

Preface

This expository monograph was written for three reasons. Firstly, we wanted to present the solution to a problem posed by Wolfgang Krull in 1932 [Krull 32]. He asked whether what we now call the “Krull-Schmidt Theorem” holds for artinian modules. The problem remained open for 63 years: its solution, a negative answer to Krull’s question, was published only in 1995 (see [Facchini, Herbera, Levy and Vámos]). Secondly, we wanted to present the answer to a question posed by Warfield in 1975 [Warfield 75]. He proved that every finitely presented module over a serial ring is a direct sum of uniserial modules, and asked if such a decomposition was unique. In other words, Warfield asked whether the “Krull-Schmidt Theorem” holds for serial modules. The solution to this problem, a negative answer again, appeared in [Facchini 96]. Thirdly, the solution to Warfield’s problem shows interesting behavior, a rare phenomenon in the history of Krull-Schmidt type theorems. Essentially, the Krull-Schmidt Theorem holds for some classes of modules and not for others. When it does hold, any two indecomposable decompositions are uniquely determined up to a permutation, and when it does not hold for a class of modules, this is proved via an example. For serial modules the Krull-Schmidt Theorem does not hold, but any two indecomposable decompositions are uniquely determined up to *two* permutations. We wanted to present such a phenomenon to a wider mathematical audience.

Apart from these three reasons, we present in this book various topics of module theory and ring theory, some of which are now considered classical (like Goldie dimension, semiperfect rings, Krull dimension, rings of quotients, and their applications) whereas others are more specialized (like dual Goldie dimension, semilocal endomorphism rings, serial rings and modules, exchange property, Σ -pure-injective modules).

We now consider the three reasons above in more detail.

1) *Krull’s problem.* The classical Krull-Schmidt Theorem says that if M_R is a right module of finite composition length over a ring R and

$$M_R = A_1 \oplus \cdots \oplus A_n = B_1 \oplus \cdots \oplus B_m$$

are two decompositions of M_R as direct sums of indecomposable modules, then

$n = m$ and, after a suitable renumbering of the summands, $A_i \cong B_i$ for every $i = 1, \dots, n$. In the paper [Krull 32, pp. 37–38] Krull recalls this theorem and asks whether the result remains true for artinian modules; that is, whether $A_1 \oplus \dots \oplus A_n = B_1 \oplus \dots \oplus B_m$ with each A_i and B_j an indecomposable artinian right module over a ring R , implies that $m = n$ and, after a renumbering of the summands, $A_i \cong B_i$ for each i .

During the years Krull's question was not forgotten (see for instance [Levy, p. 660]), and various partial results were proved. For instance, [Warfield 69a, Proposition 5] showed that the answer is “yes” when the ring R is either right noetherian or commutative (Proposition 2.63). He did this by showing that, over any ring, every artinian indecomposable module with Loewy length $\leq \omega$ has a local endomorphism ring. By the Krull-Schmidt-Remak-Azumaya Theorem, direct sums of indecomposable modules with local endomorphism rings have unique direct sum decompositions, even when the direct sum contains infinitely many terms, so that Warfield could conclude that the answer to Krull's question was positive if the base ring R was right noetherian or commutative.

Since direct sum decompositions of modules correspond to decompositions of their endomorphism ring in a natural way, Krull's problem is a particular case of the problem of determining what kinds of rings can occur as endomorphism rings of artinian modules. Rosa Camps and Warren Dicks [Camps and Dicks] showed that the endomorphism ring of any artinian module is semilocal, i.e., semisimple artinian modulo its Jacobson radical (Theorem 4.12). This allowed them to prove that artinian modules cancel from direct sums; that is, if $M \oplus A \cong M \oplus B$ with A and B arbitrary modules and M artinian, then $A \cong B$ (Corollary 4.6). All module-finite algebras over a semilocal noetherian commutative ring are semilocal, and applying a result of Camps and Menal it is possible to prove that all such module-finite algebras are isomorphic to endomorphism rings of artinian modules (Corollary 8.18). Therefore all decompositions of noetherian modules over the semilocal rings that occur in integral representation theory yield corresponding decompositions of artinian modules over suitable rings. Using this, it is possible to construct various examples. For instance, fix an integer $n \geq 2$. Then there is a ring R and an artinian right module M_R which is the direct sum of 2 indecomposable modules, and also the direct sum of 3 indecomposable modules, and also the direct sum of 4 indecomposable modules, and \dots , and also the direct sum of n indecomposable modules (Example 8.21). There exists another ring R with four indecomposable, pairwise nonisomorphic, artinian modules M_1, M_2, M_3, M_4 such that $M_1 \oplus M_2 \cong M_3 \oplus M_4$ (Example 8.20).

These examples answer Krull's question: the Krull-Schmidt Theorem fails for artinian modules.

2) *Warfield's problem.* Recall that a module is said to be *uniserial* if for any submodules A and B of M either $A \subseteq B$ or $B \subseteq A$. A *serial* module is a module that is a direct sum of uniserial modules, and a ring R is *serial*

if the two modules R_R and ${}_R R$ are both serial modules. Important classes of rings yield examples of serial rings. For instance, semisimple artinian rings, commutative valuation rings, and rings of triangular matrices over a field are serial rings.

In 1975 R. B. Warfield published a paper in which he described the structure of serial rings and proved that every finitely presented module over a serial ring is a direct sum of uniserial modules [Warfield 75]. On page 189 of that paper, talking of the problems that remained open, he said that "... perhaps the outstanding open problem is the uniqueness question for decompositions of a finitely presented module into uniserial summands (proved in the commutative case and in one noncommutative case by [Kaplansky 49])." In other words, Warfield asked whether the Krull-Schmidt Theorem holds for direct sums of uniserial modules.

Warfield's problem was solved completely in [Facchini 96] by giving a counterexample: Krull-Schmidt fails for serial modules. For instance, fix an integer $n \geq 2$. Then there exist $2n$ pairwise non-isomorphic finitely presented uniserial modules $U_1, U_2, \dots, U_n, V_1, V_2, \dots, V_n$ over a suitable serial ring such that $U_1 \oplus U_2 \oplus \dots \oplus U_n \cong V_1 \oplus V_2 \oplus \dots \oplus V_n$ (Example 9.21).

3) *The weak Krull-Schmidt Theorem for serial modules.* As we have just said, the Krull-Schmidt Theorem does not hold for direct sums of uniserial modules. Nevertheless, a weak form of the Krull-Schmidt Theorem still holds for these modules. If A and B are modules, write $[A]_m = [B]_m$ if A and B are in the same *monogeny class*, that is, if there is a monomorphism $A \rightarrow B$ and a monomorphism $B \rightarrow A$. Similarly, write $[A]_e = [B]_e$ if A and B are in the same *epigeny class*, that is, if there is an epimorphism $A \rightarrow B$ and an epimorphism $B \rightarrow A$. If $U_1, \dots, U_n, V_1, \dots, V_t$ are uniserial modules, then $U_1 \oplus \dots \oplus U_n \cong V_1 \oplus \dots \oplus V_t$ if and only if $n = t$ and there are two permutations σ, τ of $\{1, \dots, n\}$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i = 1, \dots, n$. This is a rare phenomenon: the isomorphism class of a serial module is completely determined up to *two* permutations.

Such a weak form of the Krull-Schmidt Theorem holds not only for direct sums of uniserial modules, but, more generally, for direct sums of *biuniform* modules, i.e., modules that are uniform and couniform (Chapter 9).

This book also deals with a number of other topics as well. For instance, we study the class of the rings that can be realized as endomorphism rings of artinian modules, and serial rings belonging to this class are characterized. We introduce Σ -pure-injective modules, because every artinian module is Σ -pure-injective as a module over its endomorphism ring. In order to determine whether a ring can be realized as the endomorphism ring of an artinian module, we may look for sufficient information about the structure of its Σ -pure-injective modules. We consider modules with the exchange property, semiperfect rings, serial rings, their Krull dimension and their quotient rings.

The last chapter contains some open problems.

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