corollary 31.6.2. *The cut polytope* $\text{CUT}_n^\square$ *is* 3*-neighborly.* ∎

Corollary 31.6.3.

(i) *For* $n \geq 4$*, every face of* $\text{CUT}_n^\square$ *of dimension* $d \leq 5$ *is a simplex.*

(ii) $\phi_d(\text{CUT}_n^\square) \subseteq \phi_d(\text{MET}_n^\square)$*, for* $d = 0, 1, 2$. ∎

The results from Corollaries 31.6.2 and 31.6.3 (i) are best possible; that is, $\mathrm{CUT}_n^\square$ is not 4-neighborly and there exists a 6-dimensional face of $\mathrm{CUT}_n^\square$ ($n \geq 4$) which is not a simplex. Indeed, for $n = 4$, $\mathrm{CUT}_4^\square$ itself is a nonsimplex 6-dimensional face. For $n \geq 5$, consider the face F of $\mathrm{CUT}_n^\square$ which is defined by the inequality:

$$\sum_{4 \leq i < j \leq n} x_{ij} \geq 0.$$

Then, F contains the following eight cut vectors $\delta(S)$ for $S = \emptyset$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, and $\{1,2,3\}$. They are not affinely independent as they satisfy:

$$\delta(\{1\}) + \delta(\{2\}) + \delta(\{3\}) + \delta(\{1,2,3\}) = \delta(\{1,2\}) + \delta(\{1,3\}) + \delta(\{2,3\}).$$

Hence, F is a nonsimplex face of dimension 6 of $\mathrm{CUT}_n^\square$. (In fact, one can check that F is also a face of $\mathrm{MET}_n^\square$.) Hence, the four cut vectors $\delta(\emptyset)$, $\delta(\{1,2\})$, $\delta(\{1,3\})$, and $\delta(\{2,3\})$ do not form a face of $\mathrm{CUT}_n^\square$. This shows that $\mathrm{CUT}_n^\square$ is not 4-neighborly.

The result of Corollary 31.6.3 (ii) is also best possible, i.e., there exists a 3-dimensional face of $\mathrm{CUT}_n^\square$ which is not a face of $\mathrm{MET}_n^\square$ (for $n \geq 5$). The following example is given in Deza and Deza [1995]. Let $n \geq 5$. Consider the face F of $\mathrm{CUT}_n^\square$ which is defined by $F := \mathrm{CUT}_5^\square$ for $n = 5$ and

$$F := \{x \in \mathrm{CUT}_n^\square \mid x_{1i} + x_{2i} + x_{12} = 2 \text{ and } x_{1i} - x_{2i} - x_{12} = 0 \text{ for } i = 6, \ldots, n\}$$

for $n \geq 6$. The cut vectors lying in F are of the form $\delta(S \cup \{1\})$, where $S \subseteq \{2,3,4,5\}$. Therefore, $F \approx \mathrm{CUT}_5^\square$ is a 10-dimensional face of $\mathrm{CUT}_n^\square$ which is not a face of $\mathrm{MET}_n^\square$. Consider the set

$$G := \mathrm{Conv}(\delta(\{1,2\}), \delta(\{1,3\}), \delta(\{1,4\}), \delta(\{1,5\}))$$

and let H denote the face of $\mathrm{MET}_n^\square$ which is defined by the triangle inequalities:

$$x_{1i} + x_{2i} + x_{12} = 2,\ x_{1i} - x_{2i} - x_{12} = 0\ (i = 6, \ldots, n)$$

$$\text{and}\ \ x_{1i} + x_{1j} + x_{ij} = 2\ (2 \leq i < j \leq 5).$$

Then, G is a 3-dimensional face of $\mathrm{CUT}_n^\square$ as

$$G = \{x \in F \mid \sum_{1 \leq i < j \leq 5} x_{ij} = 6 \text{ and } x_{1i} + x_{1j} + x_{ij} = 2 \text{ for } 2 \leq i < j \leq 5\}.$$

But, G is not a face of $\mathrm{MET}_n^\square$. To see it, consider the point $x \in \mathbb{R}^{E_n}$ defined by $x_{1i} = 1$, $x_{2i} = x_{3i} = x_{4i} = x_{5i} = \frac{1}{3}$ for $i = 6, \ldots, n$, $x_{ij} = 0$ for $6 \leq i < j \leq n$, and $x_{ij} = \frac{2}{3}$ for $1 \leq i < j \leq 5$. Then, $x \in H \setminus G$. If G is a face of $\mathrm{MET}_n^\square$, then G is a face of H and, thus, there exists a triangle inequality valid for G and violated by x. Now one can easily check that no such inequality exists. This shows that G is not a face of $\mathrm{MET}_n^\square$.

Even though not every d-dimensional face of $\mathrm{CUT}_n^\square$ is a face of $\mathrm{MET}_n^\square$ when $d \geq 3$, the next result shows that a lot of them remain faces of $\mathrm{MET}_n^\square$ when $d \leq \log_2 n$.

Given $S_1, \ldots, S_k \subseteq V_n$, recall that the cut vectors $\delta(S_1), \ldots, \delta(S_k)$ are said to be in general position if the set

$$\bigcap_{i \in I} S_i \cap \bigcap_{i \notin I} (V_n \setminus S_i)$$

is nonempty, for every subset $I \subseteq \{1, \ldots, k\}$. Then, $k \leq \log_2 n$ and the cut vectors $\delta(S_1), \ldots, \delta(S_k)$ are linearly independent. Deza, Laurent and Poljak [1992] show that cuts in general position form a face; the proof of this result is along the same lines as that of Theorem 31.6.1, but with more technical details. Compare the results in Theorems 31.6.4 and 31.5.9.

Theorem 31.6.4. *Let $\delta(S_1), \ldots, \delta(S_k)$ be k cut vectors in general position. Then, the set* $\mathrm{Conv}(\delta(S_1), \ldots, \delta(S_k))$ *is a face of* $\mathrm{MET}_n^\square$ *and, thus, of* $\mathrm{CUT}_n^\square$. ∎

Therefore, $\mathrm{CUT}_n^\square$ and $\mathrm{MET}_n^\square$ share a lot of common faces, at least up to dimension $\lfloor \log_2 n \rfloor$. This is an indication that the semimetric polytope is wrapped quite tightly around the cut polytope.

31.6.2 Small Polytopes

We group here some results and questions related to facets/vertices of the cut polytope $\mathrm{CUT}_n^\square$ and the semimetric polytope $\mathrm{MET}_n^\square$, especially for the small values of n, $n \leq 7$. The reader may consult Deza [1994, 1996] for a detailed survey on various combinatorial and geometric properties of these polyhedra.

n	# facets of CUT_n	# facets of $\mathrm{CUT}_n^\square$	# orbits of facets
3	3	4	1
4	12	16	1
5	40	56	2
6	210	368	3
7	38,780	116,764	11
8	49,604,520	217,093,472	147

Figure 31.6.5: Number of facets of cut polyhedra for $n \leq 8$

All the facets of the cut cone CUT_n and the cut polytope $\mathrm{CUT}_n^\square$ are known for $n \leq 7$; they were described in Section 30.6. The extreme rays of MET_n and the vertices of $\mathrm{MET}_n^\square$ are also known for $n \leq 7$; the extreme rays of MET_7 were computed by Grishukhin [1992a] and the vertices of $\mathrm{MET}_7^\square$ by Deza, Deza and Fukuda [1996]. For $n \leq 6$, they are very simple. Namely, besides the cut vectors (that are all the integral vertices), all of them arise from the vector $(2/3, \ldots, 2/3)$ after possibly applying switching[10] and gate 0-extensions[11]. Fig-

[10]The metric polytope being preserved under the switching operation, its set of vertices is partitioned into switching classes. Namely, if x is a vertex of $\mathrm{MET}_n^\square$, then all vectors in its switching class $\{r_{\delta(A)}(x) \mid A \subseteq V_n\}$ are also vertices of $\mathrm{MET}_n^\square$. For instance, the cut vectors form a single switching class.

[11]Given $x \in \mathbb{R}^{E_n}$, we remind that its gate 0-extension is the vector $y \in \mathbb{R}^{E_{n+1}}$ defined by $y_{ij} := x_{ij}$ for $ij \in E_n$, $y_{1,n+1} := 0$, $y_{i,n+1} := x_{1i}$ for $i = 2, \ldots, n$. It can be easily verified that

ures 31.6.5 and 31.6.6 summarize information on the number of facets/vertices of the cut and semimetric polyhedra. Data for CUT_8 and $\mathrm{CUT}_8^\square$ come from Christof and Reinelt [1996]. (We remind that orbits are obtained by action of switching and permutations.)

n	# extreme rays of MET_n	# vertices of $\mathrm{MET}_n^\square$	# orbits of vertices
3	3	4	1
4	7	8	1
5	25	32	2
6	296	544	3
7	55,226	275,840	13

Figure 31.6.6: Number of extreme rays/vertices of semimetric polyhedra for $n \leq 7$

Much information is known about the 1-skeleton graph of $\mathrm{MET}_n^\square$ and about the ridge graphs[12] of $\mathrm{MET}_n^\square$ and $\mathrm{CUT}_n^\square$. We quote here some facts and questions and refer to the original papers or to the survey by Deza [1996] for more details.

The 1-Skeleton Graph of the Semimetric Polytope. As the semimetric polytope $\mathrm{MET}_n^\square$ is preserved under the switching operation, this induces a partition of its vertices into switching classes. The cut vectors form a single switching class, which is a clique in the 1-skeleton graph of $\mathrm{MET}_n^\square$ (by Theorem 31.6.1). On the other hand, it is shown in Laurent [1996c] that every other switching class of vertices is a stable set in the 1-skeleton graph of $\mathrm{MET}_n^\square$; that is, no two nonintegral switching equivalent vertices of $\mathrm{MET}_n^\square$ form an edge on $\mathrm{MET}_n^\square$. The following conjecture is posed by Laurent and Poljak [1992].

Conjecture 31.6.7. *Every fractional vertex of* $\mathrm{MET}_n^\square$ *is adjacent to some cut vector (i.e., to some integral vertex of* $\mathrm{MET}_n^\square$*). Equivalently, for every fractional vertex x of* $\mathrm{MET}_n^\square$*, some switching $r_{\delta(S)}(x)$ of it lies on an extreme ray of* MET_n.

This can be seen as an analogue of the following property, shared by the facets of the cut polytope: For every facet of the cut polytope there exists a switching of it that contains the origin. A consequence of Conjecture 31.6.7 would be that the 1-skeleton graph of $\mathrm{MET}_n^\square$ has diameter ≤ 3. Conjecture 31.6.7 has been verified for several classes of vertices (see Laurent [1996c]) and for $n \leq 7$ (see Deza, Deza and Fukuda [1996]).

Adjacency has been analyzed in detail for some classes of vertices. Given a subset $S \subseteq V_n$, let $d(K_{S,V_n \setminus S})$ denote the path metric of the complete bipartite graph with node bipartition $(S, V_n \setminus S)$. Then, $x_S := \frac{1}{3} d(K_{S,V_n \setminus S})$ is a vertex of $\mathrm{MET}_n^\square$ (taking value $\frac{1}{3}$ on the edges of the bipartition and value $\frac{2}{3}$ elsewhere).

y is a vertex of $\mathrm{MET}_{n+1}^\square$ whenever x is a vertex of $\mathrm{MET}_n^\square$.

[12]Let P be a d-dimensional polyhedron. Its *ridge graph* is the graph with node set the set of facets of P and with two facets being adjacent if their intersection has dimension $d-2$.

The vertices x_S ($S \subseteq V_n$) form a switching class. The adjacency relations between the cut vectors $\delta(S)$ and the vertices x_T (for $S, T \subseteq V_n$) are described in Deza and Deza [1994b]. Namely, the two vertices $\delta(S)$ and x_T are adjacent on $\mathrm{MET}_n^\square$ if and only if the cut vectors $\delta(S)$ and $\delta(T)$ are not adjacent in the folded n-cube graph, i.e., if $|S\triangle T| \neq 1, n-1$. In the case $n = 5$, the cut vectors $\delta(S)$ and the vectors x_T (for $S, T \subseteq V_5$) form all the vertices of $\mathrm{MET}_5^\square$. Hence, the 1-skeleton graph of $\mathrm{MET}_5^\square$ is completely known; its diameter is equal to 2.

Deza and Deza [1994b] analyze adjacency among further vertices of the form: cut vectors $\delta(S)$, x_T ($S, T \subseteq V_n$) and their gate extensions. This permits, in particular, to describe the 1-skeleton graph of $\mathrm{MET}_6^\square$, whose diameter is equal to 2.

The vertices of $\mathrm{MET}_7^\square$ and their adjacencies are described in Deza, Deza and Fukuda [1996]; in particular, the 1-skeleton graph of $\mathrm{MET}_7^\square$ has diameter 3. Figure 31.6.8 shows the 13 orbits of vertices of $\mathrm{MET}_7^\square$; for each orbit O_i, a representative vertex v_i is given as well as its cardinality $|O_i|$, the number A_i of neighbors of v_i in the 1-skeleton graph and the number I_i of triangle facets containing v_i.

Orbit	Representative vertex v_i	$\lvert O_i\rvert$	I_i	A_i
O_1	$(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$	64	105	55 226
O_2	$\frac{2}{3}(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$	64	35	896
O_3	$\frac{2}{3}(1,1,1,1,1,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$	1 344	40	763
O_4	$\frac{2}{3}(1,1,1,1,0,1,1,1,1,1,0,1,1,1,1,1,1,1,1,1,1)$	6 720	45	594
O_5	$\frac{2}{3}(1,1,1,1,0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0)$	2 240	49	496
O_6	$\frac{1}{4}(1,2,3,1,2,1,1,2,2,1,2,1,1,2,3,2,3,2,1,2,1)$	20 160	30	96
O_7	$\frac{1}{3}(1,1,1,1,1,1,2,2,1,1,1,2,1,1,1,1,1,1,2,2,2)$	4 480	26	76
O_8	$\frac{2}{5}(2,1,1,1,1,2,2,1,1,1,1,2,1,1,1,2,1,1,2,1,2)$	23 040	28	57
O_9	$\frac{1}{3}(2,2,1,1,1,2,2,1,1,1,1,2,1,1,1,2,1,1,2,1,2)$	40 320	22	46
O_{10}	$\frac{1}{3}(1,1,1,1,1,1,2,2,1,1,1,2,1,1,1,2,1,1,2,2,2)$	40 320	23	39
O_{11}	$\frac{2}{7}(1,2,3,2,1,2,1,2,1,2,1,1,2,1,1,1,2,2,1,1,1)$	40 320	25	30
O_{12}	$\frac{1}{5}(3,2,3,3,1,1,1,2,2,2,2,3,3,3,3,4,4,2,2,4,2)$	16 128	25	27
O_{13}	$\frac{1}{6}(1,2,4,2,2,2,1,3,3,3,3,2,2,2,4,2,2,2,4,4,4)$	80 640	23	24
Total		275 840		

Figure 31.6.8: The orbits of vertices of $\mathrm{MET}_7^\square$

The Ridge Graph of the Semimetric Polytope. The ridge graph G_n of the semimetric polytope $\mathrm{MET}_n^\square$ is studied in detail in Deza and Deza [1994b]. The graph G_n has $4\binom{n}{3}$ vertices and, for $n \geq 4$, two triangle facets are adjacent in G_n if and only if they are nonconflicting. (Two triangle inequalities are said to be *conflicting* if there exists a pair ij such that the two inequalities have nonzero coordinates of distinct signs at the position ij.) For instance, $G_3 = K_4$ and

$G_4 = K_4 \times K_4$. More generally, for $n \geq 4$ the complement of G_n is locally[13] the bouquet[14] of $n-3$ copies of $K_3 \times K_3$ along a common K_3; its valency is $k = 3(2n-5)$, two adjacent nodes have $\lambda \in \{2(n-2), 4\}$ common neighbors, while two nonadjacent nodes have μ common neighbors with $\mu \in \{4,6\}$ for $n=5$ and $\mu \in \{0,4,6\}$ for $n \geq 6$. In particular, the diameter of G_n is equal to 2 for $n \geq 4$. Note that the complement of the ridge graph G_5 of $\mathrm{MET}_5^\square$ provides an example[15] of a regular graph of diameter 2 in which the number of common neighbors to two arbitrary nodes belongs to $\{\lambda, \mu\} = \{4,6\}$.

Deza and Deza [1994b] also describe the ridge graph G'_n of the semimetric cone MET_n, which is an induced subgraph of G_n. Namely, for $n \geq 4$, the complement of G'_n is locally the bouquet of $n-3$ copies of the circuit C_6 along a common edge. The graph G'_n has diameter 2 for $n \geq 4$.

The Ridge Graph of the Cut Polytope. The ridge graph of the cut polytope $\mathrm{CUT}_n^\square$ is studied in Deza and Deza [1994a]. (It suffices to consider the case $n \geq 5$ as $\mathrm{CUT}_n^\square = \mathrm{MET}_n^\square$ for $n \leq 4$.) The ridge graph of $\mathrm{CUT}_n^\square$ is described there for $n \leq 7$. In particular, two facets of $\mathrm{CUT}_5^\square$ are adjacent in the ridge graph if and only if they are nonconflicting, but this is not true for $n \geq 6$. The ridge graph of $\mathrm{CUT}_n^\square$ has diameter 2 for $n = 4,5$, diameter 3 for $n = 6$ and its diameter belongs to $\{3,4\}$ for $n = 7$. The following conjecture is posed in Deza and Deza [1994a].

Conjecture 31.6.9. *Every facet of* $\mathrm{CUT}_n^\square$ *is adjacent to at least one triangle facet in the ridge graph of* $\mathrm{CUT}_n^\square$.

This conjecture would imply that the ridge graph of $\mathrm{CUT}_n^\square$ has diameter ≤ 4. The conjecture is shown to hold for $n \leq 7$. Further properties and questions, also concerning the ridge graph of the cut cone, can be found in Deza and Deza [1994a].

n	vol $\mathrm{MET}_n^\square$	vol $\mathrm{CUT}_n^\square$	ratio ρ_n
3	1/3	1/3	100%
4	2/45	2/45	100%
5	4/1701	32/14,175	$\sim$ 96%
6	71,936/1,477,701,225	2384/58,046,625	$\sim$ 84%

Figure 31.6.10: Volumes of cut and semimetric polytopes for $n \leq 6$

Further combinatorial properties of cut and semimetric polyhedra have been studied. For instance, Deza and Deza [1995] have completely described the face lattices of both the cut polytope $\mathrm{CUT}_n^\square$ and the semimetric polytope $\mathrm{MET}_n^\square$

[13] The local structure of a graph G is the subgraph induced by the neighbors of any given vertex, assuming that these induced subgraphs are the same at all the vertices.

[14] Let $G = (V, E)$ be a graph and, for $U \subset V$, let $H = G[U]$ be an induced subgraph of G. Let $G_i = (V_i, E_i)$ $(i = 1, \ldots, k)$ be k isomorphic copies of G such that $V_i \cap V_j = U$ for all $i \neq j$. Then, the graph $(\cup_{i=1}^k V_i, \cup_{i=1}^k E_i)$ is called the bouquet of the k copies of G along H.

[15] This generalization of the notion of strongly regular graph is studied in Erickson et al. [1996].

for $n \leq 5$. Deza, Deza and Fukuda [1996] give the edge connectivity of the adjacency and ridge graphs for cut and semimetric polytopes. To conclude we mention some facts about the volume of cut and semimetric polyhedra.

A way of measuring the tightness of the relaxation of $\mathrm{CUT}_n^\square$ by $\mathrm{MET}_n^\square$ could be by considering the ratio

$$\rho_n := \frac{\mathrm{vol}\ \mathrm{CUT}_n^\square}{\mathrm{vol}\ \mathrm{MET}_n^\square}$$

of their volumes. Unfortunately, computing the volume of a polytope is a hard task in general. These volumes have been computed in the case $n \leq 6$ in Deza, Deza and Fukuda [1996]; we report the results in Figure 31.6.10.

31.7 Distance of Facets to the Barycentrum

We are interested here in evaluating what is the minimum possible distance of a facet to the barycentrum of $\mathrm{CUT}_n^\square$. Most of the results here come from Deza, Laurent and Poljak [1992].

Let $b := \left(\sum_{S\subseteq V_n | 1\notin S} \delta(S)\right)/2^{n-1}$ denote the barycentrum of $\mathrm{CUT}_n^\square$. Then,

$$b = (1/2, \ldots, 1/2).$$

The Euclidean distance from b to the hyperplane defined by the equation: $v^T x = \alpha$ is given by the formula:

$$\frac{|v^T b - \alpha|}{\| v \|_2}.$$

It can be easily checked that the distance from b to a facet F remains invariant if we replace F by a switching of it. In particular, the distance from b to any triangle facet is equal to $\frac{1}{2\sqrt{3}}$. The following conjecture is posed by Deza, Laurent and Poljak [1992].

Conjecture 31.7.1. *The distance from the barycentrum b to any facet of* $\mathrm{CUT}_n^\square$ *is greater than or equal to $\frac{1}{2\sqrt{3}}$, this smallest distance being attained precisely by the triangle facets.*

They show that this conjecture holds for all pure facets, i.e., for all the facets that are defined by an inequality with $0, \pm 1$-coefficients.

Theorem 31.7.2. *Let $v^T x \leq \alpha$ be an inequality defining a facet of* $\mathrm{CUT}_n^\square$ *and such that $v \in \{-1, 0, 1\}^{E_n}$. Then, the distance from this facet to the barycentrum b is greater than or equal to $(2\sqrt{3})^{-1}$. Moreover, this smallest distance is realized precisely when $v^T x \leq \alpha$ is a triangle inequality.* ∎

The proof of Theorem 31.7.2 relies on establishing a good lower bound for the max-cut problem in the graph K_n with edge weights v. For a vector $v \in \mathbb{R}^{E_n}$, set

$$\mathrm{mc}(K_n, v) := \max(v^T \delta(S) \mid S \subseteq V_n).$$

Then, it shown in Deza, Laurent and Poljak [1992] that

$$\mathrm{mc}(K_n, v) \geq \left(\sum_{1 \leq i < j \leq n} v_{ij} \right) /2 + \parallel v \parallel_2 (2\sqrt{3})^{-1}$$

for every $v \in \{0, \pm 1\}^{E_n}$. Note that, if one can prove that this inequality remains valid for *any* $v \in \mathbb{R}^{E_n}$, then Conjecture 31.7.1 would follow.

It may be instructive to evaluate the exact distance to the barycentrum for some concrete classes of facets. For instance, let $D(r,p)$ denote the distance from the barycentrum b to the hyperplane defined by the clique-web inequality:

$$\mathrm{CW}^r_{2p-2r-1}(1, \ldots, 1, -1, \ldots, -1)^T x \leq 0$$

(with p coefficients $+1$ and $p - 2r - 1$ coefficients -1). Then,

$$D(r,p) = \frac{r+1}{2} \sqrt{\frac{p-2r-1}{2p-r-1}}.$$

Hence, for $r = 0$ (hypermetric case), $D(0,p) = \frac{1}{2}\sqrt{\frac{p-1}{2p-1}}$, which is asymptotically $\frac{1}{2\sqrt{2}} (> \frac{1}{2\sqrt{3}})$ when $p \longrightarrow \infty$. In the case $p = 2r + 3$ (the case of the bicycle odd wheel inequality), $D(r, 2r+3) = \frac{r+1}{\sqrt{6r+10}}$, which tends to $\frac{1}{\sqrt{6}} (> \frac{1}{2\sqrt{3}})$ as $r \longrightarrow \infty$.

One can also check that the distance from b to the hyperplane defined by the (nonpure) clique-web inequality:

$$\mathrm{CW}^r_{p(r+1)-2r-1}(r, \ldots, r, -1, \ldots, -1)^T x \leq 0$$

(with p coefficients r and $pr - 2r - 1$ coefficients -1) is asymptotically $\frac{1}{\sqrt{2}}$ as $r, p \longrightarrow \infty$.

We show in Figure 31.7.3 what is the exact distance to the barycentrum for each of the eleven types of facets of $\mathrm{CUT}^{\square}_7$. These eleven types of facets are listed as in Section 30.6 as F_i for $i = 1, \ldots, 11$. The second row in Figure 31.7.3 gives the exact value for the distance $D(F)$ from the barycentrum b to the hyperplane containing the facet F. The third row gives an approximate value for $D(F) \cdot 2\sqrt{3}$, that is, the ratio $\frac{D(F)}{D(F_1)}$, where F_1 is the triangle facet. Hence, the pentagonal facet is the next closest facet, while the facet F_8 is the farthest one.

F	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}
dist.	$\frac{1}{2\sqrt{3}}$	$\frac{1}{\sqrt{10}}$	$\frac{2}{\sqrt{31}}$	$\frac{1}{2}\sqrt{\frac{3}{7}}$	$\frac{1}{2}\sqrt{\frac{6}{11}}$	$\frac{7}{2\sqrt{69}}$	$\frac{1}{2}$	$\frac{5}{2\sqrt{11}}$	$\frac{9}{2\sqrt{133}}$	$\sqrt{\frac{2}{7}}$	$\frac{5}{2\sqrt{29}}$
ratio	1	1.09	1.24	1.13	1.28	1.46	1.73	2.61	1.35	1.85	1.61

Figure 31.7.3: Distance to the barycentrum of the facets of CUT_7

We conclude with two related results, concerning the width and the diameter of the cut polytope. Given a polytope P, its *width* is defined as

$$\mathrm{width}(P) := \min_{\|c\|_2=1} (\max_{x\in P} c^T x - \min_{x\in P} c^T x).$$

The diameter of P has already been defined in Section 31.1 as $\max_{x,y\in P} \| x-y \|_2$. It is easy to see that it can be alternatively defined as

$$\mathrm{diam}(P) = \max_{\|c\|_2=1} (\max_{x\in P} c^T x - \min_{x\in P} c^T x).$$

In other words, the width and the diameter are, respectively, the smallest and the largest distance between two supporting hyperplanes for P. G. Rote (personal communication) has computed the width of $\mathrm{CUT}_n^\square$.

Proposition 31.7.4. *The width of the cut polytope* $\mathrm{CUT}_n^\square$ *is equal to* 1.

Proof. The proof is based on the following inequality: Let $a_1, \ldots, a_N \in \mathbb{R}$ be N scalars such that $\sum_{i=1}^N a_i = 0$ and $\sum_{i=1}^N a_i^2 = N$. Then,

$$\max_i a_i - \min_i a_i \geq 2.$$

(Indeed, say, $a_1 \leq \ldots \leq a_N$ and set $s := \frac{a_1+a_N}{2}$. If $a_N - a_1 < 2$ then $|a_i - s| < 1$ for all i. Hence, $a_i^2 + s^2 - 2a_i s < 1$ for all i. By summing over i, we obtain that $N + Ns^2 < N$, a contradiction.)
Let $c \in \mathbb{R}^{E_n}$ with $\| c \|_2 = 1$. For $S \subseteq V_n$, set $x_S := e - 2\delta(S)$ (where e denotes the all-ones vector) and set $a_S := c^T x_S$. Then, it is easy to check that $\sum_S a_S = 0$ and $\sum_S (a_S)^2 = N (:= 2^{n-1})$. Applying the above inequality, we obtain that $2 \leq \max_S c^T x_S - \min_S c^T x_S$. This shows that $1 \leq \max_S c^T \delta(S) - \min_S c^T \delta(S)$. Hence, the width of $\mathrm{CUT}_n^\square$ is greater than or equal to 1. The value 1 is attained, for instance, by taking for c a coordinate vector. Hence, $\mathrm{CUT}_n^\square$ has width 1. ∎

Hence, the cut polytope has the same width as the unit hypercube. Poljak and Tuza [1995] have computed the diameter of the cut polytope.

Proposition 31.7.5. *The diameter of* $\mathrm{CUT}_n^\square$ *is equal to* $\sqrt{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil}$, *that is, to* $\frac{n}{2}$ *if n is even and to* $\frac{\sqrt{n^2-1}}{2}$ *if n is odd.*

Proof. Set $\alpha := \sqrt{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil}$. Let $c \in \mathbb{R}^{E_n}$ with Euclidean norm 1. Let $\delta(S)$ and $\delta(T)$ be two cut vectors realizing, respectively, the maximum and the minimum of $c^T x$ over $x \in \mathrm{CUT}_n^\square$. Define $c' := c^{\delta(T)}$, i.e., $c'_{ij} := -c_{ij}$ if $|T \cap \{i,j\}| = 1$ and $c'_{ij} = c_{ij}$ otherwise. Then,

$$c^T \delta(S) - c^T \delta(T) = (c')^T \delta(S \triangle T) \leq \sqrt{|\delta(S \triangle T)|} \leq \alpha$$

(the last but one inequality follows from the fact that $\sum_{1\leq i\leq n} u_i \leq \sqrt{n}$ for any vector $u \in \mathbb{R}^n$ of Euclidean norm 1). On the other hand, the following vector

c realizes equality. Let $S \subseteq V_n$ with $|S| = \lfloor \frac{n}{2} \rfloor$. Set $c_{ij} := \frac{1}{\alpha}$ if $\delta(S)_{ij} = 1$ and $c_{ij} := 0$ if $\delta(S)_{ij} = 0$. Then, $\max c^T x = \alpha$ is attained at $\delta(S)$ and $\min c^T x = 0$ is attained at $\delta(\emptyset)$. ∎

31.8 Simplex Facets

We give here some more information on the simplex faces of $\mathrm{CUT}_n^\square$. We have seen in Section 31.6 that $\mathrm{CUT}_n^\square$ has lots of simplex faces of dimension up to $\lfloor \log_2 n \rfloor$. In fact, $\mathrm{CUT}_n^\square$ has also fairly many simplex facets.

Let us summarize the known classes of simplex facets of $\mathrm{CUT}_n^\square$; for more details, we refer to Deza and Laurent [1993a].

For $n \geq 3$, the hypermetric inequality:

$$Q_n(n-4, 1, 1, -1, \ldots, -1)^T x \leq 0 \tag{31.8.1}$$

defines a simplex facet of $\mathrm{CUT}_n^\square$ (Deza and Rosenberg [1984]; recall Corollary 28.2.12).

For $n \geq 6$, the clique-web inequality:

$$\mathrm{CW}_n^{n-6}(n-4, n-5, n-5, -1, \ldots, -1)^T x \leq 0 \tag{31.8.2}$$

defines a simplex facet of $\mathrm{CUT}_n^\square$ (Deza and Laurent [1992c]). For $n = 6$, the two inequalities (31.8.1) and (31.8.2) coincide. Actually, for $n \leq 6$, all the simplex facets of $\mathrm{CUT}_n^\square$ arise from (31.8.2) (up to permutation and switching).

For $n = 7$, in addition to the simplex facets that can be derived from (31.8.1) and (31.8.2) by permutation and switching, there are four more groups of simplex facets; namely, the clique-web facets defined by the two inequalities:

$$\mathrm{CW}_7^1(2, 2, 1, 1, -1, -, 1, -1)^T x \leq 0,$$

$$\mathrm{CW}_7^1(1, 1, 1, 1, 1, -1, -1)^T x \leq 0,$$

the facet defined by the parachute inequality $(\mathrm{Par}_7)^T x \leq 0$ (recall (30.4.1)), and the facet defined by Grishukhin's inequality $(\mathrm{Gr}_7)^T x \leq 0$ (recall (30.5.1)).

Hence, among the eleven types of facets of $\mathrm{CUT}_7^\square$, six of them are simplices, namely, the ones numbered 6 to 11 in Section 30.6. Therefore, using the data from Figure 30.6.1, one can count the exact number of simplex facets of $\mathrm{CUT}_7^\square$. Among its 116764 facets, $\mathrm{CUT}_7^\square$ has 113536 simplex facets. Hence, about 97.2% of the total number of facets are simplices ! Deza and Deza [1994a] conjecture that this phenomenon is general, i.e., that the great majority of facets of $\mathrm{CUT}_n^\square$ are simplices. They state the following as an attempt to understand the global shape of the cut polytope:

> "We think that the shape of the cut polytope is essentially given by the nonsimplex facets, in particular, by its triangle facets, and that the huge majority of the facets of $\mathrm{CUT}_n^\square$ are simplices which only 'polish' it."

Interestingly, each of the simplex facets described above has the following property (31.8.3) (see Deza and Laurent [1993a] for a proof). Let F denote such a simplex facet and let $\delta(S_k)$ $(1 \leq k \leq \binom{n}{2})$ denote its roots. Let $d \in F$ with decomposition $d = \sum_{1 \leq k \leq \binom{n}{2}} \lambda_k \delta(S_k)$ where $\lambda_k \geq 0$ for all k. Then,

$$\begin{array}{c} d \text{ belongs to the cut lattice } \mathcal{L}_n \\ \text{(i.e., if } d \in \mathbb{Z}^{E_n} \text{ and satisfies the parity condition (24.1.1)),} \\ \Downarrow \\ \text{all } \lambda_k' s \text{ are integers.} \end{array} \tag{31.8.3}$$

In other words, in the terminology of Part IV, the parity condition suffices for ensuring hypercube embeddability for the class of distances $d \in F$.

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