

Preface

This book is intended to be an introduction to the fascinating theory of generalized polygons for both the graduate student and the specialized researcher in the field. It gathers together a lot of basic properties (some of which are usually referred to in research papers as belonging to folklore) and very recent and sometimes deep results. I have chosen a fairly strict geometrical approach, which requires some knowledge of basic projective geometry. Yet, it enables one to prove some typically group-theoretical results such as the determination of the automorphism groups of certain Moufang polygons. As such, some basic group-theoretical knowledge is required of the reader.

The notion of a generalized polygon is a relatively recent one. But it is one of the most important concepts in incidence geometry. Generalized polygons are the building bricks of Tits buildings. They are the prototypes and precursors of more general geometries such as partial geometries, partial quadrangles, semi-partial geometries, near polygons, Moore geometries, etc. The main examples of generalized polygons are the natural geometries associated with groups of Lie type of relative rank 2. This is where group theory comes in and we come to the historical *raison d'être* of generalized polygons.

In 1959 Jacques Tits discovered the simple groups of type 3D_4 by classifying the trialities with at least one absolute point of a D_4 -geometry. The method was predominantly geometric, and so not surprisingly the corresponding geometries (the twisted triality hexagons) came into play. Generalized hexagons were born. In an appendix to his paper on trialities, Tits introduced for the first time the notion of a generalized polygon, remarking that generalized quadrangles are the geometries belonging to the classical groups of type B_2 and C_2 . Of course, this birth date is only official, because these geometries were already implicitly present in some of Tits' earlier papers. Also, generalized quadrangles have already been around for a while as quadrics of Witt index 2, or as line systems corresponding to symplectic polarities in three-dimensional projective space over a field (and as such, some characterizations of the symplectic quadrangles already existed). But the explicit idea of studying geometries like generalized polygons is due to Tits. By the way, the class of generalized polygons includes the class of projective planes (and these had already been studied in depth).

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The starting point of this book was a remark made by Bill Kantor in the (Como) Springer Lecture Notes **1181** of 1986 (page 82): “It would be tempting to write a book on this subject”. At that time, though, I was not thinking at all about polygons. But I was some five years later. In 1991, during a visit to Braunschweig, Theo Grundhöfer brought up this very line of Kantor’s again. I felt attracted and we decided to write a book on polygons together. The actual start was delayed for a while. From 1992 on I occasionally worked on this book, and by 1995 I had written a fair amount. Due to various circumstances, Theo had to drop out. I would never have started the project on my own, but it was already in such an advanced state that I could not give up. And I kept receiving valuable help from Theo. So I decided to go on *alone* and the result is this book. The word *alone* is not to be taken too literally. Indeed, there are some people whom I would like to thank explicitly for their very kind help.

Of course I am indebted to Theo Grundhöfer to start with. Half of the book was written while I was staying in Würzburg, happily enjoying a grant from the DAAD, and the warm hospitality of Theo and his associates, especially Linus Kramer.

Then there is Jacques Tits, who gave some magical lectures at the Collège de France on Moufang polygons. I learned a lot there and I used it in many places in this book. But I am especially grateful to Tits for the time he took to listen to me and answer my questions. He also read part of a preliminary manuscript and needless to say his remarks were both very valuable and stimulating.

Francis Buekenhout, Frank De Clerck, Michael Joswig, Bill Kantor, Norbert Knarr, Linus Kramer, Bernhard Mühlerr, Allen Offer, Stanley Payne, Anja Steinbach, Jef Thas, Richard Weiss (in alphabetical order) all read (large) parts of the manuscript and provided very helpful remarks.

I am also greatly indebted to my students Leen Brouns and Eline Govaert for detecting numerous typos and other mistakes while they were reading a preliminary draft of this book for their master thesis. In this connection, I also have to thank the students attending my course in “Buildings and the Geometry of Diagrams” at the University of Ghent in the fall of 1996. My course was based on the manuscript of this book, and while preparing for their exam they discovered some more deficiencies. I would like especially to mention Anne-Marie Acou, Bram Desmet, Francis Gardeyn, Inge Oosterlinck, Nick Sabbe and Valéry Vermeulen.

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As I already mentioned, I have chosen a geometrical approach to the theory of generalized polygons. Of course, not everything can be proved with synthetic geometry. Therefore, I have also chosen to work with coordinates, as this is in my opinion the middle way between geometry and algebra. On the one hand it is a very elementary technique, but on the other hand strong enough, for instance, to understand the geometry of the Ree–Tits octagons, or to establish the structure of some automorphism groups. Since the emphasis is on geometry, I have ignored a lot of other features of polygons; this is motivated by the fact that the present

book is already voluminous enough. So you will find for instance neither the construction of hexagons with Cayley algebras due to Schellekens, nor the explicit description of all Jordan algebras giving rise to Moufang hexagons. Even worse: there is no rigorous existence proof for the Moufang octagons in this book. But coordinatization and commutation relations provide an elementary description and that is enough for the purpose of this book. In addition, I have given geometric evidence of the existence by proving that a polarity with at least one absolute point in a building of type F_4 gives rise to a generalized octagon.

Since the main goal of this book is to develop a geometric theory for polygons, I have dedicated the first chapter to a lot of elementary geometric properties, every one of which is known, but few of which are in fact explicitly written down somewhere. This should provide a good reference for this material. I have reserved no space for developing matrix techniques, and so some results in that direction are not proved here (the bulk of the theorem of Feit and Higman — which requires such matrix techniques — is proved in an appendix). A consequence of the geometric approach is the fact that I occasionally use notions and results from the theory of buildings. It is not necessary to have a full understanding of that theory, but it helps if one knows some of the major highlights. For instance, I use polar spaces and D_4 -geometries to introduce some examples of Moufang hexagons in Chapter 2, but I have tried to make the exposition as self-contained as possible. Of course, projective geometry is a necessary prerequisite and I expect the reader to be familiar with the general theory of projective spaces over skew fields. I do, however, occasionally review some important elementary aspects of projective geometry, and also of group theory. In Chapter 3, the coordinatization of polygons is introduced and some additional examples are explicitly given. The first three chapters can be considered as the basis of the whole book.

From there on, one can read every chapter on its own. Of course there are links between the chapters, but these are *horizontal* rather than *vertical*. For instance, Chapter 4 basically investigates the properties of the automorphism groups of some classical and mixed polygons. In particular, it is proved that those polygons satisfy the *Moufang condition*. Chapter 5 treats the converse: given a Moufang polygon, can we say that it is necessarily a classical or mixed one? However, the classification of Moufang polygons is not proved in this work, as this will be the major theme of a forthcoming book of Tits and Weiss. Chapter 6 brings together the conditions under which we may conclude that a certain polygon is of a certain type (namely, a well-defined subclass of the class of Moufang polygons). So it helps if one knows about Moufang polygons, but this is not required in order to understand the arguments in this chapter. Chapter 7 deals with polarities, ovoids and spreads. We focus on the Suzuki–Tits ovoids and the Ree–Tits unitals. We devote a few words to the recently discovered Moufang quadrangles. In Chapter 8 we gather some projective properties of polygons, such as the determination of some projectivity groups and some little projective groups, and the classification of some projective embeddings. Finally, Chapter 9 is an overview of the topological

counterpart of most of what has been done in the other chapters, and it is mainly expository, unlike Chapters 1 through 8, where I have tried to prove most things within a reasonable restriction of space. Some appendices conclude the book. For instance, the only explicit result concerning projective planes is in Appendix B: it is a proof by Tits of the classification of non-associative alternative division rings, independent of the characteristic. Some links between the Moufang quadrangles and the theory of algebraic groups are explained in Appendix C via Tits diagrams. I close the book with ten open problems.

Let me also point out that this book is meant to be complementary to the monograph of Payne and Thas, *Finite Generalized Quadrangles*. That is why I have neither included nor used much of the machinery developed in *op. cit.* That is also the reason why I sometimes skip proofs concerning quadrangles. But it is not necessary to have read *op. cit.* in order to understand this book. The arguments that I use are self-contained and the proofs I omit are not necessary to the rest. So primarily this book is concerned with hexagons and octagons, but it is actually fascinating to see how one can unify some things for generalized polygons, and that is my main goal.

More generalities, motivation and history are contained in the introduction to each chapter. I would like to mention one more point. There are a lot of theorems on generalized n -gons that indicate that the more interesting values of n are 3, 4, 6, and also 8. The illustration preceding this preface serves to stress this fact. The 15 little circles together with their 15 adjoining lines form a generalized 4-gon. Drawn like that, it inspired Karen De Jonghe to turn it into a little work of art. Thanks, Karen.

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I close this preface with some technical remarks. The notions that I define are written in **boldface**, while an undefined (but in general obvious, or irrelevant, or to be defined) notion that I use for the first time is written in *italics*. The strings $A := B$ or $B =: A$ define the new symbol A as the old one B . In the statements of the lemmas, propositions, theorems and corollaries, and sometimes in the text, I have written the words **if . . . then** and **if and only if** in boldface, or, if logically subordinate to some other **if . . . then** or **if and only if**, then I have underlined them. This contributes to the clarity. Some more logical structure: the end of a proof, or of a statement without proof, is marked by a little box. But when it concerns a lemma within another proof, then I use the abbreviation “QED”. On top of that, the word lemma is in this case put in a box, as are other titles that contribute to the logical structure of the proof, such as “STEP I”, etc. Also, I usually denote a map exponentially (but sometimes for clarity, I break my own rules), which means that I write a composition of maps from left to right.

I have included a few illustrations that might help the reader through an argument. Unfortunately, most proofs require either too many diagrams, or too complicated ones. Therefore, I would advise the reader to draw his/her own diagrams.

I have not included an author index. Instead, I have listed in the Bibliography the pages where every reference is mentioned in the book, including unpublished references.

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Van Maldeghem, H.

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