Chapter 1. Outline of the Book

This chapter gives an overview of the topics treated in this book. The central objects in the book are polytopes and cones related to cuts and metrics. Interesting problems concerning these polyhedra arise in many different areas of mathematics and its applications. Surprisingly, these polyhedra have been considered independently by a number of authors with various mathematical backgrounds and motivations. One of our objectives is to show on the one hand, the richness and diversity of the results in connection with these polyhedra, and on the other hand, how they can be treated in a unified way as various aspects of a common set of objects. Research on cuts and metrics profits greatly from the variety of subjects where the problems arise. Observations made in different areas by independent authors turn out to be equivalent, facts are not isolated, and views from different perspectives provide new interpretations, connections and insights.

This book is subdivided into five parts, each treating seemingly diverse topics. Namely, Parts I to V contain results relevant to the following areas:

1. the theory of metrics; more precisely, isometric embeddings into the Banach ℓ_1-space,

2. the geometry of numbers; more precisely, lattices and Delaunay polytopes,

3. graph theory; more precisely, the hypercube and its isometric subgraphs,

4. design theory; more precisely, the designs arising in connection with the various hypercube embeddings of the equidistant metric, together with complexity aspects of the hypercube embeddability problem,

5. geometry of polyhedra; more precisely, geometric questions on the cut and metric polyhedra (e.g., description of their facets, adjacencies, symmetries, etc.); applications to the solution of some problems such as Borsuk's problem, or completion problems for positive semidefinite matrices and Euclidean distance matrices.

We have made each of the five parts as self-contained as possible. For this reason, some notions and definitions may be repeated in different parts if they are central there. In principle, a reader who is interested, for instance, only in the aspects of geometry of numbers of cuts may consult Part II without any prior reading of

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Part I. Chapter 2, however, contains some basic notation on graphs, polyhedra, matrices and algorithms that will be used throughout the book.

In what follows we give a brief overview of the material covered in Parts I to V. This introductory treatment is meant to provide an orientation map through the book for the reader. We already define here several notions, but all of them will be redefined later in the text as they are needed.

1.1 Outline of Part I. Measure Aspects: ℓ_1-Embeddability and Probability

In Part I we study the distance spaces that can be isometrically embedded into the ℓ_1-space $(\mathbb{R}^m, d_{\ell_1})$ for some integer $m \geq 1$. Here, d_{ℓ_1} denotes the ℓ_1-distance defined by

$$d_{\ell_1}(x,y) := \sum_{1 \leq i \leq m} |x_i - y_i| \text{ for } x, y \in \mathbb{R}^m.$$

One of the basic results is a characterization in terms of cut semimetrics. Given a subset S of the set $V_n := \{1, \ldots, n\}$, the *cut semimetric* $\delta(S)$ is the distance on V_n where two elements $i \in S$, $j \in V_n \setminus S$ are at distance 1, while two elements $i, j \in S$, or $i, j \in V_n \setminus S$, are at distance 0. Every cut semimetric is obviously isometrically ℓ_1-embeddable. In fact, a distance d is isometrically ℓ_1-embeddable if and only if it can be decomposed as a nonnegative linear combination of cut semimetrics. In other words, if CUT_n denotes the cone generated by the cut semimetrics on V_n, then

$$d \text{ is isometrically } \ell_1\text{-embeddable} \iff d \in \mathrm{CUT}_n.$$

The cone CUT_n is called the *cut cone.* We also consider isometric embeddings into the hypercube. Call a distance d on V_n *hypercube embeddable* if the distance space (V_n, d) can be isometrically embedded into the space $(\{0,1\}^m, d_{\ell_1})$ (for some $m \geq 1$), i.e., if we can find n binary vectors $v_1, \ldots, v_n \in \{0,1\}^m$ such that

$$d(i,j) = d_{\ell_1}(v_i, v_j) \text{ for all } i, j \in V_n.$$

In fact, the hypercube embeddable distances on V_n are the members of the cut cone CUT_n that can be written as a nonnegative integer combination of cut semimetrics.

Let $\mathrm{CUT}_n^{\square}$ denote the *cut polytope*, which is defined as the convex hull of the cut semimetrics $\delta(S)$ for $S \subseteq V_n$. That is, $\mathrm{CUT}_n^{\square}$ consists of the distances that can be decomposed as convex combinations of cut semimetrics. The cut cone and polytope also admit the following characterization in terms of measure spaces: A distance d belongs to the cut cone CUT_n (resp. the cut polytope $\mathrm{CUT}_n^{\square}$) if and only if there exist a measure space (resp. a probability space) $(\Omega, \mathcal{A}, \mu)$ and n events $A_1, \ldots, A_n \in \mathcal{A}$ such that

$$d(i,j) = \mu(A_i \triangle A_j) \text{ for all } i, j \in V_n.$$

(See Section 4.2 for the above results.)

There is another set of polyhedra that are closely related to cut polyhedra and for which the above interpretation in terms of measure spaces takes a nice form. Given a subset S of V_n, its *correlation vector* $\pi(S)$ is defined by $\pi(S)_{ij} = 1$ if both i, j belong to S and $\pi(S)_{ij} = 0$ otherwise, for $i, j \in V_n$. The cone generated by the correlation vectors $\pi(S)$ for $S \subseteq V_n$ is called the *correlation cone* and is denoted by COR_n. Similarly, $\mathrm{COR}_n^\square$ denotes the *correlation polytope*, defined as the convex hull of the correlation vectors. These polyhedra admit the following characterization (see Section 5.3): A vector p belongs to the correlation cone COR_n (resp. the correlation polytope $\mathrm{COR}_n^\square$) if and only if there exist a measure space (resp. a probability space) $(\Omega, \mathcal{A}, \mu)$ and n events $A_1, \ldots, A_n \in \mathcal{A}$ such that

$$p_{ij} = \mu(A_i \cap A_j) \text{ for all } i, j \in V_n.$$

Hence, the members of the correlation polytope are nothing but the pairwise joint correlations of a set of n events; this explains the name "correlation" polyhedra.

In fact, this result is an analogue of the similar result mentioned above for the cut polyhedra. The point is that the correlation polyhedron COR_n (or $\mathrm{COR}_n^\square$) is the image of the cut polyhedron CUT_{n+1} (or $\mathrm{CUT}_{n+1}^\square$) under a linear bijective mapping (the covariance mapping; see Section 5.2). This is a simple but interesting correspondence as it permits to translate results between cut polyhedra and correlation polyhedra. One of our objectives in this book will be to bring together and give a unified presentation for results that have been obtained by different authors in these two contexts (cut/correlation).

The correlation polytope provides the right setting for a classical question in probability theory, often referred to as the Boole problem and which can be stated as follows: Given n events $A_1, \ldots, A_n$ in a probability space $(\Omega, \mathcal{A}, \mu)$ find a good estimate of the probability $\mu(A_1 \cup \ldots \cup A_n)$ that at least one of these events occurs using the fact that the pairwise correlations $\mu(A_i \cap A_j)$ are known. Tight lower bounds for $\mu(A_1 \cup \ldots \cup A_n)$ can be derived from the valid inequalities for $\mathrm{COR}_n^\square$ (see Section 5.4).

We have now seen that the ℓ_1-embeddable distances on V_n are the members of the cut cone CUT_n. Hence, testing ℓ_1-embeddability amounts to testing membership in the cut cone. This problem turns out to be NP-complete. Moreover, characterizing ℓ_1-embeddability amounts to finding a description of the cone CUT_n by a set of linear inequalities. As CUT_n is a polyhedral cone (since it is generated by the finite set of cut semimetrics), we know that it can be described by a finite list of inequalities. However, finding the full list for arbitrary n is an 'impossible' task if NP $\neq$ co-NP. Nevertheless, large classes of valid inequalities for CUT_n (or $\mathrm{CUT}_n^\square$) are known. We give an up-to-date survey of what is known about the linear description of the cut polyhedra in Part V. Among the known inequalities, the most important ones are the hypermetric inequalities and the negative type inequalities, which are introduced in Section 6.1. They are the inequalities of the form:

$$\sum_{1 \le i < j \le n} b_i b_j x_{ij} \le 0$$

where $b_1, \dots, b_n$ are integers with sum $\sum_{i=1}^n b_i = 1$ (*hypermetric* case) or $\sum_{i=1}^n b_i = 0$ (*negative type* case). Part II will be entirely devoted to hypermetric inequalities and, more specifically, to their link with Delaunay polytopes in lattices. The hypermetric inequalities provide necessary conditions for ℓ_1-embedability. In fact, hypermetricity turns out to be a sufficient condition for ℓ_1-embeddability for several classes of metrics. Several such classes are presented in Chapter 8; they consist of metrics arising from valuated poset lattices, semigroups and normed vector spaces. The negative type inequalities are implied by the hypermetric inequalities. Hence, they provide a weaker necessary condition for ℓ_1-embeddability.

Negative type inequalities are classical inequalities in analysis. They were already used by Schoenberg in the thirties for characterizing the distance spaces that are isometrically ℓ_2-embeddable; namely, Schoenberg proved that a distance d is isometrically ℓ_2-embeddable if and only if the squared distance d^2 satisfies the negative type inequalities. Moreover, the negative type inequalities define a cone which is nothing but the image of the cone of positive semidefinite symmetric matrices (of order $n-1$ if the inequalities are on n points) under a linear bijective mapping (in fact, the same mapping that made the link between cut and correlation polyhedra). These results are presented in Sections 6.2 and 6.3, together with further basic facts on ℓ_2-spaces.

Several additional aspects are treated in Part I, including: operations and functional transforms of distance spaces preserving some metric properties such as ℓ_1-embeddability, hypermetricity, etc. (see Chapters 7 and 9); for given n, the minimum dimension of an ℓ_1-space permitting to embed any ℓ_1-embeddable distance on n points; for given m, the minimum number of points to check in a distance space in order to ensure embeddability in the m-dimensional ℓ_1-space (see Chapter 11).

We consider in Chapter 10 the question of finding Lipschitz embeddings where a small distortion of the distances is allowed. Bourgain [1985] shows that every semimetric on n points can be embedded into some ℓ_1-space with a distortion in $O(\log_2 n)$. We present this result together with an application by Linial, London and Rabinovich [1994] to approximations of multicommodity flows.

1.2 Outline of Part II. Hypermetric Spaces: an Approach via Geometry of Numbers

Part II is entirely devoted to the study of hypermetric inequalities and of their link with some objects of the geometry of numbers, namely, lattices and Delaunay polytopes.

Hypermetric inequalities are the inequalities of the form:

$$\sum_{1 \le i < j \le n} b_i b_j x_{ij} \le 0$$

where $b_1, \dots, b_n$ are integers with sum $\sum_{i=1}^n b_i = 1$. They define a cone, called the *hypermetric cone* and denoted by HYP_n. Note that triangle inequalities are a special case of hypermetric inequalities (obtained by taking all components of

b equal to 0 except two equal to 1 and one equal to -1). Hence, hypermetricity is a strengthening of the notion of semimetric. As every cut semimetric satisfies the hypermetric inequalities, we have

$$\mathrm{CUT}_n \subseteq \mathrm{HYP}_n.$$

This inclusion holds at equality if $n \leq 6$ and is strict for $n \geq 7$. For instance, the path metric of the graph $K_7 \backslash P_3$ is hypermetric but not ℓ_1-embeddable. Actually, the graphs whose path metric is hypermetric are characterized in Chapter 17.

A typical example of a hypermetric space arises from point lattices. Let L be a point lattice in $\mathbb{R}^k$, that is, a discrete subgroup of $\mathbb{R}^k$. Take a sphere $S \subseteq \mathbb{R}^k$ in one of the interstices of L, i.e., such that no point from L lies in the closed ball with boundary S. Blow up S until it is 'held rigidly' by lattice points. Then, the set of lattice points lying on S endowed with the square of the Euclidean distance forms a distance space which is semimetric and, moreover, hypermetric. The convex hull of the set $S \cap L$ of lattice points lying on S forms a polytope, called a *Delaunay polytope*. Hence, Delaunay polytopes have the interesting property that their set of vertices can be endowed with a metric structure which is hypermetric. In other words, for any Delaunay polytope P with set of vertices $V(P)$, the distance space $(V(P), (d_{\ell_2})^2)$ is a hypermetric space. Even more striking is the fact that, conversely, every hypermetric distance space on n points can be isometrically embedded into the space $(V(P), (d_{\ell_2})^2)$ for some Delaunay polytope P of dimension $k \leq n-1$. These results are presented in Section 14.1.

The hypermetric cone HYP_n is defined by infinitely many inequalities. However, it can be shown that a finite number of them suffices to describe HYP_n. In other words, HYP_n is a polyhedral cone. See Section 14.2 where several proofs are given for this result. One of them relies essentially on the above link between hypermetrics and Delaunay polytopes and on Voronoi's finiteness result for the number of types of lattices in fixed dimension.

The correspondence between hypermetrics and Delaunay polytopes permits the translation of several notions from the hypermetric cone to Delaunay polytopes. For instance, one can define the *rank* of a Delaunay polytope as the dimension of the smallest face of the hypermetric cone containing the corresponding hypermetric distance. One can then define, in particular, extreme Delaunay polytopes which correspond to extreme rays of the hypermetric cone. This notion of rank and the correspondence between Delaunay polytopes and faces of the hypermetric cone are investigated in Chapter 15.

The various types of Delaunay polytopes that may arise in root lattices are described in Section 14.3. The extreme Delaunay polytopes among them are classified; there are three of them, namely, the 1-dimensional simplex, the Schläfli polytope 2_{21} (of dimension 6) and the Gosset polytope 3_{21} (of dimension 7) (see Section 16.2). Further examples of extreme Delaunay polytopes are described in Sections 16.3 and 16.4; they arise from other lattices such as the Leech lattice and the Barnes-Wall lattice. Some connections between extreme Delaunay polytopes and equiangular sets of lines or perfect lattices are also mentioned in Sections 16.1 and 16.5.

Chapter 17 studies *hypermetric graphs* in detail, i.e., the graphs whose path metric is hypermetric. These graphs are characterized as the isometric subgraphs of Cartesian products of three types of graphs, namely, half-cube graphs, cocktail-party graphs and the Gosset graph G_{56}. Moreover, ℓ_1-graphs are those for which no Gosset graph occurs in the Cartesian product. Several refined results are presented; in particular, for suspension graphs and for graphs having some regularity properties. Further characterizations are discussed for bipartite graphs equipped with the truncated distance (taking value 1 on edges and value 2 on non-edges).

We encounter in this context the class consisting of the connected regular graphs whose adjacency matrix has minimum eigenvalue greater than or equal to -2. This class is well studied in the literature. Beside line graphs and cocktail-party graphs, it contains a list of 187 graphs, which is subdivided into three groups. Each of these three groups is characterized by some parameter. Interestingly, this parameter has an interpretation in terms of some associated Delaunay polytope using hypermetricity (see Section 17.2). Hence this is an example of a situation where a new approach: hypermetricity, sheds new light on a classical notion.

1.3 Outline of Part III. Embeddings of Graphs

In Part III we study various metric and embeddability properties of graphs. For a connected graph $G = (V, E)$ we consider the associated *path metric* d_G defined on the node set V of G, where the distance between two nodes $i, j \in V$ is defined as the length of a shortest path connecting i and j in G. Our objective in Part III is to investigate the structure of the graphs whose path metric enjoys some metric properties such as ℓ_1-embeddability, hypercube embeddability, hypermetricity, etc.

The graphs which are isometric subgraphs of hypercubes are well understood. Several characterizations are presented in Chapter 19. One of them states that the isometric subgraphs of the hypercube are precisely the bipartite graphs whose path metric satisfies a restricted class of hypermetric inequalities, namely, the pentagonal inequalities (hypermetric inequalities on five points). Being an isometric subgraph of a hypercube means being an isometric subgraph of a Cartesian product of copies of K_2. In Chapter 20 we consider isometric embeddings into arbitrary Cartesian products. The following is a well-known result in the metric theory of graphs: Every graph can be isometrically embedded in a canonical way into a (smallest) Cartesian product, called the *canonical metric representation* of the graph. For bipartite graphs, this representation permits to obtain a decomposition of the path metric as a linear combination of primitive semimetrics. One of the main tools underlying these various results is an equivalence relation defined on the edge set of the graph. The number of equivalence classes is an invariant of the graph, called its *isometric dimension.* The number of factors in the canonical representation is precisely the isometric dimension. Moreover, for a bipartite graph G the isometric dimension of G is equal to the (linear) dimension of the smallest face of the semimetric cone that contains d_G.

In Chapter 21 we study *ℓ_1-graphs* in detail, i.e., the graphs whose path metric is ℓ_1-embeddable. This constitutes a relaxation of hypercube embeddability. Indeed a graph G is an ℓ_1-graph if and only if its path metric d_G is hypercube embeddable up to scale, i.e., if ηd_G is hypercube embeddable for some integer η. The smallest such η is called the *minimum scale* of the graph. It is shown that the minimum scale of an ℓ_1-graph is equal to 1 or to an even number and that it is less than or equal to $n-2$ (n is the number of nodes of the graph). For ℓ_1-graphs the factors in the canonical representation are of a very special type; indeed, they are either half-cube graphs or cocktail-party graphs. This result is already proved in Section 14.3, using the connection with Delaunay polytopes. Another proof is given in Chapter 21 which is elementary and has several important applications. In particular, it yields a polynomial time algorithm for recognizing ℓ_1-graphs as well as as a characterization for ℓ_1-rigid graphs (the graphs having an essentially unique ℓ_1-embeding).

The ℓ_1-graphs with minimum scale 1 or 2 are precisely those that can be isometrically embedded into some half-cube graph. They can, moreover, be characterized in terms of some forbidden isometric subspaces (see Section 21.4).

1.4 Outline of Part IV. Hypercube Embeddings and Designs

In Part IV we investigate in detail the hypercube embeddability problem. Given a distance d on V_n, one may ask the following questions: Is d hypercube embeddable ? If yes, does d admit a unique hypercube embedding ? If this is the case we say that d is h-rigid. (Here, "unique" means unique up to certain trivial operations.) If d is not h-rigid, then what are the possible hypercube embeddings of d ?

There are some classes of metrics for which the first question has trivially a positive answer. This is the case, for instance, for the *equidistant metric* $2t\mathbb{1}_n$ where $t \geq 1$ is an integer; $2t\mathbb{1}_n$ denotes the metric on V_n taking value $2t$ for every pair of distinct points. Then, only the last two questions about the number of hypercube embeddings are of interest.

On the other hand, there are some classes of metrics for which deciding hypercube embeddability is a hard task. In fact, testing hypercube embeddability for general metrics is an NP-hard problem. Nevertheless, for some classes of metrics, one is able to characterize their hypercube embeddability by a set of conditions which can be tested in polynomial time. Several such classes are presented in Chapter 24. Among them, we examine the classes of metrics taking two distinct values of the form: a, $2a$ ($a \geq 1$ integer), or three distinct values of the form: $a, b, a+b$ ($a, b \geq 1$ integers not both even). For instance, testing hypercube embeddability for the class of distances on n points with values in the set $\{2,4\}$, or $\{1,2,3\}$, or $\{3,5,8\}$ can be done in time polynomial in n. On the other hand, this same problem is NP-complete for the class of distances with values in the set $\{2,3,4,6\}$. We also examine the class of metrics having a "bipartite structure"; by this we mean the metrics on V_n for which there exists a subset S of points such that any two points of S (or of its complement) are at distance 2. One of the main tools used for recognizing hypercube embeddability for the above classes of metrics is that they contain large equidistant submetrics that are h-rigid; this fact allows us to infer information on the structure of the metrics from the local structure of some of their submetrics.

Chapters 22 and 23 deal essentially with the equidistant metric $2t\mathbb{1}_n$. In Chapter 22 we give some conditions on n and t under which the metric $2t\mathbb{1}_n$ is h-rigid. For instance, if $n \geq t^2+t+3$, then $2t\mathbb{1}_n$ is h-rigid. Moreover, for $n = t^2+t+2$ with $t \geq 3$, the metric $2t\mathbb{1}_n$ is h-rigid if and only if there does not exist a projective plane of order t. In Chapter 23 we examine the possible hypercube embeddings of the metric $2t\mathbb{1}_n$ when it is not h-rigid. An easy observation is that the possible hypercube embeddings of $2t\mathbb{1}_n$ correspond to the $(2t,t,n-1)$-designs (a $(2t,t,n-1)$-design being a collection $\mathcal{B}$ of subsets of V_{n-1} such that every point of V_{n-1} belongs to $2t$ members of $\mathcal{B}$ and every two points of V_{n-1} belong to t common members of $\mathcal{B}$). This leads to the question of finding such designs with specified parameters. This topic is treated in detail in Chapter 23. For instance, a well-known result by Ryser asserts that any design corresponding to a hypercube embedding of $2t\mathbb{1}_n$ has at least $n-1$ blocks, with equality if and only if $n = 4t$ and there exists a Hadamard matrix of order $4t$. Hence, two important classes of designs: projective planes and Hadamard designs, play an important role in the study of the variety of embeddings of the equidistant metric. An explicit description of all the possible hypercube embeddings of $2t\mathbb{1}_n$ is given in Section 23.4 for the following restricted parameters: $t \leq 2$ and $(t=3, n=5)$.

In Chapter 25 we group results related to cut lattices. The cut lattice consists of the vectors that can be written as an integer combination of cut semimetrics. Note that belonging to both the cut cone and to the cut lattice is a necessary condition for a distance d to be hypercube embeddable. In Section 25.3 we study the graphs whose family of cuts forms a Hilbert basis; this amounts to studying (in the context of arbitrary graphs) the case when the above necessary condition is also sufficient. In Section 25.1 we give a description of the cut lattice and of several related lattices. Constructions are presented in Section 25.2 for distances that belong to the cut cone and to the cut lattice but that are not hypercube embeddable.

1.5 Outline of Part V. Facets of the Cut Cone and Polytope

In Part V we survey known results about the facial structure and the geometry of the cut cone and of the cut polytope.

A fundamental property is that all the facets of the cut polytope $\mathrm{CUT}_n^\square$ can be obtained from the facets of $\mathrm{CUT}_n^\square$ that contain a given vertex; this is derived by the so-called switching operation. In particular, all the facets of $\mathrm{CUT}_n^\square$ can be derived from the facets of the cut cone CUT_n. Therefore, for the purpose of investigating the facial structure, it suffices to consider the cut cone. As we have already mentioned, finding a complete linear description of the cut polyhedra is probably a hopeless task. Nevertheless, large classes of inequalities are known. Two classes have already been introduced; they are the hypermetric inequalities and the negative type inequalities. The negative type inequalities never define facets of the cut cone as they are implied by the hypermetric inequalities. On the other hand, the hypermetric inequalities contain large subclasses of facets; they are investigated in Chapter 28.

Triangle inequalities, a very special case of hypermetric inequalities, are considered in detail in Chapter 27. Despite their simplicity, the triangle facets already contain a considerable amount of information about the cut polyhedra. For instance, they provide an integer programming formulation for cuts. Moreover, the triangle inequalities provide the complete linear description of the cut cone CUT_n for $n \leq 4$. Their projections suffice to describe the cut polyhedron of an arbitrary graph G if G does not have K_5 as a graph minor (and, hence, if G is planar).

We make in Section 27.4 a detour to cycle polyhedra of binary matroids. Cycle spaces of binary matroids are nothing but set families that are closed under taking symmetric differences. Hence, the family of cuts in a graph is an instance of cycle space. The switching operation applies in the general framework of binary matroids and there are analogues of the triangle inequalities (in fact, of their projections) for the cycle polyhedra. Hence, several questions that are raised for cut polyhedra can be posed in the general setting of binary matroids; for instance, about linear relaxations by the triangle inequalities or about Hilbert bases. We review in Section 27.4 the main results in this area.

Hypermetric and negative type inequalities belong to the larger class of gap inequalities, described in Section 28.4. Although gap inequalities themselves are not well understood, a weakening of them (obtained by loosening their right-hand sides) serves as a basis for obtaining very good approximations for the max-cut problem (see Section 28.4.1).

In Chapter 29 we study the clique-web inequalities, that constitute a generalization of hypermetric inequalities. In Chapter 30 we present several other classes of inequalities: suspended tree inequalities, path-block-cycle inequalities, circulant inequalities, parachute inequalities, etc.. Section 30.6 contains the complete linear description of the cone CUT_n for $n \leq 7$.

Chapter 31 contains several geometric properties of the cut polytope $\mathrm{CUT}_n^\square$ and of its relaxation by the semimetric polytope $\mathrm{MET}_n^\square$ (defined by the triangle inequalities). In Section 31.6 we study adjacency properties of these polytopes. For instance, any two cuts are adjacent on both the cut polytope and the semimetric polytope. Hence, the 1-skeleton graph of the cut polytope is the complete graph. Moreover, $\mathrm{CUT}_n^\square$ has many simplex faces in common with $\mathrm{MET}_n^\square$ of dimension up to $\lfloor \log_2 n \rfloor$. This indicates that $\mathrm{MET}_n^\square$ is wrapped quite tightly around $\mathrm{CUT}_n^\square$. In Section 31.7, the Euclidean distance from the hyperplane supporting a facet of $\mathrm{CUT}_n^\square$ to the barycentrum of $\mathrm{CUT}_n^\square$ is considered. It is conjectured that this distance is minimized by triangle facets. The conjecture is verified for all facets defined by an inequality with coefficients in $\{0, 1, -1\}$ and asymptotically for some other cases. Simplex facets are considered in Section 31.8. It turns out that for $n \leq 7$ the great majority of facets of $\mathrm{CUT}_n^\square$ are simplices. In fact about 97% of the facets of $\mathrm{CUT}_7^\square$ are simplices ! This may well be a general phenomenon for any n.

Further geometric results are presented in Sections 31.1-31.4. Borsuk [1933] asked whether it is possible to partition every set X of points in $\mathbb{R}^d$ into $d+1$ subsets, each having a smaller diameter than X. This question was answered in the negative by Kahn and Kalai [1993] by a construction using cuts, that we present in Section 31.1. The result in Section 31.2 indicates how to obtain valid inequalities for pairwise angles among a set of vectors from the valid inequalities for the cut polytope. This permits in particular to answer an old question of Fejes Tóth [1959] concerning the maximum value for the sum of the pairwise angles among a set of n vectors. Section 31.3 deals with the completion problem for partial positive semidefinite matrices. It turns out that necessary conditions for this completion problem can be obtained from the valid inequalities for the cut polytope, as a reformulation of the result in Section 31.2. Finally, Section 31.4 deals with the completion problem for partial Euclidean matrices; that is, with the study of the projections of the negative type cone. These two completion problems are closely related and have intimate links with the polyhedra under investigation in this book.

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