

## Chapter II

### Extension of the Integral

**Historical Note.** Riemann introduced his integral in the context of *Fourier series*. It happens in numerous contexts, from engineering to number theory, that one wishes to write a function  $f$  defined on the interval  $(-\pi, \pi]$  as a superposition of simple trigonometric functions:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx), \quad (1)$$

i.e. as a *Fourier series*. The reader might well have come across them in advanced Calculus or in a course on differential equations. The problem is to determine which functions can be represented as in Equation (1); in which sense they are so represented, that is to say, in which sense the partial sums, the *trigonometric polynomials*

$$f_N(x) = \frac{a_0}{2} + \sum_{\nu=0}^N a_{\nu} \cos(\nu x) + b_{\nu} \sin(\nu x)$$

converge to  $f$  as  $N \rightarrow \infty$ ; and then to find the coefficients  $a_n, b_n$ . Assume for argument's sake that the  $f_N$  converge, say pointwise, to  $f$ . The coefficients should then be given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \quad \text{and} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx. \quad (2)$$

The reason is this: for  $N \geq n$

$$\int_{-\pi}^{\pi} f_N(x) \cdot \cos(nx) dx = \pi a_n \quad \text{and} \quad \int_{-\pi}^{\pi} f_N(x) \cdot \sin(nx) dx = \pi b_n. \quad (3)$$

This follows from the well-known orthogonality relations

$$\int_{-\pi}^{\pi} \cos(\nu x) \cos(nx) dx = \int_{-\pi}^{\pi} \sin(\nu x) \sin(nx) dx = \begin{cases} 0 & \text{if } \nu \neq n \\ \pi & \text{if } \nu = n \end{cases}$$

$$\text{and} \quad \int_{-\pi}^{\pi} \cos(\nu x) \sin(nx) dx = 0.$$

If we let  $N \rightarrow \infty$  in Equation (3), the integrals tend to

$$\int_{-\pi}^{\pi} f(x) \cos(nx) dx \quad \text{and} \quad \int_{-\pi}^{\pi} f(x) \sin(nx) dx ,$$

respectively. Voilà.

This argument uses a theorem about the interchange of limit and integral that does not exist! What we need here is a theorem like this: “If  $(f_n)$  is a sequence of integrable functions that converges pointwise to  $f$  then  $f$  is integrable and  $\int f_n \xrightarrow{n \rightarrow \infty} \int f$ .” As mathematicians in the late 19<sup>th</sup> century were looking for results of this nature, it became soon clear that any theorems one might conjecture and that would really help are simply not true (see Exercise I.6.6). Lebesgue cut the Gordian knot in 1902 by proposing a notion of integral that encompasses Riemann’s and features limit theorems strong enough to advance the theory of Fourier series considerably. For example, the limit Theorem I.6.5 stays, even when uniform convergence in (a) is replaced by mere pointwise convergence — this is Lebesgue’s celebrated Dominated Convergence Theorem (DCT). (The theorem quoted at the beginning of this paragraph is not true in his theory, though, without the domination condition (b); there exists no notion of integral for which it would be.) The DCT produces vast numbers of new integrable functions, namely, the pointwise limit of any dominated sequence of functions that have been shown to be integrable. In fact, good limit theorems go hand-in-hand with abundance of integrable functions.

**Our strategy** — or rather Daniell’s strategy — towards an integral extension that provides more integrable functions is to replace Jordan’s mean  $\| \cdot \|^\natural$  with a smaller semi-norm  $\| \cdot \|_*$ , Daniell’s mean, that still majorizes the elementary integral; this makes it easier for a function  $f$  to be approximable by step functions:  $f$  may be such that  $\|f - \phi\|^\natural > 1$  for all  $\phi \in \mathcal{E}$ , yet there may be a sequence  $(\phi_n)$  in  $\mathcal{E}$  with  $\|f - \phi_n\|_* \rightarrow 0$ ;  $f$  is then not Riemann integrable, yet is integrable in the new sense.

Daniell’s construction of  $\| \cdot \|_*$  is quite similar to that of  $\| \cdot \|^\natural$ . The Riemann upper integral  $\int^\natural$  is simply replaced by a better suited one, the Daniell upper integral  $\int^*$ . The Daniell mean  $\| \cdot \|_*$  is then defined by  $\|f\|_* = \int^* |f|$ , just as  $\| \cdot \|^\natural$  was defined by  $\|f\|^\natural = \int^\natural |f|$ , and from there we proceed as in the previous chapter. Not only will  $\| \cdot \|_*$  be finite on some functions that are unbounded or have unbounded support — thus subsuming, for instance, positive improperly Riemann integrable functions into the theory —, functions much wilder than Riemann integrable ones can be approximated by step functions in the sense of  $\| \cdot \|_*$ . In fact, the closure of the step functions under Daniell’s mean  $\| \cdot \|_*$ , the new class of integrable functions, is so rich that it is quite hard to exhibit a function not in it (Section 8).

Daniell’s seminorm  $\| \cdot \|_*$  is again solid and majorizes the elementary integral. It has an additional continuity property that Jordan’s seminorm  $\| \cdot \|^\natural$  does not

share: for every sequence  $(f_n)$  of positive functions,

$$\left\| \sum_{n=1}^{\infty} f_n \right\|^* \leq \sum_{n=1}^{\infty} \|f_n\|^* . \quad (\text{CSA})$$

This is reasonably called **countable subadditivity**. It is the countable subadditivity of  $\| \|^*$  in conjunction with the other properties that leads to the beautiful limit theorems of Lebesgue's integral.

It turns out that countably subadditive solid function seminorms arise in other contexts, not necessarily from Daniell upper integrals, so that the limit theorems are available for them, as well. We shall make it a point to state in every section which properties of  $\| \|^*$  are actually used in the proofs. This will allow us later to invoke the results for means such as the  $p$ -norms, without having to prove them again.

## II.1 $\Sigma$ -additivity

Before revamping Riemann's theory of integration with all its intuitive appeal one would try, of course, to prove limit theorems better than the Dominated Uniform Convergence Theorem I.6.5. One might imagine that the next result was discovered this way. It is an extremely modest result, applying as it does only to a sequence of step functions whose pointwise and monotone limit is *a priori* known to be a step function again. Yet the whole development of the Lebesgue integral rests on it.

**Lemma 1.1** (i) *Let  $(\phi_n)$  be an increasing sequence of step functions on the line whose pointwise supremum is again a step function  $\phi$ . Then*

$$\lim_n \int \phi_n = \int \lim_n \phi_n = \int \phi .$$

(ii) *Let  $(\delta_n)$  be a sequence of positive step functions on the line whose pointwise sum exists and is a step function  $\phi$  again. Then*

$$\sum_n \int \delta_n = \int \sum_n \delta_n = \int \phi .$$

(iii) *Let  $(\psi_n)$  be a decreasing sequence of step functions with pointwise infimum zero. Then*

$$\int \psi_n \xrightarrow{n \rightarrow \infty} 0 .$$

**Proof.** These three statements are in a trivial way consequences of each other: if (iii) is true then we apply it to the sequence  $(\phi - \phi_n)$  and obtain (i); (i) applied to the sequence of partial sums  $\sum_{\nu=1}^n \delta_\nu$  yields (ii); and (ii) applied to the decreasing sequence  $(\psi_n - \psi_{n+1})$  says that

$$\int \psi_1 = \sum_{\nu=1}^{\infty} \int (\psi_\nu - \psi_{\nu+1}) = \lim_n \sum_{\nu=1}^n \int (\psi_\nu - \psi_{\nu+1}) = \int \psi_1 - \lim_n \int \psi_{n+1},$$

and implies  $\int \psi_n \xrightarrow{n \rightarrow \infty} 0$ .

So we prove (iii). Let  $\epsilon > 0$  be given and let  $(-M, M]$  be an interval containing the carrier  $[\psi_1 > 0]$  of  $\psi_1$ . All the  $\psi_n$  vanish off this interval. Let  $x_n^1, \dots, x_n^{I(n)}$  be the points where  $\psi_n$  jumps,  $n = 1, 2, \dots$ , and consider the open sets

$$B_n = \bigcup_{m=0}^n \bigcup_{i=1}^{I(m)} (x_m^i - \epsilon 2^{-m-i-2}, x_m^i + \epsilon 2^{-m-i-2}).$$

They clearly increase with  $n$ . Next consider the sets

$$U_n = B_n \cup [\psi_n < \epsilon].$$

These sets are again open. For if  $x \in B_n$  then  $B_n$  itself is a neighborhood of  $x$  contained in  $U_n$ ; and if  $x \in U_n \setminus B_n$  then  $\psi_n$  does not jump at  $x$ , so that  $\psi_n < \epsilon$  in a whole neighborhood of  $x$ . The sets  $U_n$  clearly increase with  $n$ . Their union is the whole line. For if  $x \in \mathbb{R}$  then  $x \in [\psi_n < \epsilon] \subset U_n$  provided the index  $n$  is so high that  $\psi_n(x) < \epsilon$ . Such indices exist since  $\psi_n(x) \downarrow_{n \rightarrow \infty} 0$ . There will exist an index  $N$  such that  $U_N$  contains the compact set  $[-M, M]$ . Thus for all  $n \geq N$

$$[-M, M] \subset [\psi_n < \epsilon] \cup \tilde{B}_n, \quad (1.1)$$

where

$$\tilde{B}_n = \bigcup_{m=0}^n \bigcup_{i=1}^{I(m)} (x_m^i - \epsilon 2^{-m-i-2}, x_m^i + \epsilon 2^{-m-i-2})$$

is a set of  $\mathcal{E}[\mathbb{R}]$  slightly bigger than  $B_n$ . Equation (1.1) says that  $\psi_n < \epsilon$  off  $\tilde{B}_n$  for  $n \geq N$ . Now  $\tilde{B}_n$  is a set of measure less than

$$\sum_{n=0}^{\infty} \sum_{i=1}^{\infty} 2^{-n-i-1} \epsilon = \epsilon.$$

The set  $G_n = (-M, M] \setminus \tilde{B}_n$  belongs to  $\mathcal{E}[\mathbb{R}]$ , and  $\psi_n = G_n \cdot \psi_n + \tilde{B}_n \cdot \psi_n$ . Thus

$$\begin{aligned} \int \psi_n(x) dx &= \int_{G_n} \psi_n(x) dx + \int_{\tilde{B}_n} \psi_n(x) dx \\ &\leq \epsilon \cdot \lambda(G_n) + \epsilon \cdot \sup_x \psi_n(x) \leq \epsilon \cdot (M + \|\psi_1\|_u) \end{aligned}$$

for  $n \geq N$ . Since  $\epsilon > 0$  was arbitrary, (iii) follows. ▀

Statement (i) is commonly paraphrased as “*the elementary Riemann integral is  $\sigma$ -continuous,*” and statement (ii) as “*the elementary Riemann integral is  $\sigma$ -additive.*” (The prefix  $\sigma$ - connotes countable behavior in increasing circumstances.) Statement (iii) can be read as saying “*the elementary Riemann integral is  $\delta$ -continuous at zero.*” (The prefix  $\delta$ - connotes countable behavior in decreasing circumstances.)

**Exercise 1.2** Properties (i) – (iii) are equivalent with  $\sigma$ -continuity at zero.

Lemma 1.1 is a rather modest limit result. Yet it is fair to say that Lebesgue’s theory amounts to parlaying it into most satisfying limit theorems for his new extension of the integral.

## II.2 Elementary Integrals

It was soon noticed that Lebesgue’s methods cover not only the integral on the line, but also the integral on the plane or on  $\mathbb{R}^n$ . They make use only of the fact that the step functions  $\mathcal{E}[\mathbb{R}]$  form a lattice ring and that the “elementary integral”  $\int : \mathcal{E}[\mathbb{R}] \rightarrow \mathbb{R}$  is linear, positive, and  $\sigma$ -additive. They do, in particular, not rely on the nature of the “elementary functions” in  $\mathcal{E}[\mathbb{R}]$ . In other words, they apply to the general positive  $\sigma$ -additive elementary integral:

**Definition 2.1** (i) An **elementary integral** is a pair  $(\mathcal{E}, m)$  consisting of a lattice ring  $\mathcal{E}$  of bounded functions on some set, and of a linear map

$$m : \mathcal{E} \rightarrow \mathbb{R} .$$

The functions in  $\mathcal{E}$  are the **elementary integrands** or **elementary functions**. If we need to refer to the set on which they live, we call it the **ambient set**. The linear map  $m$  by itself is — in slight abuse of the language — again called the **elementary integral**. It is sometimes convenient to write  $\int \phi \, dm$  or  $\int \phi(x) m(dx)$  for  $m(\phi)$ , or even simply  $\int \phi$  if there is no doubt which linear map  $m : \mathcal{E} \rightarrow \mathbb{R}$  is meant:

$$m(\phi) = \int \phi \, dm = \int \phi(x) m(dx) = \int \phi , \quad \phi \in \mathcal{E} .$$

The letter  $\lambda$  is reserved for the Lebesgue integral. Whenever we want to stress that we are in the particular context of the usual integral on the line we write  $(\mathcal{E}[\mathbb{R}], \lambda)$ ,  $(\mathcal{E}[\mathbb{R}], \int dx)$ ,  $(\mathcal{E}[\mathbb{R}], \int d\lambda)$ ,  $(\mathcal{E}[\mathbb{R}], \int \lambda(dx))$ , and talk about the **elementary Lebesgue integral**.

(ii) The elementary integral  $(\mathcal{E}, m) = (\mathcal{E}, \int)$  is **positive** if  $0 \leq \phi \in \mathcal{E}$  implies that  $0 \leq m(\phi)$  or, equivalently, if  $\int \phi \leq \int \psi$  for  $\phi \leq \psi$  in  $\mathcal{E}$ .

(iii) The elementary integral  $(\mathcal{E}, \int)$  is  **$\sigma$ -additive** if

$$\int \sum \delta_n = \sum \int \delta_n$$

for every sequence  $(\delta_n)$  of positive elementary integrands whose pointwise sum exists and is an elementary integrand — the proof of Lemma 1.1 shows that this is equivalent with  $\delta$ -continuity at zero, and also with  $\sigma$ -continuity: for every increasing sequence  $(\phi_n)$  of elementary integrands whose pointwise supremum is again an elementary integrand

$$\lim_n \int \phi_n = \int \lim_n \phi_n = \int \phi. \quad (2.1)$$

**Exercise 2.2\*** The  $\sigma$ -additivity,  $\sigma$ -continuity, and  $\delta$ -continuity of an elementary integral are equivalent, whether it is positive or not.

**Exercise 2.3** The elementary Riemann integral on the plane (Exercise I.4.8) is  $\sigma$ -additive. So is the elementary Riemann integral on  $\mathbb{R}^n$ .

Lemma 1.1 states that the usual integral  $(\mathcal{E}[\mathbb{R}], \int dx)$  of step functions on the line is a positive  $\sigma$ -additive elementary integral. As pointed out above, Lebesgue's or Daniell's extension theories of the elementary Lebesgue integral on the line use only this fact. The Lebesgue integral does have special properties having to do with the structure of the ambient space  $\mathbb{R}$  as a topological field, but these do not enter the extension theory. The latter applies to every positive  $\sigma$ -additive elementary integral. We shall therefore develop right away the integration theory of the general positive  $\sigma$ -additive elementary integral. Apart from the obvious gain in generality, the arguments become clearer this way.

The reader interested only in the Lebesgue integral on the line can skip the remainder of this section, provided he understands  $\mathcal{E}$  to mean  $\mathcal{E}[\mathbb{R}]$  and  $m(\phi) = \int \phi$  to mean  $\int \phi(x) dx$  in the sequel. For the reader interested in the full generality it may still be helpful to visualize the usual integral  $(\mathcal{E}[\mathbb{R}], \int dx)$  on the line during the arguments, or better yet the integral  $(\mathcal{E}[\mathbb{R}^2], \int dx dy)$  in the plane — especially if he has done Exercise 2.3.

Mathematicians, too, are subject to temptations, and one of the more insidious ones is to generalize for generalization's sake — thus producing what in the community is called “generalized abstract nonsense.” We owe it to the reader to dispel his suspicion that Definition 2.1 does just that. To this end we exhibit in the remainder of this section two rich classes of examples of positive  $\sigma$ -additive elementary integrals, the Radon measures and the (linear extensions of) set functions. Both occur abundantly in nature.

## Radon Measures

Let  $S$  be a locally compact Hausdorff space<sup>1</sup>.  $C_{00}[S]$  denotes the collection of continuous functions on  $S$  that have compact support. This is plainly a lattice ring of bounded functions and thus qualifies for the domain  $\mathcal{E}$  of an elementary integral. A positive linear functional  $m : C_{00}[S] \rightarrow \mathbb{R}$  is called a **positive Radon measure** — they abound, as we shall see later (Example 2.21 and Exercise IV.4.8). In keeping with the notation of Definition 2.1 we also write  $m(\phi) = \int \phi dm$ . Radon measures are the prime tools in the analysis of the ubiquitous spaces  $C[K]$ ,  $K$  compact.

**Lemma 2.4** *A positive Radon measure  $m$  is  $\sigma$ -additive.*

**Proof.** Let  $(\phi_n)$  be an increasing sequence in  $C_{00+}[S]$  with pointwise supremum  $\phi \in C_{00}[S]$ . By Dini's Lemma I.3.1, the sequence  $(\phi_n)$  converges to  $\phi$  uniformly on the support of  $\phi$ . There exists a positive function  $\psi \in C_{00}[S]$  that equals 1 on the support of  $\phi^2$  — in the language of Proposition I.4.3,  $\phi$  is confined by  $\psi$ . From here on we simply repeat the proof of the Dominated Uniform Convergence Theorem I.6.5: the order relation  $|\phi - \phi_n| \leq \psi \cdot \|\phi - \phi_n\|_u$  in conjunction with the positivity of  $m$  implies

$$|m(\phi) - m(\phi_n)| = m(\phi - \phi_n) \leq m(\psi) \cdot \|\phi - \phi_n\|_u \xrightarrow{n \rightarrow \infty} 0 . \quad \blacksquare$$

**Exercise 2.5** Let  $S$  be an open or closed subset of  $\mathbb{R}^n$  and  $K \subset S$  compact. There exists a continuous function  $\psi : S \rightarrow [0, 1]$  of compact support which equals 1 on  $K$ .

**Exercise 2.6** The Riemann integral restricted to  $C_{00}[\mathbb{R}^n]$  is a positive Radon measure.

**Exercise 2.7\*** The proof of the  $\sigma$ -additivity in Lemma 2.4 actually yields a little more: Let  $\Phi$  be an increasingly directed subfamily of  $C_{00+}$ ; that is to say, given any two members of  $\Phi$  there is a third one that exceeds both. Assume the pointwise supremum of  $\Phi$  is a function  $\psi \in C_{00}[S]$ . Then  $m(\sup \Phi) = \sup_{\phi \in \Phi} m(\phi) = \lim_{\phi \in \Phi} m(\phi)$ .

The behavior of Radon measures shown in Exercise 2.7 gives rise to the following

**Definition 2.8 (Order-continuous elementary integrals)** *An elementary integral  $(\mathcal{E}, \int)$  is **order-continuous** if for every increasingly directed subset  $\Phi$  of elementary integrands whose pointwise supremum  $\sup \Phi$  is also an elementary integrand*

$$m(\sup \Phi) = \lim_{\phi \in \Phi} m(\phi) .$$

<sup>1</sup> The reader not familiar with general topology may take this to mean an open or a closed subset of  $\mathbb{R}^n$ . In such a set any two distinct points clearly have disjoint compact neighborhoods — this accounts for the words “Hausdorff” and “locally compact.”

<sup>2</sup> Urysohn's lemma provides it — see any textbook on point set topology. The reader not versed in point set topology should do Exercise 2.5.

Order-continuous elementary integrals have a slightly superior integration theory (see, however, Exercise III.5.11 on p. 90). It is left to the reader to establish this in the exercises.

## Set Functions

The next class of examples, the positive  $\sigma$ -additive set functions, occur naturally on  $\mathbb{R}^n$  and in elementary models of probability theory.

A non-void collection  $\mathcal{R}$  of subsets of some set  $S$  is a **ring of sets** if it is closed under taking finite unions and relative complements — and then under taking finite intersections:  $A \cap B = A \setminus (A \setminus B)$ . If the ring  $\mathcal{R}$  contains the ambient set  $S$  then it is called an **algebra of sets**.

A **measure** on  $\mathcal{R}$  is a map  $\mu : \mathcal{R} \rightarrow \mathbb{R}$  that is additive: for any two disjoint sets  $A, B \in \mathcal{R}$ ,

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

$\mu$  is **positive** if  $\mu(A) \geq 0$  for every  $A \in \mathcal{R}$  and  **$\sigma$ -additive** if

$$\mu\left(\bigcup A_n\right) = \sum \mu(A_n)$$

for any countable collection  $\{A_n\}$  of mutually disjoint sets in  $\mathcal{R}$  whose union belongs to  $\mathcal{R}$ . A simple argument as in Lemma 1.1 or Exercise 2.2 lets us rephrase the  $\sigma$ -additivity as

$$\sigma\text{-continuity:} \quad \mathcal{R} \ni A_n \uparrow A \in \mathcal{R} \implies \mu(A_n) \rightarrow \mu(A)$$

$$\text{or as } \delta\text{-continuity at } \emptyset: \quad \mathcal{R} \ni A_n \downarrow \emptyset \implies \mu(A_n) \rightarrow 0.$$

**Exercise 2.9\*** The two previous conditions are reasonably called  $\sigma$ -continuity and  $\delta$ -continuity at zero, respectively. Show that  $\sigma$ -additivity,  $\sigma$ -continuity, and  $\delta$ -continuity of a measure are equivalent, whether it is positive or not.

**Exercise 2.10** A ring contains the empty set, and the empty set has measure zero.

**Exercise 2.11** A non-void collection of subsets closed under taking finite unions is a ring if it is also closed under taking relative complements and an algebra if it is also closed under taking complements.

**Exercise 2.12** Let  $S$  be a set. (i) The intersection of any collection of rings of subsets of  $S$  is a ring of subsets. (ii) Let  $\mathcal{C}$  be a non-void family of subsets of  $S$ . The collection of all rings containing  $\mathcal{C}$  is not void. The intersection of this collection is called the ring generated by  $\mathcal{C}$ . Let us denote it by  $[\mathcal{C}]$ . (iii) If  $\mathcal{C}$  is finite then so is  $[\mathcal{C}]$ , and if  $\mathcal{C}$  is countable then so is  $[\mathcal{C}]$ .

**A measure  $(\mathcal{R}, \mu)$  gives rise to an elementary integral** by the simple expedient of extension by linearity: A *step function over  $\mathcal{R}$*  is, of course, a function  $\phi$  that takes only finitely many values, each non-zero value on a set from  $\mathcal{R}$ . That

is to say, the “level sets”  $[\phi = r]$  belong to  $\mathcal{R}$  for every  $r \neq 0$ , all but finitely many of them being void. Let us denote by  $\mathcal{E}[\mathcal{R}]$  the collection of step functions over  $\mathcal{R}$ . There is an immediate and intuitive way to define the  $\mu$ -integral of a step function, namely by the time-honored formula  $\sum \text{height-of-step} \times \mu\text{-size-of-step}$ :

$$\int \phi d\mu = \sum_{r \in \mathbb{R}} r \cdot \mu([\phi = r]) . \quad (2.2)$$

This is a finite sum. It is not quite obvious that  $\mathcal{E}[\mathcal{R}]$  is, indeed, a lattice ring and that the integral is linear.

**Proposition 2.13** *Let  $\mu$  be a measure on the ring  $\mathcal{R}$  of sets.*

(i) *The step functions  $\mathcal{E}[\mathcal{R}]$  are exactly the linear span of the (indicator functions of the) sets in  $\mathcal{R}$  and form a lattice ring.*

(ii) *The map  $\int d\mu$  of Equation (2.2) is the unique linear extension of  $\mu$  to  $\mathcal{E}[\mathcal{R}]$ . That is to say,  $\int d\mu : \mathcal{E}[\mathcal{R}] \rightarrow \mathbb{R}$  is linear, agrees on  $\mathcal{R} \subset \mathcal{E}[\mathcal{R}]$  with  $\mu$ , and is the only map with these two properties. For this reason we also write  $\mu(\phi)$  for  $\int \phi d\mu$  when  $\phi \in \mathcal{E}[\mathcal{R}]$ .*

For the proof a little lemma is needed:

**Lemma 2.14** *For every finite collection  $\mathcal{I} = \{A_1, \dots, A_I\}$  of sets in  $\mathcal{R}$  there is another finite collection  $\mathcal{C} = \{C_1, \dots, C_N\}$  of **mutually disjoint** sets in  $\mathcal{R}$  such that every  $A_i \in \mathcal{I}$  is the union of sets in  $\mathcal{C}$ :*

$$A_i = \bigcup \{C_n : C_n \subset A_i\} \text{ for } 1 \leq i \leq I . \quad (2.3)$$

**Proof.** This is obvious if  $I = 1$ . If it is true for  $I - 1$ , and if  $\mathcal{C}' = \{C_1, \dots, C_{N'}\}$  is the finite collection of mutually disjoint sets in  $\mathcal{R}$  such that (2.3) is satisfied for  $1 \leq i \leq I - 1$ , then

$$\mathcal{C} = \{C_n \cap A_I : 1 \leq n \leq N'\} \cup \{C_n \setminus A_I : 1 \leq n \leq N'\} \cup \left\{ A_I \setminus \bigcup_{n=1}^{N'} C_n \right\}$$

is a collection of at most  $2N' + 1$  mutually disjoint sets from  $\mathcal{R}$  satisfying (2.3).

Now to the proof of Proposition 2.13. To show that the step functions over  $\mathcal{R}$  form a lattice ring let  $\phi_1, \phi_2$  be two of them. Let  $\mathcal{I}$  be the collection of non-void level sets of  $\phi_1$  and  $\phi_2$  and  $\mathcal{C}$  the disjoint collection provided by Lemma 2.14. Note that we can write<sup>3</sup>

$$\begin{aligned} \phi_i &= \sum_{r \in \mathbb{R}} r \cdot [\phi_i = r] = \sum_{r \in \mathbb{R}} r \cdot \sum \{C \in \mathcal{C} : C \subset [\phi_i = r]\} \\ &= \sum_{r, C \subset [\phi_i = r]} r \cdot C = \sum_{r, C \subset [\phi_i = r]} \phi_i(C) \cdot C = \sum_{C \in \mathcal{C}} \phi_i(C) \cdot C \end{aligned}$$

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<sup>3</sup>  $\phi_i$  is clearly constant on each  $C \in \mathcal{C}$ , and  $\phi_i(C)$  denotes the value it takes there.

for  $i = 1, 2$ . That is to say,  $\phi_i$  is a linear combination of (the indicator functions of) the sets in  $\mathcal{C}$ . Since the latter are disjoint, it is evident that the sum, infimum, etc. of  $\phi_1$  and  $\phi_2$  are step functions over  $\mathcal{C}$ :  $\mathcal{E}[\mathcal{R}]$  is a lattice ring. There is a simple expression for the integral in terms of  $\mathcal{C}$ :

$$\int \phi_i = \sum_r r \cdot \mu([\phi_i = r]) = \sum_r r \cdot \sum_{C \subset [\phi_i = r]} \mu(C) = \sum_{C \in \mathcal{C}} \phi_i(C) \cdot \mu(C) .$$

The linearity of the integral is obvious from this. \_\_\_\_\_■

**Exercise 2.15\*** The measures on a fixed ring of sets form a vector space under pointwise operations. So do the elementary integrals on the step functions over that ring. The map that associates with a measure on the ring its extension on the step functions is a linear isomorphism that preserves the order:  $\mu \geq 0 \iff \int d\mu \geq 0$ .

The extension by linearity of a measure on a ring to an elementary integral on the step functions over that ring is such a simple procedure that we shall perform it automatically and without mention whenever we encounter a measure  $\mu$ . We take the liberty to denote the extension again by  $\mu$  if that is notationally convenient:

$$\mu : \mathcal{R} \xrightarrow{\text{additive}} \mathbb{R} \quad \text{gives rise to} \quad \mu : \mathcal{E}[\mathcal{R}] \xrightarrow{\text{linear}} \mathbb{R} .$$

The extension preserves  $\sigma$ -additivity, at least for positive measures:

**Proposition 2.16** *Let  $\mu$  be a positive measure on a ring  $\mathcal{R}$  of sets. Then its linear extension  $\int d\mu$  is  $\sigma$ -additive if and only if  $\mu$  is.*

**Proof.** The necessity is obvious. We shall prove the sufficiency in the form of Equation (2.1): if  $(\phi_n)$  is a sequence in  $\mathcal{E}[\mathcal{R}]$  with pointwise supremum  $\phi \in \mathcal{E}[\mathcal{R}]$  then  $\int \phi_n d\mu \xrightarrow{n \rightarrow \infty} \int \phi d\mu$ . This is evident from the inequality

$$\begin{aligned} \int (\phi - \phi_n) d\mu &= \int (\phi - \phi_n) \cdot [\phi - \phi_n \leq \epsilon] d\mu + \int (\phi - \phi_n) \cdot [\phi - \phi_n > \epsilon] d\mu \\ &\leq \mu([\phi > 0]) \cdot \epsilon + \|\phi\|_u \cdot \mu([\phi - \phi_n > \epsilon]) . \end{aligned}$$

This number can be made arbitrarily small by the choice first of  $\epsilon > 0$  and then of  $n$ . Indeed, the sets  $[\phi - \phi_n > \epsilon] \in \mathcal{R}$  decrease to the void set, which has measure zero, and their  $\mu$ -measure thus tends to zero. \_\_\_\_\_■

**Exercise 2.17\*** Why did we stipulate that the elementary sets that can be measured should form a ring, and that the measure be additive on it? The following is meant to shed some light on this question. Let  $\mathcal{C}$  be a collection of subsets of some set  $S$ . A step function or simple function over  $\mathcal{C}$  is, naturally, a function  $f$  that takes only finitely many values, all of them finite, and such that the sets  $[f = r]$  belong to  $\mathcal{C}$  for  $0 \neq r \in \mathbb{R}$ .

Show that the step functions over  $\mathcal{C}$  form a vector space precisely if  $\mathcal{C}$  is a ring; and that they then form a lattice ring. Show that this ring of functions contains the constant 1, i.e. is an algebra of functions, if and only if  $\mathcal{C}$  is an algebra of sets.

A map  $\mu : \mathcal{C} \rightarrow \mathbb{R}$  has a linear extension to the step functions over  $\mathcal{C}$  if and only if it is additive on  $\mathcal{C}$ .

**Example 2.18** The usual length function  $\lambda$  on  $\mathcal{A}[\mathbb{R}]$  is an instance of a positive  $\sigma$ -additive measure on a ring, by Lemma 1.1; so is the area on the ring  $\mathcal{A}[\mathbb{R}^2]$  of sets in  $\mathcal{E}[\mathbb{R}^2]$  of Exercise I.4.8 and the volume of boxes and their finite unions in  $\mathbb{R}^n$ .

**Exercise 2.19\* (Distribution Functions)** Let  $F$  be an increasing function on the line. For any  $a \leq b$  set  $\mu((a, b]) = F(b) - F(a)$ . This “interval function” has a unique extension to a positive measure  $\mu = dF$  on all of  $\mathcal{A}[\mathbb{R}]$ .  $\mu$  is  $\sigma$ -additive if and only if  $F$  is right-continuous. The function  $F$  is called a **distribution function** of  $\mu$ . The corresponding elementary integral is denoted by  $\int \phi(x) F(dx)$  or  $\int \phi(x) dF(x)$ .

**Example 2.20 (Probability)** Here is another instance in which measures on rings of sets arise naturally. Many elementary probabilistic models for a physical system start out with modelling the states of the system by the points of a set  $\Omega$ , stipulating a correspondence of observable events with an algebra  $\mathcal{A}$  of subsets of  $\Omega$ , and assuming that the probability of an event is modelled by the value  $\mathbb{P}(A)$  of a positive measure  $\mathbb{P}$  on the subset  $A \subset \Omega$  corresponding to the event. The step functions  $\phi$  over  $\mathcal{A}$  are called observables and are identified with measurements on the system, and the elementary integral  $\int \phi d\mathbb{P}$  can be defined on them as in Proposition 2.13 and is called the *expectation* and written as  $\mathbb{E}[\phi]$ . The problem arises to extend  $\mathbb{E}$  to sufficiently many functions of the state (more *observables*) to obtain a rich calculus. To this end one requires that  $\mathbb{P}$  be  $\sigma$ -additive—a requirement hard to justify except by its results—and then applies the integration theory below.

**Example 2.21 (Probability)** There is another way to make a probabilistic model for a physical system, one that comes with the  $\sigma$ -additivity built in. For concreteness’ sake assume the system is an ear of corn. There are many measurements one can perform: length, diameter, number of kernels, weight, length plus weight, diameter plus 3 times the number of kernels, etc. These measurements form in an obvious way an “abstract” algebra  $\mathcal{M}$ . For every  $m \in \mathcal{M}$  let  $\|m\|$  denote the supremum of all values that the measurement  $m$  could ever take, over all possible ears of corn. Clearly  $\| \cdot \|$  is a norm on  $\mathcal{M}$  with  $\|m_1 \cdot m_2\| \leq \|m_1\| \cdot \|m_2\|$  and  $\|m^2\| = \|m\|^2$ . A commutative normed algebra  $(\mathcal{M}, \| \cdot \|)$  having this property is called a real  $C^*$ -algebra, and there is a theorem that it can be realized as the space of continuous functions on a compact space, with  $\| \cdot \|$  corresponding to the uniform norm  $\| \cdot \|_u$ . In other words, there is a compact space  $K$  and an isometric isomorphism of  $\mathcal{M}$  with (a dense subalgebra of)  $C[K]$ . Now the expected value of a measurement<sup>4</sup>  $m$  evidently should be increasing and depend linearly on  $m \in \mathcal{M}$ . Transported to  $C[K]$ , it is modelled by a positive Radon measure  $\phi \mapsto \mathbb{E}[\phi]$  on  $K$ . As such it is automatically  $\sigma$ -additive.

**Example 2.22** Here is an example of an elementary integral that can be viewed both as a Radon measure and as coming from a set function. Let  $S$  be any set, let  $\mathcal{R}$  denote the collection of all finite subsets, and  $\mathcal{E}$  the collection of functions that vanish at all but finitely many points. Clearly,  $\mathcal{E}$  are the step functions over  $\mathcal{R}$ , and  $\mathcal{R}$  are the idempotent functions of  $\mathcal{E}$ . For  $A \in \mathcal{R}$  let  $\mu(A)$  be the number of points in  $A$ . This is plainly a  $\sigma$ -

<sup>4</sup> In the so-called frequency model this would be the limit of the average of the measurement  $m$  taken over ever bigger samples of ears of corn.

additive measure. Its linear extension to  $\mathcal{E}$  is given by the formula

$$\int \phi \, d\mu = \sum_{s \in S} \phi(s) .$$

If  $S$  is given the discrete topology by declaring every set to be open, then  $\mathcal{E} = C_{00}[S]$  and  $\int d\mu$  is a Radon measure. In any case,  $\mu$  is known as *counting measure*.

**Remark 2.23** In the examples where we established  $\sigma$ -additivity (Lemma 1.1, Exercise I.4.8, Lemma 2.4, Exercise 2.19, and Example 2.21), it was, with mind-numbing monotonicity, due to compactness in the ambient space  $S$ . There are, in fact no natural instances of  $\sigma$ -additivity that are not ultimately due to compactness. This has led Bourbaki [2] to identify integration theory with the theory of Radon measures. This view does not have a large following, because keeping track of the compact sets in the ambient space is often too cumbersome, in particular in the case of probability. The preponderant view today is to establish  $\sigma$ -additivity using compactness, rejoice in it, and to invoke the topology thereafter only when it is inevitable.

## II.3 The Daniell Mean

In the remainder of the chapter,  $(\mathcal{E}, \int)$  is a fixed positive  $\sigma$ -additive elementary integral<sup>5</sup>. If there is need to mention the set on which the functions of  $\mathcal{E}$  live we refer to it as the *ambient set*.

**The Construction of the Daniell Upper Integral**  $\int^*$ , replacement for the Riemann upper integral  $\int^\natural$ , is in two steps. In the first step it is defined only for functions  $h$  that are pointwise suprema of countable families in  $\mathcal{E}$ . Let  $\mathcal{E}^\uparrow$  denote their collection: a function  $h$  belongs to  $\mathcal{E}^\uparrow$  if there exists a sequence  $(\phi_n)$  of elementary functions whose pointwise supremum equals  $h$ . Replacing  $\phi_n$  by  $\phi_1 \vee \dots \vee \phi_n$  we see that such a sequence can be chosen to be increasing. It is understood that a function in  $\mathcal{E}^\uparrow$  may take the value  $\infty$ . The upper integral of a function  $h$  in  $\mathcal{E}^\uparrow$  is defined by

$$\int^* h = \sup \left\{ \int \phi : h \geq \phi \in \mathcal{E} \right\} . \quad (3.1)$$

If the set of numbers in the brackets is unbounded this is to mean the symbol  $+\infty$ . For an arbitrary function  $f$  set

$$\int^* f = \inf \left\{ \int^* h : f \leq h \in \mathcal{E}^\uparrow \right\} . \quad (3.2)$$

---

<sup>5</sup> As pointed out at the beginning of Section 2, the reader may take this to be the usual integral  $(\mathcal{E}[\mathbb{R}], \int dx)$  on the line.

If the set of numbers in the brackets is void, i.e., if  $f$  is not majorized by any function in  $\mathcal{E}^\uparrow$ , this is to mean the symbol  $+\infty$ . We refer to this up-and-down way to define the Daniell upper integral as “*Daniell’s up-and-down procedure*.”

The reader might be wondering why we should want to make these definitions or how anybody came to dream them up if they really should turn out useful. A little patience; we shall address these questions on page 45. It will facilitate the discussion to know the properties of  $\int^*$ .

We start with the behavior of  $\int^*$  on  $\mathcal{E}^\uparrow$ .

**Lemma 3.1** (i)  $\mathcal{E}^\uparrow$  is closed under addition, multiplication with positive scalars, and under taking finite infima and countable suprema.

(ii)  $\int^*$  is continuous along increasing sequences of  $\mathcal{E}^\uparrow$ ; that is to say, for any increasing sequence  $(h_n)$  in  $\mathcal{E}^\uparrow$  with pointwise supremum  $h$  — which by (i) belongs to  $\mathcal{E}^\uparrow$  —

$$\int^* h = \lim_{n \rightarrow \infty} \int^* h_n = \sup_n \int^* h_n . \quad (*)$$

(iii)  $h \mapsto \int^* h$  is positive-homogeneous and additive on  $\mathcal{E}^\uparrow$ .

**Proof.** (i): Let  $h_1, h_2, \dots$  be a countable collection of functions in  $\mathcal{E}^\uparrow$ . Every one of the  $h_n$  is the supremum of a countable collection in  $\mathcal{E}$ :

$$h_n = \sup\{\phi_{n,k} : k = 1, 2, \dots\} , \quad \text{say.}$$

Then  $h_1 + h_2$  is the supremum of  $\{\phi_{1,k} + \phi_{2,k} : k \in \mathbb{N}\} \subset \mathcal{E}$ , while  $r \cdot h_1$  is the supremum of  $\{r \cdot \phi_{1,k}\} \subset \mathcal{E}$ , provided  $r \in \mathbb{R}$  is positive. Next,  $\sup_n h_n$  is the pointwise supremum of the countable collection  $\{\phi_{n,k} : k, n = 1, 2, \dots\} \subset \mathcal{E}$ . Lastly, observe that  $h_n$  is the pointwise limit of the *increasing* sequence

$$\widehat{\phi}_{n,k} = \bigvee_{i=1}^k \phi_{n,i} \in \mathcal{E} , \quad k = 1, 2, \dots$$

Clearly  $h_1 \wedge h_2 = \sup_k \widehat{\phi}_{1,k} \wedge \widehat{\phi}_{2,k}$  belongs to  $\mathcal{E}^\uparrow$ .

(ii): Since  $\int^*$  is clearly increasing, the limit in (\*) equals the supremum. The inequality  $\int^* h \geq \sup \int^* h_n$  is obvious. Only the reverse inequality needs proving. Let  $\int^* h > r \in \mathbb{R}$ . There are an elementary function  $\phi \leq h$  with  $\int \phi > r$  and countable collections  $\{\phi_{n,k} : k \in \mathbb{N}\}$  of elementary functions with pointwise supremum  $h_n$ . The elementary functions

$$\widehat{\phi}_n = \phi \wedge \left( \bigvee_{1 \leq m, k \leq n} \phi_{m,k} \right) \leq h_n$$

increase pointwise to  $\phi$ . By Lemma 1.1,

$$\lim_{n \rightarrow \infty} \int \widehat{\phi}_n > r .$$

As  $\int^* h_n \geq \int \widehat{\phi}_n$ , we have  $\sup_n \int^* h_n > r$ , and thus the desired inequality

$$\sup_n \int^* h_n \geq \int^* h .$$

(iii): The positive-homogeneity is obvious. Let then  $h, h' \in \mathcal{E}^\uparrow$ . There are sequences  $(\phi_n), (\phi'_n)$  of elementary functions increasing pointwise to  $h, h'$ , respectively. Clearly  $(\phi_n + \phi'_n)$  increases pointwise to  $h + h'$ . From (ii),

$$\int^* h + h' = \lim_{n \rightarrow \infty} \int^* \phi_n + \phi'_n = \lim_{n \rightarrow \infty} \left( \int \phi_n + \int \phi'_n \right) = \int^* h + \int^* h' . \quad \blacksquare$$

We are in position to investigate the behavior of  $\int^*$  on arbitrary functions.

**Proposition 3.2** *The Daniell upper integral is increasing and positive-homogeneous. Moreover, it is countably subadditive; that is to say, for any sequence  $(f_n)$  of positive numerical functions*

$$\int^* \sum_{n=1}^{\infty} f_n \leq \sum_{n=1}^{\infty} \int^* f_n .$$

Lastly

$$\int^* \phi = \int \phi \text{ for } \phi \in \mathcal{E} .$$

**Proof.** The monotonicity and positive-homogeneity are obvious. As to the countable subadditivity, if  $\sum \int^* f_n < r$  then there are functions  $h_n \geq f_n$  in  $\mathcal{E}^\uparrow$  with  $\sum \int^* h_n < r$ . By (ii) and (iii) of Lemma 3.1,

$$\begin{aligned} \int^* \sum_{n=1}^{\infty} f_n &\leq \int^* \sum_{n=1}^{\infty} h_n = \int^* \lim_{N \rightarrow \infty} \sum_{n=1}^N h_n \\ &= \lim_{N \rightarrow \infty} \int^* \sum_{n=1}^N h_n = \lim_{N \rightarrow \infty} \sum_{n=1}^N \int^* h_n < r . \end{aligned} \quad \text{—} \blacksquare$$

**Exercise 3.3** (i) Prove the monotonicity and the positive-homogeneity of  $\int^*$  in all detail. (ii) Trace the countable subadditivity of  $\int^*$  to the  $\sigma$ -additivity of  $\int$ . (iii) Show that  $\int^\natural$  is not countably subadditive.

The countable subadditivity is the only one of these features of  $\int^*$  that the Riemann upper integral  $\int^\natural$  does not share. The celebrated limit theorems of the Lebesgue integral all are its consequences.

Let us take a time out to address the questions on page 43 about the provenance of the definition of  $\int^*$ . It takes some experience, but not much, to see that the “Little Limit Lemma” 1.1 will result in the additivity of the extension of the integral to  $\mathcal{E}^\uparrow$ , if that extension is defined by (3.1) — see Lemma 3.1 (iii). It would be too early to rejoice, because  $\mathcal{E}^\uparrow$ , though large, is not a vector space so that there is no way to so much as talk about the linearity of the extended integral on it. We have learned in Proposition I.4.5 on p. 19, though, that taking an infimum as in (I.4.3) over a functional that behaves additively results in a *subadditive* functional on *all* functions. Now, our experience with Riemann’s lower integral  $\int_\natural$  leads us to expect that replacing the “up-and-down-procedure” by a “down-and-up-procedure” will produce a *Daniell lower integral*  $\int_*$  that is defined on *all* functions and is *superadditive*<sup>6</sup>. The set of functions on which  $\int^*$  and  $\int_*$  agree will be a vector space, and the common value will be both subadditive and superadditive there; i.e., it will be additive. In this way we will arrive at an extension of the integral that is defined on a large vector space of functions and is linear there. In other words, once we have realized that Lemma 1.1 will result in the additivity of the extension (3.1) on  $\mathcal{E}^\uparrow$  we simply follow in Riemann’s footsteps. This program succeeds — see Exercise 6.2 on p. 59. To summarize: some analysis of what we want to accomplish and what we have learned leads to the definitions (3.1) and (3.2) in a rather straightforward way.

Actually, we have learned in Section I.5 that it suffices to define the upper integral and to establish its subadditivity and solidity. There is no need to worry about the lower integral and its properties. A simple absolute-value sign will take care of such worries very nicely:

**Definition 3.4** *As in Section I.5 we define the seminorm  $\| \cdot \|$  associated with the upper integral  $\int^*$ , the **Daniell mean**, by*

$$\|f\|^* = \int^* |f| \in \overline{\mathbb{R}}_+.$$

*We say the sequence  $(f_n)$  **converges in  $\| \cdot \|$ -mean** to  $f$  if  $\|f - f_n\|^* \rightarrow 0$ .*

*Later we shall consider several elementary integrals  $\mu = \int d\mu, \nu = \int d\nu, \dots$  on  $\mathcal{E}$ . The corresponding Daniell means will differ, of course, so we denote them by  $\| \cdot \|^\mu, \| \cdot \|^\nu, \dots$ . In particular,  $\| \cdot \|^\lambda$  denotes the Daniell mean for Lebesgue measure  $\lambda$ .*

We heed the lesson learned in Section I.5. Instead of using

$$-\infty < \int_* f = \int^* f < +\infty$$

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<sup>6</sup> The reader can make sure by going through Exercise 3.14 on p. 47.

to define the integrability of  $f$  we call  $f$  integrable if it is the  $\|\cdot\|^*$ -mean limit of elementary functions. The two definitions should, of course, be equivalent, and they are (Exercise 6.2 on p. 59); the second definition is just more expeditious<sup>7</sup>. Before investigating integrability, though, let us list the pertinent features of  $\|\cdot\|^*$ :

**Theorem 3.5 (Properties of the Daniell Mean)** (i) The number  $\|f\|^* \in [0, \infty]$  is defined for every numerical function  $f$  on the ambient space. The functional  $\|\cdot\|^*$  is absolute-homogeneous and solid<sup>8</sup>. Furthermore,  $\|\cdot\|^*$  is countably sub-additive; that is to say, for any sequence  $(f_n)$  of positive numerical functions

$$\left\| \sum_{n=1}^{\infty} f_n \right\|^* \leq \sum_{n=1}^{\infty} \|f_n\|^* . \quad (CSA)$$

(ii) The functional  $\|\cdot\|^*$  is finite on elementary functions. Moreover, for every sequence  $(\phi_n)$  of positive elementary functions

$$\sup_N \left\| \sum_{n=1}^N \phi_n \right\|^* < \infty \quad \text{implies} \quad \|\phi_n\|^* \xrightarrow{n \rightarrow \infty} 0 . \quad (M)$$

(iii)  $\|\cdot\|^*$  majorizes the elementary integral: for all elementary functions  $\phi$

$$\left| \int \phi \right| \leq \|\phi\|^* .$$

**Proof.** (i) and (iii) are immediate from Proposition 3.2. As for (M), if  $\left\| \sum_{n=1}^N \phi_n \right\|^* = \sum_{n=1}^N \int \phi_n$  is bounded independently of  $N$  then clearly  $\|\phi_n\|^* = \int \phi_n \xrightarrow{n \rightarrow \infty} 0$ . ■

The reader might be wondering why (M) is included in the list 3.5, perfectly obvious as its proof shows it to be. The reason is that countably subadditive solid seminorms that satisfy (M) without being additive on  $\mathcal{E}_+$  occur frequently. In fact, they appear with such frequency that they deserve their own name:

**Definition 3.6** Let  $\mathcal{E}$  be a ring<sup>9</sup> of bounded functions, to be called the **elementary functions**, on some set. Any functional  $\|\cdot\|^*$  having properties (i) and (ii) of Theorem 3.5 is called a **mean** for  $\mathcal{E}$ .

The designation “Jordan mean” of Definition I.5.2 is really a misnomer, because  $\|\cdot\|^*$  is not a mean in the sense of the definition above: it is not countably subadditive. This will not cause confusion once it has been noted.

<sup>7</sup> It has the additional advantage of applying in more general circumstances: to signed measures, which we shall meet, and to stochastic integrals and spectral measures, which the reader might meet elsewhere.

<sup>8</sup> Recall that this means that  $\|f\|^* \leq \|g\|^*$  whenever  $|f| \leq |g|$ . Together with (ii) it implies in particular that  $\|\phi\|^* = \|\phi\|^* < \infty$  for elementary integrands  $\phi$ .

<sup>9</sup> Not necessarily a vector lattice!

In the remainder of this and all of the next chapter we generally use only the fact that  $\mathcal{E}$  is a ring of bounded functions and that  $\|\cdot\|^*$  is a mean on it. In the later chapters we shall meet many other such pairs, so that we may then use the results established here for them as well.

## Supplements and Additional Exercises

**Exercise 3.7\*** (**Chebyshev's Inequality**) For any mean  $\|\cdot\|^*$ , any function  $f$ , and any  $r > 0$

$$\| [f > r] \|^* \leq \frac{1}{r} \cdot \|f\|^*.$$

**Exercise 3.8** Let  $S$  be any set, and  $\mathcal{E}$  any ring of bounded functions on  $S$  that contains the functions of finite carrier. Show that  $\|\cdot\|_u$  satisfies (i) of Theorem 3.5 but is a mean if and only if  $S$  is finite.

**Exercise 3.9** Suppose  $S$  is a locally compact metric space and  $\mathcal{E} = C_{00}[S]$  is the lattice ring of continuous functions of compact support. Then a function  $h$  belongs to  $\mathcal{E}^\uparrow$  if and only if it has  $\sigma$ -compact carrier and is lower semicontinuous. (A function  $h$  is *lower semicontinuous* if the sets  $[h > r]$  are open for all  $r \in \mathbb{R}$ . A subset of  $S$  is  $\sigma$ -compact if it is contained in the countable union of compact sets.)

**Exercise 3.10** The Daniell upper integral has a property somewhat sharper than countable subadditivity. Let  $f_n$  be an increasing sequence of arbitrary functions — possibly taking the values  $\pm\infty$  — with pointwise supremum  $f$ . Then

$$\int^* f = \lim_n \int^* f_n :$$

“The Daniell upper integral is *continuous along arbitrary increasing sequences*.”

**Exercise 3.11\*** Prove the countable subadditivity of the Daniell mean using its finite subadditivity and the result of Exercise 3.10.

**Exercise 3.12** Try to show that  $\int^*$  is continuous along arbitrary decreasing sequences:  $f_n \downarrow f$  implies  $\int^* f_n \downarrow \int^* f$ .

**Exercise 3.13\*** Compare the Jordan and Daniell means:  $\int^* f \leq \int^\natural f$  and  $\|f\|^* \leq \|f\|^\natural$  for all numerical functions  $f$ .

**Exercise 3.14\*** (i) Define  $\mathcal{E}_\downarrow$  and the lower Daniell integral  $\int_*$ . (ii) Show that  $\int_*$  is positive-homogeneous, increasing, continuous along *decreasing* sequences of  $\mathcal{E}_\downarrow$ , and countably superadditive. (iii) Show that  $\mathcal{E}_\downarrow = -\mathcal{E}^\uparrow$  and  $\int_* f = -\int^*(-f)$  and that  $\int_*$  is continuous along arbitrary decreasing sequences. (iv) Show that  $\int_* f \leq \int^* f \quad \forall f$ .

**Exercise 3.15** For all positive numerical functions  $f$ ,

$$\int^* f d\lambda = \sup \left\{ \int^* \phi \cdot f d\lambda : \phi \in \mathcal{E}_+[\mathbb{R}], \phi \leq 1 \right\},$$

and consequently for all positive numerical functions  $f$

$$\|f\|^* = \sup \{ \|\phi \cdot f\|^* : \phi \in \mathcal{E}_+[\mathbb{R}], \phi \leq 1 \}.$$

**The Order-Continuous Case.** For the remainder of this section  $(\mathcal{E}, \int)$  is a positive order-continuous elementary integral (see Definition 2.8). Denote by  $\mathcal{E}^\uparrow$  the collection of all functions on the ambient space that are pointwise suprema of

*arbitrary*, not necessarily countable, families of elementary integrands. Define a new upper integral  $\int^\bullet$ , the **Daniell–Stone upper integral**, first on functions  $h \in \mathcal{E}^\uparrow$  by

$$\int^\bullet h = \sup\left\{\int \phi : h \geq \phi \in \mathcal{E}\right\}$$

and then on arbitrary functions  $f$  by

$$\int^\bullet f = \inf\left\{\int^\bullet h : f \leq h \in \mathcal{E}^\uparrow\right\}.$$

Finally, the **Daniell–Stone mean** is defined by

$$\|f\|^\bullet = \int^\bullet |f| \in \overline{\mathbb{R}}_+. \quad (3.3)$$

The following exercises develop the properties of these notions in complete analogy with the main body of this section.

**Exercise 3.16\*** (i)  $\mathcal{E}^\uparrow$  is closed under addition, multiplication with positive scalars, and under taking finite infima and arbitrary suprema.

(ii)  $\int^\bullet$  is **continuous along increasingly directed subsets** of  $\mathcal{E}^\uparrow$ ; that is to say, for any increasingly directed subset  $H \subset \mathcal{E}^\uparrow$  with pointwise supremum  $h$  — which by (i) belongs to  $\mathcal{E}^\uparrow$  —

$$\int^\bullet h = \sup_{h' \in H} \int^\bullet h'.$$

(iii)  $h \mapsto \int^\bullet h$  is positive-homogeneous and additive on  $\mathcal{E}^\uparrow$ .

**Exercise 3.17\***  $\|\cdot\|^\bullet$  is a mean and majorizes the elementary integral. Furthermore,  $\|\cdot\|^\bullet$  is order-continuous in the following sense:

**Definition 3.18** A mean  $\|\cdot\|^*$  is **order-continuous** if it is continuous along increasingly directed subsets of  $\mathcal{E}^\uparrow$ .

**Exercise 3.19**  $\int$  is clearly also  $\sigma$ -additive, so we can construct both  $\|\cdot\|^*$  and  $\|\cdot\|^\bullet$ . Show that  $\|\cdot\|^\bullet \leq \|\cdot\|^*$ . Suppose  $\mathcal{E}$  contains a countable subset that is uniformly dense; then  $\|\cdot\|^\bullet = \|\cdot\|^*$ .

## II.4 Negligible Functions and Sets

Recall the state of affairs: we are facing a mean  $\|\cdot\|^*$  on a ring  $\mathcal{E}$  of elementary functions, for instance the Daniell mean of an elementary integral on  $\mathcal{E}$ . Our goal is the integration theory of  $(\mathcal{E}, \|\cdot\|^*)$ , i.e., the investigation of structure of the closure of  $\mathcal{E}$  under  $\|\cdot\|^*$ . In this section only the solidity and countable subadditivity of the mean  $\|\cdot\|^*$  are exploited.

**Definition 4.1** A function  $f$  is called **negligible** if its mean  $\|f\|^*$  is zero. A set is negligible if its indicator function is negligible.

A property of the points of the underlying set is said to hold **almost everywhere**, or **a.e.** for short, if the set of points where it fails to hold is negligible.

If we want to emphasize that the definition refers to the solid and countably subadditive seminorm  $\|\cdot\|^*$  we talk about  $\|\cdot\|^*$ -negligible functions and  $\|\cdot\|^*$ -a.e. convergence, etc. If  $\|\cdot\|^*$  is the Daniell mean  $\|\cdot\|^\lambda$  for Lebesgue measure  $\lambda$  and if we want to emphasize that fact then we use the words “**Lebesgue negligible**.”

**Exercise 4.2\*** Page 27 contains another definition of a Lebesgue negligible set.

(i) Show that it agrees with the present one. (ii) Show that the set  $\mathbb{Q}$  of rational numbers is Lebesgue negligible. (iii) Given  $\epsilon > 0$ , construct a dense open set  $U \subset \mathbb{R}$  whose Lebesgue outer measure  $\lambda^*(U) \stackrel{\text{def}}{=} \int^* U d\lambda$  is less than  $\epsilon$ .

Here are the permanence properties of negligibility:

**Proposition 4.3** (i) The sum of countably many positive negligible functions is negligible. The union of countably many negligible sets  $N_n$  is negligible. Any subset of a negligible set is negligible.

(ii) A function  $f$  is negligible if and only if it vanishes almost everywhere, that is to say, if and only if its carrier  $[f \neq 0]$  is negligible.

(iii) If  $f'$  and  $f''$  agree almost everywhere then they have the same mean.

(iv) A function with finite mean is finite almost everywhere.

**Proof.** (i): If  $f_n$ ,  $n = 1, 2, \dots$ , are negligible functions then

$$\left\| \bigvee_{n=1}^{\infty} |f_n| \right\|^* \leq \left\| \sum_{n=1}^{\infty} |f_n| \right\|^* \leq \sum_{n=1}^{\infty} \|f_n\|^* = \sum_{n=1}^{\infty} 0 = 0.$$

The second claim is a particular instance of this, since sets are positive (idempotent) functions (Convention I.2.6). The third claim is immediate from the solidity of  $\|\cdot\|^*$ .

(ii):  $[f \neq 0](x) \leq \sum_{n=1}^{\infty} |f(x)|$ . Thus if  $\|f\|^* = 0$  then

$$\|[f \neq 0]\|^* \leq \sum_{n=1}^{\infty} \|f\|^* = \sum_{n=1}^{\infty} 0 = 0.$$

Conversely,  $|f| \leq \sum_{n=1}^{\infty} [f \neq 0]$ , so that  $\|[f \neq 0]\|^* = 0$  implies

$$\|f\|^* \leq \sum_{n=1}^{\infty} \|[f \neq 0]\|^* = \sum_{n=1}^{\infty} 0 = 0.$$

(iii):  $\|[f' \neq f'']\|^* = 0$  implies  $\|f' - f''\|^* = 0$ , and by Exercise I.7.2

$$\left| \|f'\|^* - \|f''\|^* \right| \leq \|f' - f''\|^* = 0.$$

(iv):  $n \cdot [|f| = \infty] \leq |f| \quad \forall n \in \mathbb{N}$ , and so

$$n \cdot [|f| = \infty] \| \cdot \| \leq \|f\| \quad \forall n \in \mathbb{N}.$$

If  $[|f| = \infty]$  is not negligible,  $\|f\|$  must be infinite. ▀

**Functions Defined Almost Everywhere.** The only functions of interest for the purpose at hand are, of course, those with finite mean. We should like to argue that the sum of any two of them has finite mean again, in view of the subadditivity of  $\| \cdot \|$ . A technical difficulty appears: even if  $f$  and  $g$  have finite mean there may be points  $x$  where  $f(x) = +\infty$  and  $g(x) = -\infty$  or vice versa; then  $f(x) + g(x)$  is not defined.

The solution to this tiny quandary is to notice that such ambiguities may happen at most in a negligible set of arguments  $x$  (see Proposition 4.3 (iv)). We simply extend  $\| \cdot \|$  to functions that are defined merely almost everywhere:

**Definition 4.4** *Let  $f$  be a function defined almost everywhere, that is to say such that the complement of  $\text{dom}(f)$  is negligible. We set  $\|f\| = \|f'\|$ , where  $f'$  is any function defined everywhere and coinciding with  $f$  almost everywhere in the points where  $f$  is defined.*

Part (iii) of the previous proposition shows that this definition is good: it does not matter which function  $f'$  we choose to agree a.e. with  $f$ ; any two will differ negligibly and thus have the same mean. Given two functions  $f$  and  $g$  with finite mean that are merely almost everywhere defined we define their sum  $f + g$  to equal  $f(x) + g(x)$  where both  $f(x)$  and  $g(x)$  are finite. This function is almost everywhere defined, as the set of points where  $f$  or  $g$  are infinite or not defined is negligible. It is clear how to define the infimum, maximum, product, scalar multiples, etc. of functions that are merely a.e. defined.

From now on, therefore, “**function**” will stand for “almost everywhere defined function” if the context permits it.

**Definition 4.5** *An almost everywhere defined function  $f$  is said to have finite mean, or to be **finite in mean**, if  $\|f\| < \infty$ .  $\mathcal{F}^*$  denotes the collection of a.e. defined functions with finite mean. We write  $\mathcal{F}^*[\| \cdot \|]$  if we want to stress the point that the mean is  $\| \cdot \|$ .*

**Theorem 4.6**  *$\mathcal{F}^*$  is closed under taking finite linear combinations, finite maxima and minima, and under chopping; and  $\| \cdot \|$  is a solid and countably subadditive seminorm on  $\mathcal{F}^*$ . The pair  $(\mathcal{F}^*, \| \cdot \|)$  is a complete seminormed space. Of the utmost importance is this fact: **Every mean–Cauchy sequence has a subsequence that converges pointwise almost everywhere to a mean–limit.***

**Proof.** The first statement is left as an exercise. For the remaining two claims let  $(f_n)$  be a mean-Cauchy sequence in  $\mathcal{F}^*$ ; that is to say

$$\sup_{m,n \geq N} \|f_m - f_n\|^* \xrightarrow{N \rightarrow \infty} 0.$$

For  $n = 1, 2, \dots$  there exists a function  $f'_n$  that is everywhere defined and finite and agrees with  $f_n$  a.e. Let  $N_n$  denote the negligible set of points where  $f_n$  is not defined or does not agree with  $f'_n$ . There is an increasing sequence  $(n_k)$  of indices such that

$$\|f'_n - f'_{n_k}\|^* \leq 2^{-k-1} \text{ for } n \geq n_k.$$

Clearly  $\|f'_{n_{k+1}} - f'_{n_k}\|^* \leq 2^{-k}$ , so that  $g \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} |f'_{n_{k+1}} - f'_{n_k}|$

has finite mean, by countable subadditivity. Therefore the “bad set”

$$B = \bigcup_{n=1}^{\infty} N_n \cup [g = \infty]$$

is negligible (Proposition 4.3 (iv)). If  $x$  belongs to the “good set”  $G = B^c$  then  $g(x) < \infty$  and

$$f(x) = f'_{n_1}(x) + \sum_{k=1}^{\infty} (f'_{n_{k+1}}(x) - f'_{n_k}(x)) = \lim_{k \rightarrow \infty} f'_{n_k}(x)$$

exists, since the infinite sum above converges absolutely. Also,

$$\begin{aligned} \|f - f_{n_K}\|^* &= \|f - f'_{n_K}\|^* = \|G \cdot (f - f'_{n_K})\|^* = \|G \cdot \sum_{k=K}^{\infty} f'_{n_{k+1}} - f'_{n_k}\|^* \\ &\leq \left\| \sum_{k=K}^{\infty} |f'_{n_{k+1}} - f'_{n_k}| \right\|^* \leq 2^{-K} \xrightarrow{K \rightarrow \infty} 0. \end{aligned}$$

Thus  $(f'_{n_k})_{k=1}^{\infty}$  converges to  $f$  both in mean and pointwise on  $G$ . So does the subsequence  $(f_{n_k})$  of the original sequence  $(f_n)$ . Given  $\epsilon > 0$ , let  $K$  be so large that

$$\|f - f_{n_k}\|^* = \|f - f'_{n_k}\|^* < \epsilon/2 \quad \text{for } k \geq K$$

$$\text{and} \quad \|f_m - f_n\|^* < \epsilon/2 \quad \text{for } m, n \geq n_K.$$

$$\text{So if } n \geq n_K \text{ then} \quad \|f - f_n\|^* < \|f - f_{n_K}\|^* + \|f_{n_K} - f_n\|^* < \epsilon.$$

This shows that  $f_n \xrightarrow{n \rightarrow \infty} f$  in mean and  $f_{n_k} \xrightarrow{n \rightarrow \infty} f$  both in mean and almost everywhere. ■

From now on we shall not be so excruciatingly punctilious. If we have to perform algebraic or limit arguments on a sequence of functions that are defined merely almost everywhere we replace without mention every one of them with a function that is defined and finite everywhere and perform the arguments on the resulting sequence; this affects neither the means of the functions nor their convergence in mean or almost everywhere.

**Exercise 4.7\*** Let  $(f_n)$  be a mean-convergent sequence with limit  $f$ . Any function differing negligibly from  $f$  is also a mean limit of  $(f_n)$ , and any two mean limits of  $(f_n)$  differ negligibly.

**Exercise 4.8**  $\mathcal{F}^*$  is not in general a ring.

## II.5 Integrable Functions

In this section  $\|\cdot\|^*$  is a mean on  $\mathcal{E}$ . For the time being the elementary integrands  $\mathcal{E}$  are merely assumed to form a ring of bounded functions on the ambient set. This generality costs a little but comes in handy in connection with Fubini's theorem—see page 128.

**Definition 5.1** *An almost everywhere defined function  $f$  is called **integrable** if there is a sequence  $(\phi_n)$  of elementary functions converging in mean to it:  $\|f - \phi_n\|^* \rightarrow 0$ . In other words,  $f$  is integrable if it belongs to the mean-closure of  $\mathcal{E}$  in  $\mathcal{F}^*$ . The collection of integrable functions is denoted by  $\mathcal{L}^1$ .*

*If we want to stress the point that the definition refers to the pair  $(\mathcal{E}, \|\cdot\|^*)$  we talk about  $(\mathcal{E}, \|\cdot\|^*)$ -integrability and  $\mathcal{L}^1[\mathcal{E}, \|\cdot\|^*]$ . If  $\|\cdot\|^*$  is the Daniell mean  $\|\cdot\|^\mu$  for the positive  $\sigma$ -additive elementary integral  $\int = \int d\mu$  then the functions in  $\mathcal{L}^1$  are called the  **$\mu$ -integrable functions**, and we write  $\mathcal{L}^1 = \mathcal{L}^1[\mu]$ .*

The integrable functions are thus defined with the mean  $\|\cdot\|^*$  just as the Riemann integrable functions were characterized with  $\|\cdot\|^{\natural}$  in Proposition I.5.1.

## The Permanence Properties

As discussed in Section I.6 one does not want to apply the definition to discover whether a given function  $f$  is integrable. Rather one establishes once and for all the permanence properties of integrability and checks that  $f$  is made up from functions known to be integrable via constructions that preserve integrability. It is expedient to do this first, and then to define and examine the integral with these tools at one's disposal.

**Theorem 5.2 (Mean Completeness)** (i) If  $(f_n)$  is a sequence of integrable functions converging in mean to  $f$  then  $f$  is integrable.

(ii)  $\mathcal{L}^1$  is complete in mean. Moreover, every mean-Cauchy sequence  $(f_n)$  has a subsequence  $(f_{n_k})$  that converges almost everywhere to a mean limit of  $(f_n)$ .

(iii) If  $\mathcal{L}^1 \ni f_n \rightarrow f$  in mean and  $g = f$  a.e. then  $f_n \rightarrow g \in \mathcal{L}^1$  in mean as well.

**Proof.** For (i) copy the proof of Theorem I.6.2, and (ii) is then immediate from Theorem 4.6 on p. 50. (iii) is obvious, just a reminder. ▀

The existence of a pointwise a.e. convergent subsequence of a mean-convergent sequence  $(f_n)$  is frequently very helpful in identifying the limit, as we shall presently see. One might be inclined to hope that a mean convergent sequence  $(f_n)$ , itself, converges pointwise a.e. Alas, this is not so. Here is an example of a sequence  $(\phi_n)$  of elementary functions that converges in mean to zero but fails to converge at any single point of  $(0, 1]$ : every natural number  $n$  can be written uniquely in the form  $n = 2^m + k$ , with  $m, k \in \mathbb{N}$  and  $0 \leq k < 2^m$ . This being done, let  $\phi_n$  be the indicator function of the interval  $(k2^{-m}, (k+1)2^{-m}]$ . Clearly  $\|\phi_n\|^\lambda = 2^{-n} \rightarrow 0$  as  $n \rightarrow \infty$ . Nevertheless,  $\phi_n(x)$  takes the values 0 and 1 again and again, for every  $x \in (0, 1]$ . (It might help to draw the graphs of the first few of the  $\phi_n$ .)

**Lemma 5.3** (i) If  $\phi \in \mathcal{E}$  then  $|\phi|$  is integrable. Therefore a positive integrable function  $f$  is the mean limit of positive elementary integrands.

(ii) Property (M) extends to positive integrable functions: if a sequence  $(f_n)$  of them satisfies  $\sup_N \|\sum_{n=1}^N f_n\|^* < \infty$  then  $\|f_n\|^* \rightarrow 0$ .

**Proof.** (i): Set  $M = \sup_x |\phi(x)|$ , and let  $(P_n)$  be a sequence of positive polynomials with zero constant term that converges increasingly to the absolute value function  $t \mapsto |t|$  on  $[-M, M]$ . Such is provided by Corollary I.3.5 on p. 10. Then  $\phi_n = P_n \circ \phi \in \mathcal{E}_+$  increases to  $|\phi|$ . The sequence  $(\phi_n)$  is mean-Cauchy. Indeed, if it were not then there would exist an  $\epsilon > 0$  and a subsequence  $(\phi_{n_k})$  so that the positive elementary integrands  $\psi_k \stackrel{\text{def}}{=} \phi_{n_{k+1}} - \phi_{n_k}$  would have  $\|\psi_k\|^* > \epsilon$ . This is impossible, though, by property (M) of a mean, since  $\|\sum_k \psi_k\|^* \leq \|\phi\|^* < \infty$ . Theorem 5.2 now provides a mean limit  $g \in \mathcal{L}^1$  of  $(\phi_n)$ , which must equal  $|\phi|$  almost everywhere, since some subsequence of  $(\phi_n)$  converges almost everywhere both to  $|\phi|$  and to  $g$ . To prove the second claim of (i) let an  $\epsilon > 0$  be given. There is an  $\phi \in \mathcal{E}$  with  $\|f - \phi\|^* < \epsilon/2$ . Since  $|f - |\phi|| \leq |f - \phi|$ , this implies  $\|f - |\phi|\|^* < \epsilon/2$ . Then there is a positive  $\phi_n \in \mathcal{E}$  with  $\| |\phi| - \phi_n \|^* < \epsilon/2$ , and the subadditivity of  $\|\cdot\|^*$  yields  $\|f - \phi_n\|^* < \epsilon$ .

(ii): There are positive elementary functions  $\phi_n$  with  $\|f_n - \phi_n\|^* < 2^{-n}$ .

Since

$$\begin{aligned} \sup_N \left\| \sum_{n=1}^N \phi_n \right\|^* &\leq \sup_N \left\| \sum_{n=1}^N (\phi_n - f_n) \right\|^* + \sup_N \left\| \sum_{n=1}^N f_n \right\|^* \\ &\leq \sum_{n=1}^{\infty} 2^{-n} + \sup_N \left\| \sum_{n=1}^N f_n \right\|^* < \infty, \end{aligned}$$

we have  $\|\phi_n\|^* \xrightarrow{n \rightarrow \infty} 0$  and thus  $\|f_n\|^* \xrightarrow{n \rightarrow \infty} 0$ . ▀

**Theorem 5.4 (The Monotone Convergence Theorem or MCT)** *Let  $(f_n)$  be an increasing or decreasing sequence of integrable functions whose means form a bounded set of reals. Then  $(f_n)$  converges to its pointwise limit  $f$  in mean.*

**Proof.** As the sequence  $(f_n(x))$  of reals is monotone it has a limit  $f(x)$ , possibly equal to  $\pm\infty$ . First the case that the sequence  $(f_n)$  is increasing. Then  $(f_n)$  is mean-Cauchy. Indeed, assume it were not. There would then exist an  $\epsilon > 0$  and a subsequence  $(f_{n_k})$  with  $\|f_{n_{k+1}} - f_{n_k}\|^* > \epsilon$ . However, this sequence of positive integrable functions satisfies clearly

$$\sup_K \left\| \sum_{k=1}^K f_{n_{k+1}} - f_{n_k} \right\|^* = \sup_K \|f_{n_K} - f_{n_1}\|^* \leq \|f_{n_1}\|^* + \sup_N \|f_N\|^* < \infty.$$

By Lemma 5.3 (iii), though,  $(f_{n_{k+1}} - f_{n_k})$  must converge to zero in mean.

Now that we know that  $(f_n)$  is Cauchy, we employ Theorem 5.2: there is a mean-limit  $f'$  and a subsequence  $(f_{n_k})$ <sup>10</sup> so that  $f_{n_k}(x) \rightarrow f'(x)$  for all  $x$  outside some negligible set  $N$ . For every single  $x$ , though,  $f_n(x) \rightarrow f(x)$ . Thus

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = \lim_{k \rightarrow \infty} f_{n_k}(x) = f'(x) \text{ for } x \notin N.$$

Hence  $f$  is equal almost everywhere to the mean-limit  $f'$  and thus is a mean-limit itself.

If  $(f_n)$  is decreasing rather than increasing then  $(-f_n)$  increases pointwise—and by the above in mean—to  $-f$ ; again  $f_n \xrightarrow{n \rightarrow \infty} f$  in mean. ▀

**Theorem 5.5 (Permanence Properties concerning Algebra and Order)**

*Let  $f, f'$  be integrable functions and  $r$  a real number. Then  $f + f'$ ,  $rf$ ,  $|f|$ ,  $f \vee f'$ ,  $f \wedge f'$ , and  $f \wedge 1$  are integrable; so is  $f \cdot f'$  if  $f$  or  $f'$  is bounded.*

*In particular,  $\mathcal{L}^1$  is a vector lattice closed under chopping, and the bounded functions in  $\mathcal{L}^1$  form both a ring and a vector lattice closed under chopping.*

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<sup>10</sup> Not the same as in the previous argument, which was, after all, shown not to exist.

**Proof.** If  $\mathcal{E}$  is a lattice ring then the proof of Theorem I.6.3 on p. 24 applies word-for-word, provided  $\|\cdot\|^{\natural}$  is replaced everywhere by  $\|\cdot\|^{*}$ . Nothing was used in that proof beyond the subadditivity and solidity of the seminorm and the fact that  $\mathcal{E}$  is a lattice ring.

If  $\mathcal{E}$  is merely known to be a ring then  $\mathcal{L}^1$  is a vector space as the very first argument in the proof of Theorem 6.1 shows. The fact that the product of two integrable functions is integrable provided one of them is bounded(!) can again be taken literally from Theorem I.6.3. This leaves the permanence properties concerning the order. Assume then  $f$  is integrable. There exists a sequence  $(\phi_n)$  of elementary integrands converging in mean to  $f$ . Since  $||f| - |\phi_n|| \leq |f - \phi_n|$ ,  $||f| - |\phi_n||^{*} \leq ||f - \phi_n||^{*} \xrightarrow{n \rightarrow \infty} 0$ , so  $|f|$  is integrable, inasmuch as the  $|\phi_n|$  are —see Lemma 5.3 (i) and Theorem 5.2 (i).

By Exercise I.2.13 on p. 8,  $\mathcal{L}^1$  is a vector lattice.

To see that it is closed under chopping, the only thing not yet proved, let  $f \in \mathcal{L}^1$ . In order to show that  $f \wedge 1 \in \mathcal{L}^1$  it suffices to show that  $(f \wedge 1)_+ = f_+ \wedge 1$  is integrable, because  $(f \wedge 1)_- = f \wedge 0$  certainly is, and  $f \wedge 1$  is the sum of these two functions. In other words, we may assume that  $f$  is positive. Now if  $\phi \in \mathcal{E}_+$  then  $\phi \wedge 1$  is the uniform limit of an increasing sequence of positive elementary integrands (Exercise I.3.18) and thus is integrable. If  $f \geq 0$  is merely integrable, we approximate it in mean by a sequence  $(\phi_n)$  of *positive* elementary integrands —so that  $\phi_n \wedge 1$  is integrable—and use the solidity of  $\|\cdot\|^{*}$  on the inequality  $|f \wedge 1 - \phi_n \wedge 1| \leq |f - \phi_n|$  to show that  $f \wedge 1$  is integrable as well. ▀

The permanence properties concerning order and algebraic operations are thus not any better than for the Jordan norm  $\|\cdot\|^{\natural}$ . The ones concerning limits are much superior, though, and permit the construction of vast numbers of new integrable functions. The best is yet to come; compare the following result with Theorem I.6.5 on p. 26.

**Theorem 5.6 (The Dominated Convergence Theorem or DCT)** *Let  $(f_n)$  be a sequence of integrable functions and assume that*

- (i)  $f_n \xrightarrow{n \rightarrow \infty} f$  *pointwise almost everywhere and*
- (ii)  $|f_n| \leq g$  *for some function  $g$  with finite mean and all indices  $n$ .*

*Then  $(f_n)$  converges to  $f$  in mean, and consequently  $f$  is integrable.*

**Proof.** As in the proof of the Monotone Convergence Theorem we begin by showing that the sequence  $(f_n)$  is Cauchy. To this end consider the positive function

$$g_N = \sup\{|f_n - f_m| : m, n \geq N\} = \lim_{K \rightarrow \infty} \bigvee_{m, n=N}^K |f_n - f_m| \leq 2g.$$

By Theorem 5.5 and the Monotone Convergence Theorem,  $g_N$  is integrable. Moreover,  $g_N(x)$  converges decreasingly to zero at all points  $x$  at which  $f_n(x)$  converges, that is, almost everywhere. By Exercise 5.7,

$$\|g_N\|^* \xrightarrow{N \rightarrow \infty} 0.$$

Now  $\|f_n - f_m\|^* \leq \|g_N\|^*$  for  $m, n \geq N$ , so  $(f_n)$  is indeed Cauchy in mean. By Theorem 5.2, the sequence has a mean limit  $f'$  and a subsequence  $(f_{n_k})$  that converges pointwise a.e. to  $f'$ . Since  $(f_{n_k})$  also converges to  $f$  a.e.,  $f$  and  $f'$  agree almost everywhere, namely, at all points  $x$  at which both  $(f_n(x))$  and  $(f_{n_k}(x))$  converge. Thus  $\|f_n - f\|^* = \|f_n - f'\|^* \xrightarrow{n \rightarrow \infty} 0$ . ■

The Dominated Convergence Theorem is central. Most other results in integration theory follow from it. It is false without some domination condition such as (ii), as the following example shows. Let  $\phi_n$  be the elementary function equal to  $n$  on  $(0, 1/n]$  and zero elsewhere; the sequence  $(\phi_n)$  converges to zero at every single point, yet  $\|\phi_n\|^\lambda = \int \phi_n(x) dx = 1$  for all  $n$ . However, a slightly weaker condition, uniform integrability, suffices (Proposition 5.20). The assumption of pointwise a.e. convergence in Theorem 5.6 and Proposition 5.20 can also be replaced with a weaker condition, convergence in measure. These matters are deferred until later (Corollary III.4.12 on p. 84).

## Supplements and Additional Exercises

For the while,  $\|\cdot\|^*$  is still an arbitrary mean on a ring  $\mathcal{E}$ , not necessarily the Daniell mean for some  $\sigma$ -additive elementary integral.

**Exercise 5.7\*** Show in detail that the conclusion of Theorem 5.4 continues to hold if the  $f_n$  are merely defined a.e. and are increasing or decreasing merely a.e.

**Exercise 5.8\*** A function  $f$  is integrable if and only if there is a sequence  $(\phi_n)$  of elementary functions with  $\sum_{n=1}^{\infty} \|\phi_n\|^* < \infty$  and  $f = \sum_{n=1}^{\infty} \phi_n$  almost everywhere.

**Exercise 5.9\*** A function  $h \in \mathcal{E}^\dagger$  is integrable if and only if  $\|h\|^* < \infty$ . The mean is continuous along increasing sequences of  $\mathcal{E}^\dagger$ .

**Exercise 5.10** The collection  $\mathcal{L}_b^1$  of bounded integrable functions is a lattice ring closed under chopping and mean-dense in  $\mathcal{L}^1$ . So is the collection  $\mathcal{L}_{b0}^1 = (\mathcal{L}_b^1)_0$  of  $\mathcal{L}_b^1$ -confined functions in  $\mathcal{L}_b^1$ .

**Exercise 5.11** Suppose  $\mathcal{E}$  is a lattice ring. The collection  $\mathcal{E}_{00}$  of  $\mathcal{E}$ -confined functions in  $\mathcal{E}$  then is a self-confining lattice ring uniformly dense in  $\mathcal{E}$  and mean-dense in  $\mathcal{L}^1$ .

**Exercise 5.12** (i)  $\mathcal{L}^1$  is not in general a ring; (ii) in fact, it is a ring precisely if it is finite-dimensional, and then it is an algebra. (iii) Discuss when this can happen.

**Exercise 5.13\* (Fatou's lemma)** The following is an easy but very useful corollary of the Monotone Convergence Theorem: let  $(f_n)$  be a sequence of positive integrable functions.

Then

$$\|\liminf_{n \rightarrow \infty} f_n\|^* \leq \liminf_{n \rightarrow \infty} \|f_n\|^*;$$

and if  $\liminf_{n \rightarrow \infty} \|f_n\|^*$  is finite then  $\liminf_{n \rightarrow \infty} f_n$  is integrable.

**Exercise 5.14** The composition  $\gamma \circ f$  of a continuous function  $\gamma$  with an integrable function  $f$  is integrable, provided there is a function  $h \in \mathcal{F}^*$  with  $|\gamma \circ f| \leq h$  a.e.

**Exercise 5.15** A function  $g : \mathbb{R} \rightarrow \mathbb{R}$  is called a **Lipschitz function** if there is a constant  $K$  such that  $|g(s) - g(t)| \leq K|s - t|$  for all  $s, t \in \mathbb{R}$ . Show that the composition  $g \circ f$  of a Lipschitz function  $g$  that vanishes at zero with a bounded integrable function  $f$  is integrable.

**Exercise 5.16 (On Order-continuous Means)** Assume the mean  $\| \cdot \|$  is order-continuous, say  $\| \cdot \|$  is the Daniell-Stone mean  $\| \cdot \|^\bullet$  of an order-continuous elementary integral (see page 48). Prove: if  $\Phi$  is an increasingly directed or decreasingly directed family of elementary integrands with  $\sup_{\phi \in \Phi} \|\phi\|^\bullet < \infty$  then  $\sup \Phi$  or  $\inf \Phi$ , respectively, is integrable and  $\Phi \rightarrow \lim \Phi$  in mean. The conclusion continues to hold if  $\Phi$  is an increasingly directed subset of  $\mathcal{E}^\uparrow$  with  $\sup_{h \in \Phi} \|h\|^\bullet < \infty$ .

**Uniform Integrability.** To discuss this notion, let us first introduce **order intervals**: Let  $g, g' \in \mathcal{L}^1$ . The order interval  $[g, g']$  is defined by

$$[g, g'] = \{f \in \mathcal{L}^1 : g \leq f \leq g'\}.$$

The domination condition (ii) in the Dominated Convergence Theorem amounts to requiring that the  $f_n$  all lie in some fixed order interval  $[-g, g]$ .

If  $f$  is any function in  $\mathcal{L}^1$  there is a function  $f_g^{g'}$  in  $[g, g']$  closest to  $f$  in mean, to wit:

$$f_g^{g'} = g \vee f \wedge g'.$$

It is obtained by chopping  $f$  below with  $g$  and above with  $g'$ . It is advisable to draw a picture.

**Exercise 5.17** Show that  $\|f - f_g^{g'}\|^* \leq \|f - f'\|^* \quad \forall f' \in [g, g']$ .

**Definition 5.18** A collection  $\mathcal{S}$  of integrable functions is called **uniformly integrable** if for every  $\epsilon > 0$  there is an order interval  $[g, g'] \subset \mathcal{L}^1$  such that the mean distance from  $f$  to  $[g, g']$ :

$$\inf_{f' \in [g, g']} \|f - f'\|^* = \|f - f_g^{g'}\|^* = \|f - (g \vee f \wedge g')\|^*$$

is less than  $\epsilon$  for all  $f \in \mathcal{S}$ .

**Exercise 5.19**  $\mathcal{S}$  is uniformly integrable if and only if there is, for every  $\epsilon > 0$ , an elementary function  $\psi$  such that for all  $f \in \mathcal{S}$

$$\|f - (-\psi \vee f \wedge \psi)\|^* < \epsilon.$$

Uniform integrability can replace condition (ii) of the Dominated Convergence Theorem; and this much domination is also necessary. We leave the proof of this fact as an exercise:

**Proposition 5.20** *Let  $(f_n)$  be a sequence of integrable functions converging point-wise a.e. to some function  $f$ . Then  $(f_n)$  converges in mean to  $f$  if and only if the family  $\{f_n : n \in \mathbb{N}\}$  is uniformly integrable.*

## II.6 Extending the Integral

In this section we assume that the mean  $\|\cdot\|^*$  majorizes the elementary integral:

$$\left| \int \phi \right| \leq \|\phi\|^* .$$

This holds, for instance, if  $\|\cdot\|^*$  is the Daniell mean for  $\int$ , but there are other instances (Section V.2). The elementary integrands are assumed to form a ring  $\mathcal{E}$ .

The **extended integral** is now defined exactly as in Proposition I.5.1 on p. 21, with  $\|\cdot\|^*$  replacing the Jordan mean<sup>11</sup>  $\|\cdot\|^{\natural}$ . Namely, if  $f$  is a  $\|\cdot\|^*$ -integrable function there is a sequence  $(\phi_n)$  of elementary integrands converging in mean to  $f$ , and the integral of  $f$  is defined as

$$\int f = \lim \int \phi_n .$$

Why is the integral well-defined?<sup>12</sup> First, since  $\|\cdot\|^*$  majorizes the integral,

$$\begin{aligned} \left| \int \phi_n - \int \phi_m \right| &= \left| \int (\phi_n - \phi_m) \right| \\ &\leq \|\phi_n - \phi_m\|^* \leq \|\phi_n - f\|^* + \|f - \phi_m\|^* \xrightarrow{m,n \rightarrow \infty} 0 . \end{aligned}$$

Thus  $(\int \phi_n)$  is a Cauchy sequence of reals and does have a limit. Next, if  $(\phi'_n)$  is a second sequence of elementary functions converging in mean to  $f$  then

$$\|\phi_n - \phi'_n\|^* \leq \|\phi_n - f\|^* + \|f - \phi'_n\|^* \xrightarrow{n \rightarrow \infty} 0 ;$$

so  $|\int \phi_n - \int \phi'_n| = |\int \phi_n - \phi'_n| \rightarrow 0$ , and  $(\int \phi_n)$  and  $(\int \phi'_n)$  have the same limit: the integral is well defined.

Here are some basic properties of the integral:

**Theorem 6.1** (i) *The extended integral is linear and increasing; that is to say, for any  $f, g \in \mathcal{L}^1$  and  $r, s \in \mathbb{R}$*

$$\int (rf + sg) = r \int f + s \int g ,$$

and

$$f \leq g \implies \int f \leq \int g .$$

<sup>11</sup> which is no mean, of course, see page 46

<sup>12</sup> If you did Exercise I.7.17 on p. 29 you know.

(ii) The integral is still majorized by the mean: for  $f \in \mathcal{L}^1$ ,  $|\int f| \leq \|f\|^*$ . If  $\|\cdot\|^*$  agrees with  $\int$  on the positive elementary integrands then

$$\left| \int f \right| \leq \int |f| = \int^* |f| = \|f\|^* \quad \forall f \in \mathcal{L}^1.$$

(iii) If the sequence  $(f_n)$  of integrable functions converges in mean to  $f$  then  $f$  is integrable, and  $\int f_n \xrightarrow{n \rightarrow \infty} \int f$ .

(iv) Suppose the sequence  $(f_n)$  of integrable functions increases or decreases a.e. to a function  $f$  and  $\lim \int f_n$  is finite **or**  $(f_n)$  converges a.e. to  $f$  and  $\int^* \sup |f_n|$  is finite. Then  $f_n \rightarrow f$  in mean,  $f$  is integrable, and  $\int f_n \rightarrow \int f$ .

**Proof.** The first statement is an instance of Exercise I.7.17: Let  $(\phi_n), (\psi_n)$  be sequences in  $\mathcal{E}$  converging in mean to  $f, g \in \mathcal{L}^1$ , respectively. Then  $(r\phi_n + s\psi_n)$  converges to  $rf + sg$  in mean, since

$$\|(rf + sg) - (r\phi_n + s\psi_n)\|^* \leq |r| \cdot \|f - \phi_n\|^* + |s| \cdot \|g - \psi_n\|^* \xrightarrow{n \rightarrow \infty} 0.$$

Thus

$$\int(rf + sg) = \lim_n \int(r\phi_n + s\psi_n) = \lim_n \left( r \int \phi_n + s \int \psi_n \right) = r \int f + s \int g.$$

This shows that the integral is linear. By taking differences the monotonicity is seen to be equivalent to

$$f \geq 0 \implies \int f \geq 0 \quad \forall f \in \mathcal{L}^1.$$

By Lemma 5.3, there are positive elementary functions converging in mean to  $f$ . The limit of their integrals,  $\int f$ , is then positive as well.

(ii): Let  $(\phi_n)$  be a sequence of elementary functions with  $\|f - \phi_n\|^* \xrightarrow{n \rightarrow \infty} 0$ . As  $||f| - |\phi_n|| \leq |f - \phi_n|$ ,  $||f| - |\phi_n||^* \xrightarrow{n \rightarrow \infty} 0$  and  $\int |f| = \lim \int \phi_n$ . The elementary inequality  $|\int \phi_n| \leq \int |\phi_n|$  produces

$$\left| \int f \right| = \lim_n \left| \int \phi_n \right| \leq \lim_n \int |\phi_n| \leq \lim_n \|\phi_n\|^*.$$

Now  $\|\phi_n\|^* \xrightarrow{n \rightarrow \infty} \|f\|^*$  by Exercise I.7.2 on p. 28, and the desired inequalities follow. (iii) is evident and so is (iv) when  $\|\cdot\|^*$  is the Daniell mean for  $\int$ ; the general case is left as a (non-trivial) exercise. ▀

**Exercise 6.2\*** Assume not only that  $\|\cdot\|^*$  majorizes the elementary integral  $\int$ , but that it is, specifically, the Daniell mean  $\|\cdot\|^\mu$  for the  $\sigma$ -additive elementary integral  $\int d\mu$ . Show that a function  $f$  is then  $(\mathcal{E}, \|\cdot\|^*)$ -integrable if and only if

$$-\infty < \int_* f = \int^* f < +\infty \quad (6.1)$$

(see Exercise 3.14). In this case the common value  $\int^* f = \int_* f$  is the integral. Compare this with Proposition I.5.1 on p. 21.

**Remark 6.3** The Dominated Convergence Theorem is often stated as follows: *If the sequence  $(f_n)$  of integrable functions is dominated and converges to the function  $f$  a.e. then  $f$  is integrable and  $\int f_n \xrightarrow{n \rightarrow \infty} \int f$ .*

This is evident, since  $f_n \xrightarrow{n \rightarrow \infty} f$  in mean and the integral is continuous in mean. This statement seems genuinely weaker than the DCT. However, it is not really, given some argument: Prove the following generalization of the DCT: Let  $(g_n)$  be a sequence of integrable functions converging almost everywhere to an integrable function  $g$  and assume that  $\lim_n \int g_n = \int g$ . If  $(f_n)$  is a sequence of integrable functions converging to  $f$  almost everywhere and if  $|f_n| \leq g_n \quad \forall n$  then  $(f_n)$  converges to  $f$  in mean.

## The Lebesgue Integral.

If  $\|\cdot\|^*$  is the Daniell mean  $\|\cdot\|^\lambda$  for the usual integral on the line we write  $\mathcal{L}^1[\lambda]$  or  $\mathcal{L}^1[\mathbb{R}]$  for  $\mathcal{L}^1$  and talk about **Lebesgue integrable functions**. The integral of an integrable function  $f$  is then called its **Lebesgue integral** and written as  $\int f d\lambda$  or  $\int f(x) dx$ . We shall even mark the upper integral and write  $\int^* f(x) d\lambda(x) = \int^* f(x) dx$  instead of just  $\int^*$ .

How does Lebesgue's integral compare with Riemann's?

**Proposition 6.4** *A Riemann integrable function is Lebesgue integrable, and its Riemann integral coincides with its Lebesgue integral.*

**Proof.** From Exercise 3.13,  $\|\cdot\|^* \leq \|\cdot\|^{\natural}$ . So if  $(\phi_n)$  is a sequence of elementary functions that converges to the Riemann integrable function  $f$  in Jordan mean then it converges in Daniell-mean:

$$\|f - \phi_n\|^* \leq \|f - \phi_n\|^{\natural} \xrightarrow{n \rightarrow \infty} 0.$$

The integral in either sense is the limit of the sequence  $(\int \phi_n)$  of reals. —■

## Supplements and Additional Exercises

The mean  $\|\cdot\|^*$  is now Daniell's mean  $\|\cdot\|^\mu$  for the  $\sigma$ -additive elementary integral  $\int d\mu$ .

**Exercise 6.5** For any  $f \in \mathcal{F}^*[\|\cdot\|^*]$  there is an integrable function  $h$  with  $|f| \leq h$ .

**Exercise 6.6** A function  $f$  with finite upper integral  $\int^* f$  is majorized by an integrable function  $\bar{f}$  with  $\int^* f = \int \bar{f}$ . Such  $\bar{f}$  is called an integrable **upper envelope** of  $f$ . A function  $f$  with finite lower integral  $\int_* f$  majorizes an integrable function  $\underline{f}$  with  $\int_* f = \int \underline{f}$ . Such  $\underline{f}$  is called a **lower envelope** for  $f$ . If  $f$  is a set then  $\bar{f}$  and  $\underline{f}$  can be chosen to be sets.

**Exercise 6.7**  $\int^*(f + f') = \int^* f + \int^* f'$  if one of  $f, f'$  is integrable.

**Exercise 6.8** Let  $\|\cdot\|$  be a mean on  $\mathcal{E}[\mathbb{R}]$  having  $\|\phi\| = \int \phi$  for all positive elementary integrands. Then  $\|f\| \leq \|f\|^*$  for all  $f: S \rightarrow \mathbb{R}$ .

**Exercise 6.9** Suppose someone has somehow constructed a vector space  $\mathcal{L}^*$  of functions containing  $\mathcal{E}$  and a linear and positive extension  $\int$  of the elementary integral to  $\mathcal{L}^*$  such that the MCT and DCT hold; that is to say, if  $(f_n)$  is a sequence in  $\mathcal{L}^*$  that either increases with  $\sup \int f_n < \infty$  or that converges pointwise being dominated by a function in  $\mathcal{L}^*$ , then the limit function  $f$  belongs to  $\mathcal{L}^*$  and  $\int f = \lim \int f_n$ . Show that then there is a mean  $\|\cdot\|$  majorizing the elementary integral and such that  $\mathcal{L}^*$  is the closure of  $\mathcal{E}$  with respect to it.

**Exercise 6.10\***  $C_{00}[\mathbb{R}]$  is dense in  $\mathcal{L}^1[\mathbb{R}]$ .  $(\mathcal{L}^1[\lambda], \|\cdot\|^*)$  is separable.

**Exercise 6.11\*** Lebesgue integral is *translation invariant* — the meaning of this will become clear during the exercise. For  $a \in \mathbb{R}$  and any function  $f$  on the line denote by  $f_a$  the translated function  $x \mapsto f(x - a)$ . Then for all  $a \in \mathbb{R}$ ,  $\phi \in \mathcal{E}$ , and  $f: \mathbb{R} \rightarrow \mathbb{R}$

- (i)  $\phi \in \mathcal{E}[\mathbb{R}]$  if and only if  $\phi_a \in \mathcal{E}[\mathbb{R}]$ : the collection  $\mathcal{E}[\mathbb{R}]$  is translation invariant.
- (ii)  $\int \phi_a(x) \lambda(dx) = \int \phi(x) \lambda(dx)$ : the elementary integral is translation invariant.
- (iii)  $\int^* f_a(x) \lambda(dx) = \int^* f(x) \lambda(dx)$ : the upper integral is translation invariant.
- (iv)  $f \in \mathcal{L}^1[\lambda]$  if and only if  $f_a \in \mathcal{L}^1[\lambda]$ , and then  $\int f(x) \lambda(dx) = \int f_a(x) \lambda(dx)$ : the class of integrable functions and the extended integral are translation invariant.

**Exercise 6.12\*** Let  $f$  be a Lebesgue integrable function such that  $\int f(x)\phi(x)dx$  vanishes for every continuous function  $\phi$  of compact support. Show that  $f = 0$  a.e.

Can we draw the same conclusion if  $\int f(x)\phi(x)dx = 0$  for every  $\phi \in C_{00}^\infty[\mathbb{R}]$ ?

## II.7 Integrable Sets

Again, the arguments of the section apply to any mean.

According to our Convention I.2.6 on p. 6, sets are identified with their indicator functions: *A set is integrable if its indicator function is integrable*. The permanence properties of the class of integrable sets are immediate from those of the class of integrable functions:

**Proposition 7.1** *The union and set difference of two integrable sets are integrable. The intersection of a countable family of integrable sets is integrable. The union of a countable family of integrable sets is integrable provided it is contained in an integrable set  $C$ .*

**Proof.** Let  $A_1, A_2, \dots$  be a countable family of integrable sets. Then

$$\begin{aligned} A_1 \cup A_2 &= A_1 \vee A_2, \\ A_1 \setminus A_2 &= A_1 - (A_1 \wedge A_2), \end{aligned}$$

$$\bigcap_{n=1}^{\infty} A_n = \bigwedge_{n=1}^{\infty} A_n = \lim_{N \rightarrow \infty} \bigwedge_{n=1}^N A_n ,$$

and

$$\bigcup_{n=1}^{\infty} A_n = C - \bigwedge_{n=1}^{\infty} (C - A_n) ,$$

in the sense that in every line the set on the left has — or *is*, see Convention I.2.6—the indicator function on the right, which is integrable by the permanence properties of Section 5. ■

**Some Notation** will facilitate the discussion of integrable sets and their connection with integrable functions. Recall from Proposition 2.13 on p. 39 that a collection of subsets of some set is a ring of sets if it is closed under taking finite unions and relative complements — and then under taking finite intersections. A ring of sets that is closed under taking countable intersections is called a  **$\delta$ -ring**. Proposition 7.1 can thus be expressed by saying that the integrable sets form a  $\delta$ -ring.

A finite linear combination of integrable sets is clearly integrable. It is an integrable function that takes only finitely many values, every nonzero one in an integrable set. Such functions are called ***integrable simple functions***. In other words, the integrable simple functions are the step functions over the  $\delta$ -ring of integrable sets.

A set  $A$  is called  **$\sigma$ -finite** if it can be covered by countably many integrable sets. In many instances the whole ambient set is  $\sigma$ -finite. In that case we say that the ***mean is  $\sigma$ -finite***. If the mean is Daniell's mean for some elementary integral and is  $\sigma$ -finite, then we also say the elementary integral or measure is  $\sigma$ -finite. This is the same as saying the ambient space is the union of countable many integrable sets. Most means are  $\sigma$ -finite. The only example of a mean that is not and can still be considered a natural example is that of the Daniell mean of a Radon measure on a locally compact space so huge that it cannot be covered by countably many compact sets.

**Exercise 7.2\*** If there is a sequence of elementary integrands whose pointwise supremum is strictly positive then any mean is  $\sigma$ -finite. Thus Lebesgue integral on the line or on  $\mathbb{R}^n$  is  $\sigma$ -finite. Conversely, if the Daniell mean of  $\int$  on  $\mathcal{E}$  is  $\sigma$ -finite then there exists a sequence of elementary integrands  $\phi_n \in \mathcal{E}_+$  with  $\sup_n \phi_n = 1$ .

**Exercise 7.3** If  $\| \cdot \| \neq 0$  is  $\sigma$ -finite then there is a countable collection of disjoint non-negligible integrable sets whose union is the ambient space.

**Exercise 7.4\*** If the mean is  $\sigma$ -finite then any collection  $\mathcal{M}$  of mutually disjoint non-negligible integrable sets is at most countable.

Recall from Proposition 2.13 on p. 39 that a collection of subsets of some set is an algebra of sets if it is a ring and contains the whole ambient space. An algebra of sets that is closed under taking countable intersections is automatically closed under taking countable unions and is called a  **$\sigma$ -algebra**. The integrable sets do not, in general, form a  $\sigma$ -algebra — the line  $\mathbb{R}$ , for instance, is not Lebesgue

integrable. If they do, then we say the mean is **finite** or **totally finite**. If the mean is Daniell's mean for some elementary integral and is totally finite, then we also say the elementary integral or measure is totally finite.

Let us now investigate the relation of the integrable sets to the integrable functions.

**Proposition 7.5** *Let  $f$  be an integrable function. (i) For any strictly positive real  $r$ ,  $[f > r]$ ,  $[f \geq r]$ ,  $[f < -r]$ , and  $[f \leq -r]$  are integrable sets. (ii)  $f$  is the limit almost everywhere and in mean of a sequence  $(f_n)$  of simple integrable functions with  $|f_n| \leq |f|$ . (iii)  $f$  has  $\sigma$ -finite carrier.*

**Proof.** For the first claim, note that

$$[f > 1] = \lim_{n \rightarrow \infty} 1 \wedge (n(f - f \wedge 1))$$

is integrable: the functions  $f_n = 1 \wedge (n(f - f \wedge 1))$  are integrable by Theorem 5.5 and are dominated by  $|f|$ . By the DCT, their limit is integrable. This limit is zero at any point  $x$  where  $f(x) \leq 1$  and 1 at any point  $x$  where  $f(x) > 1$ ; in other words, it is (the indicator function of) the set  $[f > 1]$ , which is therefore integrable. *This is the first and only place where we use the fact that  $\mathcal{L}^1$  is closed under chopping.*

The set  $[f > r]$  equals  $[f/r > 1]$  and is thus integrable as well. Finally,

$$[f \geq r] = \left\{ \bigcap [f > r - 1/n] : \mathbb{N} \ni n > 1/r \right\}$$

is integrable, and  $[f < -r] = [-f > r]$  and  $[f \leq -r] = [-f \geq r]$ .

For (ii), let  $f_n$  be the simple integrable function

$$\begin{aligned} f_n = & \sum_{k=1}^{2^{2n}} k 2^{-n} \cdot [k 2^{-n} < f \leq (k+1) 2^{-n}] \\ & - \sum_{k=1}^{2^{2n}} k 2^{-n} \cdot [-k 2^{-n} > f \geq -(k+1) 2^{-n}]. \end{aligned} \tag{7.1}$$

By (i), the sets

$$[k 2^{-n} < f \leq (k+1) 2^{-n}] = [f > k 2^{-n}] \setminus [f > (k+1) 2^{-n}]$$

are integrable if  $k \neq 0$ . Thus  $f_n$ , being a linear combination of integrable functions, is integrable. Now  $(f_n)$  converges pointwise to  $f$  and is dominated by  $|f|$ , and the claim follows.

As for (iii), the carrier  $[f \neq 0]$  of  $f$  equals the union of the integrable sets  $[|f| > 1/n]$ . ■

**Remark 7.6** Suppose the mean  $\|\cdot\|^*$  is Daniell's mean  $\|\cdot\|^\mu$  for some elementary integral  $\int d\mu$ . Let us write  $\mu(A)$  for  $\int A d\mu$  when  $A$  is an integrable set. According to Equation (7.1), the integral of an integrable function  $f$  is the limit of sums of the form

$$I_n = \sum_{1 \leq |k| \leq 2^n} k 2^{-n} \mu(A_n), \quad (7.2)$$

where  $A_n \stackrel{\text{def}}{=} \{x : f(x) \text{ lies between } k 2^{-n} \text{ and } (k+1) 2^{-n}\}.$

Proposition 7.5 can then be read as follows. Instead of chopping the domain into little pieces as is done in the Calculus to approximate the integral of  $f$  by a Riemann sum, subdivide its range into intervals of length  $2^{-n}$  and replace the Riemann sum by the sums (7.2). It is sometimes said that this “transition from the  $x$ -axis to the  $y$ -axis” is Lebesgue's great contribution; this makes things sound too easy to do him justice. There is, after all, the not-so-small matter of attaching a measure to the sets  $A_n$ .

**Exercise 7.7\*** Let  $f_n$  be a sequence of integrable functions, all vanishing off the same integrable set  $A$  and converging uniformly to  $f$ . Then  $f_n \rightarrow f$  in mean and  $f$  is integrable.

**Exercise 7.8** Let  $f$  be a positive function. The following are equivalent:

(i)  $f$  is integrable; (ii) there exists an increasing sequence  $(s_n)$  of simple integrable functions with  $f = \sup s_n$  a.e. and  $\sup \int s_n < \infty$ . In this case the previous supremum is the integral of  $f$ , provided  $\|\cdot\|^*$  is the Daniell mean of  $\int$ ; (iii) there exists a sequence  $(s_n)$  of simple integrable functions with  $f = \sum s_n$  a.e. and  $\sum \int s_n < \infty$ . In this case the previous sum is the integral of  $f$  if  $\|\cdot\|^*$  is the Daniell mean of  $\int$ .

Assume henceforth that an elementary integral  $\int = \int d\mu$  is given and that the mean majorizes the integral:

$$\left| \int \phi d\mu \right| \leq \|\phi\|^* \quad \forall \phi \in \mathcal{E}. \quad \text{Then} \quad \left| \int f(x) \mu(dx) \right| \leq \|f\|^* \quad \forall f \in \mathcal{L}^1.$$

The integral of an integrable set  $A$ ,  $\int A(x) \mu(dx)$ , is usually denoted by  $\mu(A)$ . Let  $f$  be an integrable function. An integrable set  $A$  being a bounded function, the product  $A \cdot f = 1_A \cdot f$  is integrable. Its integral is usually denoted by

$$\int_A f \quad \text{or} \quad \int_A f d\mu \quad \text{or} \quad \int_A f(s) \mu(ds).$$

**Exercise 7.9\* (A Borel–Cantelli Lemma)** Given a sequence  $(A_n)$  of integrable sets that are all contained in some other integrable set, let  $A$  denote the set of points that lie in infinitely many of them.  $A$  is integrable. If  $\sum_n \mu(A_n) < \infty$  then  $A$  is negligible.

**Exercise 7.10\* (The Riesz Representation Theorem)** The restriction of the integral to the integrable sets is a positive  $\sigma$ -additive measure. The step functions over this ring, the simple integrable functions, are dense in  $\mathcal{L}^1$ .

**Exercise 7.11** Exercise 7.10 produces a  $\sigma$ -additive set function  $\mu$  on the  $\delta$ -ring  $\mathcal{R} \subset \mathcal{L}^1$  of integrable sets. We are in the situation of page 38. We would then produce an elementary integral by linear extension and then extend this, for instance by constructing its Daniell mean, or by using statement (ii) of Proposition 7.5 as the definition of integrability, to

a new class of integrable functions. Show that either way we end up right back at  $\mathcal{L}^1$ : much ado with nothing to show for it.

The remainder of the section deals specifically with the Lebesgue integral. We write  $\int_A f(x) \lambda(dx)$  for  $\int_A f$  in order to emphasize that it is Lebesgue measure we are integrating with. The  $\delta$ -ring of Lebesgue integrable sets is denoted by  $\mathcal{A}[\lambda]$ . The integral  $\int 1_A(x) \lambda(dx)$  of a Lebesgue integrable set is traditionally written as  $\lambda(A)$  and called the **Lebesgue measure** of  $A$ . Similarly, the upper integral of (the indicator function of) a subset  $A$  of the line is denoted by  $\lambda^*(A)$  and called the **outer Lebesgue measure** of  $A$ .

**Proposition 7.12 (Regularity of Lebesgue Measure)** *Recall that an open subset of the line is the union of at most countably many mutually disjoint open intervals  $(a_i, b_i)$ , among which may or may not be one or both of the infinite intervals  $(-\infty, b)$  and  $(a, \infty)$ . For the purpose of this proposition let us call this representation of an open set its **canonical representation**.*

- (i) *The outer Lebesgue measure of an open set is the sum of the lengths of the intervals in its canonical representation.*
- (ii) *A bounded open subset of the line is Lebesgue integrable, and its Lebesgue measure is then the sum of the lengths of the intervals in its canonical representation.*
- (iii) *A compact subset of the line is Lebesgue integrable.*

*Now let  $A$  be any Lebesgue integrable set, and let  $\epsilon > 0$ .*

- (iv) *There exists an open integrable set  $U$  containing  $A$  with  $\lambda(U \setminus A) < \epsilon$ . This fact is called **outer regularity** of Lebesgue measure.*
- (v) *There exists a compact set  $K \subset A$  with  $\lambda(A \setminus K) < \epsilon$ . This fact is called **inner regularity** of Lebesgue measure.*

## Additional Exercises

**Exercise 7.13** Every  $\sigma$ -additive measure on the line is inner and outer regular.

The remaining exercises deal specifically with the Lebesgue integral.

**Exercise 7.14** (See Exercise 3.15) Show that

$$\int^* f d\lambda = \sup \left\{ \int_A^* f : A \text{ integrable} \right\}$$

for all positive numerical functions  $f$ , and consequently

$$\|f\|^* = \sup \{ \|A \cdot f\|^* : A \text{ integrable} \}.$$

**Exercise 7.15** Let  $A$  be a subset of the line, and let  $a \in \mathbb{R}$ . Show that the right translate  $A + a = \{x + a : x \in A\}$  has the same outer measure as  $A$ :  $\lambda^*(A + a) = \int^* 1_{A+a} = \int^* 1_A = \lambda^*(A)$ . Show that if  $A$  is Lebesgue integrable then so is  $A + a$  and has the same measure. [Hint:  $A + a = A_a$  in the notation of Exercise 6.11 on p. 61.]

The following project permits the reader to check her understanding of the material presented so far.

**Project 7.16** For  $\phi \in C_{00}[\mathbb{R}]$  let  $\int \phi$  denote its Riemann integral. The triple  $(\mathbb{R}, C_{00}[\mathbb{R}], \int)$  has the same properties as the triple  $(\mathbb{R}, \mathcal{E}[\mathbb{R}], \int)$  we have been considering so far. Namely,  $C_{00}[\mathbb{R}]$  is a lattice ring of bounded functions, and  $\int : C_{00}[\mathbb{R}] \rightarrow \mathbb{R}$  is linear, positive, and  $\sigma$ -additive (Use Lemma I.3.1). Define the upper integral for this triple as in Section 3, replacing  $\mathcal{E}$  everywhere by  $C_{00}[\mathbb{R}]$ . Define the corresponding mean, the integrable functions, the extended integral, and produce the permanence properties of the past sections. Then show that the integrable functions so obtained are exactly the Lebesgue integrable ones, and that the integrals coincide.

## II.8 Example of a Non-Integrable Function

Let us try to think of a non-integrable function  $f$  on the line. There is  $f(x) = x^2$ , which can't be integrable because  $\|f\|^* = \infty$ : “ $f$  is too big.” We understand this obstruction to integrability very well. Let us accept it and look for a “small” function  $f$  that is not integrable. Let us try to find one that is bounded and of compact support, say, so that its mean is finite. This is not easy: how would we describe such a function? How does one usually describe a function? Well, one starts with simple ones, polynomials or step functions, say, and produces more complicated ones as algebraic or order combinations or as limits of these. The permanence properties of the class  $\mathcal{L}^1$  are so good, though, that we have no chance at producing our non-integrable function that way. We might be led to the conjecture that every function  $f$  with  $\|f\|^* < \infty$  is integrable. Alas, this is not true. This section exhibits an example of a bounded set  $B \subset \mathbb{R}$  that is not Lebesgue integrable. However, this is not easy and, in fact, requires a logical device not openly employed so far, the axiom of choice.

We start the construction of  $B$  by defining an equivalence relation  $\sim$  on  $\mathbb{R}$

by

$$r \sim r' \iff r - r' \in \mathbb{Q}.$$

Every equivalence class is countable, and there are uncountably many of them. Every equivalence class clearly has an element between  $-1$  and  $1$  in it. Using the axiom of choice, we pick from every equivalence class a number between  $-1$  and  $1$  to make the “bad” set  $B$ .

(i) The sets  $B+q$  and  $B+q'$  are disjoint if  $q \neq q' \in \mathbb{Q}$ . Indeed, if their intersection has a number  $b+q = b'+q'$  in it,  $b, b' \in B$ ,  $q, q' \in \mathbb{Q}$ , then  $b \sim b'$  and consequently  $b = b'$ , since only one representative of every equivalence class is in  $B$ . In that case  $q$  equals  $q'$ , and the two classes are identical.

Let us enumerate the rationals in  $(-2, 2)$ :  $\mathbb{Q} \cap (-2, 2) = \{q_1, q_2, \dots\}$  and let  $B_n$  be  $B$  translated by  $q_n$ :  $B_n = B + q_n$ ,  $n \in \mathbb{N}$ . The sets  $B_n$  are countable in number, mutually disjoint, and contained in the interval  $(-3, 3)$ .

(ii) The interval  $(-1, 1)$  is contained in the union of the  $B_n$ . Indeed, if  $r \in (-1, 1)$  then  $r$  is in one of the equivalence classes and differs from the representative chosen from this class by a rational  $q$  which cannot be bigger than 2 in absolute value.  $q$  is one of the  $q_n$  and consequently  $r \in B_n$ . From the *countable subadditivity* of the upper integral we deduce now

$$2 = \int^* (-1, 1) \leq \int^* \bigcup_{n=1}^{\infty} B_n \leq \sum_{n=1}^{\infty} \int^* B_n.$$

Thus one of the numbers  $\int^* B_n$  must be strictly positive. By Exercise 7.15, all the  $B_n$  have the same upper integral, and so

$$\int^* B = \int^* B_n > 0 \quad \forall n \in \mathbb{N}.$$

Now assume, by way of contradiction, that  $B$  is Lebesgue integrable. Then so are all the  $B_n$  (Exercise 7.15), and the countable *additivity* of the measure on integrable sets yields

$$\int^* \left( \bigcup_{n=1}^{\infty} B_n \right) = \sum_{n=1}^{\infty} \int B_n = \sum_{n=1}^{\infty} \int B = +\infty.$$

This is impossible, since  $\bigcup_{n=1}^{\infty} B_n \subset (-3, 3)$  and thus  $\lambda^*(\bigcup_{n=1}^{\infty} B_n) \leq 6$ . This contradiction shows that  $B$  cannot be Lebesgue integrable. ▀

**Exercise 8.1** Construct a decreasing sequence  $f_n$  of functions with pointwise infimum zero such that  $\int^* f_n \not\rightarrow 0$ .



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