

## Chapter 2

# The Krull-Schmidt-Remak-Azumaya Theorem

### 2.1 The exchange property

In the first two sections of this chapter we shall consider modules with the exchange property. Making use of the exchange property we shall study refinements of direct sum decompositions (Sections 2.3 and 2.10), prove the Krull-Schmidt-Remak-Azumaya Theorem (Section 2.4) and prove that every finitely presented module over a serial ring is serial (Section 3.5). If  $A, B, C$  are submodules of a module  $M$  and  $C \leq A$ , then  $A \cap (B + C) = (A \cap B) + C$ . This is called the *modular identity*. We begin with an immediate consequence of the modular identity that will be used repeatedly in the sequel.

**Lemma 2.1** *If  $A \subseteq B \subseteq A \oplus C$  are modules, then  $B = A \oplus D$ , where  $D = B \cap C$ .*

*Proof.* Application of the modular identity to the modules  $A \subseteq B$  and  $C$  yields  $B \cap (C + A) = (B \cap C) + A$ , that is,  $B = A + D$ . This sum is direct because  $A \cap D \subseteq A \cap C = 0$ .  $\square$

Given a cardinal  $\aleph$ , an  $R$ -module  $M$  is said to have the  $\aleph$ -*exchange property* if for any  $R$ -module  $G$  and any two direct sum decompositions

$$G = M' \oplus N = \oplus_{i \in I} A_i,$$

where  $M' \cong M$  and  $|I| \leq \aleph$ , there are  $R$ -submodules  $B_i$  of  $A_i$ ,  $i \in I$ , such that  $G = M' \oplus (\oplus_{i \in I} B_i)$ .

In this notation an application of Lemma 2.1 to the modules

$$B_i \subseteq A_i \subseteq B_i \oplus (M' \oplus (\oplus_{j \neq i} B_j))$$

yields  $A_i = B_i \oplus D_i$ , where  $D_i = A_i \cap (M' \oplus (\oplus_{j \neq i} B_j))$ . Hence the submodules  $B_i$  of  $A_i$  in the definition of module with the  $\aleph$ -exchange property are necessarily direct summands of  $A_i$ .

A module  $M$  has the *exchange property* if it has the  $\aleph$ -exchange property for every cardinal  $\aleph$ . A module  $M$  has the *finite exchange property* if it has the  $\aleph$ -exchange property for every finite cardinal  $\aleph$ .

A finitely generated module has the exchange property if and only if it has the finite exchange property.

**Lemma 2.2** *If  $G, M', N, P, A_i$  ( $i \in I$ ),  $B_i$  ( $i \in I$ ) are modules,  $B_i \subseteq A_i$  for every  $i \in I$ ,*

$$G = M' \oplus N \oplus P = (\oplus_{i \in I} A_i) \oplus P \quad (2.1)$$

and

$$G/P = ((M' + P)/P) \oplus (\oplus_{i \in I} ((B_i + P)/P)), \quad (2.2)$$

then

$$G = M' \oplus (\oplus_{i \in I} B_i) \oplus P.$$

*Proof.* From (2.2) it follows immediately that  $G = M' + (\sum_{i \in I} B_i) + P$ . In order to show that this sum is direct, suppose  $m' + (\sum_{i \in I} b_i) + p = 0$  for some  $m' \in M'$ ,  $b_i \in B_i$  almost all zero, and  $p \in P$ . From (2.2) we have that  $(m' + P) + (\sum_{i \in I} (b_i + P)) = 0$  in  $G/P$ , so that  $m' \in P$  and  $b_i \in P$  for every  $i \in I$ . Then by (2.1) we get  $m' \in M' \cap P = 0$  and  $b_i \in B_i \cap P \subseteq A_i \cap P = 0$ . Therefore  $p = 0$ .  $\square$

The proof of the next corollary follows immediately from the definitions and Lemma 2.2.

**Corollary 2.3** *If  $G, M', N, P, A_i$  ( $i \in I$ ) are modules,  $|I| \leq \aleph$ ,*

$$G = M' \oplus N \oplus P = (\oplus_{i \in I} A_i) \oplus P$$

and  $M'$  has the  $\aleph$ -exchange property, then for every  $i \in I$  there exists a direct summand  $B_i$  of  $A_i$  such that

$$G = M' \oplus (\oplus_{i \in I} B_i) \oplus P. \quad \square$$

The rest of this section is devoted to proving the first properties of modules with the exchange property. The following result shows that the class of modules with the  $\aleph$ -exchange property is closed under direct summands and finite direct sums.

**Lemma 2.4** *Suppose  $\aleph$  is a cardinal and  $M = M_1 \oplus M_2$ . The module  $M$  has the  $\aleph$ -exchange property if and only if both  $M_1$  and  $M_2$  have the  $\aleph$ -exchange property.*

*Proof.* Suppose  $M = M_1 \oplus M_2$  has the  $\aleph$ -exchange property,

$$G = M'_1 \oplus N = \oplus_{i \in I} A_i,$$

$M'_1 \cong M_1$  and  $|I| \leq \aleph$ . Then  $G' = M_2 \oplus G = M' \oplus N = M_2 \oplus (\oplus_{i \in I} A_i)$ , where  $M' = M'_1 \oplus M_2 \cong M$ . Fix an element  $k \in I$ , and set  $I' = I \setminus \{k\}$ . Then  $G' = M' \oplus N = (M_2 \oplus A_k) \oplus (\oplus_{i \in I'} A_i)$ . Hence there exist submodules  $B \subseteq M_2 \oplus A_k$  and  $B_i \subseteq A_i$  for every  $i \in I'$  such that

$$G' = M' \oplus B \oplus (\oplus_{i \in I'} B_i). \quad (2.3)$$

Since  $M_2 \subseteq M_2 \oplus B \subseteq M_2 \oplus A_k$ , it follows from Lemma 2.1 that

$$M_2 \oplus B = M_2 \oplus B_k,$$

where  $B_k = (M_2 \oplus B) \cap A_k$ . Thus  $M' \oplus B = (M'_1 \oplus M_2) \oplus B = M'_1 \oplus M_2 \oplus B_k$ . Substituting this into (2.3) we obtain

$$G' = M'_1 \oplus M_2 \oplus (\oplus_{i \in I} B_i). \quad (2.4)$$

Application of the modular identity to the modules  $M'_1 \oplus (\oplus_{i \in I} B_i) \subseteq G$  and  $M_2$  yields  $G \cap (M_2 + (M'_1 \oplus (\oplus_{i \in I} B_i))) = (G \cap M_2) + (M'_1 \oplus (\oplus_{i \in I} B_i))$ , that is,  $G = M'_1 \oplus (\oplus_{i \in I} B_i)$ . Thus  $M_1$  has the  $\aleph$ -exchange property.

Conversely, suppose  $M_1$  and  $M_2$  have the  $\aleph$ -exchange property and

$$G = M'_1 \oplus M'_2 \oplus N = \oplus_{i \in I} A_i,$$

where  $M'_1 \cong M_1$ ,  $M'_2 \cong M_2$  and  $|I| \leq \aleph$ . Since  $M_1$  has the  $\aleph$ -exchange property, there are submodules  $A'_i \subseteq A_i$  such that  $G = M'_1 \oplus M'_2 \oplus N = M'_1 \oplus (\oplus_{i \in I} A'_i)$ . Since  $M_2$  also has the  $\aleph$ -exchange property, from Corollary 2.3 it follows that for every  $i \in I$  there exists a submodule  $B_i \subseteq A'_i$  such that

$$G = M'_2 \oplus (\oplus_{i \in I} B_i) \oplus M'_1.$$

Thus  $M = M_1 \oplus M_2$  has the  $\aleph$ -exchange property.  $\square$

Clearly, every module has the 1-exchange property. Modules the with 2-exchange property have the finite exchange property, as the next lemma shows.

**Lemma 2.5** *If a module  $M$  has the 2-exchange property, then  $M$  has the finite exchange property.*

*Proof.* It is sufficient to show, for an arbitrary integer  $n \geq 2$ , that if  $M$  has the  $n$ -exchange property, then  $M$  has the  $(n+1)$ -exchange property. Let  $M$  be a module with the  $n$ -exchange property ( $n \geq 2$ ) and suppose

$$G = M' \oplus N = A_1 \oplus A_2 \oplus \cdots \oplus A_{n+1},$$

where  $M' \cong M$ . Set  $P = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ , so that  $G = M' \oplus N = P \oplus A_{n+1}$ . Since  $M$  has the 2-exchange property, there exist submodules  $P' \subseteq P$  and  $B_{n+1} \subseteq A_{n+1}$  such that  $G = M' \oplus P' \oplus B_{n+1}$ . An application of Lemma 2.1 to the modules  $P' \subseteq P \subseteq P' \oplus (M' \oplus B_{n+1})$  and  $B_{n+1} \subseteq A_{n+1} \subseteq B_{n+1} \oplus (M' \oplus P')$  yields  $P = P' \oplus P''$  and  $A_{n+1} = B_{n+1} \oplus A'_{n+1}$ , where  $P'' = P \cap (M' \oplus B_{n+1})$  and  $A'_{n+1} = A_{n+1} \cap (M' \oplus P')$ . From the decompositions

$$G = M' \oplus P' \oplus B_{n+1} = (P'' \oplus A'_{n+1}) \oplus (P' \oplus B_{n+1})$$

we infer that  $P''$  is isomorphic to a direct summand of  $M'$ . Therefore  $P''$  has the  $n$ -exchange property by Lemma 2.4. Since

$$P = P' \oplus P'' = A_1 \oplus A_2 \oplus \cdots \oplus A_n,$$

there exist submodules  $B_i \subseteq A_i$  ( $i = 1, 2, \dots, n$ ) such that

$$P = P'' \oplus B_1 \oplus B_2 \oplus \cdots \oplus B_n.$$

Application of Lemma 2.1 to the modules

$$P'' \subseteq M' \oplus B_{n+1} \subseteq G = P'' \oplus (P' \oplus A_{n+1})$$

yields  $M' \oplus B_{n+1} = P'' \oplus P'''$ , where  $P''' = (M' \oplus B_{n+1}) \cap (P' \oplus A_{n+1})$ . Therefore

$$\begin{aligned} G &= M' \oplus P' \oplus B_{n+1} = P' \oplus P'' \oplus P''' = P \oplus P''' \\ &= B_1 \oplus \cdots \oplus B_n \oplus P'' \oplus P''' = B_1 \oplus \cdots \oplus B_n \oplus B_{n+1} \oplus M', \end{aligned}$$

that is,  $M$  has the  $(n+1)$ -exchange property. □

## 2.2 Indecomposable modules with the exchange property

The aim of this section is to show that the indecomposable modules with the (finite) exchange property are exactly those with a local endomorphism ring. First we prove two elementary lemmas that will be used often in the sequel.

**Lemma 2.6** *Let  $A$  be a module and let  $M_1, M_2, M'$  be submodules of  $A$ . Suppose  $A = M_1 \oplus M_2$ . Let  $\pi_2: A = M_1 \oplus M_2 \rightarrow M_2$  denote the canonical projection. Then  $A = M_1 \oplus M'$  if and only if  $\pi_2|_{M'}: M' \rightarrow M_2$  is an isomorphism. If these equivalent conditions hold, then the canonical projection  $\pi_{M'}: A \rightarrow M'$  with respect to the decomposition  $A = M_1 \oplus M'$  is  $(\pi_2|_{M'})^{-1} \circ \pi_2$ .*

*Proof.* The mapping  $\pi_2|_{M'}$  is injective if and only if  $M' \cap M_1 = 0$ , and is surjective if and only if for every  $x_2 \in M_2$  there exists  $x' \in M'$  such that

$x' = x_1 + x_2$  for some  $x_1 \in M_1$ , that is, if and only if  $M_2 \subseteq M' + M_1$ , i.e., if and only if  $M_1 + M_2 = M' + M_1$ . This proves the first part of the statement. For the second part suppose that the equivalent conditions hold. Given an arbitrary element  $a \in A$ , one has  $a = x_1 + \pi_{M'}(a)$  for a suitable element  $x_1 \in M_1$ . Hence  $\pi_2(a) = \pi_2(x_1) + \pi_2|_{M'}(\pi_{M'}(a))$ , from which  $(\pi_2|_{M'})^{-1} \pi_2(a) = \pi_{M'}(a)$ .  $\square$

**Lemma 2.7** *Let  $M, N, P_1, \dots, P_n$  be modules with  $M \oplus N = P_1 \oplus \dots \oplus P_n$ . If  $M$  is an indecomposable module with the finite exchange property, then there is an index  $j = 1, 2, \dots, n$  and a direct sum decomposition  $P_j = B \oplus C$  of  $P_j$  such that  $M \oplus N = M \oplus B \oplus (\oplus_{i \neq j} P_i)$ ,  $M \cong C$  and  $N \cong B \oplus (\oplus_{i \neq j} P_i)$ .*

*Proof.* Since  $M$  has the finite exchange property, for every  $i = 1, 2, \dots, n$  there exists a decomposition  $P_i = B_i \oplus C_i$  of  $P_i$  such that

$$M \oplus B_1 \oplus \dots \oplus B_n = P_1 \oplus \dots \oplus P_n.$$

If we factorize modulo  $B_1 \oplus \dots \oplus B_n$  we find that  $M \cong C_1 \oplus \dots \oplus C_n$ . But  $M$  is indecomposable. Hence there exists an index  $j$  such that  $M \cong C_j$  and  $C_i = 0$  for every  $i \neq j$ . Then  $B_i = P_i$  for  $i \neq j$ , hence

$$M \oplus N = M \oplus B_1 \oplus \dots \oplus B_n = M \oplus B_j \oplus (\oplus_{i \neq j} P_i).$$

In particular,  $N \cong B_j \oplus (\oplus_{i \neq j} P_i)$ . Thus  $B = B_j$  and  $C = C_j$  have the required properties.  $\square$

**Theorem 2.8** *The following conditions are equivalent for an indecomposable module  $M_R$ .*

- (a) *The endomorphism ring of  $M_R$  is local.*
- (b)  *$M_R$  has the finite exchange property.*
- (c)  *$M_R$  has the exchange property.*

*Proof.* (a)  $\Rightarrow$  (b). Let  $M_R$  be a module with local endomorphism ring  $\text{End}(M_R)$ . By Lemma 2.5 in order to prove that the finite exchange property holds it suffices to show that  $M$  has the 2-exchange property. Let  $G, N, A_1, A_2$  be modules such that  $G = M \oplus N = A_1 \oplus A_2$ . Let  $\varepsilon_M, \varepsilon_{A_1}, \varepsilon_{A_2}, \pi_M, \pi_{A_1}, \pi_{A_2}$  be the embeddings of  $M, A_1, A_2$  into  $G$  and the canonical projections of  $G$  onto  $M, A_1, A_2$  with respect to these two decompositions. We must show that there are submodules  $B_1 \subseteq A_1$  and  $B_2 \subseteq A_2$  such that  $G = M \oplus B_1 \oplus B_2$ . Now

$$1_M = \pi_M \varepsilon_M = \pi_M (\varepsilon_{A_1} \pi_{A_1} + \varepsilon_{A_2} \pi_{A_2}) \varepsilon_M = \pi_M \varepsilon_{A_1} \pi_{A_1} \varepsilon_M + \pi_M \varepsilon_{A_2} \pi_{A_2} \varepsilon_M.$$

Since  $\text{End}(M)$  is local, one of these two summands, say  $\pi_M \varepsilon_{A_1} \pi_{A_1} \varepsilon_M$ , must be an automorphism of  $M$ . Let  $H$  be the image of the monomorphism

$$\varepsilon_{A_1} \pi_{A_1} \varepsilon_M: M \rightarrow G,$$

so that  $\varepsilon_{A_1}\pi_{A_1}\varepsilon_M$  induces an isomorphism  $M \rightarrow H$  and  $\pi_M|_H: H \rightarrow M$  is an isomorphism. From Lemma 2.6 we have that  $G = N \oplus H$  and the projection  $G \rightarrow H$  with respect to this decomposition is  $(\pi_M|_H)^{-1}\pi_M$ . Since

$$H = \varepsilon_{A_1}\pi_{A_1}\varepsilon_M(M) \subseteq A_1 \subseteq N \oplus H,$$

it follows from Lemma 2.1 that  $A_1 = H \oplus B_1$ , where  $B_1 = A_1 \cap N$ , and the projection  $A_1 \rightarrow H$  with respect to this decomposition is  $(\pi_M|_H)^{-1}\pi_M|_{A_1}$ . Therefore  $G = A_1 \oplus A_2 = H \oplus (B_1 \oplus A_2)$ . With respect to this last decomposition of  $G$  the projection  $G \rightarrow H$  is  $(\pi_M|_H)^{-1}\pi_M|_{A_1}\pi_{A_1} = (\pi_M|_H)^{-1}\pi_M\varepsilon_{A_1}\pi_{A_1}$ , and this mapping restricted to  $M$  is  $(\pi_M|_H)^{-1}\pi_M\varepsilon_{A_1}\pi_{A_1}\varepsilon_M$ . This is an isomorphism. Again by Lemma 2.6 we get that  $G = M \oplus B_1 \oplus A_2$ .

(b)  $\Rightarrow$  (c). Let  $M_R$  be an indecomposable module with the finite exchange property and suppose  $G = M \oplus N = \oplus_{i \in I} A_i$ . Fix a non-zero element  $x \in M$ . There is a finite subset  $F$  of  $I$  such that  $x \in \oplus_{i \in F} A_i$ . Set  $A' = \oplus_{i \in I \setminus F} A_i$ , so that  $G = M \oplus N = (\oplus_{i \in F} A_i) \oplus A'$ . By Lemma 2.7 either there is an index  $j \in F$  and a direct sum decomposition  $A_j = B \oplus C$  of  $A_j$  such that

$$G = M \oplus B \oplus (\oplus_{i \in F, i \neq j} A_i) \oplus A',$$

or there is a direct sum decomposition  $A' = B' \oplus C'$  of  $A'$  such that

$$G = M \oplus B' \oplus (\oplus_{i \in F} A_i).$$

The second possibility cannot occur because  $M \cap (\oplus_{i \in F} A_i) \neq 0$ . Hence there is an index  $j \in F$  and a submodule  $B$  of  $A_j$  such that

$$G = M \oplus B \oplus (\oplus_{i \in F, i \neq j} A_i) \oplus A' = M \oplus B \oplus (\oplus_{i \in I, i \neq j} A_i).$$

(c)  $\Rightarrow$  (a). Let  $M$  be an indecomposable module and suppose that  $\text{End}(M)$  is not a local ring. Then there exist two elements  $\varphi, \psi \in \text{End}(M)$  which are not automorphisms of  $M$ , such that  $\varphi - \psi = 1_M$ . Let  $A = M_1 \oplus M_2$  be the external direct sum of two modules  $M_1, M_2$  both equal to  $M$ , and let  $\pi_i: A \rightarrow M_i$ ,  $i = 1, 2$  be the canonical projections. The composite of the mappings

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix}: M \rightarrow M_1 \oplus M_2 \text{ and } (1_M - 1_M): M_1 \oplus M_2 \rightarrow M$$

is the identity mapping of  $M$ , so that if  $M'$  denotes the image of  $\begin{pmatrix} \varphi \\ \psi \end{pmatrix}$  and  $K$  denotes the kernel of  $(1_M - 1_M)$ , then  $A = M' \oplus K$ . If the exchange property were to hold for  $M$ , there would be direct summands  $B_1$  of  $M_1$  and  $B_2$  of  $M_2$  such that  $A = M' \oplus K = M' \oplus B_1 \oplus B_2$ . Since  $M_1$  and  $M_2$  are indecomposable, we would have either  $A = M' \oplus M_1$  or  $A = M' \oplus M_2$ . If  $A = M' \oplus M_1$ , then  $\pi_2|_{M'}: M' \rightarrow M_2$  is an isomorphism (Lemma 2.6). Then the composite mapping  $\pi_2 \circ \begin{pmatrix} \varphi \\ \psi \end{pmatrix}: M \rightarrow M_2$  is an isomorphism. But  $\pi_2 \circ \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \psi$ , contradiction. Similarly if  $A = M' \oplus M_2$ . This shows that  $M$  does not have the exchange property.  $\square$

## 2.3 Isomorphic refinements of finite direct sum decompositions

Let  $M$  be a module over a ring  $R$ . Suppose that  $\{M_i \mid i \in I\}$  and  $\{N_j \mid j \in J\}$  are two families of submodules of  $M$  such that  $M = \bigoplus_{i \in I} M_i = \bigoplus_{j \in J} N_j$ . Then these two decompositions are said to be *isomorphic* if there exists a one-to-one correspondence  $\varphi: I \rightarrow J$  such that  $M_i \cong N_{\varphi(i)}$  for every  $i \in I$ , and the second decomposition is a *refinement* of the first if there is a surjective map  $\varphi: J \rightarrow I$  such that  $N_j \subseteq M_{\varphi(j)}$  for every  $j \in J$  (equivalently, if there is a surjective map  $\varphi: J \rightarrow I$  such that  $\bigoplus_{j \in \varphi^{-1}(i)} N_j = M_i$  for every  $i \in I$ ). The first theorem of this section gives a criterion that assures the existence of isomorphic refinements of two direct sum decompositions.

**Theorem 2.9** *Let  $\aleph$  be a cardinal, let  $M$  be a module with the  $\aleph$ -exchange property, and let  $M = \bigoplus_{i \in I} M_i = \bigoplus_{j \in J} N_j$  be two direct sum decompositions of  $M$  with  $I$  finite and  $|J| \leq \aleph$ . Then these two direct sum decompositions of  $M$  have isomorphic refinements.*

*Proof.* We may assume  $I = \{0, 1, 2, \dots, n\}$ . We shall construct a chain  $N_j \supseteq N'_{0,j} \supseteq N'_{1,j} \supseteq \dots \supseteq N'_{n-1,j} \supseteq N'_{n,j}$  for every  $j \in J$  such that

$$M = \left( \bigoplus_{i=0}^k M_i \right) \oplus \left( \bigoplus_{j \in J} N'_{k,j} \right)$$

for every  $k = 0, 1, 2, \dots, n$ . The construction of the  $N'_{k,j}$  is by induction on  $k$ . For  $k = 0$  the module  $M_0$  has the  $\aleph$ -exchange property (Lemma 2.4). Hence there are submodules  $N'_{0,j}$  of  $N_j$  such that  $M = M_0 \oplus \left( \bigoplus_{j \in J} N'_{0,j} \right)$ . Suppose  $1 \leq k \leq n$  and that the modules  $N'_{k-1,j}$  with  $M = \left( \bigoplus_{i=0}^{k-1} M_i \right) \oplus \left( \bigoplus_{j \in J} N'_{k-1,j} \right)$  have been constructed. Apply Corollary 2.3 to the decompositions

$$M = M_k \oplus \left( \bigoplus_{i=k+1}^n M_i \right) \oplus \left( \bigoplus_{i=0}^{k-1} M_i \right) = \left( \bigoplus_{j \in J} N'_{k-1,j} \right) \oplus \left( \bigoplus_{i=0}^{k-1} M_i \right)$$

(note that  $M_k$  has the  $\aleph$ -exchange property by Lemma 2.4). Then there exist submodules  $N'_{k,j}$  of  $N'_{k-1,j}$  such that  $M = M_k \oplus \left( \bigoplus_{j \in J} N'_{k,j} \right) \oplus \left( \bigoplus_{i=0}^{k-1} M_i \right)$ , which is what we had to prove.

For  $k = n$  we have that  $M = \left( \bigoplus_{i=0}^n M_i \right) \oplus \left( \bigoplus_{j \in J} N'_{n,j} \right)$ , so that  $N'_{n,j} = 0$  for every  $j \in J$ . Since the  $N'_{k,j}$  are direct summands of  $M$  contained in  $N'_{k-1,j}$ , there is a direct sum decomposition  $N'_{k-1,j} = N'_{k,j} \oplus N_{k,j}$  for every  $k$  and  $j$  (Lemma 2.1). Similarly,  $N_j = N'_{0,j} \oplus N_{0,j}$ . Hence  $N_j = N_{0,j} \oplus N_{1,j} \oplus \dots \oplus N_{n,j}$  for every  $j \in J$ , so that  $M = \bigoplus_{j \in J} \bigoplus_{i=0}^n N_{i,j}$  is a refinement of the decomposition  $M = \bigoplus_{j \in J} N_j$ .

As  $M = \left( \bigoplus_{i=0}^{k-1} M_i \right) \oplus \left( \bigoplus_{j \in J} N'_{k-1,j} \right) = \left( \bigoplus_{i=0}^k M_i \right) \oplus \left( \bigoplus_{j \in J} N'_{k,j} \right)$  for  $k = 1, 2, \dots, n$ , factorizing modulo  $\left( \bigoplus_{i=0}^{k-1} M_i \right) \oplus \left( \bigoplus_{j \in J} N'_{k,j} \right)$  we obtain that  $\bigoplus_{j \in J} N_{k,j} \cong M_k$  for  $k = 1, 2, \dots, n$ . Similarly  $\bigoplus_{j \in J} N_{0,j} \cong M_0$ . Hence for every

$i = 0, 1, 2, \dots, n$  there is a decomposition  $M_i = \bigoplus_{j \in J} N''_{i,j}$  of  $M_i$  with  $N''_{i,j} \cong N_{i,j}$  for every  $i$  and  $j$ . Thus  $\bigoplus_{i=0}^n \bigoplus_{j \in J} N''_{i,j}$  is a refinement of the decomposition  $M = \bigoplus_{i=0}^n M_i$  isomorphic to the decomposition  $M = \bigoplus_{j \in J} \bigoplus_{i=0}^n N_{i,j}$ .  $\square$

A second case in which it is possible to find isomorphic refinements is that of direct sum decompositions into countably many direct summands with the  $\aleph_0$ -exchange property. This is proved in the next theorem.

**Theorem 2.10** *Let  $M$  be a module with two direct sum decompositions*

$$M = B_0 \oplus B_1 \oplus B_2 \oplus \dots \quad (2.5)$$

$$M = C_0 \oplus C_1 \oplus C_2 \oplus \dots \quad (2.6)$$

*with countably many direct summands, where all the summands  $B_i$  and  $C_j$  have the  $\aleph_0$ -exchange property. Then the two direct sum decompositions (2.5) and (2.6) have isomorphic refinements.*

*Proof.* Set  $B'_{i,-1} = B_i$  and  $C'_{i,-1} = C_i$  for every  $i = 0, 1, 2, \dots$ . By induction on  $j = 0, 1, 2, \dots$  we shall construct direct summands  $B_{i,j}, B'_{i,j}$  of  $B_i$  for every  $i = j+1, j+2, j+3, \dots$  and direct summands  $C_{i,j}, C'_{i,j}$  of  $C_i$  for every  $i = j, j+1, j+2, \dots$  such that the following properties hold for every  $j = 0, 1, 2, \dots$ :

- (a)  $B'_{i,j-1} = B_{i,j} \oplus B'_{i,j}$  for every  $i = j+1, j+2, j+3, \dots$ ;
- (b)  $C'_{i,j-1} = C_{i,j} \oplus C'_{i,j}$  for every  $i = j, j+1, j+2, \dots$ ;
- (c)  $M = (B'_{0,-1} \oplus C'_{0,0}) \oplus (B'_{1,0} \oplus C'_{1,1}) \oplus \dots \oplus (B'_{j,j-1} \oplus C'_{j,j}) \oplus (\bigoplus_{k=j+1}^{\infty} C'_{k,j})$ ;
- (d)  $M = (B'_{0,-1} \oplus C'_{0,0}) \oplus (B'_{1,0} \oplus C'_{1,1}) \oplus \dots \oplus (B'_{j,j-1} \oplus C'_{j,j}) \oplus (\bigoplus_{k=j+1}^{\infty} B'_{k,j})$ ;
- (e)  $C_{j,j} \oplus C_{j+1,j} \oplus C_{j+2,j} \oplus \dots \cong B'_{j,j-1}$ ;
- (f)  $B_{j+1,j} \oplus B_{j+2,j} \oplus B_{j+3,j} \oplus \dots \cong C'_{j,j}$ .

Case  $j = 0$ . Since  $B_0$  has the  $\aleph_0$ -exchange property, there exists a decomposition  $C_i = C_{i,0} \oplus C'_{i,0}$  of  $C_i$  for every  $i = 0, 1, 2, \dots$  such that

$$M = B_0 \oplus C'_{0,0} \oplus C'_{1,0} \oplus C'_{2,0} \oplus \dots \quad (2.7)$$

From (2.6) and (2.7) it follows that

$$B_0 \cong C_{0,0} \oplus C_{1,0} \oplus C_{2,0} \oplus \dots,$$

that is, (e) holds.

The direct summand  $C'_{0,0}$  of  $C_0$  has the  $\aleph_0$ -exchange property by Lemma 2.4. Applying Corollary 2.3 to (2.5) and (2.7) we obtain direct sum decompositions  $B_i = B_{i,0} \oplus B'_{i,0}$  of  $B_i$  for every  $i = 1, 2, 3, \dots$  such that

$$M = B_0 \oplus C'_{0,0} \oplus B'_{1,0} \oplus B'_{2,0} \oplus B'_{3,0} \oplus \dots \quad (2.8)$$



From (2.5) and (2.8) we get that  $C'_{0,0} \cong B_{1,0} \oplus B_{2,0} \oplus B_{3,0} \oplus \dots$ , that is, (f) holds. This concludes the construction of the submodules  $C_{i,0}, C'_{i,0}$  of  $C_i$  for every  $i = 0, 1, 2, \dots$  and the submodules  $B_{i,0}, B'_{i,0}$  of  $B_i$  for every  $i = 1, 2, 3, \dots$ . Now properties (a) and (b) hold because  $B'_{i,-1} = B_i$  and  $C'_{i,-1} = C_i$  for every  $i = 0, 1, 2, \dots$ . Property (c) is given by equation (2.7), and (d) is given by equation (2.8). This concludes the first inductive step  $j = 0$ .

Now fix an integer  $\ell > 0$  and suppose we have already constructed the direct summands  $C_{i,j}, C'_{i,j}$  of  $C_i$  with the required properties for  $j < \ell$  and  $i \geq j$  and the direct summands  $B_{i,j}, B'_{i,j}$  of  $B_i$  for  $j < \ell$  and  $i > j$ . In particular we suppose that (d) holds for  $j = \ell - 1$ , that is, we suppose that

$$M = B'_{\ell,\ell-1} \oplus \left( \bigoplus_{k=\ell+1}^{\infty} B'_{k,\ell-1} \right) \oplus \left( (B'_{0,-1} \oplus C'_{0,0}) \right. \\ \left. \oplus (B'_{1,0} \oplus C'_{1,1}) \oplus \dots \oplus (B'_{\ell-1,\ell-2} \oplus C'_{\ell-1,\ell-1}) \right), \quad (2.9)$$

and we suppose that (c) holds for  $j = \ell - 1$ , that is,

$$M = \left( \bigoplus_{k=\ell}^{\infty} C'_{k,\ell-1} \right) \oplus \left( (B'_{0,-1} \oplus C'_{0,0}) \right. \\ \left. \oplus (B'_{1,0} \oplus C'_{1,1}) \oplus \dots \oplus (B'_{\ell-1,\ell-2} \oplus C'_{\ell-1,\ell-1}) \right). \quad (2.10)$$

Since  $B'_{\ell,\ell-1}$  is a direct summand of  $B_{\ell}$ , it has the  $\aleph_0$ -exchange property by Lemma 2.4. Hence we can apply Corollary 2.3 to the two decompositions (2.9) and (2.10), and obtain that there exist direct sum decompositions

$$C'_{k,\ell-1} = C_{k,\ell} \oplus C'_{k,\ell}$$

of  $C'_{k,\ell-1}$  for every  $k = \ell, \ell + 1, \ell + 2, \dots$  such that

$$M = B'_{\ell,\ell-1} \oplus \left( \bigoplus_{k=\ell}^{\infty} C'_{k,\ell} \right) \oplus \left( (B'_{0,-1} \oplus C'_{0,0}) \right. \\ \left. \oplus (B'_{1,0} \oplus C'_{1,1}) \oplus \dots \oplus (B'_{\ell-1,\ell-2} \oplus C'_{\ell-1,\ell-1}) \right). \quad (2.11)$$

This is property (c) for the integer  $\ell$ . Now equality (2.10) can be rewritten as

$$M = \left( \bigoplus_{k=\ell}^{\infty} (C_{k,\ell} \oplus C'_{k,\ell}) \right) \oplus \left( (B'_{0,-1} \oplus C'_{0,0}) \right. \\ \left. \oplus (B'_{1,0} \oplus C'_{1,1}) \oplus \dots \oplus (B'_{\ell-1,\ell-2} \oplus C'_{\ell-1,\ell-1}) \right).$$

This and (2.11) yield

$$B'_{\ell,\ell-1} \cong \bigoplus_{k=\ell}^{\infty} C_{k,\ell},$$

that is, property (e) holds.

Now  $C'_{\ell,\ell}$  is a direct summand of  $C_{\ell}$ . Hence it has the  $\aleph_0$ -exchange property. Equality (2.11) can be rewritten as

$$M = C'_{\ell,\ell} \oplus \left( \bigoplus_{k=\ell+1}^{\infty} C'_{k,\ell} \right) \oplus \left( (B'_{0,-1} \oplus C'_{0,0}) \right. \\ \left. \oplus (B'_{1,0} \oplus C'_{1,1}) \oplus \dots \oplus (B'_{\ell-1,\ell-2} \oplus C'_{\ell-1,\ell-1}) \oplus B'_{\ell,\ell-1} \right) \quad (2.12)$$

and (2.9) can be rewritten as

$$M = \left( \bigoplus_{k=\ell+1}^{\infty} B'_{k,\ell-1} \right) \oplus \left( (B'_{0,-1} \oplus C'_{0,0}) \oplus (B'_{1,0} \oplus C'_{1,1}) \oplus \cdots \oplus (B'_{\ell-1,\ell-2} \oplus C'_{\ell-1,\ell-1}) \oplus B'_{\ell,\ell-1} \right). \quad (2.13)$$

Applying Corollary 2.3 to (2.12) and (2.13) we find that there exists a direct sum decomposition  $B'_{k,\ell-1} = B_{k,\ell} \oplus B'_{k,\ell}$  for every  $k = \ell+1, \ell+2, \dots$  such that

$$M = C'_{\ell,\ell} \oplus \left( \bigoplus_{k=\ell+1}^{\infty} B'_{k,\ell} \right) \oplus \left( (B'_{0,-1} \oplus C'_{0,0}) \oplus (B'_{1,0} \oplus C'_{1,1}) \oplus \cdots \oplus (B'_{\ell-1,\ell-2} \oplus C'_{\ell-1,\ell-1}) \oplus B'_{\ell,\ell-1} \right). \quad (2.14)$$

This proves property (d) for the integer  $\ell$ . From (2.13) and (2.14) it follows that  $C'_{\ell,\ell} \cong \bigoplus_{k=\ell+1}^{\infty} B_{k,\ell}$ . Hence (f) holds, and this concludes the construction by induction.

From (e) we infer that there exist modules  $B_{i,j}$  for  $i \leq j$  such that  $B_{i,j} \cong C_{j,i}$  and

$$B_{j,j} \oplus B_{j,j+1} \oplus B_{j,j+2} \oplus \cdots = B'_{j,j-1} \quad (2.15)$$

for every  $j \geq 0$ . From (a) we have that

$$\begin{aligned} B_i &= B'_{i,-1} = B_{i,0} \oplus B'_{i,0} = B_{i,0} \oplus B_{i,1} \oplus B'_{i,1} \\ &= \cdots = B_{i,0} \oplus B_{i,1} \oplus \cdots \oplus B_{i,i-1} \oplus B'_{i,i-1}, \end{aligned}$$

so that

$$B_i = \bigoplus_{k=0}^{\infty} B_{i,k} \quad (2.16)$$

by (2.15).

Similarly, from (f) we get that there exist modules  $C_{i,j}$  for  $i < j$  such that  $C_{i,j} \cong B_{j,i}$  and

$$C_{j,j+1} \oplus C_{j,j+2} \oplus C_{j,j+3} \oplus \cdots = C'_{j,j} \quad (2.17)$$

for every  $j \geq 0$ . From (b) it follows that  $C_j = C_{j,0} \oplus C_{j,1} \oplus \cdots \oplus C_{j,j} \oplus C'_{j,j}$ , so that

$$C_j = \bigoplus_{k=0}^{\infty} C_{j,k} \quad (2.18)$$

by (2.17).

Since  $B_{i,j} \cong C_{j,i}$  for every  $i, j = 0, 1, 2, \dots$ , (2.16) and (2.18) yield the required isomorphic refinements of the decompositions (2.5) and (2.6).  $\square$

## 2.4 The Krull-Schmidt-Remak-Azumaya Theorem

The Krull-Schmidt-Remak-Azumaya Theorem is one of the main topics of this volume. We shall obtain a proof of it using the exchange property. We begin with a lemma that is of independent interest.

**Lemma 2.11** *If a module  $M$  is a direct sum of modules with local endomorphism rings, then every indecomposable direct summand of  $M$  has local endomorphism ring.*

*Proof.* Suppose  $M = A \oplus B = \bigoplus_{i \in I} M_i$ , where  $A$  is indecomposable and all the modules  $M_i$  have local endomorphism ring. Let  $F$  be a finite subset of  $I$  with  $A \cap \bigoplus_{i \in F} M_i \neq 0$ , and set  $C = \bigoplus_{i \in F} M_i$ . The module  $C$  has the exchange property (Lemma 2.4 and Theorem 2.8). Hence there exist direct sum decompositions  $A = A' \oplus A''$  of  $A$  and  $B = B' \oplus B''$  of  $B$  such that  $M = C \oplus A' \oplus B'$ . Note that  $A'$  is a proper submodule of  $A$ , because  $A \cap C \neq 0$  and  $A' \cap C = 0$ . Since  $A$  is indecomposable, it follows that  $A' = 0$ . Thus  $M = C \oplus B'$ . From  $M = C \oplus B' = A \oplus B' \oplus B''$  it follows that  $C \cong A \oplus B''$ . Thus  $A$  is isomorphic to a direct summand of  $C$ . Hence  $A$  has the exchange property by Lemma 2.4. Therefore  $A$  has local endomorphism ring by Theorem 2.8.  $\square$

We are ready for the proof of the Krull-Schmidt-Remak-Azumaya Theorem.

**Theorem 2.12** (Krull-Schmidt-Remak-Azumaya Theorem) *Let  $M$  be a module that is a direct sum of modules with local endomorphism rings. Then any two direct sum decompositions of  $M$  into indecomposable direct summands are isomorphic.*

*Proof.* Suppose that  $M = \bigoplus_{i \in I} M_i = \bigoplus_{j \in J} N_j$ , where all the  $M_i$  and  $N_j$  are indecomposable. By Lemma 2.11 all the modules  $M_i$  and  $N_j$  have local endomorphism rings. For  $I' \subseteq I$  and  $J' \subseteq J$  let

$$M(I') = \bigoplus_{i \in I'} M_i \quad \text{and} \quad N(J') = \bigoplus_{j \in J'} N_j.$$

By Lemma 2.4 and Theorem 2.8 the modules  $M(I')$  and  $N(J')$  have the exchange property whenever  $I'$  and  $J'$  are finite. Since the summands  $N_j$  are indecomposable, for every finite subset  $I' \subseteq I$  there exists a subset  $J' \subseteq J$  such that  $M = M(I') \oplus N(J \setminus J')$ . From  $M = M(I') \oplus N(J \setminus J') = N(J') \oplus N(J \setminus J')$ , we get that  $M(I') \cong N(J')$ . By Theorem 2.9 applied to the decompositions  $M(I') \cong N(J')$ , the two decompositions  $M(I') = \bigoplus_{i \in I'} M_i$  and  $N(J') = \bigoplus_{j \in J'} N_j$  have isomorphic refinements. From the indecomposability of the  $M_i$  and  $N_j$ , it follows that there is a one-to-one correspondence  $\varphi: I' \rightarrow J'$  such that  $M_i \cong N_{\varphi(i)}$  for every  $i \in I'$ . For every  $R$ -module  $A$  set

$$I_A = \{i \in I \mid M_i \cong A\} \quad \text{and} \quad J_A = \{j \in J \mid N_j \cong A\}.$$

From what we have just seen it follows that if  $I_A$  is finite, then  $|I_A| \leq |J_A|$ , and if  $I_A \neq \emptyset$ , then  $J_A \neq \emptyset$ . By symmetry, if  $J_A$  is finite, then  $|J_A| \leq |I_A|$ , and  $J_A \neq \emptyset$  implies  $I_A \neq \emptyset$ . In order to prove the theorem it is sufficient to show that  $|I_A| = |J_A|$  for every  $R$ -module  $A$ .

Suppose first that  $I_A$  is finite. In this case we argue by induction on  $|I_A|$ . If  $|I_A| = 0$ , then  $|J_A| = 0$ . If  $|I_A| \geq 1$ , fix an index  $i_0 \in I_A$ . Then there is an index  $j_0 \in J$  such that  $M = M(\{i_0\}) \oplus N(J \setminus \{j_0\})$ . If we factorize the module  $M(\{i_0\}) \oplus N(J \setminus \{j_0\}) = M(I)$  modulo  $M(\{i_0\})$  we obtain that

$$N(J \setminus \{j_0\}) \cong M(I \setminus \{i_0\}).$$

From the induction hypothesis we get that  $|I_A \setminus \{i_0\}| = |J_A \setminus \{j_0\}|$ , so that  $|I_A| = |J_A|$ .

By symmetry we can conclude that if  $J_A$  is finite, then  $|I_A| = |J_A|$  as well.

Hence we can suppose that both  $I_A$  and  $J_A$  are infinite sets. By symmetry it is sufficient to show that  $|J_A| \leq |I_A|$  for an arbitrary module  $A$ .

For each  $i \in I_A$  set  $J(i) = \{j \in J \mid M = M_i \oplus N(J \setminus \{j\})\}$ . Obviously  $J(i) \subseteq J_A$ . If  $x$  is a non-zero element of  $M_i$ , then there is a finite subset  $J''$  of  $J$  such that  $x \in N(J'')$ . Hence  $M_i \cap N(K) \neq 0$  for every  $K \subseteq J$  that contains  $J''$ . Thus  $J(i) \subseteq J''$ , so that  $J(i)$  is finite.

We claim that  $\bigcup_{i \in I_A} J(i) = J_A$ . In order to prove the claim, fix  $j \in J_A$ . Then there exists a finite subset  $I'$  of  $I$  such that  $N_j \cap M(I') \neq 0$ . Hence there exists a finite subset  $J' \subseteq J$  such that  $M = M(I') \oplus N(J \setminus J')$ . Note that  $j \in J'$ . Since  $N(J' \setminus \{j\})$  has the exchange property, we can apply Corollary 2.3 to the decompositions  $M = N(J' \setminus \{j\}) \oplus N_j \oplus N(J \setminus J') = (\oplus_{i \in I'} M_i) \oplus N(J \setminus J')$ . Then for every  $i \in I'$  there exists a direct summand  $M'_i$  of  $M_i$  such that  $M = N(J' \setminus \{j\}) \oplus (\oplus_{i \in I'} M'_i) \oplus N(J \setminus J')$ . Then  $N_j \cong \oplus_{i \in I'} M'_i$ , so that there exists an index  $k \in I'$  with  $M'_k = M_k$  and  $M'_i = 0$  for every  $i \in I'$ ,  $i \neq k$ . Note that  $M_k \cong N_j \cong A$ , so that  $k \in I_A$ . Thus

$$M = N(J' \setminus \{j\}) \oplus M_k \oplus N(J \setminus J') = M_k \oplus N(J \setminus \{j\}),$$

that is,  $j \in J(k)$ . Hence  $j \in \bigcup_{i \in I_A} J(i)$ , which proves the claim.

It follows that

$$|J_A| = \left| \bigcup_{i \in I_A} J(i) \right| \leq |I_A| \cdot \aleph_0 = |I_A|.$$

□

## 2.5 Applications

In this section we apply the Krull-Schmidt-Remak-Azumaya Theorem to some important classes of modules.

A first immediate application of the Krull-Schmidt-Remak-Azumaya Theorem 2.12 can be given to the class of semisimple modules.

**Lemma 2.13** (Schur) *The endomorphism ring  $\text{End}(M)$  of a simple module  $M$  is a division ring.*

*Proof.* If  $M$  is a simple module and  $f$  is a non-zero endomorphism of  $M$ , then  $\ker f$  and  $f(M)$  must be submodules of  $M$ . Hence they are either 0 or  $M$ . If  $\ker f = M$ , then  $f = 0$ , contradiction. Therefore  $\ker f = 0$  and  $f$  is injective. If  $f(M) = 0$ , then  $f = 0$ , contradiction. Therefore  $f(M) = M$  and  $f$  is surjective. Thus  $f$  is an automorphism of  $M$ , that is,  $f$  is invertible in  $\text{End}(M)$ .  $\square$

Since division rings are local rings, we get the Krull-Schmidt-Remak-Azumaya Theorem for semisimple modules:

**Theorem 2.14** *Any two direct sum decompositions of a semisimple module into simple direct summands are isomorphic.*  $\square$

The next result describes the structure of the submodules and the homomorphic images of a semisimple module.

**Proposition 2.15** *Let  $M$  be a semisimple  $R$ -module and  $\{M_i \mid i \in I\}$  a family of simple submodules of  $M$  such that  $M = \bigoplus_{i \in I} M_i$ . Then for every submodule  $N$  of  $M$  there is a subset  $J$  of  $I$  such that  $N \cong \bigoplus_{i \in J} M_i$  and  $M/N \cong \bigoplus_{i \in I \setminus J} M_i$ .*

*Proof.* By Proposition 1.1 the submodule  $N$  of the semisimple module  $M$  is a direct summand of  $M$ , so that  $M = N \oplus N'$  for a submodule  $N' \cong M/N$  of  $M$ . By Proposition 1.1 again, both  $N$  and  $N'$  are semisimple. Hence  $N = \bigoplus_{\lambda \in \Lambda} N_\lambda$  and  $N' = \bigoplus_{\mu \in \Lambda'} N'_\mu$  for suitable simple submodules  $N_\lambda, N'_\mu$ . By Theorem 2.14 the two decompositions  $\bigoplus_{i \in I} M_i = (\bigoplus_{\lambda \in \Lambda} N_\lambda) \oplus (\bigoplus_{\mu \in \Lambda'} N'_\mu)$  of  $M$  are isomorphic. Therefore there are a subset  $J$  of  $I$  and one-to-one correspondences  $\varphi: J \rightarrow \Lambda$  and  $\psi: I \setminus J \rightarrow \Lambda'$  such that  $M_i \cong N_{\varphi(i)}$  for every  $i \in J$  and  $M_i \cong N'_{\psi(i)}$  for every  $i \in I \setminus J$ . The conclusion follows immediately.  $\square$

As a second application of Theorem 2.12, we study the uniqueness of decomposition of some particular artinian or noetherian modules.

**Lemma 2.16** *Let  $M$  be a module and  $f$  an endomorphism of  $M$ .*

(a) *If  $n$  is a positive integer such that  $f^n(M) = f^{n+1}(M)$ , then*

$$\ker(f^n) + f^n(M) = M.$$

- (b) If  $M$  is an artinian module, then  $f$  is an automorphism if and only if  $f$  is injective.

*Proof.* (a) If  $n$  is such that  $f^n(M) = f^{n+1}(M)$ , then  $f^t(M) = f^{t+1}(M)$  for every  $t \geq n$ , so that  $f^n(M) = f^{2n}(M)$ . Let us show that

$$\ker(f^n) + f^n(M) = M.$$

If  $x \in M$ , then  $f^n(x) \in f^n(M) = f^{2n}(M)$ , so that  $f^n(x) = f^n(y)$  for some  $y \in f^n(M)$ . Therefore  $z = x - y \in \ker(f^n)$ , and  $x = z + y \in \ker(f^n) + f^n(M)$ .

(b) If  $f$  an injective endomorphism of the artinian module  $M$ , the descending chain

$$f(M) \supseteq f^2(M) \supseteq f^3(M) \supseteq \dots$$

is stationary, so that  $\ker(f^n) + f^n(M) = M$  for some positive integer  $n$  by part (a). As  $f^n$  is injective,  $\ker(f^n) = 0$ , and therefore  $f^n(M) = M$ . In particular,  $f$  is surjective.  $\square$

Similarly it can be proved that

**Lemma 2.17** *Let  $M$  be a module and  $f$  an endomorphism of  $M$ .*

- (a) *If  $n$  is a positive integer such that  $\ker f^n = \ker f^{n+1}$ , then*

$$\ker(f^n) \cap f^n(M) = 0.$$

- (b) *If  $M$  is a noetherian module, then  $f$  is an automorphism if and only if  $f$  is surjective.*  $\square$

A submodule  $N$  of a module  $M_R$  is *fully invariant* if  $\varphi(N) \subseteq N$  for every  $\varphi \in \text{End}(M_R)$ , that is, if  $N$  is a submodule of the left  $\text{End}(M_R)$ -module  $M$ . A submodule  $N$  of  $M_R$  is *essential* in  $M_R$  if  $N \cap P \neq 0$  for every non-zero submodule  $P$  of  $M_R$ . The *socle* of a module  $M_R$  is the sum of all simple submodules of  $M_R$ . It is a semisimple fully invariant submodule of  $M_R$  and it will be denoted  $\text{soc}(M_R)$ . Since every non-zero artinian module has a simple submodule, the socle is an essential submodule in every artinian module. If  $N_R$  is an artinian module and its socle  $\text{soc}(N_R)$  is a simple module, by Lemma 2.16(b) an endomorphism  $f \in \text{End}(N_R)$  is not an automorphism if and only if  $f(\text{soc}(N_R)) = 0$ . It follows that  $\text{End}(N_R)$  is a local ring with Jacobson radical  $J(\text{End}(N_R)) = \{f \in \text{End}(N_R) \mid f(\text{soc}(N_R)) = 0\}$ . Therefore Theorem 2.12 yields

**Theorem 2.18** *Let  $M_R$  be an  $R$ -module that is a direct sum of artinian modules with simple socle. Then any two direct sum decompositions of  $M_R$  into indecomposable direct summands are isomorphic.*  $\square$

The hypothesis that the artinian modules have simple socle is essential in Theorem 2.18 as we shall see in Section 8.2.

Recall that a module  $M$  is *local* if it has a greatest proper submodule. Hence noetherian local modules are the “duals” of artinian modules with a simple socle. The next result is the dual of Theorem 2.18 and is proved similarly.

**Theorem 2.19** *Let  $M$  be an  $R$ -module that is a direct sum of noetherian local modules. Then any two direct sum decompositions of  $M$  into indecomposable direct summands are isomorphic.*  $\square$

Our third application of the Krull-Schmidt-Remak-Azumaya Theorem will be to the class of modules of finite composition length.

**Lemma 2.20** (Fitting’s Lemma) *If  $M$  is a module of finite length  $n$  and  $f$  is an endomorphism of  $M$ , then  $M = \ker(f^n) \oplus f^n(M)$ .*

*Proof.* Since  $M$  is of finite length  $n$ , both the chains

$$\ker f \subseteq \ker f^2 \subseteq \ker f^3 \subseteq \dots$$

and

$$f(M) \supseteq f^2(M) \supseteq f^3(M) \supseteq \dots$$

are stationary at the  $n$ -th step, so that  $\ker(f^n) \oplus f^n(M) = M$  by Lemmas 2.16(a) and 2.17(a).  $\square$

We shall say that a module  $M_R$  is a *Fitting module* if for every  $f \in \text{End}(M_R)$  there is a positive integer  $n$  such that  $M = \ker(f^n) \oplus f^n(M)$ . Thus by Lemma 2.20 every module of finite length is a Fitting module. It is easily seen that direct summands of Fitting modules are Fitting modules.

**Lemma 2.21** *The endomorphism ring of an indecomposable Fitting module is a local ring.*

*Proof.* If  $M$  is a Fitting module and  $f$  is an endomorphism of  $M$ , there exists a positive integer  $n$  such that  $M = \ker(f^n) \oplus f^n(M)$ . If  $M$  is indecomposable, two cases may occur. In the first case  $\ker(f^n) = 0$  and  $f^n(M) = M$ . Then  $f^n$  is an automorphism of  $M$ , so that  $f$  is an automorphism of  $M$ . In the second case  $\ker(f^n) = M$ , that is,  $f$  is nilpotent. Hence every endomorphism of  $M$  is either invertible or nilpotent.

In order to show that the endomorphism ring  $\text{End}(M)$  of  $M$  is local, we must show that the sum of two non-invertible endomorphisms is non-invertible. Suppose that  $f$  and  $g$  are two non-invertible endomorphisms of  $M$ , but  $f + g$  is invertible. If  $h = (f + g)^{-1}$  is the inverse of  $f + g$ , then  $fh + gh = 1$ . Since  $f$  and  $g$  are not automorphisms, neither  $fh$  nor  $gh$  are automorphisms. Therefore, as we have just seen in the previous paragraph, there exists a positive integer  $n$  such that  $(gh)^n = 0$ . Since  $1 = (1 - gh)(1 + gh + (gh)^2 + \dots + (gh)^{n-1})$ , the endomorphism  $1 - gh = fh$  is invertible. This contradiction proves the lemma.  $\square$

Theorem 2.12 and Lemma 2.21 yield

**Theorem 2.22** *Let  $M$  be an  $R$ -module that is a direct sum of indecomposable Fitting modules. Then any two direct sum decompositions of  $M$  into indecomposable direct summands are isomorphic.*  $\square$

In particular, from Lemma 2.20 it follows that

**Corollary 2.23** (The Krull-Schmidt Theorem) *Let  $M$  be an  $R$ -module of finite length. Then  $M$  is the direct sum of a finite family of indecomposable modules, and any two direct sum decompositions of  $M$  as direct sums of indecomposable modules are isomorphic.*  $\square$

A further class of modules to which the Krull-Schmidt-Remak-Azumaya Theorem can be applied immediately is the class of indecomposable injective modules. The proof of the following lemma is an easy exercise left to the reader.

**Lemma 2.24** *Let  $M \neq 0$  be an  $R$ -module. The following conditions are equivalent:*

- (a) *The intersection of any two non-zero submodules of  $M$  is non-zero.*
- (b) *The injective envelope of  $M$  is indecomposable.*
- (c) *Every non-zero submodule of  $M$  is essential in  $M$ .*
- (d) *Every non-zero submodule of  $M$  is indecomposable.*  $\square$

An  $R$ -module  $M \neq 0$  is said to be *uniform* if it satisfies the equivalent conditions of Lemma 2.24. For instance, an artinian module is uniform if and only if it has a simple socle.

We state the next lemma in the language of Grothendieck categories. The reader who is not used to this language may think of the case of an indecomposable injective  $R$ -module.

**Lemma 2.25** *Let  $M$  be an indecomposable injective object of a Grothendieck category  $\mathcal{C}$ . Then*

- (a) *An endomorphism of  $M$  is an automorphism if and only if it is a monomorphism.*
- (b) *The endomorphism ring of  $M$  is a local ring.*

*In particular, the endomorphism ring of every indecomposable injective module is a local ring.*



*Proof.* (a) If  $f: M \rightarrow M$  is a monomorphism, then  $\text{im}(f)$  is a subobject of  $M$  isomorphic to  $M$ . In particular,  $\text{im}(f)$  is a non-zero direct summand of  $M$ . Since  $M$  is indecomposable,  $\text{im}(f) = M$  and  $f$  is an automorphism.

(b) Let  $\text{End}_{\mathcal{C}}(M)$  denote the endomorphism ring of  $M$ . We must show that the sum of two non-invertible elements of  $\text{End}_{\mathcal{C}}(M)$  is non-invertible. Suppose that  $f$  and  $g$  are two non-invertible endomorphisms of  $M$ . In (a) we have seen that  $f$  and  $g$  are not monomorphisms, that is,  $\ker f \neq 0$  and  $\ker g \neq 0$ . Since  $M$  is coirreducible (= uniform), we have  $\ker f \cap \ker g \neq 0$ . Now

$$\ker f \cap \ker g \subseteq \ker(f + g),$$

so that  $\ker(f + g) \neq 0$ . Therefore  $f + g$  is not invertible in  $\text{End}_{\mathcal{C}}(M)$ .  $\square$

As an immediate application of Lemma 2.25 to the category  $\mathcal{C} = \text{Mod-}R$  and the Krull-Schmidt-Remak-Azumaya Theorem we have:

**Theorem 2.26** *Let  $M$  be an  $R$ -module that is a direct sum of injective indecomposable modules. Then any two direct sum decompositions of  $M$  into indecomposable direct summands are isomorphic.*  $\square$

As a second application of Lemma 2.25 to the category  $\mathcal{C} = ({}_R\text{FP}, \text{Ab})$  we find:

**Corollary 2.27** *The endomorphism ring of an indecomposable pure-injective module is a local ring.*

*Proof.* Let  $M_R$  be an indecomposable pure-injective module, let  ${}_R\text{FP}$  be the full subcategory of  $R\text{-Mod}$  whose objects are the finitely presented left  $R$ -modules, let  $\text{Ab}$  be the category of abelian groups, and let  $\mathcal{C} = ({}_R\text{FP}, \text{Ab})$  be the category of all additive functors from  ${}_R\text{FP}$  to  $\text{Ab}$ . Then

$$M \otimes_R -: {}_R\text{FP} \rightarrow \text{Ab}$$

is an indecomposable injective object in the Grothendieck category  $\mathcal{C}$  (Proposition 1.39). Since  $\text{End}_R(M) \cong \text{End}_{\mathcal{C}}(M)$ , the ring  $\text{End}_R(M)$  is local by Lemma 2.25(b).  $\square$

We conclude this section showing that the Krull-Schmidt Theorem holds for  $\Sigma$ -pure-injective modules. We need a preliminary proposition.

**Proposition 2.28** *Let  $R$  be a ring and let  $\mathcal{B} \subseteq \mathcal{C}$  be classes of non-zero right  $R$ -modules. Suppose that every module in  $\mathcal{C}$  has a direct summand in  $\mathcal{B}$  and that for every proper pure submodule  $P$  of any module  $M \in \mathcal{C}$  there exists a submodule  $D$  of  $M$  such that  $D \in \mathcal{B}$ ,  $P \cap D = 0$  and  $P + D = P \oplus D$  is pure in  $M$ . Then every module  $M \in \mathcal{C}$  is a direct sum of modules belonging to  $\mathcal{B}$ .*

*Proof.* Suppose  $M_R \in \mathcal{C}$ . Let  $\mathcal{I}$  be the set of submodules of  $M$  that belong to  $\mathcal{B}$  and put  $\mathcal{S} = \{I \mid I \subseteq \mathcal{I}, \sum_{N \in I} N \text{ is a pure submodule of } M \text{ and the sum } \sum_{N \in I} N \text{ is direct}\}$ . Since every module in  $\mathcal{C}$  has a direct summand in  $\mathcal{B}$ , the set  $\mathcal{S}$  is non-empty. Partially order  $\mathcal{S}$  by set inclusion. Then the union of a chain of elements of  $\mathcal{S}$  is an element of  $\mathcal{S}$  by Proposition 1.31(a). By Zorn's Lemma  $\mathcal{S}$  has a maximal element  $J$ . Set  $P = \sum_{N \in J} N = \oplus_{N \in J} N$ , so that  $P$  is a pure submodule of  $M$ . If  $P$  is a proper submodule of  $M$ , then there is a submodule  $D$  of  $M$  such that  $D \in \mathcal{B}$ ,  $P \cap D = 0$  and  $P + D$  is pure in  $M$ . Thus  $J \cup \{D\} \in \mathcal{S}$ , and  $D \notin J$  because  $P \cap D = 0$  and the module  $D \in \mathcal{B}$  is non-zero. This contradicts the maximality of  $J$ . Hence  $M = P$  is a direct sum of modules belonging to  $\mathcal{B}$ .  $\square$

**Theorem 2.29** *Let  $M_R$  be a  $\Sigma$ -pure-injective  $R$ -module. Then  $M_R$  is a direct sum of modules with local endomorphism ring, so that any two direct sum decompositions of  $M_R$  as direct sums of indecomposables are isomorphic.*

*Proof.* We shall apply Proposition 2.28 to the class  $\mathcal{C}$  of  $\Sigma$ -pure-injective non-zero  $R$ -modules and the class  $\mathcal{B}$  of indecomposable  $\Sigma$ -pure-injective  $R$ -modules. Firstly, we must show that every  $\Sigma$ -pure-injective non-zero module  $M_R$  has an indecomposable direct summand  $N$ . To see this, let  $x$  be a non-zero element of  $M_R$ . Let  $\mathcal{P}$  be the set of all pure submodules of  $M_R$  that do not contain  $x$ . Then  $\mathcal{P}$  is non-empty and the union of every chain in  $\mathcal{P}$  is an element of  $\mathcal{P}$  (Proposition 1.31(a)). By Zorn's Lemma  $\mathcal{P}$  has a maximal element  $Q$ . By Corollary 1.42 there exists a submodule  $N$  of  $M$  such that  $M = Q \oplus N$ . If  $N = N' \oplus N''$  with  $N', N'' \neq 0$ , then  $Q \oplus N'$  and  $Q \oplus N''$  are direct summands of  $M$  that do not belong to  $\mathcal{P}$ . Hence  $x \in (Q \oplus N') \cap (Q \oplus N'') = Q$ , a contradiction. The contradiction proves that  $N$  is indecomposable, as we wanted to prove.

Secondly, we must show that for every proper pure submodule  $P$  of a  $\Sigma$ -pure injective module  $M_R$  there exists an indecomposable  $\Sigma$ -pure-injective submodule  $D$  of  $M$  such that  $P \cap D = 0$  with  $P + D = P \oplus D$  pure in  $M$ . For such a pure submodule  $P$  we know that  $M_R = P \oplus P'$  for a submodule  $P'$  (Corollary 1.42), and by the first part of the proof  $P' = D \oplus P''$  for suitable submodules  $D$  and  $P''$  with  $D$  indecomposable. The module  $D$  has the required properties.

Hence by Proposition 2.28 every  $\Sigma$ -pure-injective module  $M$  is a direct sum of indecomposable modules.

Finally, every indecomposable direct summand of  $M$  is pure-injective, hence every indecomposable direct summand of  $M$  has a local endomorphism ring (Corollary 2.27).  $\square$

## 2.6 Goldie dimension of a modular lattice

The notion of Goldie dimension of a module concerns the lattice  $\mathcal{L}(M)$  of all submodules of  $M$ , which is a modular lattice. Modular lattices seem to be the proper setting for the definition of Goldie dimension. Hence in this section we shall consider arbitrary modular lattices and their Goldie dimension.

Throughout this section  $(L, \vee, \wedge)$  will denote a modular lattice with 0 and 1, that is, a lattice with a smallest element 0 and a greatest element 1 such that  $a \wedge (b \vee c) = (a \wedge b) \vee c$  for every  $a, b, c \in L$  with  $c \leq a$ . If  $a, b \in L$  and  $a \leq b$  let  $[a, b] = \{x \in L \mid a \leq x \leq b\}$  be the *interval* between  $a$  and  $b$ .

A finite subset  $\{a_i \mid i \in I\}$  of  $L \setminus \{0\}$  is said to be *join-independent* if  $a_i \wedge (\bigvee_{j \neq i} a_j) = 0$  for every  $i \in I$ . The empty subset of  $L \setminus \{0\}$  is join-independent. An arbitrary subset  $A$  of  $L \setminus \{0\}$  is *join-independent* if all its finite subsets are join-independent.

**Lemma 2.30** *Let  $A \subseteq L \setminus \{0\}$  be a join-independent subset of a modular lattice  $L$ . If  $B, C$  are finite subsets of  $A$  and  $B \cap C = \emptyset$ , then  $(\bigvee_{b \in B} b) \wedge (\bigvee_{c \in C} c) = 0$ .*

*Proof.* By induction on the cardinality  $|B|$  of  $B$ . The case  $|B| = 0$  is trivial. Suppose the lemma holds for subsets of cardinality  $< |B|$ . Fix an element  $b \in B$  and set  $B' = B \setminus \{b\}$  and  $a = (\bigvee_{x \in B'} x) \wedge (\bigvee_{y \in C} y)$ . By the induction hypothesis

$$(b \vee a) \wedge \left( \bigvee_{x \in B'} x \right) \leq \left( \bigvee_{y \in \{b\} \cup C} y \right) \wedge \left( \bigvee_{x \in B'} x \right) = 0, \quad (2.19)$$

and by the definition of join-independent set

$$\left( \bigvee_{x \in B'} x \vee a \right) \wedge b \leq \left( \bigvee_{x \in B' \cup C} x \right) \wedge b = 0. \quad (2.20)$$

Then

$$\begin{aligned} a &\leq (b \vee a) \wedge \left( \bigvee_{x \in B'} x \right) \\ &= (b \vee a) \wedge \left( \left( \bigvee_{x \in B'} x \right) \vee b \right) \quad (\text{by the modular identity}) \\ &= ((b \vee a) \wedge \left( \bigvee_{x \in B'} x \right)) \vee b \quad (\text{by (2.19)}) \\ &= 0 \vee b = b, \end{aligned} \quad (2.21)$$

and

$$\begin{aligned} a &\leq \left( \bigvee_{x \in B'} x \vee a \right) \wedge \left( \bigvee_{x \in B} x \right) \\ &= \left( \bigvee_{x \in B'} x \vee a \right) \wedge \left( b \vee \bigvee_{x \in B'} x \right) \quad (\text{by the modular identity}) \\ &= \left( \left( \bigvee_{x \in B'} x \vee a \right) \wedge b \right) \vee \left( \bigvee_{x \in B'} x \right) \quad (\text{by (2.20)}) \\ &\leq 0 \vee \bigvee_{x \in B'} x = \bigvee_{x \in B'} x. \end{aligned} \quad (2.22)$$

From (2.21) and (2.22) it follows that  $a \leq b \wedge \bigvee_{x \in B'} x = 0$ , as desired.  $\square$

**Proposition 2.31** *Let  $A \subseteq L \setminus \{0\}$  be a join-independent subset of a modular lattice  $L$ . Let  $a \in L$  be a non-zero element such that  $a \wedge (\bigvee_{b \in B} b) = 0$  for every finite subset  $B$  of  $A$ . Then  $A \cup \{a\}$  is a join-independent subset of  $L$ .*

*Proof.* We must prove that every finite subset of  $A \cup \{a\}$  is join-independent. This is obvious for finite subsets of  $A$ . Hence it suffices to show that if  $B$  is a finite subset of  $A$ , then  $B \cup \{a\}$  is join-independent. Since  $a \wedge (\bigvee_{b \in B} b) = 0$ , we have to prove that  $b \wedge (a \vee \bigvee_{x \in B \setminus \{b\}} x) = 0$  for each  $b \in B$ . Now

$$\begin{aligned} & \left( \bigvee_{y \in B} y \right) \wedge \left( a \vee \left( \bigvee_{x \in B \setminus \{b\}} x \right) \right) && \text{(by the modular identity)} \\ &= \left( \left( \bigvee_{y \in B} y \right) \wedge a \right) \vee \left( \bigvee_{x \in B \setminus \{b\}} x \right) && \text{(by hypothesis)} \\ &= 0 \vee \left( \bigvee_{x \in B \setminus \{b\}} x \right) = \bigvee_{x \in B \setminus \{b\}} x, \end{aligned} \tag{2.23}$$

so that

$$\begin{aligned} & b \wedge \left( a \vee \bigvee_{x \in B \setminus \{b\}} x \right) && \text{(since } b \leq \bigvee_{y \in B} y \text{)} \\ &= b \wedge \left( \bigvee_{y \in B} y \right) \wedge \left( a \vee \bigvee_{x \in B \setminus \{b\}} x \right) && \text{(by (2.23))} \\ &= b \wedge \left( \bigvee_{x \in B \setminus \{b\}} x \right) = 0 && \text{(because } B \text{ is join-independent).} \end{aligned} \quad \square$$

By Zorn's Lemma every join-independent subset of  $L \setminus \{0\}$  is contained in a maximal join-independent subset of  $L \setminus \{0\}$ .

An element  $a \in L$  is *essential* in  $L$  if  $a \wedge x \neq 0$  for every non-zero element  $x \in L$ . Thus 0 is essential in  $L$  if and only if  $L = \{0\}$ . If  $a, b$  are elements of  $L$ ,  $a \leq b$  and  $a$  is essential in the lattice  $[0, b]$ , then  $a$  is said to be *essential* in  $b$ . In particular, 0 is essential in  $b$  if and only if  $b = 0$ .

**Lemma 2.32** *Let  $a, b, c$  be elements of  $L$ . If  $a$  is essential in  $b$  and  $b$  is essential in  $c$ , then  $a$  is essential in  $c$ .*

*Proof.* Let  $x$  be a non-zero element of  $[0, c]$ . We must show that  $a \wedge x \neq 0$ . Now  $b \wedge x \neq 0$  because  $b$  is essential in  $c$ , hence  $a \wedge (b \wedge x) \neq 0$  because  $a$  is essential in  $b$ . But  $a \wedge (b \wedge x) = a \wedge x$ .  $\square$

**Lemma 2.33** *Let  $a, b, c, d$  be elements of  $L$  such that  $b \wedge d = 0$ . If  $a$  is essential in  $b$  and  $c$  is essential in  $d$ , then  $a \vee c$  is essential in  $b \vee d$ .*

*Proof.* If any of the four elements  $a, b, c, d$  is zero, the statement of the lemma is trivial. Hence we shall assume that  $a, b, c, d$  are all non-zero.

We claim that if the hypotheses of the lemma hold for the four elements  $a, b, c, d \in L \setminus \{0\}$ , then  $a \vee d$  is essential in  $b \vee d$ . Assume the contrary. Then there exists a non-zero element  $x \in L$  such that  $x \leq b \vee d$  and

$$(a \vee d) \wedge x = 0.$$

Since  $\{a, d\}$  is join-independent, the set  $\{a, d, x\}$  is join-independent by Proposition 2.31. In particular,  $a \wedge (d \vee x) = 0$ , so that  $a \wedge b \wedge (d \vee x) = 0$ . This implies that  $b \wedge (d \vee x) = 0$ , because  $a$  is essential in  $b$  and  $b \wedge (d \vee x) \leq b$ . Now  $\{d, x\} \subseteq \{a, d, x\}$  is join-independent, and thus  $b \wedge (d \vee x) = 0$  forces that  $\{b, d, x\}$  is join-independent (Proposition 2.31). In particular,  $x \wedge (b \vee d) = 0$ . But  $x \leq b \vee d$ , so that  $x = 0$ . This contradiction proves the claim.

If we apply the claim to the four elements  $c, d, a, a$ , we obtain that  $c \vee a$  is essential in  $d \vee a$ , that is,  $a \vee c$  is essential in  $a \vee d$ . The conclusion now follows from this, the claim and Lemma 2.32.  $\square$

By an easy induction argument we obtain

**Corollary 2.34** *Let  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$  be elements of  $L$  such that  $\{b_1, b_2, \dots, b_n\}$  is join-independent. If  $a_i$  is essential in  $b_i$  for every  $i = 1, 2, \dots, n$ , then  $a_1 \vee a_2 \vee \dots \vee a_n$  is essential in  $b_1 \vee b_2 \vee \dots \vee b_n$ .*  $\square$

A lattice  $L \neq \{0\}$  is *uniform* if all its non-zero elements are essential in  $L$ , that is, if  $x, y \in L$  and  $x \wedge y = 0$  imply  $x = 0$  or  $y = 0$ . An element  $a$  of a modular lattice  $L$  is *uniform* if  $a \neq 0$  and the lattice  $[0, a]$  is uniform.

**Lemma 2.35** *If a modular lattice  $L$  does not contain infinite join-independent subsets, then for every non-zero element  $a \in L$  there exists a uniform element  $b \in L$  such that  $b \leq a$ .*

*Proof.* Let  $a \neq 0$  be an element of a modular lattice  $L$  such that every  $b \leq a$  is not uniform. We shall define a sequence  $a_1, a_2, a_3, \dots$  of non-zero elements of  $[0, a]$  such that for every  $n \geq 1$  the set  $\{a_1, a_2, \dots, a_n\}$  is join-independent and  $a_1 \vee \dots \vee a_n$  is not essential in  $[0, a]$ . The construction of the elements  $a_n$  is by induction on  $n$ . For  $n = 1$  note that  $a$  is not uniform, hence there exist non-zero elements  $a_1, a'_1 \in [0, a]$  such that  $a_1 \wedge a'_1 = 0$ , i.e.,  $a_1$  has the required properties. Suppose  $a_1, \dots, a_{n-1}$  have been defined. Since  $a_1 \vee \dots \vee a_{n-1}$  is not essential in  $[0, a]$ , there exists a non-zero  $b \in [0, a]$  such that  $b \wedge (a_1 \vee \dots \vee a_{n-1}) = 0$ . The element  $b$  is not uniform. Hence there exist  $a_n, a'_n \in [0, b]$ , where  $a_n, a'_n$  are non-zero, such that  $a_n \wedge a'_n = 0$ . Then  $a_n \wedge (a_1 \vee \dots \vee a_{n-1}) = 0$ , so that  $\{a_1, a_2, \dots, a_n\}$  is join-independent by Proposition 2.31. Moreover

$$\begin{aligned}
 a'_n \wedge (a_1 \vee \dots \vee a_n) & \quad (\text{since } a'_n \leq b) \\
 &= a'_n \wedge b \wedge ((a_1 \vee \dots \vee a_{n-1}) \vee a_n) \quad (\text{by the modular identity}) \\
 &= a'_n \wedge ((b \wedge (a_1 \vee \dots \vee a_{n-1})) \vee a_n) \\
 &= a'_n \wedge (0 \vee a_n) = 0.
 \end{aligned}$$

This completes the construction. Now  $\{a_n \mid n \geq 1\}$  is an infinite join-independent set.  $\square$

**Theorem 2.36** *The following conditions are equivalent for a modular lattice  $L$  with 0 and 1.*

- (a)  *$L$  does not contain infinite join-independent subsets.*
- (b)  *$L$  contains a finite join-independent subset  $\{a_1, a_2, \dots, a_n\}$  with  $a_i$  uniform for every  $i = 1, 2, \dots, n$  and  $a_1 \vee a_2 \vee \dots \vee a_n$  essential in  $L$ .*
- (c) *The cardinality of the join-independent subsets of  $L$  is  $\leq m$  for a non-negative integer  $m$ .*
- (d) *If  $a_0 \leq a_1 \leq a_2 \leq \dots$  is an ascending chain of elements of  $L$ , then there exists  $i \geq 0$  such that  $a_i$  is essential in  $a_j$  for every  $j \geq i$ .*

Moreover, if these equivalent conditions hold and  $\{a_1, a_2, \dots, a_n\}$  is a finite join-independent subset of  $L$  with  $a_i$  uniform for every  $i = 1, 2, \dots, n$  and  $a_1 \vee a_2 \vee \dots \vee a_n$  essential in  $L$ , then any other join-independent subset of  $L$  has cardinality  $\leq n$ .

*Proof.* (a)  $\Rightarrow$  (b). Let  $\mathcal{F}$  be the family of all join-independent subsets of  $L$  consisting only of uniform elements. The family  $\mathcal{F}$  is non-empty (Lemma 2.35). By Zorn's Lemma  $\mathcal{F}$  has a maximal element  $X$  with respect to inclusion. By (a) the set  $X$  is finite, say  $X = \{a_1, a_2, \dots, a_n\}$ . The element  $a_1 \vee a_2 \vee \dots \vee a_n$  is essential in  $L$ , otherwise there would exist a non-zero element  $x \in L$  such that  $(a_1 \vee a_2 \vee \dots \vee a_n) \wedge x = 0$ , and by Lemma 2.35 there would be a uniform element  $b \in L$  such that  $b \leq x$ . Then  $\{a_1, a_2, \dots, a_n, b\}$  would be join-independent by Proposition 2.31, a contradiction.

(b)  $\Rightarrow$  (c). Suppose that (b) holds, so that there exists a finite join-independent subset  $\{a_1, a_2, \dots, a_n\}$  of  $L$  with  $a_i$  uniform for every  $i$  and

$$a_1 \vee \dots \vee a_n$$

essential in  $L$ . Assume that there exists a join-independent subset  $\{b_1, b_2, \dots, b_k\}$  of  $L$  of cardinality  $k > n$ . For every  $t = 0, 1, \dots, n$  we shall construct a subset  $X_t$  of  $\{a_1, a_2, \dots, a_n\}$  of cardinality  $t$  and a subset  $Y_t$  of  $\{b_1, b_2, \dots, b_k\}$  of cardinality  $k - t$  such that  $X_t \cap Y_t = \emptyset$  and  $X_t \cup Y_t$  is join-independent. For  $t = 0$  set  $X_0 = \emptyset$  and  $Y_0 = \{b_1, b_2, \dots, b_k\}$ . Suppose that  $X_t$  and  $Y_t$  have been constructed for some  $t$ ,  $0 \leq t < n$ . We shall construct  $X_{t+1}$  and  $Y_{t+1}$ . Since  $|Y_t| = k - t > n - t > 0$ , there exists  $j = 1, 2, \dots, k$  with  $b_j \in Y_t$ . Set

$$c = \bigvee_{y \in (X_t \cup Y_t) \setminus \{b_j\}} y.$$

We claim that  $c \wedge a_\ell = 0$  for some  $\ell = 1, 2, \dots, n$ . Otherwise, if  $c \wedge a_i \neq 0$  for every  $i = 1, 2, \dots, n$ , then  $c \wedge a_i$  is essential in  $a_i$  because  $a_i$  is uniform, so that  $\bigvee_{i=1}^n c \wedge a_i$  is essential in  $\bigvee_{i=1}^n a_i$  by Corollary 2.34. Since  $\bigvee_{i=1}^n a_i$  is

essential in 1, it follows that  $\bigvee_{i=1}^n c \wedge a_i$  is essential in 1 (Lemma 2.32). Then  $c \geq \bigvee_{i=1}^n c \wedge a_i$  is essential in 1, so that  $c \wedge b_j \neq 0$ . This contradicts the fact that  $X_t \cup Y_t$  is join-independent and the contradiction proves the claim. From Proposition 2.31 and the claim it follows that  $(X_t \cup \{a_\ell\}) \cup (Y_t \setminus \{b_j\})$  is join-independent, so that  $X_{t+1} = X_t \cup \{a_\ell\}$  and  $Y_{t+1} = Y_t \setminus \{b_j\}$  have the required properties. This completes the construction of the sets  $X_t$  and  $Y_t$ .

For  $t = n$  we have a non-empty subset  $Y_n$  of  $\{b_1, b_2, \dots, b_k\}$  such that  $\{a_1, a_2, \dots, a_n\} \cup Y_n$  is a join-independent subset of cardinality  $k$ , so that

$$(a_1 \vee a_2 \vee \dots \vee a_n) \wedge y = 0$$

for every  $y \in Y_n$ , and this contradicts the fact that  $a_1 \vee a_2 \vee \dots \vee a_n$  is essential in  $L$ . Hence every join-independent subset of  $L$  has cardinality  $\leq n$ .

(c)  $\Rightarrow$  (d). If (d) does not hold, there is a chain  $a_0 \leq a_1 \leq a_2 \leq \dots$  of elements of  $L$  such that for every  $i \geq 0$  there exists  $j(i) > i$  with  $a_i$  not essential in  $a_{j(i)}$ . Set  $j_0 = 0$  and  $j_{n+1} = j(j_n)$  for every  $n \geq 0$ . Then for every  $n \geq 0$  there exists a non-zero element  $b_n \leq a_{j_{n+1}}$  such that  $b_n \wedge a_{j_n} = 0$ . The set  $\{b_n \mid n \geq 0\}$  is join-independent by Proposition 2.31. Thus (c) does not hold.

(d)  $\Rightarrow$  (a). If (a) is not satisfied, then  $L$  contains a countable infinite join-independent subset  $\{b_i \mid i \geq 0\}$ . Set  $a_n = \bigvee_{i=0}^n b_i$ . Then  $a_0 \leq a_1 \leq a_2 \leq \dots$ , and for every  $n \geq 0$  the element  $a_n$  is not essential in  $a_{n+1}$  because

$$a_n \wedge b_{n+1} = 0.$$

Hence (d) is not satisfied.

The last part of the statement has already been seen in the proof of (b)  $\Rightarrow$  (c).  $\square$

Thus, for a modular lattice  $L$ , either there is a finite join-independent subset  $\{a_1, a_2, \dots, a_n\}$  with  $a_i$  uniform for every  $i = 1, 2, \dots, n$  and

$$a_1 \vee a_2 \vee \dots \vee a_n$$

essential in  $L$ , and in this case  $n$  is said to be the *Goldie dimension*  $\dim L$  of  $L$ , or  $L$  contains infinite join-independent subsets, in which case  $L$  is said to have *infinite Goldie dimension*. The Goldie dimension of a lattice  $L$  is zero if and only if  $L$  has exactly one element.

## 2.7 Goldie dimension of a module

In this section we shall apply the Goldie dimension of modular lattices introduced in the previous section to the lattice  $\mathcal{L}(M)$  of all submodules of a module  $M_R$ . If the lattice  $\mathcal{L}(M)$  has finite Goldie dimension  $n$ , then  $n$  will be said to be the *Goldie dimension*  $\dim M_R$  of the module  $M_R$ . Otherwise, if the lattice  $\mathcal{L}(M)$  has infinite Goldie dimension, that is, if  $M_R$  contains an infinite direct

sum of non-zero submodules, the module  $M_R$  will be said to have *infinite Goldie dimension* ( $\dim M_R = \infty$ ).

Since a module  $M$  is essential in its injective envelope  $E(M)$ ,

$$\dim(M) = \dim(E(M)).$$

In Section 2.5 we had already defined uniform modules. Obviously, a module  $M$  is uniform if and only if the lattice  $\mathcal{L}(M)$  is uniform. A module  $M$  has finite Goldie dimension  $n$  if and only if it contains an essential submodule that is the finite direct sum of  $n$  uniform submodules  $U_1, \dots, U_n$  (Theorem 2.36(b)). In this case  $E(M) = E(U_1) \oplus E(U_2) \oplus \dots \oplus E(U_n)$  is the finite direct sum of  $n$  indecomposable modules. Note that by Theorem 2.26 we already knew that if  $E(M)$  is a finite direct sum of indecomposable modules, then the number of direct summands in any indecomposable decomposition of  $E(M)$  does not depend on the decomposition. Hence a module  $M$  has finite Goldie dimension  $n$  if and only if its injective envelope  $E(M)$  is the direct sum of  $n$  indecomposable modules.

In the next proposition we collect the most important arithmetical properties of the Goldie dimension of modules. Some of these properties have already been noticed. Their proof is elementary.

**Proposition 2.37** *Let  $M$  be module.*

- (a)  $\dim(M) = 0$  if and only if  $M = 0$ .
- (b)  $\dim(M) = 1$  if and only if  $M$  is uniform.
- (c) If  $N \leq M$  and  $M$  has finite Goldie dimension, then  $N$  has finite Goldie dimension and  $\dim(N) \leq \dim(M)$ .
- (d) If  $N \leq M$  and  $M$  has finite Goldie dimension, then  $\dim(N) = \dim(M)$  if and only if  $N$  is essential in  $M$ .
- (e) If  $M$  and  $M'$  are modules of finite Goldie dimension, then  $M \oplus M'$  is a module of finite Goldie dimension and  $\dim(M \oplus N) = \dim(M) + \dim(N)$ .

□

Artinian modules and noetherian modules have finite Goldie dimension. For an artinian module  $M$ , the Goldie dimension of  $M$  is equal to the composition length of its socle  $\text{soc}(M)$ . In particular, an artinian module  $M$  has Goldie dimension 1 if and only if it has a simple socle.

The next proposition contains a first application of the Goldie dimension of a ring.

**Proposition 2.38** *Let  $R$  be a ring and suppose that  $R_R$  has finite Goldie dimension. Then every surjective endomorphism of a finitely generated projective right  $R$ -module  $P_R$  is an automorphism. In particular, every right or left invertible element of  $R$  is invertible.*



*Proof.* If  $P_R$  is a finitely generated projective  $R$ -module, then  $P_R$  has finite Goldie dimension. If  $\varphi$  is a surjective endomorphism of  $P_R$ , then

$$P_R \oplus \ker \varphi \cong P_R,$$

so that  $\dim(\ker \varphi) = 0$ , that is,  $\varphi$  is injective. For the second part of the statement we must show that if  $x, y \in R$  and  $xy = 1$ , then  $yx = 1$ . Since  $xy = 1$ , left multiplication by  $x$  is a surjective endomorphism  $\mu_x$  of  $R_R$ . From  $xy = 1$  it follows that  $yR \oplus \ker(\mu_x) = R$ . Hence  $yR = R$ , i.e.,  $y$  is also right invertible. Thus  $y$  is invertible and  $x$  is its two-sided inverse.  $\square$

## 2.8 Dual Goldie dimension of a module

We shall now apply the results on the Goldie dimension of modular lattices of Section 2.6 to the dual lattice of the lattice  $\mathcal{L}(M)$  of all submodules of a module  $M$ . If  $(L, \wedge, \vee)$  is a modular lattice, then its dual lattice  $(L, \vee, \wedge)$  is also a modular lattice. In particular, the Duality Principle holds for modular lattices, that is, if a statement  $\Phi$  expressed in terms of  $\wedge$ ,  $\vee$ ,  $\leq$  and  $\geq$  is true for all modular lattices, then the dual statement of  $\Phi$ , obtained from  $\Phi$  interchanging  $\wedge$  with  $\vee$  and  $\leq$  with  $\geq$ , is also true for all modular lattices.

Since the dual of the lattice  $\mathcal{L}(M)$  of all submodules of a module  $M$  is modular, all the results of Section 2.6 hold for this lattice. We now shall translate the results of Section 2.6 for the dual of the lattice  $\mathcal{L}(M)$  to the language of modules.

Let  $M$  be a right  $R$ -module. A finite set  $\{N_i \mid i \in I\}$  of proper submodules of  $M$  is said to be *coindependent* if  $N_i + (\bigcap_{j \neq i} N_j) = M$  for every  $i \in I$ , or, equivalently, if the canonical injective mapping  $M / \bigcap_{i \in I} N_i \rightarrow \bigoplus_{i \in I} M / N_i$  is bijective. An arbitrary set  $A$  of proper submodules of  $M$  is *coindependent* if its finite subsets are coindependent. If  $A$  is a coindependent set of proper submodules of  $M$  and  $N$  is a proper submodule of  $M$  such that  $N + (\bigcap_{X \in A} X) = M$  for every finite subset  $B$  of  $A$ , then  $A \cup \{N\}$  is a coindependent set of submodules of  $M$  (Proposition 2.31). By Zorn's Lemma, every coindependent set of submodules of  $M$  is contained in a maximal coindependent set.

The following lemma, which is dual to Lemma 2.24, has an elementary proof.

**Lemma 2.39** *Let  $M \neq 0$  be an  $R$ -module. The following conditions are equivalent:*

- (a) *The sum of any two proper submodules of  $M$  is a proper submodule of  $M$ .*
- (b) *Every proper submodule of  $M$  is superfluous in  $M$ .*
- (c) *Every non-zero homomorphic image of  $M$  is indecomposable.*  $\square$

An  $R$ -module  $M \neq 0$  is said to be *couniform*, or *hollow*, if it satisfies the equivalent conditions of the previous Lemma 2.39. Every local module is couniform, but not conversely. For instance, the  $\mathbf{Z}$ -module  $\mathbf{Z}(p^\infty)$  (the Prüfer group) is couniform and is not local. Every proper submodule of a finitely generated module  $M$  is contained in a maximal submodule of  $M$ . Hence if  $M$  is a finitely generated module,  $M$  is couniform if and only if  $M$  is local. From Theorem 2.36 we obtain

**Theorem 2.40** *The following conditions are equivalent for a right module  $M$ :*

- (a) *There do not exist infinite coindependent sets of proper submodules of  $M$ .*
- (b) *There exists a finite coindependent set  $\{N_1, N_2, \dots, N_n\}$  of proper submodules of  $M$  with  $M/N_i$  couniform for all  $i$  and  $N_1 \cap N_2 \cap \dots \cap N_n$  superfluous in  $M$ .*
- (c) *The cardinality of the coindependent sets of proper submodules of  $M$  is  $\leq m$  for a non-negative integer  $m$ .*
- (d) *If  $N_0 \supseteq N_1 \supseteq N_2 \supseteq \dots$  is a descending chain of submodules of  $M$ , then there exists  $i \geq 0$  such that  $N_i/N_j$  is superfluous in  $M/N_j$  for every  $j \geq i$ .*

Moreover, if these equivalent conditions hold and  $\{N_1, N_2, \dots, N_n\}$  is a finite coindependent set of proper submodules of  $M$  with  $M/N_i$  couniform for all  $i$  and  $N_1 \cap N_2 \cap \dots \cap N_n$  superfluous in  $M$ , then every other coindependent set of proper submodules of  $M$  has cardinality  $\leq n$ .  $\square$

The *dual Goldie dimension*  $\text{codim}(M)$  of a right module  $M$  is the Goldie dimension of the dual lattice of the lattice  $\mathcal{L}(M)$ . Hence a module  $M$  has finite dual Goldie dimension  $n$  if and only if there exists a coindependent set  $\{N_1, N_2, \dots, N_n\}$  of proper submodules of  $M$  with  $M/N_i$  couniform for all  $i$  and  $N_1 \cap N_2 \cap \dots \cap N_n$  superfluous in  $M$ . And a module  $M$  has *infinite dual Goldie dimension* if there exist infinite coindependent sets of proper submodules of  $M$ . Note that if a module  $M$  has finite dual Goldie dimension, then for every proper submodule  $N$  of  $M$  there exists a proper submodule  $P$  of  $M$  containing  $N$  with  $M/P$  couniform (Lemma 2.35).

From Theorem 2.40(d) we obtain

**Corollary 2.41** *Every artinian module has finite dual Goldie dimension.*  $\square$

The proof of the next result is straightforward.

**Proposition 2.42** *Let  $M$  be module.*

- (a)  $\text{codim}(M) = 0$  if and only if  $M = 0$ .
- (b)  $\text{codim}(M) = 1$  if and only if  $M$  is couniform.

- (c) If  $N \leq M$  and  $M$  has finite dual Goldie dimension, then  $M/N$  has finite dual Goldie dimension and  $\text{codim}(M/N) \leq \text{codim}(M)$ .
- (d) If  $M$  has finite dual Goldie dimension and  $N \leq M$ , then  $\text{codim}(M/N) = \text{codim}(M)$  if and only if  $N$  is superfluous in  $M$ .
- (e) If  $M$  and  $M'$  are modules of finite dual Goldie dimension, then  $M \oplus M'$  is a module of finite dual Goldie dimension and

$$\text{codim}(M \oplus M') = \text{codim}(M) + \text{codim}(M'). \quad \square$$

If a module  $M$  has finite dual Goldie dimension  $n$ , then there exists a set  $\{N_1, N_2, \dots, N_n\}$  of submodules of  $M$  such that  $N = N_1 \cap N_2 \cap \dots \cap N_n$  is superfluous in  $M$  and  $M/N \cong \bigoplus_{i=1}^n M/N_i$  is a direct sum of  $n$  couniform modules. Note that there is no epimorphism of such a module  $M$  onto a direct sum of  $n+1$  non-zero modules. In Section 2.7 we saw that if a module  $M$  has the property that there are no monomorphisms from a direct sum of infinitely many non-zero modules into  $M$ , then  $M$  has finite Goldie dimension. This result cannot be dualized, that is, it is not true that if  $M$  is a module and there is no homomorphic image of  $M$  that is a direct product of infinitely many non-zero modules, then  $M$  has finite dual Goldie dimension. For instance, consider the  $\mathbf{Z}$ -module  $\mathbf{Z}$ , that is, the abelian group of integers. Then there is no homomorphic image of  $\mathbf{Z}$  that is a direct product  $\prod_{i \in I} G_i$  of infinitely many non-zero abelian groups  $G_i$ . But the set of all  $p\mathbf{Z}$ ,  $p$  a prime number, is an infinite coindependent set of proper subgroups of  $\mathbf{Z}$ , so that  $\text{codim}(\mathbf{Z}) = \infty$ .

For a semisimple module the dual Goldie dimension coincides with the composition length of the module. Hence for a semisimple artinian ring  $R$ ,

$$\dim(R_R) = \dim({}_R R) = \text{codim}(R_R) = \text{codim}({}_R R).$$

We shall denote this finite dimension  $\dim(R)$ .

**Proposition 2.43** *The following conditions are equivalent for a ring  $R$ .*

- (a) *The ring  $R$  is semilocal.*
- (b) *The right  $R$ -module  $R_R$  has finite dual Goldie dimension.*
- (c) *The left  $R$ -module  ${}_R R$  has finite dual Goldie dimension.*

Moreover, if these equivalent conditions hold,

$$\text{codim}(R_R) = \text{codim}({}_R R) = \dim(R/J(R)).$$

*Proof.* (a)  $\Rightarrow$  (b). Suppose  $R_R$  has infinite dual Goldie dimension, and let  $\{I_n \mid n \geq 1\}$  be an infinite coindependent set of proper right ideals of  $R$ . Then  $R/\bigcap_{n=1}^k I_n$  is a direct sum of  $k$  non-zero cyclic modules for every  $k \geq 1$ .

If  $C$  is a non-zero cyclic module,  $C/CJ(R)$  is a non-zero module. Therefore  $R/(J(R) + \bigcap_{n=1}^k I_n)$  is a direct sum of at least  $k$  non-zero modules for every  $k \geq 1$ . In particular  $R/J(R)$  cannot have finite length, so that  $R$  cannot be semilocal.

(b)  $\Rightarrow$  (a). Suppose that  $R_R$  has finite dual Goldie dimension. Let  $\mathcal{I}$  be the set of all right ideals of  $R$  that are finite intersections of maximal right ideals. Note that if  $I, J \in \mathcal{I}$  and  $I \subset J$ , then  $R/I$  and  $R/J$  are semisimple modules of finite length and

$$\text{codim}(R/J) = \text{length}(R/J) < \text{length}(R/I) = \text{codim}(R/I).$$

Since  $\text{codim}(R/I) \leq \text{codim}(R_R)$  for every  $I$ , it follows that every descending chain in  $\mathcal{I}$  is finite, i.e., the partially ordered set  $\mathcal{I}$  is artinian. In particular  $\mathcal{I}$  has a minimal element. Since any intersection of two elements of  $\mathcal{I}$  belongs to  $\mathcal{I}$ , the set  $\mathcal{I}$  has a least element, which is the Jacobson radical  $J(R)$ . Hence  $J(R) \in \mathcal{I}$  is a finite intersection of maximal right ideals. Therefore  $R/J(R)$  is a semisimple artinian right  $R$ -module, and  $R$  is semilocal.

Since (a) is right-left symmetric, (a), (b) and (c) are equivalent. Finally,  $J(R)$  is a superfluous submodule of  $R_R$  (Nakayama's Lemma 1.4), so that if (b) holds, then  $\text{codim}(R_R) = \text{codim}(R/J(R))$  by Proposition 2.42(d).  $\square$

**Corollary 2.44** *Let  $P_R$  be a finitely generated projective module over a semilocal ring  $R$ . Then every surjective endomorphism of  $P_R$  is an automorphism. In particular, every right or left invertible element of a semilocal ring is invertible.*

*Proof.* Since  $R$  is semilocal,  $R_R$  has finite dual Goldie dimension, so that  $P_R$  has finite dual Goldie dimension (Proposition 2.42). If  $f: P_R \rightarrow P_R$  is surjective, then  $\ker f$  is a direct summand of  $P_R$ , and  $\ker f \oplus P_R \cong P_R$ . Thus  $\text{codim}(\ker f) = 0$ , i.e.,  $\ker f = 0$ . The proof of the second part of the statement is analogous to the proof of the second part of the statement of Proposition 2.38.  $\square$

We conclude this section with an example. A non-zero uniserial module is both uniform and couniform. Therefore a serial module has finite Goldie dimension if and only if it is the direct sum of a finite number of uniserial modules, if and only if it has finite dual Goldie dimension. More precisely, a serial module  $M$  has finite Goldie dimension  $n$  if and only if it is the direct sum of exactly  $n$  non-zero uniserial modules (so that the number  $n$  of direct summands of  $M$  that appear in any decomposition of  $M$  as a direct sum of non-zero uniserial modules does not depend on the decomposition), if and only if  $M$  has finite dual Goldie dimension  $n$ .

## 2.9 $\aleph$ -small modules and $\aleph$ -closed classes

Let  $R$  be an arbitrary ring. An  $R$ -module  $N_R$  is *small* if for every family

$$\{M_i \mid i \in I\}$$

of  $R$ -modules and any homomorphism  $\varphi: N_R \rightarrow \bigoplus_{i \in I} M_i$ , there is a finite subset  $F \subseteq I$  such that  $\pi_j \varphi = 0$  for every  $j \in I \setminus F$ . Here the  $\pi_j: \bigoplus_{i \in I} M_i \rightarrow M_j$  are the canonical projections.

For instance, every finitely generated module is small. Another class of small modules is given by the class of uncountably generated uniserial modules, as the next proposition shows.

**Proposition 2.45** *Every uniserial module that is not small can be generated by  $\aleph_0$  elements.*

*Proof.* Let  $U$  be a uniserial module that is not small. Then there exist modules  $M_i$ ,  $i \in I$ , and a homomorphism  $\varphi: U \rightarrow \bigoplus_{i \in I} M_i$  such that if  $\pi_j: \bigoplus_{i \in I} M_i \rightarrow M_j$  denotes the canonical projection for every  $j \in I$ , then  $\pi_j \varphi \neq 0$  for infinitely many  $j \in I$ .

For every  $x \in U$  set  $\text{supp}(x) = \{i \in I \mid \pi_i \varphi(x) \neq 0\}$ , so that  $\text{supp}(x)$  is a finite subset of  $I$  for every  $x \in U$ . Note that if  $x, y \in U$  and  $xR \subseteq yR$ , then  $\text{supp}(x) \subseteq \text{supp}(y)$ . Define by induction a sequence of elements  $x_n \in U$ ,  $n \geq 0$ , such that  $\text{supp}(x_0) \subset \text{supp}(x_1) \subset \text{supp}(x_2) \subset \dots$ . Set  $x_0 = 0$ . If  $x_n \in U$  has been defined, then  $\text{supp}(x_n)$  is finite, but  $\pi_j \varphi \neq 0$  for infinitely many  $j \in I$ . Hence there exists  $k \in I$  with  $k \notin \text{supp}(x_n)$  and  $\pi_k \varphi \neq 0$ . Let  $x_{n+1} \in U$  be an element of  $U$  with  $\pi_k \varphi(x_{n+1}) \neq 0$ . Then  $\text{supp}(x_{n+1}) \not\subseteq \text{supp}(x_n)$ , so that  $x_{n+1}R \not\subseteq x_nR$ . Hence  $x_nR \subseteq x_{n+1}R$ , from which  $\text{supp}(x_n) \subset \text{supp}(x_{n+1})$ . This defines the sequence  $x_n$ .

If the elements  $x_n$  do not generate the module  $U$ , then there exists  $v \in U$  such that  $v \notin x_nR$  for every  $n \geq 0$ . Then  $vR \supseteq x_nR$  for every  $n$ , so that  $\text{supp}(v) \supseteq \text{supp}(x_n)$  for every  $n$ . This yields a contradiction, because  $\text{supp}(v)$  is finite and  $\bigcup_{n \geq 0} \text{supp}(x_n)$  is infinite. Hence the  $x_n$  generate  $U$  and  $U$  is countably generated.  $\square$

Now we shall extend the definition of small module. Let  $\aleph$  be a cardinal number. An  $R$ -module  $N_R$  is  $\aleph$ -small if for every family  $\{M_i \mid i \in I\}$  of  $R$ -modules and any homomorphism  $\varphi: N_R \rightarrow \bigoplus_{i \in I} M_i$ , the set  $\{i \in I \mid \pi_i \varphi \neq 0\}$  has cardinality  $\leq \aleph$ .

For instance, every small module is  $\aleph_0$ -small, and every uniserial module is  $\aleph_0$ -small. It is easy to see that if  $\aleph$  is a finite cardinal number and  $N_R$  is  $\aleph$ -small, then  $N_R = 0$ .

Let  $R$  be a ring,  $\mathcal{G}$  a non-empty class of right  $R$ -modules and let  $\aleph$  be a cardinal number. We say that  $\mathcal{G}$  is  $\aleph$ -closed if:

- (a)  $\mathcal{G}$  is closed under homomorphic images, that is, if  $M_R, N_R$  are right  $R$ -modules,  $f: M_R \rightarrow N_R$  is an epimorphism and  $M_R \in \mathcal{G}$ , then  $N_R \in \mathcal{G}$ ;
- (b) every module in  $\mathcal{G}$  is  $\aleph$ -small;
- (c)  $\mathcal{G}$  is closed under direct sums of  $\aleph$  modules, that is, if  $M_i \in \mathcal{G}$  for every  $i \in I$  and  $|I| \leq \aleph$ , then  $\bigoplus_{i \in I} M_i \in \mathcal{G}$ .

**Examples 2.46** (1) For an infinite cardinal number  $\aleph$  and a ring  $R$ , let  $\mathcal{G}$  be the class of all  $\aleph$ -generated modules, that is, the right  $R$ -modules that are homomorphic images of  $R^{(\aleph)}$ . Then  $\mathcal{G}$  is an  $\aleph$ -closed class.

(2) For a cardinal number  $\aleph$  and a ring  $R$ , let  $\mathcal{G}$  be the class of all  $\aleph$ -small right  $R$ -modules. Then  $\mathcal{G}$  is an  $\aleph$ -closed class.

(3) Let  $R$  be a ring and let  $\aleph$  be a finite cardinal number. We have already remarked that every  $\aleph$ -small right  $R$ -module is zero. Hence every  $\aleph$ -closed class of right  $R$ -modules consists of all zero  $R$ -modules.

(4) Let  $R$  be a ring and let  $\mathcal{G}$  be the class of all  $\sigma$ -small  $R$ -modules, that is, the right  $R$ -modules that are countable ascending unions of small submodules. Then  $\mathcal{G}$  is an  $\aleph_0$ -closed class. Note that by Proposition 2.45 every uniserial module is  $\sigma$ -small.  $\square$

The following theorem is essentially equivalent to an extension due to C. Walker of a theorem of [Kaplansky 58, Theorem 1]. Kaplansky proved it in the case in which  $\aleph = \aleph_0$  and  $\mathcal{G}$  is the class of  $\aleph_0$ -generated modules, and Walker extended it to the class of  $\aleph$ -generated modules for an arbitrary cardinal number  $\aleph$ . [Warfield 69c] remarked that the theorem holds for the classes of  $\aleph$ -small modules and  $\sigma$ -small modules, and that suitable versions for larger cardinals were also valid.

**Theorem 2.47** *Let  $R$  be a ring, let  $\aleph$  be a cardinal number and  $\mathcal{G}$  an  $\aleph$ -closed class of right  $R$ -modules. If a module  $M_R$  is a direct sum of modules belonging to  $\mathcal{G}$ , then every direct summand of  $M_R$  is a direct sum of modules belonging to  $\mathcal{G}$ .*

*Proof.* Since the case of a finite cardinal number  $\aleph$  is trivial (Example 2.46(3)), we may suppose  $\aleph$  infinite. Let  $M_R = \bigoplus_{i \in I} M_i$ , where  $M_i \in \mathcal{G}$  for every  $i \in I$ , and assume  $M_R = N_R \oplus P_R$ . Let  $\mathcal{L}(N_R)$  and  $\mathcal{L}(P_R)$  be the sets of all submodules of  $N_R$  and  $P_R$ , respectively, and let  $\mathcal{T}$  be the set of all triples  $(J, \mathcal{A}, \mathcal{B})$  such that

- (1)  $J \subseteq I$ ,  $\mathcal{A} \subseteq \mathcal{L}(N_R) \cap \mathcal{G}$ ,  $\mathcal{B} \subseteq \mathcal{L}(P_R) \cap \mathcal{G}$ ;
- (2) the sum  $\sum_{X \in \mathcal{A}} X$  is direct, that is,  $\sum_{X \in \mathcal{A}} X = \bigoplus_{X \in \mathcal{A}} X$ ;
- (3) the sum  $\sum_{Y \in \mathcal{B}} Y$  is direct, that is,  $\sum_{Y \in \mathcal{B}} Y = \bigoplus_{Y \in \mathcal{B}} Y$ ;
- (4)  $\bigoplus_{i \in J} M_i = (\bigoplus_{X \in \mathcal{A}} X) \oplus (\bigoplus_{Y \in \mathcal{B}} Y)$ .

Note that  $\mathcal{T}$  is non-empty, because  $(\emptyset, \emptyset, \emptyset) \in \mathcal{T}$ . Define a partial ordering on  $\mathcal{T}$  by setting  $(J, \mathcal{A}, \mathcal{B}) \leq (J', \mathcal{A}', \mathcal{B}')$  whenever  $J \subseteq J'$ ,  $\mathcal{A} \subseteq \mathcal{A}'$ , and  $\mathcal{B} \subseteq \mathcal{B}'$ . It is easily seen that every chain in  $\mathcal{T}$  has an upper bound in  $\mathcal{T}$ , so that by Zorn's Lemma  $\mathcal{T}$  has a maximal element  $(K, \mathcal{C}, \mathcal{D})$ . Suppose  $K \subset I$ . Let  $i \in I \setminus K$  and let  $\varepsilon$  be the idempotent endomorphisms of  $M_R$  with  $\ker(\varepsilon) = P_R$  that is the identity on  $N_R$ .

Define an ascending chain  $I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots$  of subsets of  $I$  of cardinality at most  $\aleph$  in the following way. Set  $I_0 = \{i\}$ . Suppose  $I_n$  has been defined. Since  $\bigoplus_{j \in I_n} M_j \in \mathcal{G}$ , its homomorphic images  $\varepsilon(\bigoplus_{j \in I_n} M_j)$  and  $(1 - \varepsilon)(\bigoplus_{j \in I_n} M_j)$  belong to  $\mathcal{G}$ , so that  $\varepsilon(\bigoplus_{j \in I_n} M_j) + (1 - \varepsilon)(\bigoplus_{j \in I_n} M_j)$  is in  $\mathcal{G}$ . In particular, this module is  $\aleph$ -small, hence there exists a subset  $I_{n+1}$  of  $I$  of cardinality at most  $\aleph$  such that  $\varepsilon(\bigoplus_{j \in I_n} M_j) + (1 - \varepsilon)(\bigoplus_{j \in I_n} M_j) \subseteq \bigoplus_{j \in I_{n+1}} M_j$ . Note that  $\bigoplus_{j \in I_n} M_j \subseteq \varepsilon(\bigoplus_{j \in I_n} M_j) + (1 - \varepsilon)(\bigoplus_{j \in I_n} M_j)$ , so that  $I_n \subseteq I_{n+1}$ . This completes the construction of the subsets  $I_n$  by induction.

Let  $I'$  be the union of the countable ascending chain  $I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots$ , so that  $I'$  is a subset of  $I$  of cardinality at most  $\aleph$ . Hence  $\bigoplus_{j \in I'} M_j \in \mathcal{G}$ . Since  $i \in I'$ , it follows that  $I' \cup K \supset K$ . And  $\varepsilon(\bigoplus_{j \in I_n} M_j) \subseteq \bigoplus_{j \in I_{n+1}} M_j$  for all  $n$  implies that  $\varepsilon(\bigoplus_{j \in I'} M_j) \subseteq \bigoplus_{j \in I'} M_j$ . Similarly,  $(1 - \varepsilon)(\bigoplus_{j \in I'} M_j) \subseteq \bigoplus_{j \in I'} M_j$ .

Now  $\bigoplus_{j \in K} M_j = (\bigoplus_{X \in \mathcal{C}} X) \oplus (\bigoplus_{Y \in \mathcal{D}} Y)$  because  $(K, \mathcal{C}, \mathcal{D}) \in \mathcal{T}$ , and

$$\begin{aligned} \varepsilon(\bigoplus_{j \in I' \cup K} M_j) &= \varepsilon(\bigoplus_{j \in I'} M_j + \bigoplus_{j \in K} M_j) \\ &= \varepsilon(\bigoplus_{j \in I'} M_j + \bigoplus_{X \in \mathcal{C}} X + \bigoplus_{Y \in \mathcal{D}} Y) = \varepsilon(\bigoplus_{j \in I'} M_j) + \bigoplus_{X \in \mathcal{C}} X \\ &\subseteq \bigoplus_{j \in I'} M_j + \bigoplus_{j \in K} M_j = \bigoplus_{j \in I' \cup K} M_j. \end{aligned}$$

Hence the idempotent endomorphism  $\varepsilon$  of  $M_R$  induces an idempotent endomorphism on  $\bigoplus_{j \in I' \cup K} M_j$ , so that

$$\bigoplus_{j \in I' \cup K} M_j = \varepsilon(\bigoplus_{j \in I' \cup K} M_j) \oplus (1 - \varepsilon)(\bigoplus_{j \in I' \cup K} M_j).$$

The submodule  $\bigoplus_{X \in \mathcal{C}} X$  is a direct summand of  $\bigoplus_{j \in I' \cup K} M_j$  contained in  $\varepsilon(\bigoplus_{j \in I' \cup K} M_j)$ . Hence it is a direct summand of  $\varepsilon(\bigoplus_{j \in I' \cup K} M_j)$ , that is, there exists a submodule  $\overline{X}$  of  $N_R$  such that

$$\varepsilon(\bigoplus_{j \in I' \cup K} M_j) = (\bigoplus_{X \in \mathcal{C}} X) \oplus \overline{X}.$$

Similarly,  $\bigoplus_{Y \in \mathcal{D}} Y$  is a direct summand of  $\bigoplus_{j \in I' \cup K} M_j$  contained in

$$(1 - \varepsilon)(\bigoplus_{j \in I' \cup K} M_j),$$

so that  $(1 - \varepsilon)(\bigoplus_{j \in I' \cup K} M_j) = (\bigoplus_{Y \in \mathcal{D}} Y) \oplus \overline{Y}$  for some submodule  $\overline{Y}$  of  $P_R$ . Therefore  $\bigoplus_{j \in I' \cup K} M_j = (\bigoplus_{X \in \mathcal{C}} X) \oplus \overline{X} \oplus (\bigoplus_{Y \in \mathcal{D}} Y) \oplus \overline{Y}$ . It follows that

$$\overline{X} \oplus \overline{Y} \cong \bigoplus_{j \in I' \cup K} M_j / \bigoplus_{j \in K} M_j \cong \bigoplus_{j \in I' \setminus K} M_j \in \mathcal{G},$$

so that both  $\overline{X}$  and  $\overline{Y}$  belong to  $\mathcal{G}$ . This shows that  $(I' \cup K, \mathcal{C} \cup \{\overline{X}\}, \mathcal{D} \cup \{\overline{Y}\})$  is an element of  $\mathcal{T}$  strictly greater than the maximal element  $(K, \mathcal{C}, \mathcal{D})$ . This contradiction proves that  $K = I$ . Thus  $M_R = (\bigoplus_{X \in \mathcal{C}} X) \oplus (\bigoplus_{Y \in \mathcal{D}} Y)$ . Hence  $N_R = \bigoplus_{X \in \mathcal{C}} X$  and  $P_R = \bigoplus_{Y \in \mathcal{D}} Y$ .  $\square$

If we apply Theorem 2.47 to the  $\aleph_0$ -closed class of all countably generated (=  $\aleph_0$ -generated) modules we get a famous result of [Kaplansky 58]:

**Corollary 2.48** (Kaplansky) *Any projective module is a direct sum of countably generated modules.*  $\square$

**Corollary 2.49** *Let  $N$  be a direct summand of a serial module. Then there is a decomposition  $N = \bigoplus_{i \in I} N_i$ , where each  $N_i$  is a direct summand of the direct sum of a countable family  $\{U_{i,n} \mid n \in \mathbb{N}\}$  of uniserial modules.*

*Proof.* Let  $\mathcal{G}$  be the class of all  $\sigma$ -small modules. This is an  $\aleph_0$ -closed class that contains all uniserial modules (Example 2.46(4)). Apply Theorem 2.47. Then  $N$  is a direct sum of modules  $N_i$  belonging to  $\mathcal{G}$ . Hence it suffices to prove that a module  $N_i$  belonging to  $\mathcal{G}$  that is a direct summand of a serial module is a direct summand of the direct sum of a countable family of uniserial modules. If  $N_i$  is a direct summand of a serial module  $\bigoplus_{j \in J} U_j$ , there are two homomorphisms  $\varphi: N_i \rightarrow \bigoplus_{j \in J} U_j$  and  $\psi: \bigoplus_{j \in J} U_j \rightarrow N_i$  such that  $\psi\varphi = 1_{N_i}$ . If  $N_i \in \mathcal{G}$ ,  $N_i$  is  $\aleph_0$ -small, so that the set  $C = \{j \in J \mid \pi_j\varphi \neq 0\}$  has cardinality  $\leq \aleph_0$ . Now it is easily seen that  $N_i$  is a direct summand of the direct sum of the countable family  $\{U_j \mid j \in C\}$ .  $\square$

We conclude the section with a proposition due to [Warfield 69c, Lemma 5], who proved it not only for modules, but for objects of more general abelian categories. Here we consider the case of modules only.

**Proposition 2.50** *Let  $R$  be a ring,  $\aleph$  a cardinal number and  $\mathcal{G}$  an  $\aleph$ -closed class of right  $R$ -modules. If  $M = \bigoplus_{i \in I} A_i = \bigoplus_{j \in J} B_j$ , where  $A_i, B_j$  are non-zero modules belonging to  $\mathcal{G}$  for every  $i \in I$  and every  $j \in J$ , then there exists a partition  $\{I_\lambda \mid \lambda \in \Lambda\}$  of  $I$  and a partition  $\{J_\lambda \mid \lambda \in \Lambda\}$  of  $J$  with  $|I_\lambda| \leq \aleph$ ,  $|J_\lambda| \leq \aleph$  and  $\bigoplus_{i \in I_\lambda} A_i \cong \bigoplus_{j \in J_\lambda} B_j$  for every  $\lambda \in \Lambda$ .*

*Proof.* The case of a finite cardinal number  $\aleph$  is trivial. Hence we may suppose  $\aleph$  infinite. We claim that if  $i_0 \in I$ , then there exist subsets  $I' \subseteq I$  and  $J' \subseteq J$  of cardinality  $\leq \aleph$  such that  $i_0 \in I'$  and  $\bigoplus_{i \in I'} A_i = \bigoplus_{j \in J'} B_j$ . In order to prove the claim define sets  $I'_n$  and  $J'_n$  of cardinality  $\leq \aleph$  for every integer  $n \geq 0$  by induction as follows. Set  $I'_0 = \{i_0\}$  and  $J'_0 = \emptyset$ . Suppose  $I'_n$  and  $J'_n$  have been defined. Then  $(\bigoplus_{i \in I'_n} A_i) + (\bigoplus_{j \in J'_n} B_j) \in \mathcal{G}$ , so that there exists a subset  $J'_{n+1} \subseteq J$  of cardinality at most  $\aleph$  such that  $(\bigoplus_{i \in I'_n} A_i) + (\bigoplus_{j \in J'_n} B_j) \subseteq \bigoplus_{j \in J'_{n+1}} B_j$ . Since  $(\bigoplus_{i \in I'_n} A_i) + (\bigoplus_{j \in J'_{n+1}} B_j) \in \mathcal{G}$ , there exists a subset  $I'_{n+1} \subseteq I$  of cardinality at most  $\aleph$  such that  $(\bigoplus_{i \in I'_n} A_i) + (\bigoplus_{j \in J'_{n+1}} B_j) \subseteq \bigoplus_{i \in I'_{n+1}} A_i$ . It is now obvious that  $I' = \bigcup_{n \geq 0} I'_n$  and  $J' = \bigcup_{n \geq 0} J'_n$  have the property required in the claim.

Define a chain of subsets  $K_0 \subseteq K_1 \subseteq \dots \subseteq K_\lambda \subseteq \dots$  of  $I$  and a chain of subsets  $L_0 \subseteq L_1 \subseteq \dots \subseteq L_\lambda \subseteq \dots$  of  $J$  for each ordinal  $\lambda$  by transfinite induction in the following way. Set  $K_0 = L_0 = \emptyset$ . If  $\lambda$  is a limit ordinal set  $K_\lambda = \bigcup_{\mu < \lambda} K_\mu$  and  $L_\lambda = \bigcup_{\mu < \lambda} L_\mu$ . For every ordinal  $\mu$  such that  $K_\mu = I$



set  $K_{\mu+1} = K_\mu$  and  $L_{\mu+1} = L_\mu$ . Otherwise, if  $K_\mu \subset I$ , choose an element  $i_0 \in I \setminus K_\mu$ . By the claim there exist  $I' \subseteq I$  and  $J' \subseteq J$ , both of cardinality  $\leq \aleph$ , such that  $i_0 \in I'$  and  $\oplus_{i \in I'} A_i = \oplus_{j \in J'} B_j$ . In this case set  $K_{\mu+1} = K_\mu \cup I'$  and  $L_{\mu+1} = L_\mu \cup J'$ .

Obviously  $\oplus_{i \in K_\lambda} A_i = \oplus_{j \in L_\lambda} B_j$  for every  $\lambda$ , and there exists an ordinal  $\bar{\lambda}$  such that  $K_{\bar{\lambda}} = I$ . Then  $L_{\bar{\lambda}} = J$ . Set  $I_\lambda = K_{\lambda+1} \setminus K_\lambda$  and  $J_\lambda = L_{\lambda+1} \setminus L_\lambda$  for every  $\lambda < \bar{\lambda}$ . Then

$$\oplus_{i \in K_{\lambda+1}} A_i = \oplus_{j \in L_{\lambda+1}} B_j \quad \text{and} \quad \oplus_{i \in K_\lambda} A_i = \oplus_{j \in L_\lambda} B_j$$

imply  $\oplus_{i \in I_\lambda} A_i \cong \oplus_{j \in J_\lambda} B_j$ . □

## 2.10 Direct sums of $\aleph$ -small modules

In Section 2.3 we saw two cases in which there exist isomorphic refinements of two direct sum decompositions. The next theorem examines a third case.

**Theorem 2.51** *Let  $M_R$  be a module that is a direct sum of  $\aleph_0$ -small submodules. Then any two direct sum decompositions of  $M$  into summands having the  $\aleph_0$ -exchange property have isomorphic refinements.*

*Proof.* We have already remarked that the class  $\mathcal{G}$  of all  $\aleph_0$ -small  $R$ -modules is  $\aleph_0$ -closed (Example 2.46(2)). By Theorem 2.47 any decomposition of  $M_R$  refines into one in which the summands belong to  $\mathcal{G}$ . By Lemma 2.4 every refinement of a decomposition of  $M_R$  into summands with the  $\aleph_0$ -exchange property is a decomposition into summands with the  $\aleph_0$ -exchange property. Hence we may suppose that we have two direct sum decompositions of  $M$  into summands belonging to  $\mathcal{G}$  and having the  $\aleph_0$ -exchange property. By Proposition 2.50 we may assume that the index sets are countable. In this case the result is given by Theorem 2.10. □

A fourth case in which isomorphic refinements exist is considered in the next important result, due to [Crawley and Jónsson, Theorem 7.1], who proved it for algebraic systems more general than modules. Here we shall present the proof given by [Warfield 69c, Theorem 7]. Also the proof given by Warfield holds in a context more general than ours, that is for suitable abelian categories, but we shall restrict our attention to the case we are interested in, that is, the case of modules. Recall that a module is  $\sigma$ -small if it is a countable ascending union of small submodules (Example 2.46(4)).

**Theorem 2.52** *If a module  $M$  is a direct sum of  $\sigma$ -small modules each of which has the exchange property, then any two direct sum decompositions of  $M$  have isomorphic refinements.*

*Proof.* We claim that if  $A = \bigoplus_{i=1}^{\infty} B_i$  is a direct sum of countably many  $\sigma$ -small modules  $B_i$  each of which has the exchange property and  $A = C \oplus D$ , then  $C$  is a direct sum of  $\sigma$ -small modules with the exchange property.

In order to prove the claim note that the direct summand  $C$  of  $A$  is  $\sigma$ -small, hence there is an ascending chain  $0 = S_0 \subseteq S_1 \subseteq S_2 \subseteq \dots$  of small submodules of  $C$  whose union is  $C$  itself. We shall construct submodules  $C_k, P_k$  of  $C$  for each  $k \geq 0$  with the property that (1)  $C = C_0 \oplus C_1 \oplus \dots \oplus C_k \oplus P_k$ , (2)  $S_k \subseteq C_0 \oplus C_1 \oplus \dots \oplus C_k$  and (3)  $C_k$  has the exchange property for every  $k \geq 0$ . Set  $C_0 = 0$  and  $P_0 = C$ . Suppose  $C_0, \dots, C_{k-1}, P_0, \dots, P_{k-1}$  with the required properties have been constructed. Then  $C_0 \oplus C_1 \oplus \dots \oplus C_{k-1}$  has the exchange property. Hence there exist direct summands  $B'_i$  of  $B_i$  such that  $A = C_0 \oplus C_1 \oplus \dots \oplus C_{k-1} \oplus (\bigoplus_{i=1}^{\infty} B'_i)$ . Since  $S_k$  is small, there exists a positive integer  $n(k)$  such that  $S_k \subseteq T_{n(k)}$ , where

$$T_{n(k)} = C_0 \oplus C_1 \oplus \dots \oplus C_{k-1} \oplus \left( \bigoplus_{i=1}^{n(k)} B'_i \right).$$

The module  $T_{n(k)}$  has the exchange property, so that from

$$A = C \oplus D = T_{n(k)} \oplus \left( \bigoplus_{i=n(k)+1}^{\infty} B'_i \right),$$

we have that there exist  $P_k \subseteq C$  and a direct sum decomposition  $D_k \oplus D'_k = D$  such that  $A = T_{n(k)} \oplus P_k \oplus D_k$ . Set  $C'_k = C \cap (T_{n(k)} \oplus D_k)$ , so that  $C = C'_k \oplus P_k$  by Lemma 2.1 and  $S_k \subseteq C'_k$ . Set  $C_k = C'_k \cap P_{k-1}$ . Then

$$C_0 \oplus C_1 \oplus \dots \oplus C_{k-1} \subseteq C'_k \subseteq C = C_0 \oplus C_1 \oplus \dots \oplus C_{k-1} \oplus P_{k-1}$$

forces  $C'_k = C_0 \oplus C_1 \oplus \dots \oplus C_{k-1} \oplus C_k$  (Lemma 2.1). Hence

$$C = C_0 \oplus C_1 \oplus \dots \oplus C_k \oplus P_k$$

and  $S_k \subseteq C_0 \oplus C_1 \oplus \dots \oplus C_k$ . Finally,

$$\begin{aligned} C_0 \oplus C_1 \oplus \dots \oplus C_k \oplus P_k \oplus D_k \oplus D'_k &= C \oplus D = A \\ &= T_{n(k)} \oplus P_k \oplus D_k = C_0 \oplus C_1 \oplus \dots \oplus C_{k-1} \oplus \left( \bigoplus_{i=1}^{n(k)} B'_i \right) \oplus P_k \oplus D_k \end{aligned}$$

implies that  $C_k \oplus D'_k \cong \bigoplus_{i=1}^{n(k)} B'_i$ , so that  $C_k$  has the exchange property because it is isomorphic to a direct summand of  $\bigoplus_{i=1}^{n(k)} B'_i$ . This completes the construction by induction.

It is now obvious that  $C = \bigoplus_{k=1}^{\infty} C_k$ . Since  $B_i$  is  $\sigma$ -small,  $A$  itself is  $\sigma$ -small, so that each  $C_k$  is  $\sigma$ -small. This proves the claim.

In order to prove the theorem, suppose

$$M = \bigoplus_{i \in I} B_i \tag{2.24}$$

where, for each  $i \in I$ ,  $B_i$  is  $\sigma$ -small and has the exchange property. Since every direct summand of  $B_i$  is  $\sigma$ -small and has the exchange property, it is enough

to show that the decomposition (2.24) and any other decomposition

$$M = \oplus_{j \in J} M_j \quad (2.25)$$

have isomorphic refinements. By Theorem 2.47 the decomposition (2.25) has a refinement

$$M = \oplus_{k \in K} C_k \quad (2.26)$$

in which every  $C_k$  is  $\sigma$ -small. If we apply Proposition 2.50 to the decompositions (2.24) and (2.26) we see that we may assume  $I$  and  $K$  countable. By the claim the decomposition (2.26) has a refinement that is a direct sum of  $\sigma$ -small modules with the exchange property. Now Theorem 2.51 allows us to conclude.  $\square$

From Theorem 2.52 we immediately obtain the following three corollaries:

**Corollary 2.53** *If a module  $M$  is a direct sum of countably generated modules  $M_i$ ,  $i \in I$ , each of which has the exchange property and  $N$  is a direct summand of  $M$ , then  $N = \oplus_{i \in I} N_i$ , where each  $N_i$  is isomorphic to a direct summand of  $M_i$ .*  $\square$

**Corollary 2.54** *If a module  $M$  is a direct sum of uniserial modules each of which has a local endomorphism ring, then any two direct sum decompositions of  $M$  have isomorphic refinements.*  $\square$

**Corollary 2.55** *If  $M = \oplus_{i \in I} M_i$ , where each  $M_i$  is a countably generated module with a local endomorphism ring, then any other direct sum decomposition of  $M$  can be refined to a decomposition isomorphic to the decomposition  $M = \oplus_{i \in I} M_i$ . In particular, any direct summand of  $M$  is isomorphic to  $\oplus_{i \in J} M_i$  for a subset  $J$  of  $I$ .*  $\square$

Corollary 2.55 is clearly a strengthened form of the Krull-Schmidt-Remak-Azumaya Theorem for direct sums of countably generated modules. It is apparently still an open question whether the hypothesis of being countably generated in Corollary 2.55 can be removed, that is, whether every direct summand of a direct sum of modules with local endomorphism rings is a direct sum of modules with local endomorphism rings. See [Elliger].

A ring  $R$  is said to be an *exchange ring* [Warfield 72] if  $R_R$  has the exchange property. For a ring  $R$  the right  $R$ -module  $R_R$  has the exchange property if and only if the left module  ${}_R R$  has the exchange property [Warfield 72, Corollary 2]. We shall not need this fact, and its proof will be omitted.

**Theorem 2.56** *If  $R$  is an exchange ring, then any projective right  $R$ -module is a direct sum of right ideals generated by idempotents.*

*Proof.* A projective  $R$ -module  $N_R$  is isomorphic to a direct summand of a free module  $M_R$ . Now apply Corollary 2.53 to  $M_R$  and  $N_R$ .  $\square$

Every local ring is an exchange ring by Theorem 2.8. Hence from Theorem 2.56 we have that

**Corollary 2.57** *Any projective right module over a local ring is free.*  $\square$

## 2.11 The Loewy series

In this section, we introduce Loewy modules, which form a class containing all artinian modules. Let  $M$  be a module over an arbitrary ring  $R$ . Inductively define a well-ordered sequence of fully invariant submodules  $\text{soc}_\alpha(M)$  of  $M$  as follows:

$$\begin{aligned}\text{soc}_0(M) &= 0, \\ \text{soc}_{\alpha+1}(M)/\text{soc}_\alpha(M) &= \text{soc}(M/\text{soc}_\alpha(M)) \text{ for every ordinal } \alpha, \\ \text{soc}_\beta(M) &= \bigcup_{\alpha < \beta} \text{soc}_\alpha(M) \text{ for every limit ordinal } \beta.\end{aligned}$$

The chain

$$\text{soc}_0(M) \subseteq \text{soc}_1(M) \subseteq \text{soc}_2(M) \subseteq \cdots \subseteq \text{soc}_\alpha(M) \subseteq \cdots$$

is called the (*ascending*) *Loewy series* of  $M$ . The module  $M$  is a *Loewy module* if there is an ordinal  $\alpha$  such that  $M = \text{soc}_\alpha(M)$ , and in this case the least ordinal  $\alpha$  such that  $M = \text{soc}_\alpha(M)$  is called the *Loewy length* of  $M$ . Note that the Loewy series is always stationary, that is, for every module  $M$  there exists an ordinal  $\alpha$  such that  $\text{soc}_\beta(M) = \text{soc}_\alpha(M)$  for every  $\beta \geq \alpha$  (for instance, it is sufficient to take any ordinal  $\alpha$  whose cardinality is greater than the cardinality of  $M$ ). For such an ordinal  $\alpha$ , set  $\delta(M) = \text{soc}_\alpha(M)$ . Then  $\delta(M)$  is the largest Loewy submodule of  $M$ , and  $M/\delta(M)$  has zero socle.

**Lemma 2.58** *A module  $M$  is a Loewy module if and only if every non-zero homomorphic image of  $M$  has a non-zero socle.*

*Proof.* If  $M$  is a Loewy module and  $N$  is a proper submodule of  $M$ , consider the set of all the ordinal numbers  $\alpha$  such that  $\text{soc}_\alpha(M) \subseteq N$ . It is easily seen that this set has a greatest element  $\beta$ . Then  $M/N$  is a homomorphic image of  $M/\text{soc}_\beta(M)$ , and the image of the socle  $\text{soc}_{\beta+1}(M)/\text{soc}_\beta(M)$  of  $M/\text{soc}_\beta(M)$  in  $M/N$  is non-zero. Therefore the socle of  $M/N$  is non-zero.

Conversely, if  $M$  is not a Loewy module, then  $M/\delta(M)$  is a non-zero homomorphic image of  $M$  with zero socle.  $\square$

In particular, since every non-zero artinian module has a non-zero socle, every artinian module is Loewy. Every Loewy module is an essential extension of its socle.

Let  $\mathcal{T}$  be the class of all Loewy right  $R$ -modules and let  $\mathcal{F}$  be the class of all right  $R$ -modules with zero socle. Then  $(\mathcal{T}, \mathcal{F})$  is a *torsion theory*, that is,

- (a)  $\text{Hom}(T, F) = 0$  for all  $T \in \mathcal{T}$ ,  $F \in \mathcal{F}$ .
- (b) If  $M$  is a right  $R$ -module and  $\text{Hom}(M, F) = 0$  for all  $F \in \mathcal{F}$ , then  $M \in \mathcal{T}$ .
- (c) If  $M$  is a right  $R$ -module and  $\text{Hom}(T, M) = 0$  for all  $T \in \mathcal{T}$ , then  $M \in \mathcal{F}$ .

The case of a right noetherian ring  $R$  is particularly interesting. If  $M$  is a Loewy right module over a right noetherian ring  $R$  and  $x \in M$ , then  $xR$  is a noetherian Loewy module, so that the ascending chain  $\text{soc}_n(xR)$ ,  $n \geq 0$ , must be stationary and  $xR = \text{soc}_m(xR)$  for some  $m$ . Since the modules  $\text{soc}_{n+1}(xR)/\text{soc}_n(xR)$  are semisimple noetherian modules, it follows that  $xR$  is an  $R$ -module of finite composition length. Therefore:

**Proposition 2.59** *If  $M$  is a Loewy right module over a right noetherian ring, then  $M$  is the sum of its submodules of finite composition length. In particular,  $M$  has Loewy length  $\leq \omega$ .  $\square$*

If  $x$  is an element in a right module  $M_R$ , the *annihilator*

$$\text{ann}_R(x) = \{a \in R \mid xa = 0\}$$

of  $x$  is always a right ideal of  $R$ . In particular, if  $b \in R$ , its *right annihilator*  $\text{r.ann}_R(b) = \{a \in R \mid ba = 0\}$  is a right ideal of  $R$ . As a corollary of Proposition 2.59 we obtain

**Corollary 2.60** *Let  $R$  be a right noetherian ring and let  $\mathcal{G} = \{I \mid I \text{ is a right ideal of } R \text{ and } R/I \text{ is a right } R\text{-module of finite length}\}$ . Then*

$$\delta(M_R) = \{x \in M \mid \text{ann}_R(x) \in \mathcal{G}\}$$

*for every  $R$ -module  $M_R$ .*

*Proof.* Let  $x$  be an element of  $M$ . Then  $\text{ann}_R(x) \in \mathcal{G}$  if and only if  $xR$  is a right  $R$ -module of finite length, that is, if and only if  $x \in \text{soc}_n(M_R)$  for some positive integer  $n$ , i.e., if and only if  $x \in \text{soc}_\omega(M_R) = \delta(M_R)$ .  $\square$

Proposition 2.59 can be adapted to commutative rings, as the next lemma shows.

**Lemma 2.61** *The Loewy length of an artinian module over a commutative ring is  $\leq \omega$ .*

*Proof.* If  $\text{soc}_0(M) \subseteq \text{soc}_1(M) \subseteq \text{soc}_2(M) \subseteq \cdots \subseteq \text{soc}_\alpha(M) \subseteq \cdots$  is the Loewy series of an artinian module  $M$ , then  $\bigcup_{n \in \mathbf{N}} \text{soc}_n(M) = M$ , because if  $x \in M$ , then  $xR$  is an artinian module. Hence  $xR$  is a module of finite composition length, so that  $xR \subseteq \text{soc}_n(M)$  for some  $n \in \mathbf{N}$ . Therefore  $M = \text{soc}_\omega(M)$  has Loewy length  $\leq \omega$ .  $\square$

Uniserial artinian modules of arbitrary Loewy length can be constructed over suitable non-commutative rings [Fuchs 70b, Facchini 84].

## 2.12 Artinian right modules over commutative or right noetherian rings

In this section we prove that the Krull-Schmidt Theorem holds for artinian right modules over rings which are either right noetherian or commutative. In Chapter 8 we shall see that it can fail for artinian modules over arbitrary non-commutative rings. Note that artinian modules are always finite direct sums of artinian indecomposable modules.

**Lemma 2.62** *Let  $M_R$  be a module over an arbitrary ring  $R$  and let*

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$$

*be an ascending chain of fully invariant submodules of  $M_R$ . Suppose that each  $M_i$  has finite composition length and  $M = \bigcup_{i \geq 0} M_i$ . Then*

- (a) *If  $f \in \text{End}(M_R)$ , then  $M = M' \oplus M''$ , where  $M' = \bigcup_{n \geq 0} \ker(f^n)$  and  $M'' = \bigcup_{i \geq 0} \left( \bigcap_{n \geq 0} f^n(M_i) \right)$ . Moreover,  $f$  restricts to an automorphism of  $M''$ .*
- (b) *If  $M_R$  is indecomposable, then  $\text{End}(M_R)$  is a local ring.*

*Proof.* (a) For every  $i \geq 0$  there is a positive integer  $n_i$  such that for every  $j \geq n_i$

$$f^j(M_i) = f^{n_i}(M_i) \quad \text{and} \quad \ker(f^j) \cap M_i = \ker(f^{n_i}) \cap M_i.$$

By Lemmas 2.16(a) and 2.17(a)

$$M_i = (\ker(f^j) \cap M_i) \oplus f^j(M_i),$$

so that

$$M_i = \left( \bigcup_{n \geq 0} \ker(f^n) \cap M_i \right) \oplus \left( \bigcap_{n \geq 0} f^n(M_i) \right).$$

Further,  $f$  restricted to

$$M_i'' = \bigcap_{n \geq 0} f^n(M_i)$$

is a monomorphism, hence an automorphism of  $M_i''$ . Now (a) follows easily.

(b) If  $M_R$  is indecomposable, either  $M_R = M'$  or  $M_R = M''$ . Hence for every  $f \in \text{End}(M_R)$ , either  $M_R = \bigcup_{n \geq 0} \ker(f^n)$  or  $f$  is an automorphism. This shows that if  $f$  is a non-invertible element of  $\text{End}(M_R)$ , the restriction of  $f$  to any  $M_i$  is nilpotent. Now argue as in the second paragraph of the proof of Lemma 2.21 to show that the sum of two non-invertible elements of  $\text{End}(M_R)$  is non-invertible. Thus  $\text{End}(M_R)$  is local.  $\square$

In the next proposition we prove that if the base ring  $R$  is either right noetherian or commutative, then all direct sum decompositions of a module that is a direct sum of artinian modules have an isomorphic common refinement. Hence the Krull-Schmidt Theorem holds for artinian modules over such rings.

**Proposition 2.63** *Let  $R$  be a ring which is either right noetherian or commutative and let  $M = \oplus_{i \in I} M_i$  be a right  $R$ -module which is the direct sum of indecomposable artinian modules  $M_i$ . Then any direct sum decomposition of  $M$  refines into a decomposition isomorphic to the decomposition  $M = \oplus_{i \in I} M_i$  and any direct summand  $N$  of  $M$  is isomorphic to  $\oplus_{i \in J} M_i$  for a subset  $J \subseteq I$ .*

*Proof.* Let  $A$  be an artinian right module over a ring  $R$  that is either right noetherian or commutative. By Proposition 2.59 and Lemma 2.61 the module  $A$  has Loewy length  $\leq \omega$ , so that  $A = \text{soc}_\omega(A) = \bigcup_{n \in \mathbf{N}} \text{soc}_n(A)$ . Every  $\text{soc}_n(A)$  is an artinian module of Loewy length  $\leq n$ . Since  $\text{soc}_{n+1}(A)/\text{soc}_n(A)$  is a semisimple artinian module, every  $\text{soc}_n(A)$  is a module of finite composition length. By Lemma 2.62 every indecomposable artinian module  $A$  has a local endomorphism ring and is countably generated. Now apply Corollary 2.55.  $\square$

## 2.13 Notes on Chapter 2

The exchange property was introduced by [Crawley and Jónsson]. Actually, Crawley and Jónsson's results were proved for a wide class of algebraic structures, namely for algebras in the sense of Jónsson-Tarski. Injective modules [Warfield 69c], quasi-injective modules [Fuchs 69], pure-injective modules [Zimmermann-Huisgen and Zimmermann 84], continuous modules (Mohamed and Müller, 1989), projective modules over perfect rings (Yamagata, 1974, and Harada-Ishii, 1975), and projective modules over Von Neumann regular rings [Stock] have the exchange property. It is not known whether the exchange property and the finite exchange property are equivalent for arbitrary modules. By Theorem 2.8 they are equivalent for indecomposable modules.

A module  $M$  is *continuous* if the following two conditions hold:

- (C1) every submodule of  $M$  is essential in a direct summand of  $M$ ;
- (C2) if a submodule  $N$  of  $M$  is isomorphic to a direct summand of  $M$ , then  $N$  is a direct summand of  $M$ .

A module  $M$  is *quasi-continuous* if (C1) holds, and moreover

- (C3) if  $N_1$  and  $N_2$  are direct summands of  $M$  such that  $N_1 \cap N_2 = 0$ , then  $N_1 \oplus N_2$  is a direct summand of  $M$ .

Making use of ideas of [Oshiro and Rizvi], [Mohamed and Müller] have recently proved that the exchange property and the finite exchange property are equivalent for quasi-continuous modules. Note that there exist indecomposable quasi-continuous modules without the finite exchange property, for instance the abelian group  $\mathbf{Z}$ .

The proofs of Lemma 2.2, Corollary 2.3, Lemmas 2.4 and 2.5 and Theorems 2.9 and 2.10 are taken from [Crawley and Jónsson]. In the proof of Theorem 2.8 the implication (b)  $\Rightarrow$  (c) is taken from [Crawley and Jónsson] and the remaining implications are due to [Warfield 69a, Proposition 1].

The history of the Krull-Schmidt-Remak-Azumaya Theorem begins with two papers of [Krull 25] and [Schmidt]. The present form of the theorem appeared for the first time in [Azumaya 50]. In that paper Azumaya proved the uniqueness of decomposition for infinite direct sums of modules with local endomorphism rings. In this book, the general result (Theorem 2.12) is referred to as the “Krull-Schmidt-Remak-Azumaya Theorem”, whereas the “Krull-Schmidt Theorem” is the “classical” Krull-Schmidt Theorem, that is, the result concerning modules of finite length (Corollary 2.23). Krull himself used to term “Isomorphiesatz der direkte Zerlegung” (Isomorphism theorem of direct decomposition) for what we call the Krull-Schmidt Theorem. In [Krull 32] (last paragraph of the paper), Krull asked whether the “Isomorphiesatz der direkte Zerlegung” is independent of the descending chain condition, i.e., whether the Krull-Schmidt Theorem holds for artinian modules (cf. [Levy, p. 660]). The answer to this question appeared in [Facchini, Herbera, Levy and Vámos] and is the main topic of Section 8.2. The proofs of Lemma 2.11 and the Krull-Schmidt-Remak-Azumaya Theorem we have given here are taken from [Crawley and Jónsson].

Lemmas 2.20 and 2.21 are due to [Fitting, Satz 8], and Corollary 2.27 is due to [Zimmermann and Zimmermann-Huisgen 78, Theorem 9]. The proof of Proposition 2.28 is based on an argument of [Eisenbud and Griffith, Proof of Proposition 1.1].

The Goldie dimension for modules and rings was introduced by [Goldie 60], who called it “dimension”. The Goldie dimension of a module is also called the *uniform dimension*, or the *uniform rank*, or simply the *rank* of the module. Concepts such as having finite Goldie dimension or uniform submodules and



their basic properties go back to [Goldie 58, 60]. Goldie dimension for arbitrary modular lattices was introduced by [Grzeszczuk and Puczyłowski]. For the proof of Lemma 2.30 and Proposition 2.31 we have followed [Năstăsescu and Van Oystaeyen]. The rest of the material in Section 2.6 is taken from [Grzeszczuk and Puczyłowski].

The notion of dual Goldie dimension is due to [Varadarajan], who used the term *corank* for what we call dual Goldie dimension of a module. There are a number of different ways that one could attempt to dualize the notion of Goldie dimension; for instance, [Fleury] considers the spanning dimension of a module, a possible different dualization of the Goldie dimension. The *spanning dimension* of a module  $M$  is defined as the least integer  $k$  such that  $M$  is a sum  $N_1 + \cdots + N_k$  (not necessarily direct) of  $k$  couniform submodules  $N_i$  of  $M$ . In our presentation of dual Goldie dimension we have followed [Grzeszczuk and Puczyłowski].

Proposition 2.45 is essentially taken from [Fuchs and Salce, Lemma 24].

Theorem 2.51, Corollary 2.55, the definition of exchange ring and Theorem 2.56 are due to [Warfield 69c, 69a, 72]. He also proved that a right module  $M_R$  has the finite exchange property if and only if its endomorphism ring  $\text{End}(M_R)$  is an exchange ring [Warfield 72, Theorem 2]. From Lemma 2.4 it follows immediately that if  $e$  is an idempotent in a ring  $R$ , then  $R$  is an exchange ring if and only if  $eRe$  and  $(1-e)R(1-e)$  are exchange rings. There are further characterizations of exchange rings. For instance, [Monk] proved that a ring  $R$  is an exchange ring if and only if for every  $a \in R$  there exist  $b, c \in R$  such that  $bab = b$  and  $c(1-a)(1-ba) = 1-ba$ . Goodearl ([Goodearl and Warfield 76, p. 167]) and [Nicholson] independently proved that a ring  $R$  is an exchange ring if and only if for every  $x \in R$  there exists an idempotent  $e \in R$  such that  $e \in xR$  and  $1-e \in (1-x)R$ . This characterization has allowed the notion of exchange ring to be extended to rings without unit [Ara 97]. [Nicholson] also proved that  $R$  is an exchange ring if and only if  $R/J(R)$  is an exchange ring and idempotents lift modulo  $J(R)$ .

Corollary 2.57 is a famous result of [Kaplansky 58, Theorem 2].

Loewy started using Loewy series in 1905 in the study of representations of matrix groups. Later, in 1926, Krull defined the term “Loewy series” and in [Krull 28] he observed that transfinite Loewy series could be defined. The results in Section 2.12 (i.e., that the Krull-Schmidt Theorem holds for artinian modules over rings which are either right noetherian or commutative) are due to [Warfield 69a]. The most important case of artinian module over a commutative noetherian ring was discovered by [Matlis]. He proved that if  $R$  is a noetherian commutative ring,  $S$  is a simple  $R$ -module and  $E(S)$  is the injective envelope of  $S$ , then  $E(S)$  is an artinian  $R$ -module whose endomorphism ring is a local noetherian complete commutative ring [Matlis, Theorems 3.7 and 4.2]. Conversely, if  $M_R$  is an artinian module with simple socle over a commutative ring  $R$ , then  $E = \text{End}(M_R)$  is a local noetherian complete commutative ring and  ${}_E M$  is the injective envelope of the unique simple  $E$ -module [Facchini 81,

Theorem 2.8]. The following results hold for Loewy modules over commutative rings. Let  $M$  be a Loewy module over a commutative ring  $R$ . For each ordinal  $\alpha$  the  $\alpha$ -th Loewy factor of  $M$  is the semisimple module  $\text{soc}_{\alpha+1}(M)/\text{soc}_\alpha(M)$ , and the composition length of  $\text{soc}_{\alpha+1}(M)/\text{soc}_\alpha(M)$  is the  $\alpha$ -th Loewy invariant of  $M$ , denoted by  $d_\alpha(M)$ . The *support* of  $M$  is the set of all maximal ideals  $P$  of  $R$  such that  $(0 :_M P) \neq 0$ . If  $R$  is a commutative ring and  $M$  is a Loewy  $R$ -module with finite support  $\{P_1, P_2, \dots, P_n\}$ , then  $M = \bigoplus_{i=1}^n M_i$ , where for each  $i = 1, 2, \dots, n$ ,  $M_i$  is a Loewy module whose Loewy factors are all direct sums of copies of  $R/P_i$ . If  $M$  is a Loewy module over a commutative ring,  $\alpha$  is an ordinal and  $r$  is a positive integer such that both  $d_\alpha(M)$  and  $d_{\alpha+r}(M)$  are finite, then  $d_\beta(M)$  is finite for every  $\beta > \alpha + r$  and  $M = \text{soc}_{\alpha+\omega}(M)$  [Shores, Theorem 4.2]. From this result we again obtain that every artinian module over a commutative ring has Loewy length  $\leq \omega$ . A module  $M$  over a commutative ring  $R$  is artinian if and only if it is a Loewy module with finite Loewy invariants [Facchini 81, Theorem 2.7]. Let  $M$  be an artinian module over a commutative ring such that  $d_1(M) \leq n$ . Then  $d_r(M) \leq \binom{n+r-1}{r}$

for every  $r \geq 1$  ([Shores, Theorem 4.4] and [Facchini 81, Theorem 3.1]). Now let  $t$  be an indeterminate over the ring  $\mathbf{Z}$  of integers. If  $M$  is an artinian module over a commutative ring, define  $P(M, t) = \sum_{n=0}^{\infty} d_n(M)t^n \in \mathbf{Z}[[t]]$ . Then  $P(M, t)$  is a rational function in  $t$  of the form  $f(t)/(1-t)^s$ , where  $f(t) \in \mathbf{Z}[t]$  and  $s = d_0(M)d_1(M)$ . If  $d$  is the order of the pole of  $P(M, t)$  at  $t = 1$ , then, for all sufficiently large  $n$ ,  $d_n(M)$  and the composition length  $l(\text{soc}_n(M))$  of  $\text{soc}_n(M)$  are polynomials in  $n$  with rational coefficients of degree  $d - 1$  and  $d$  respectively [Facchini 81, Theorem 3.2].

A ring  $R$  is *right semiartinian* if  $R_R$  is a Loewy module. If  $R$  is right semiartinian, every right  $R$ -module is a Loewy module. Right semiartinian rings are exchange rings [Baccella].

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