Reader Guidelines

The basic question each author should pose him/herself, preferably in the future tense before starting, is

Why have we written this book?

In our case the motivation came from many discussions we had with mathematicians, economists, engineers and physicists, mainly working in insurance companies, banks or other financial institutions. Often, these people had as students learnt the more classical theory of stochastics (probability theory, stochastic processes and statistics) and were interested in its applications to insurance and finance. In these discussions notions like *extremes, Pareto, divergent moments, leptokurtosis, tail events, Hill estimator* and many, many more would appear. Invariably, a question would follow, "Where can I read more on this?" An answer would usually involve a relatively long list of books and papers with instructions like "For this, look here, for that, perhaps you may find those papers useful, concerning the other, why not read ...". You see the point! After years of frustration concerning the non–existence of a relevant text we decided to write one ourselves. You now hold the fruit of our efforts: a book on the modelling of extremal events with special emphasis on applications to insurance and finance. The latter fields of application were mainly motivated by our joint research and teaching at the ETH where various chapters have been used for many years as *Capita Selecta* in the ETH programme on insurance mathematics. Parts of the book have also formed the basis for a Summer School of the Swiss Society of Actuaries (1994) and the Master's Programme in Insurance and Finance at ESSEC, Paris (1995). These trials have invariably led to an increase in the size of the book, due to

P. Embrechts et al., *Modelling Extremal Events*, Stochastic Modelling and Applied Probability 33, DOI 10.1007/978-3-642-33483-2_1,

questions like “Couldn’t you include this or that?”. Therefore, dear reader, you are holding a rather hefty volume. However, as in insurance and finance where everything is about “operational time” rather than real time, we hope that you will judge the “operational volume” of this book, i.e. measure its value not in physical weight but in “information” weight.

For whom have we written this book?

As already explained in the previous paragraph, in the first place for all those working in the broader financial industry faced with questions concerning extremal or rare events. We typically think of the actuarial student, the professional actuary or finance expert having this book on a corner of the desk ready for a quick freshen–up concerning a definition, technique, estimator or example when studying a particular problem involving extremal events. At the same time, most of the chapters may be used in teaching a special–topics course in insurance or mathematical finance. As such both undergraduate as well as graduate students interested in insurance and/or finance related subjects will find this text useful: the former because of its development of specific techniques in analysing extremal events, the latter because of its comprehensive review of recent research in the larger area of extreme value theory. The extensive list of references will serve both. The emphasis on economic applications does not imply that the intended readership is restricted to those working on such problems. Indeed, most of the material presented is of a much more general nature so that anyone with a keen interest in extreme value theory, say, or more generally interested in how classical probabilistic results change if the underlying assumptions allow for larger shocks in the system, will find useful material in it. However, the reader should have a good background in mathematics, including stochastics, to benefit fully. This brings us to the key question

What is this book about?

Clearly about extremal events! But what do we mean by this?

In the introduction to their book on *Outliers in Statistics*, Barnett and Lewis [51], the authors write: “When all is said and done, the major problem in outlier study remains the one that faced the very earliest research workers in the subject – what is an outlier?” One could safely repeat this sentence for our project, replacing *outlie*r by *extremal event*. In their case, they provide methodology which allows for a possible description of outliers (influential observations) in statistical data. The same will be true for our book: we will mainly present those models and techniques that allow a precise mathematical description of certain notions of extremal events. The key question to what extent these theoretical notions correspond to specific events in practice is of a much more general (and indeed fundamental) nature, not just restricted to the methodology we present here. Having said that, we will not shy away from looking at data and presenting applied techniques designed for the user. It is all too easy for the academic to hide constantly behind the screen of theoretical research: the actuary or finance expert facing the real problems has to take important decisions based on the data at hand. We shall provide him or her with the necessary language, methods, techniques and examples which will allow for a more consistent handling of questions in the area of extremal events.

Whatever definition one takes, most will agree that Table 1, taken from *Sigma* [582] contains extremal events. When looked upon as single events, each of them exhibits some common features.

– *Their (financial) impact on the (re)insurance industry is considerable.* As stated in *Sigma* [582], at $US 150 billion, the total estimated losses in 1995 amounted to ten times the cost of insured losses – an exceptionally high amount, more than half of which was accounted for by the Kobe earthquake. Natural catastrophes alone caused insured losses of $US 12.4 billion, more than half of which were accounted for by four single disasters costing some billion dollars each; the Kobe earthquake, hurricane "Opal", a hailstorm in Texas and winter storms combined with floods in Northern Europe. Natural catastrophes also claimed 20 000 of the 28 000 fatalities in the year of the report.
– *They are difficult to predict a long time ahead.* It should be noted that 28 of the insurance losses reported in Table 1 are due to natural events and only 2 are caused by man–made disasters.
– If looked at within the larger context of all insurance claims, *they are rare events.*

Extremal events in insurance and finance have (from a mathematical point of view) the advantage that they are mostly quantifiable in units of money. However most such events have a non–quantifiable component which more and more economists are trying to take into account. Going back to the data presented in Table 1, extremal events may clearly correspond to individual (or indeed grouped) claims which by far exceed the capacity of a single insurance company; the insurance world's reaction to this problem is the creation of a reinsurance market. One does not however have to go to this grand scale. Even looking at standard claim data within a given company one is typically confronted with statements like "In this portfolio, 20% of the claims are

Losses	Date	Event	Country
16 000	08/24/92	Hurricane "Andrew"	USA
11 838	01/17/94	Northridge earthquake in California	USA
5 724	09/27/91	Tornado "Mireille"	Japan
4 931	01/25/90	Winterstorm "Daria"	Europe
4 749	09/15/89	Hurricane "Hugo"	P. Rico
4 528	10/17/89	Loma Prieta earthquake	USA
3 427	02/26/90	Winter storm "Vivian"	Europe
2 373	07/06/88	Explosion on "Piper Alpha" offshore oil rig	UK
2 282	01/17/95	Hanshin earthquake in Kobe	Japan
1 938	10/04/95	Hurricane "Opal"	USA
1 700	03/10/93	Blizzard over eastern coast	USA
1 600	09/11/92	Hurricane "Iniki"	USA
1 500	10/23/89	Explosion at Philips Petroleum	USA
1 453	09/03/79	Tornado "Frederic"	USA
1 422	09/18/74	Tornado "Fifi"	Honduras
1 320	09/12/88	Hurricane "Gilbert"	Jamaica
1 238	12/17/83	Snowstorms, frost	USA
1 236	10/20/91	Forest fire which spread to urban area	USA
1 224	04/02/74	Tornados in 14 states	USA
1 172	08/04/70	Tornado "Celia"	USA
1 168	04/25/73	Flooding caused by Mississippi in Midwest	USA
1 048	05/05/95	Wind, hail and floods	USA
1 005	01/02/76	Storms over northwestern Europe	Europe
950	08/17/83	Hurricane "Alicia"	USA
923	01/21/95	Storms and flooding in northern Europe	Europe
923	10/26/93	Forest fire which spread to urban area	USA
894	02/03/90	Tornado "Herta"	Europe
870	09/03/93	Typhoon "Yancy"	Japan
865	08/18/91	Hurricane "Bob"	USA
851	02/16/80	Floods in California and Arizona	USA

Table 1 *The* 30 *most costly insurance losses 1970–1995. Losses are in million $US at 1992 prices. For a precise definition of the notion of catastrophic claim in this context see* Sigma [582].

responsible for more than 80% of the total portfolio claim amount". This is an extremal event statement as we shall discuss more in detail in Section 8.2.

By stating above that the quantifiability of insurance claims in monetary units makes the mathematical modelling more tractable, we do not want to trivialise the enormous human suffering underlying such events. It is indeed striking that, when looking at the 30 worst catastrophes, in terms of fatalities over the same period in Table 2 only one event (the Kobe earthquake) figures

Fatalities	Date/start	Event	Country
300 000	11/14/70	Hurricane	Bangladesh
250 000	07/28/76	Earthquake in Tangshan	China
140 000	04/29/91	Hurricane "Gorky"	Bangladesh
60 000	05/31/70	Earthquake	Peru
50 000	06/21/90	Earthquake	Iran
25 000	12/07/88	Earthquake in Armenia	former USSR
25 000	09/16/78	Earthquake	Iran
23 000	11/13/85	Volcanic eruption "Nevado del Ruiz"	Columbia
22 000	02/04/76	Earthquake	Guatemala
15 000	09/19/85	Earthquake in Mexico City	Mexico
15 000	08/11/79	Damburst	India
15 000	09/01/78	Flood	India
10 800	10/31/71	Flood	India
10 000	05/25/85	Hurricane	Bangladesh
10 000	11/20/77	Tornado	India
9 500	09/30/93	Earthquake in Marashtra state	India
8 000	08/16/76	Earthquake on Mindanao	Philippines
6 304	11/05/91	Typhoons "Thelma" and "Uring"	Philippines
6 000	01/17/95	Great Hanshin earthquake in Kobe	Japan
5 300	12/28/74	Earthquake	Pakistan
5 000	04/10/72	Earthquake in Fars	Iran
5 000	12/23/72	Earthquake in Managua	Nicaragua
5 000	06/30/76	Earthquake in Westirian	Indonesia
4 800	11/23/80	Earthquake	Italy
4 500	10/10/80	Earthquake	Algeria
4 000	02/15/72	Storm; snow	Iran
4 000	11/24/76	Earthquake in Van	Turkey
3 800	09/08/92	Floods in Punjab	Pakistan
3 200	04/16/78	Tornado	Reunion
3 000	08/01/88	Flood	Bangladesh

Table 2 *The* 30 *worst catastrophes in terms of fatalities 1970–1995, taken from* Sigma [582].

on both lists. Also, Table 1 mainly involves industrialised nations, whereas Table 2 primarily concerns Third World countries.

Within the finance context, extremal events present themselves spectacularly whenever major stock market crashes like the one in 1987 occur. Or recent casualties within the realm of derivatives such as the collapse of Barings Bank, the losses of the Metallgesellschaft, Proctor & Gamble, Kashima Oil, Orange County, or Sumitomo. The full analysis of events of such grand scale again goes well beyond the prime content of this book, and any claim that the managements of financial institutions will find the means of avoiding such disasters in our book would be absurd. In most of the above cases the setting–up (both in structure as well as people) of a well–functioning risk management and control system was called for. On a much smaller scale however, questions related to the estimation of Profit–and–Loss distributions or Value–at–Risk measures have to be answered with techniques presented in some of the following chapters. Though not providing a risk manager in a bank with the final product he or she can use for monitoring financial risk on a global scale, we will provide that manager with stochastic methodology needed for the construction of various components of such a global tool.

Events that concern both branches are to be found in credit insurance, mortgage–backed securities, the recent developments around catastrophic insurance futures or indeed more generally the problem of securitisation of risk. In all of these areas, there is an increasing need for modelling of events that cause larger shocks to the underlying financial system. As an example of how knowledge of basic underlying stochastic methodology may be used, consider the problem of potential increases in both the frequency as well as (inflation–adjusted) sizes of well–defined catastrophic claims. A simple, but at the same time intuitively clear method, is to plot the successive records in the data. In Figure 3 we have plotted such records for yearly frequency and insured loss data both for man–made as well as natural catastrophes over the period 1970–1995. For a precise definition of the underlying data see *Sigma* [582]. If the data were independent and identically distributed (iid), what sort of picture would one expect? An answer to this question is given in Section 6.2.4. Intuition tells us that successive records for iid data should become more and more rare as time goes by: it becomes more and more difficult to exceed all past observations.

By now, the reader should have some idea of the type of problems we are interested in. The next step would be to dig a bit deeper and explain which mathematical models we plan to discuss and what methodology we want to introduce. Before doing so, some general comments on the format of the chapters is called for.

How is new material to be presented, and indeed how should one read this book?

As stated before, we typically think of an actuary, a finance expert or a student, working on a problem in which a technique related to rare though potentially influential events is to be used. Take as an example a finance expert in the area of risk management, concerned with Value–at–Risk estimation for a specific portfolio. The Value–at–Risk may for instance be defined as the left 5% quantile of the portfolio Profit–Loss distribution. The latter is

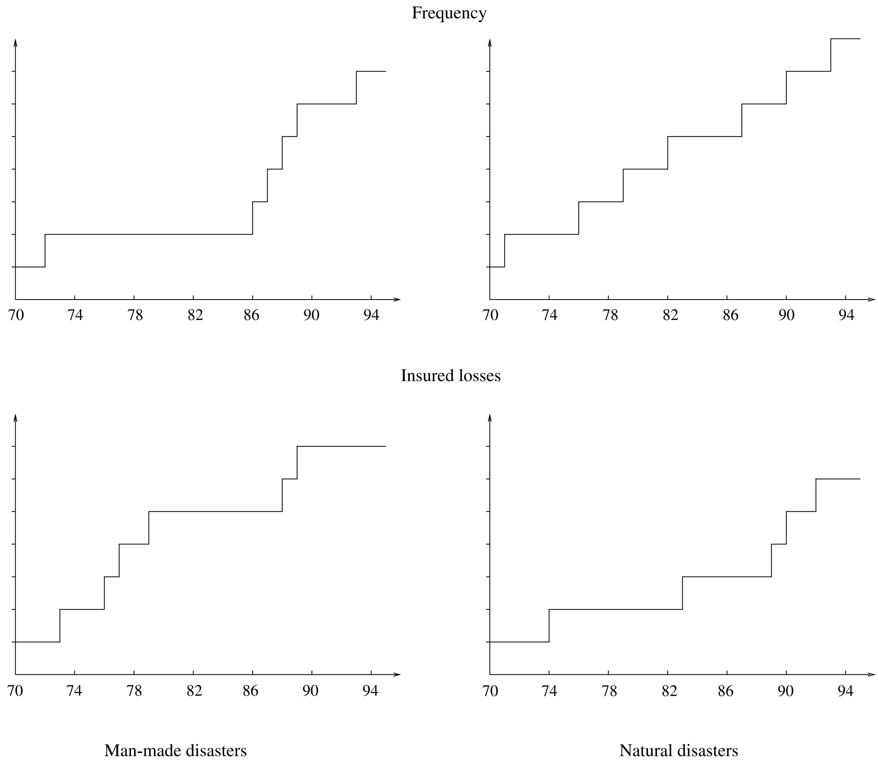

Figure 3 *Record years of catastrophic insurance claims 1970–1995: frequency and insured losses (in 1992 prices) both for man–made and natural disasters, taken from* Sigma [582]. *The graphs show a jump for each year in which a new record occurred. For instance, one observes* 8 *records for the frequency of natural disasters and* 6 *records for the insured losses.*

typically skewed with heavy tails both at left (losses) and right (gains); see Figure 4. So we end up with questions that concern finding relevant classes of Profit–Loss distributions, as well as statistical fitting and tail estimation. It is exactly for this type of problems that our book will provide the necessary background material or indeed specific techniques.

A typical chapter will introduce the new methodology in a rather intuitive (though always mathematically correct) way, stressing more the understanding of new techniques rather than following the usual theorem–proof path. We do, however, usually state theorems in their most general form, provided that this form is practically relevant. Proofs are usually given either as a sketch of the main ideas, or as a way of showing how new methods can be used in technical calculations. Sometimes we use them to highlight the instances in

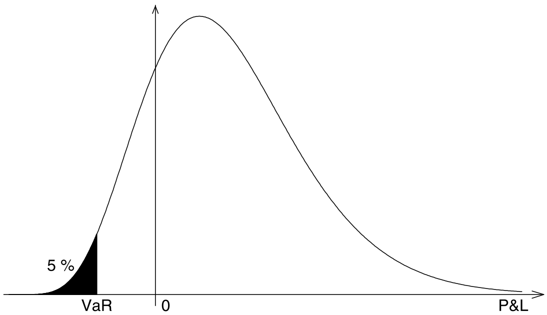

Figure 4 *Profit–Loss* ($P\&L$) *density function with related Value–at–Risk* (VaR).

the argument where classical techniques break down (explaining why), and how arguments relating to extremal events have to be handled. Each section ends with Notes and Comments giving the reader further guidance towards relevant literature on related topics. Various examples, tables and graphs have been included for illustrative purposes, but at the same time for reasons of making the text (at least optically) easier to digest. Few readers will want to read the text from cover to cover; the ideal way would be to read those sections that are necessary for the problems at hand.

Which basic models in insurance and finance do we consider?

Our main motivation comes from insurance, and consequently a bias towards problems (and topics) from that field of applications is certainly to be found in the text. On the other hand, except for Chapters 1 and 8, all chapters are aimed at a much larger audience than workers in insurance.

Mathematical modelling in finance and insurance can be traced back many centuries. For our purposes, however, history starts around the beginning of the 20th century. In 1900, Louis Bachelier showed in his thesis [35] that Brownian motion lies at the heart of any model for asset returns. Around the same time, Filip Lundberg introduced in his thesis [431] the collective risk model for insurance claim data. Lundberg showed that the homogeneous Poisson process, after a suitable time transformation, is the key model for insurance liability data. Of course, both Brownian motion and the homogeneous Poisson process are the prime examples of the wider class of stochastic processes with stationary and independent increments. We shall treat both examples more in detail and provide techniques concerning extremal events useful in

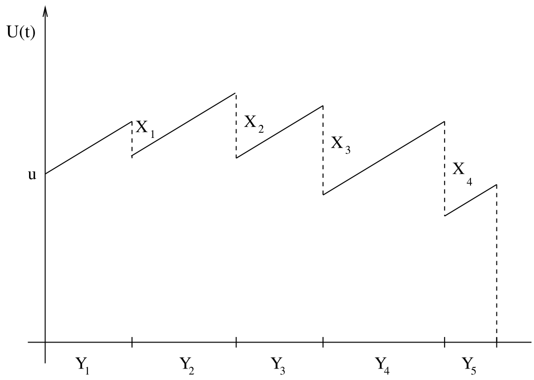

Figure 5 *One realisation of the risk process* $(U(t))$.

either case. Embedded in these processes is the structure of a random walk, i.e. the sum of iid random variables. So a more profound study of extremal events in the iid case is called for. This forms the basis for classical statistical theory and classical extreme value theory. More general models can often be transformed to the iid case; this allows us for instance to analyse general (linear) time series.

In Chapter 1 we study *the* classical model for insurance risk,

$$U(t) = u + ct - S(t)\,, \quad S(t) = \sum_{i=1}^{N(t)} X_i\,, \quad t \geq 0\,, \tag{1}$$

where u stands for initial capital, c for loaded premium rate and the total claim amount $S(t)$ consists of a random sum of iid claims X_i. Here $N(t)$ stands for the number of claims until time t. It is common to simplify this model further by assuming (as Lundberg did) that $(N(t))$ is a homogeneous Poisson process, independent of (X_i). For a realisation of $(U(t))$ see Figure 5. The process $(S(t))$ and its ramifications have been recognised as a very tractable (and reasonably realistic) model and a vast amount of literature in risk theory has been devoted to it. An important question concerns the influence of individual extremal events, i.e. large claims, on the global behaviour of $(U(t))$. In Chapter 1 the latter question will be answered via a detailed analysis of ruin probabilities associated with the process $(U(t))$. Under a condition of "small claims" (see for instance Theorem 1.2.2), the traditional Cramér–Lundberg estimate for the ruin probability yields bounds which are exponential in the initial capital u. However, in reality claims are mostly modelled by heavy–tailed distributions like Pareto, loggamma, lognormal, or heavy–tailed Weibull. See for instance Figure 6, where the left–hand picture shows those

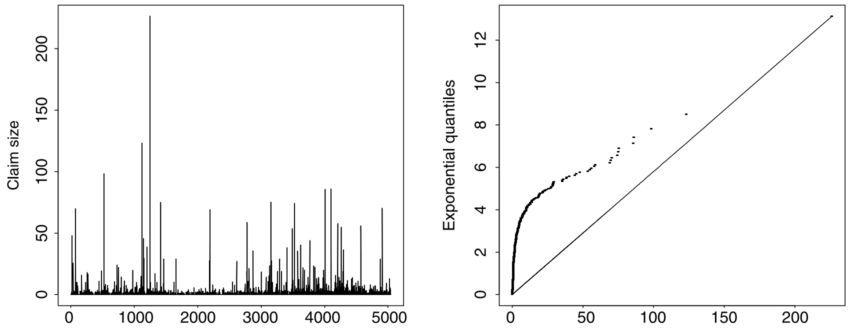

Figure 6 *Fire insurance data and corresponding exponential QQ–plot. The claim sizes are in* 1 000 *SFr.*

claim sizes of a portfolio of fire insurance that are larger than a given franchise (1 000 SFr). In the right–hand picture one finds a so–called QQ–plot of the data, measuring the fit achieved by an exponential distribution function (df). The curvature (i.e. departure from a straight line) present in the QQ–plot implies that the tails of the df of the fire data are much heavier than exponential. For a detailed discussion of these and related plotting techniques see Section 6.2.1.

Chapter 1 mainly deals with the mathematical analysis of ruin estimation under precise heavy–tailed model assumptions. Whereas Poisson processes form the basic building block underlying insurance liability processes, within finance the basic models can be transformed back to simple random walks. This is certainly true for the Cox–Ross–Rubinstein and the Black–Scholes models; see for instance Föllmer and Schweizer [242] for a nice account of the economic whys and wherefores concerning these processes.

The skeleton model in finance, corresponding to the homogeneous Poisson process in insurance, is without doubt geometric Brownian motion, i.e. the stochastic process

$$\exp\left\{\left(c-\sigma^2/2\right)t+\sigma B_t\right\}\,,\quad t\geq 0\,,$$

with (B_t) Brownian motion. Here c stands for the mean rate of return and σ for the *volatility* (riskiness). It is the solution to an Itô stochastic differential equation and provides the basis of the Black–Scholes option pricing formula and many other parts of financial theory. One of the attractions of the above model is its simplicity; indeed, as a consequence it follows that logarithmic returns are iid, normally distributed. At this point, as in insurance,

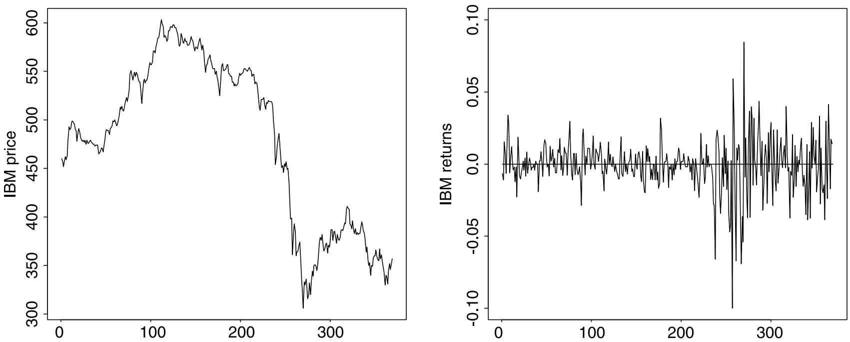

Figure 7 *Daily IBM common stock closing prices: May 17, 1961–Nov. 2, 1962.*

one should ask the question "what do the data tell us?" An answer to this question would, and indeed does, fill a book. A summary answer, fit for this introduction, is that, on the whole, geometric Brownian motion is a good first model. If however one looks more closely at data, one often finds situations as in Figure 7. In it we observe a clear change in volatility possibly triggered by some extreme returns. A multitude of models for such phenomena has been introduced including α–stable processes (as heavy–tailed alternatives to Brownian motion), and heavy–tailed time series models, for instance ARCH and GARCH models. The basic characteristics of such models will be discussed in later chapters, for instance Chapter 7, Sections 8.4 and 8.8.

From a naive point of view both fields, insurance and finance, have in common that we can observe certain financial or actuarial phenomena such as prices, exchange rates, interest rates, insurance claims, claim arrival times etc. We will later classify these observations or data, but we first want to consider them simply as a *time series* or a *continuous–time stochastic process*, i.e. we assign to each instant of time t a real random variable X_t. One of our usual requirements is that (X_t) itself or a transformed version of it (for instance the first–order differences or the log–differences) forms a *stationary* process (strictly stationary or stationary in the wide sense). In particular, this includes the important case of iid observations which provides the basis for classical fluctuation and extreme value theory, as well as for statistical estimation.

In Chapter 2 we give a general asymptotic theory for sums of iid random variables (random walk), and in Sections 2.5.1 and 2.5.3 we especially emphasize random sums like $S(t)$ in (1). This theory includes classical results such as the central limit theorem, the law of large numbers, the law of the iterated logarithm, the functional central limit theorem and their ramifications and refinements. They are important building blocks for the asymptotic theory which is a basic tool of this book. We also introduce two important classes of continuous–time stochastic processes: *Brownian motion* and *α–stable motion*. Both are continuous–time limits of appropriate partial sum processes. As such, they can be understood as *random walks in continuous time*.

After having recalled the basic partial sum theory, in Chapters 3 and 4 we turn to the analogous theory for partial maxima and order statistics. These chapters are conceived in such a way that the reader can compare and contrast results for maxima with similar ones for sums. Special attention will also be given to those questions where both theories complement one another. As a start we first present extreme value theory for iid sequences, thereby paving the way for similar results in the case of stationary sequences (X_t). In particular, we will describe and study *maxima, minima, records, record times, excesses over thresholds, the frequency of exceedances* and many other features of such sequences which are related to their extremal behaviour.

Though most of the material of this book can be found scattered over various textbooks and/or research papers, some material is presented here for the first time in textbook form. One such example is the study of linear processes

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j \, Z_{t-j} \,, \quad t \in \mathbb{Z} \,, \tag{2}$$

for iid innovations Z_t with infinite variance. Over the past 20 years methods have been developed to deal with these objects, and Chapter 7 contains a survey of the relevant results. The proofs are mostly very technical and accessible only to the specialist. This is the reason why we omitted them, but we give a very detailed reference list where the interested reader will find a wealth of extra reading material. The extreme value theory for the process (2) is dealt with in Section 5.5 under different assumptions on the innovations which include the heavy–tailed case. The extremes of more general stationary sequences are treated in Sections 4.4 and 5.3.2.

In summary, the stochastic processes of main interest can be roughly classified as follows:

- Discrete time sequences $(X_t)_{t\in\mathbb{Z}}$, in particular stationary and iid sequences as models for log–returns of prices, for exchange rates, for individual claim sizes, for inter–arrival times of claims.
- Random walk models, i.e. sums of the X_t or continuous–time models such as Brownian motion $(B_t)_{t\geq 0}$ and α–stable motion, as models for the total claim amount, aggregated returns or building blocks for price processes etc.

– Random sum processes $(S(t))_{t \geq 0}$ (see (1)) as models for the total claim amount in an insurance portfolio.
– The risk process $(U(t))_{t \geq 0}$; see (1).
– Poisson processes and Poisson random measures as means to describe rare events in space and time. The homogeneous Poisson process also serves as a basic model for claim arrival times.

After having introduced our basic models we may ask

Which distributions and stochastic processes typically describe extremal events in these models?

When we are interested in the extremal behaviour of the models described above we have to ask *how extremal events occur*. This means we have to find appropriate mathematical methods in order to explain events that occur with relatively small probability but have a significant influence on the behaviour of the whole model. For example, we may ask about the inter–relation between the iid individual claim sizes X_i and the total claim amount $S(t)$ in (1). In particular, under what assumptions and how do the values of the largest claims determine the value $S(t)$? A natural class of large claim distributions is given by the *subexponential distributions*. They are extensively treated in Chapter 1 and Appendix A3.2. Their defining property is:

$$\lim_{x \to \infty} \frac{P\left(X_1 + \cdots + X_n > x\right)}{P\left(\max\left(X_1, \ldots, X_n\right) > x\right)} = 1$$

for every $n \geq 2$. Thus the tails of the distribution of the sum and of the maximum of the first n claims are asymptotically of the same order. This clearly indicates the strong influence of the largest claim on the total claim amount.

Whereas in insurance heavy–tailed (i.e. subexponential) distributions are well recognised as standard models for individual claim sizes, the situation in finance is much more complicated. The latter is partly due to the fact that one often works with near continuous–time observed (so–called high–density) data. At the same time, marginal distributions are heavy–tailed and return data exhibit clustering of extremes and long–range dependence. There is no universally accepted nor indeed easy model that explains all these phenomena. In Section 2.4, for instance, we introduce α–stable motion $(0 < \alpha < 2)$ as a limit of partial sum processes with infinite variance. For a realisation of a 1.5–stable motion see Figure 8, where also a plot of Brownian motion is given. The α–stable processes form fundamental building blocks within more

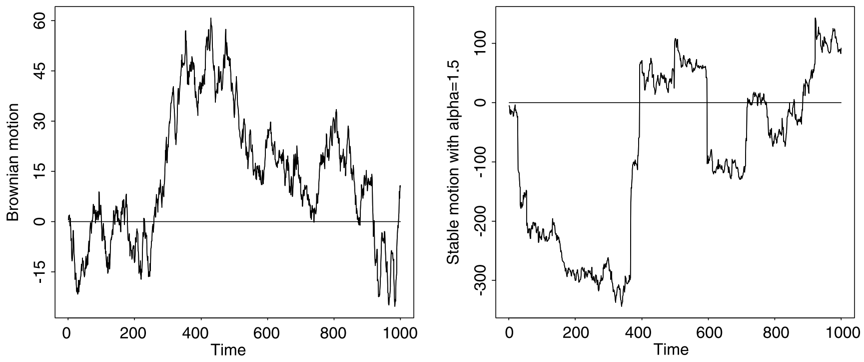

Figure 8 *Paths of Brownian motion and of a* 1.5*–stable motion.*

general model constructions and anyone interested in rare events ought to know them.

Many distributions of interest in extreme value theory turn out to be closely related to α–stable distributions. The α–stable laws are the *only* possible limit distributions for properly normalised and centred sums of iid random variables. The case $\alpha = 2$ corresponds to the normal limit; we know that a finite second moment is sufficient for the application of the central limit theorem. The case $\alpha < 2$ arises for infinite–variance iid summands. The infinite variance property has not prevented practitioners in insurance from working with such models. A quick simulation of a scenario of the total claim amount under these heavy–tailed assumptions is helpful for making a decision about the insurability of such claims. In that sense, α–stable or other heavy–tailed distributions often can be used as a worst–case scenario.

Extreme value theory is one of the main objectives of this book, and so when talking about relevant distributions in that context, we have to mention the extreme value distributions, the Gumbel law Λ, the Fréchet law Φ_α and the Weibull law Ψ_α. They are the *only* possible limit distributions for maxima of properly normalised and centred iid random variables. As such they essentially play the same role as the α–stable distributions for sums of iid random variables. Sections 3.2 and 3.3 are devoted to their study. Furthermore, in Sections 4.1 and 4.2 the theory is extended from maxima to upper order statistics.

There are of course many more distributions of interest which are somehow related to extremes. Above we have mentioned the essential ones and the way they enter applied modelling in the presence of extremal events. We will provide lists and examples of particular distributions, densities and tails in the corresponding sections.

We have already encountered the Poisson distribution in the context of risk theory. Both the Poisson distribution as well as the Poisson process are key tools in the analysis of extremal events, as we shall see on various occasions.

In sum, the following classes of distributions are of main importance in the context of extremal events:

– the subexponential distributions as realistic models for heavy–tailed random variables,
– the α–stable distributions for $\alpha < 2$ as the limit laws for sums of infinite–variance iid random variables,
– the Fréchet, the Weibull, and the Gumbel distributions, as limit laws for maxima of iid random variables,
– the normal distribution as limit law for sums of iid, finite–variance random variables,
– the Poisson distribution as limit law of binomial distributions which represent a counting measure of rare events.

As important stochastic processes we would like to mention:

– Poisson processes,
– α–stable processes ($0 < \alpha < 2$) and Brownian motion,
– more general processes using the above as input.

What are the main probabilistic tools?

Besides standard introductory probability theory and the theory of stochastic processes, many results presented will be based upon a deeper understanding of relevant asymptotic methods. One of the main tools falling into the latter category is the theory of *weak convergence of probability distributions*, both on the real line and in certain function spaces. A short summary of the methodological background is given in Appendices A1 and A2. Abstract weak–convergence techniques are needed in order to prove that suitable partial sum processes converge towards Brownian motion or α–stable processes. The strength of this process convergence is illustrated by various examples in Chapter 2. This theory allows us to characterise those distributions and processes that may arise as useful stochastic models for certain insurance and finance data.

The analysis of extremes further requires the framework of *point processes*. The general theory for the latter is rather involved, though the benefit for applications, especially those towards extremal event modelling, is considerable once the whole machinery has been set up. In Chapter 5 we give an ample number of examples of this. We have tried hard to avoid unnecessary technical details. Point process techniques are by now an unavoidable tool in modern extreme value theory, and the results are convincing and give a deep insight into the structure and occurrence of extremes.

The basic idea of weak convergence of point processes is analogous to Poisson's classical limit theorem. Weak limits of the point processes under consideration (as analogues of binomial random variables) are quite often (general) *Poisson processes* or *Poisson random measures* (as analogues to the Poisson distribution). These notions will be made precise in Sections 5.1 and 5.2.

Limit theory for sums, maxima or point processes is closely related to the power law behaviour of tails, of normalising constants, of characteristic functions in the neighbourhood of the origin etc. Exact power laws mainly occur in the very limit, but if, for instance, we discuss domains of attraction of stable laws or of extreme value distributions, power laws do not appear in "pure" form, but slightly disturbed by *slowly varying functions.* A power law times a slowly varying function is called *regularly varying.* The theory of regularly varying functions and their generalisations and extensions are important analytical tools throughout this book. Their basic properties are given in Appendix A3.1.

In Chapter 7 we provide an analysis of time series with heavy tails. A lean introduction to the relevant notions of time series analysis is given, but the reader without the necessary background will certainly have to consult some of the standard textbooks. The main objects in Chapter 7 are linear processes with heavy–tailed innovations. That chapter and Section 5.5, where extreme value theory for linear processes is treated, give quite a complete picture about this kind of process with heavy–tailed innovations.

To sum up, besides the basic classical techniques and facts from probability theory our main probabilistic tools are the following:

- weak convergence of distributions of random variables such as sums, random sums and maxima of random variables,
- weak convergence of sum processes and maximum processes to their limits in appropriate function spaces,
- point processes for describing the random distribution of points in space and time with applications to extreme value theory.

What are the appropriate statistical tools?

Insurers and bankers are interested in assessing, pricing and hedging their risks. They calculate premiums and price financial instruments including coverage against major risks. The probable maximal loss of a risk or investment portfolio is determined by extremal events. The problem we want to solve may therefore be described in its broadest terms as how to make statistical inference about the extreme values in a population or a random process. Quantities like the following may serve as indicators:

– the distribution of the annual extremes,
– the distribution of the largest values in a portfolio,
– the return period of some rare event,
– the frequency of extremal events,
– the mean excess over a given threshold,
– the distribution of the excesses,
– the time development of records.

Every piece of knowledge we can acquire about these quantities from our data helps us to predict extremal events, and hence potentially protect ourselves against adverse effects caused by them. In Chapter 6 we present a collection of methods for statistical inference based on extreme values in a sample.

Some simple exploratory data–analytical methods can be extremely useful at a descriptive stage. An example has been given in Figure 3 where a plot of the records manifests a trend in the frequency of natural disasters. Methods based on probability plots, estimated return periods or empirical mean excess functions provide first information about the extremes of a data set.

For iid data the classical extreme value distributions, the Gumbel $\varLambda$, the Fréchet $\varPhi_\alpha$ and the Weibull distribution $\varPsi_\alpha$, are the obvious candidates to model the largest values of a sample. We review parameter estimation methods for extreme value distributions, investigate their asymptotic properties and discuss their different merits and weaknesses. Extensions to upper order statistics of a sample are also treated.

Our interest focusses on extremal events of the form $\{X > x\}$ for some random variable X and large x, i.e. we want to estimate tails in their far regions and, also, high quantiles. We survey various tail and quantile estimators which are only to be found rather scattered through the literature. We also describe a variety of statistical methods based on upper order statistics and on so–called threshold methods.

Before you start!

We think it a bad idea for a methodological book like this one to distinguish too strongly between those readers working in insurance and those working more in finance. It would be especially bad to do so at the present time, when experts from both fields are increasingly collaborating either on questions of related interest (risk management say) or on new product development involving both insurance and finance features (for instance index–linked life insurance, catastrophe futures and options, securitisation of insurance risk). It is important for both sides to learn more about each other's basic models and tools. We therefore hope that a broad spectrum of readers will find various interesting facts in this book.

We start with a somewhat specialised chapter on *risk theory*; however, the basic model treated in it reappears in many fields of applications as for instance queueing theory, dam theory, inventory systems, shock models etc. Its main purpose is that it provides an ideal vehicle for the introduction of the important class of *subexponential distributions*. At the same time, the liability model that is fundamental to insurance is also discussed. From Chapter 2 onwards, standard theory is first of all reviewed (Chapter 2 on *sums*) before the core material on *probabilistic modelling of extremes* together with their *statistical analysis* are treated in Chapters 3–6. A mathematically more demanding, though with respect to applications rewarding, excursion to *point process methods* is presented in Chapter 5. Typically you would start with Chapters 2 and 3 and embark first on the *statistical methods* in Chapter 6 before coming back for a more detailed analysis of some of the techniques from Chapter 5. Chapter 7 treats the more specialised topic of *heavy–tailed time series* models. It fits into the framework of *extremes for dependent data* which earlier appears in Sections 4.4, 5.3 and 5.5. Together, Chapters 1 through 7 give a sound introduction to one–dimensional extremal event modelling. Having this methodology at our finger tips, we may start using it for understanding and solving various related problems. This is exactly what is presented in Chapter 8 on *special topics*. In it, we have brought together various problems, all of which use the foregoing theory in some form or another. Take for instance Section 8.2 where a *large claim index* is discussed, describing mathematically the 20–80 rule of thumb used by actuaries to specify the dangerousness of certain portfolios. Chapter 8 is also used to discuss briefly those extensions of the theory which should come next, such as for instance Sections 8.1 (on the *extremal index*), 8.4 (on *perpetuities and ARCH processes*) and 8.7 (on *reinsurance treaties*). This chapter could have grown considerably; somewhere however we had to stop. Therefore, most of the sections presented reflect somehow our own teaching, research and/or consulting experience. We have based an extreme value theory course for mathematics students specialising in actuarial mathematics on most of the material presented in Chapters 3 to 6, together with some sections in Chapter 8. Naturally, the Appendix is there for reviewing those tools from mathematics used most often throughout the text and which may not belong to everybody's basic toolkit.

Epilogue

You are now ready to start: good luck!

March 1997 P.E., C.K. and T.M.

Note added on the occasion of the 8th printing.

Since its first appearance in 1997, a tremendous amount of work on the modelling of extremes in finance and insurance has emerged. The following websites contain numerous relevant research papers and further useful links:

http://www.math.ethz.ch/~embrechts

http://www-m4.ma.tum.de

http://www.math.ku.dk/~mikosch

Some recent development on extreme value software can be found at

http://www-m4.ma.tum.de/pers/cklu/BookEKM.html

March 2007 P.E., C.K. and T.M.

http://www.springer.com/978-3-540-60931-5

Modelling Extremal Events
for Insurance and Finance
Embrechts, P.; Klüppelberg, C.; Mikosch, Th.
1997, XV, 648 p., Hardcover
ISBN: 978-3-540-60931-5