

Preface

This text originated as notes for the introductory graduate Real Analysis course at the University of Texas at Austin. They were intended for students who have had a first course in Real Analysis (ϵ – δ –proofs) and know the topology of the real line, in particular the notion of compactness, and the Riemann integral.

The main topic in such a course traditionally is integration theory, and so it is here. The approach taken is different from the usual one, though. Usually the development follows history. First the content is extended to a measure on a σ –algebra, — which is identified through Carathéodory’s cut condition — then to the integrable functions through another set of limit operations. Daniell observed that the extensions have much in common and can be done in one effort, saving half the labor. We follow in his footsteps. The emphasis is not, however, on saving time so much as it is on functionality: we ask what the purpose and expected properties of the prospective integral are and then fashion definitions and procedures accordingly—history and the usual treatment are reviewed, though, in Section III.7, so as to give the reader the connection of the historical development with the present one.

Functionality also guides the treatment of measurability. The text does not set down the notion of a σ –algebra as a gift from our betters, followed by a (mysterious) definition of measurability on them; this would pressure the gentle reader into accepting authoritarian mathematics or leave the rebellious one with many questions: where do these definitions come from, where do they lead, and how did anyone ever dream them up? Rather, we ask what the obstruction to the integrability of a function might be besides being too big, in other words, what the local structure of the integrable functions is. Two simple observations in answer to this lead to Littlewood’s Principles, which are used to *define* measurability. The permanence properties of measurability are rather easy to establish this way, and once they are available so is the fact that the definition agrees with the traditional one.

A third uncommon aspect of the development is the use of seminorms. Instead of arguing with the upper and lower integrals of Daniell, analogs of Lebesgue’s outer and inner measure, we observe that already in the case of the Riemann integral a simple absolute–value sign under the upper integral simplifies matters considerably, by rendering superfluous the study of the lower integral and providing a seminorm, the Daniell mean. The analyst will often attack a problem by defining a suitable seminorm with which to identify a function space in which a solution reasonably can be sought. It seems to me that it is not too early to introduce this tool in the context of integration: the problem is to extend the elementary integral from step functions to larger classes, and the Jordan and Daniell means are perfectly suitable seminorms towards that goal.

The exposition starts with a review of Weierstraß' theorem and the Riemann integral. The latter review is designed to show that the integral of Lebesgue–Carathéodory emerges from rather a minute alteration to Riemann's; it is not the mystery but the plethora of wonderful properties of the new integral that ask for another 120 pages just to write them down and prove them.

The exercises are an integral part of the material. Those marked by an asterisk* are used later on and thus must be done. For the majority of them there are answers or at least hints in the appendix.

The first version of these notes was replete with mistakes, and I thank my students for pointing them out. I hope the remaining mistakes are small in number; they are all mine.

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May 1998



<http://www.springer.com/978-3-0348-0054-9>

Integration - A Functional Approach

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1998, VIII, 197 p., Softcover

ISBN: 978-3-0348-0054-9

A product of Birkhäuser Basel