Characteristics of a System of Equations of Two-Phase Flow of Gas-Particle Mixture

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■ Impressum

This Mathematica-Notebook is part of the book entitled

S.P. Kiselev, E.V. Vorozhtsov, and V.M. Fomin
Foundations of Fluid Mechanics with Applications
Problem Solving Using *Mathematica.*
Birkhauser Boston, Basel, 1999.

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■ General Description

The program prog7-1.nb, which follows below, enables one to investigate the zeroes of the characteristic equation (7.2.5) (see Section 7.2 of the above book). The program finds the analytic expression for all four zeroes of equation (7.2.5) and shows the region of real roots in the (e, M_{12}) plane, where $e = m_2 \, r_{11} / (r_{22} \, m_1)$.

A more detailed discussion of the properties of the zeroes of equation (7.2.5) may be found in Section 7.2 of the above mentioned book.

■ User's Guide

■ Step 1

Load and compile the program file beginning with the line

```
f[x_, e_, m_] := x^2 (1 - (m - x)^2) + e (m - x)^2
```

■ Step 2

Specify the input data by entering them in the line (see also Section "Examples of the Input Data" below)

```
gaspart[1, 1, 1, 1, 1, 31, 31]
```

Then click in this line and wait for the result of numerical computation. The meaning of the input parameters is explained in the Section "Parameters Used in Program prog7-1.nb".

■ Program Listing

```
ClearAll[f];
f[x_, e_, m_] := x^2 (1 - (m - x)^2) + e (m - x)^2

gaspart[nroot_, np_, npict_, epsmax_, M12_, neps_, nM12_] :=
( sol = Solve[f[x, e, m] == 0, x];
x1 = x /. sol[[nroot]];
l1 = v2 + c x1 /. {m -> M12, e -> m2 r11 / (r22 m1), p -> m1, q -> m2}
If[np == 1, Print["lambda = ", l1]];
  xs[e_, m_] = x1;
xx = {}; yy = {};
de = epsmax / (neps - 1); dm = M12 / (nM12 - 1);
Do[ej = N[10^-4 + (j - 1) de];
   Do[mk = N[10^-5 + (k - 1) dm]; x1 = xs[ej, mk];
 If[Abs[Im[x1]] < 10^-5, AppendTo[xx, ej]; AppendTo[yy, mk]],
                      {k, nM12}], {j, neps}];
nx = Length[xx];
Print["The number of real points = ", nx];
If[nx > 0, xy = MapThread[List, {xx, yy}];
gr1 = ListPlot[xy, PlotStyle -> {PointSize[0.02]},
                   AspectRatio -> Automatic,
                   PlotRange -> All,
                   AxesLabel -> {"e", "M12"},
                   DisplayFunction -> Identity];
ne1 = 101; de1 = epsmax / (ne1 - 1);
x2 = Table[N[(j - 1) de1], {j, ne1}];
y2 = Table[N[1 + 1.85 x2[[j]]^0.37], {j, ne1}];
xy = MapThread[List, {x2, y2}];
gr2 = ListPlot[xy, PlotJoined -> True,
                   AspectRatio -> Automatic,
                   DisplayFunction -> Identity];
Show[gr1, gr2, DisplayFunction -> $DisplayFunction]] );
```

■ Parameters Used in Program prog7-1.nb

Parameter	Description
nroot	the number of the root of equation (6.2.5); nroot is a positive integer, 1 <= nroot <= 4
np	if np = 1, then the analytical expression for the root l_{nroot} is printed
npict	if npict = 1, then the picture of the region of real roots l_j, j= 1,2,3,4, is plotted
epsmax	the size of the region in the (e, M_{12}) plane along the e-axis, where e = m_2 r_{11} /(r_{22} m_1); epsmax > 0
M12	the size of the region in the (e, M_{12}) plane along the M_{12} -axis, where e = m_2 r_{11} /(r_{22} m_1); M12 > 0
neps	the number of nodes of an uniform grid along the e axis; neps is a positive integer
nM12	the number of nodes of an uniform grid along the M_{12} axis; nM12 is a positive integer

■ Examples of the Input Data

■ Example 1

Root No. 1

gaspart[1, 1, 1, 1, 3, 21, 31]

lambda =

$$v_2 + c\left(\frac{M_{12}}{2} - \frac{1}{2}\sqrt{\left(1 + M_{12}^2 + \frac{1}{3}\left(1 - M_{12}^2 + \frac{m_2 r_{11}}{m_1 r_{22}}\right) - \left(8M_{12}^3 - 8M_{12}\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right) - \frac{16 m_2 M_{12} r_{11}}{m_1 r_{22}}\right)\Big/\right.\right.$$
$$\left(4\sqrt{\left(1 + \frac{1}{3}\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right) + \left(2^{1/3}\left(1 - M_{12}^2 + \frac{m_2 r_{11}}{m_1 r_{22}}\right)^2\right)\Big/\right.\right.$$
$$\left(3\left(2\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right)^3 + 12\sqrt{3}\, M_{12}\sqrt{\left(1 - 3M_{12}^2 + 3M_{12}^4 - \right.\right.\right.\right.$$
$$M_{12}^6 + \frac{m_2^3 r_{11}^3}{m_1^3 r_{22}^3} + \frac{3 m_2^2 r_{11}^2}{m_1^2 r_{22}^2} - \frac{3 m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2} + \frac{3 m_2 r_{11}}{m_1 r_{22}} +$$
$$\left.\left.\left.\left.\frac{21 m_2 M_{12}^2 r_{11}}{m_1 r_{22}} + \frac{3 m_2 M_{12}^4 r_{11}}{m_1 r_{22}}\right)\sqrt{\frac{m_2 r_{11}}{m_1 r_{22}}} + \frac{108 m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2} - \right.\right.\right.$$
$$\left.\left.\left.\frac{108 m_2 M_{12}^4 r_{11}}{m_1 r_{22}} + \frac{108 m_2 M_{12}^2 r_{11}\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right)}{m_1 r_{22}}\right)\text{^}(1/3)\right) +$$

$$\frac{1}{3\,2^{1/3}}\left(\left(2\left(-1+M_{12}^2-\frac{m_2 r_{11}}{m_1 r_{22}}\right)^3+12\sqrt{3}\,M_{12}\sqrt{\left(1-3\right.\right.\right.$$

$$M_{12}^2+3M_{12}^4-M_{12}^6+\frac{m_2^3 r_{11}^3}{m_1^3 r_{22}^3}+\frac{3m_2^2 r_{11}^2}{m_1^2 r_{22}^2}-\frac{3m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2}+$$

$$\left.\left.\left.\frac{3m_2 r_{11}}{m_1 r_{22}}+\frac{21m_2 M_{12}^2 r_{11}}{m_1 r_{22}}+\frac{3m_2 M_{12}^4 r_{11}}{m_1 r_{22}}\right)\sqrt{\frac{m_2 r_{11}}{m_1 r_{22}}}+\frac{108m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2}-\right.\right.$$

$$\left.\left.\frac{108m_2 M_{12}^4 r_{11}}{m_1 r_{22}}+\frac{108m_2 M_{12}^2 r_{11}\left(-1+M_{12}^2-\frac{m_2 r_{11}}{m_1 r_{22}}\right)}{m_1 r_{22}}\right)\wedge(1/3)\right)+\frac{m_2 r_{11}}{m_1 r_{22}}\right)\Bigg)-$$

$$\left(2^{1/3}\left(1-M_{12}^2+\frac{m_2 r_{11}}{m_1 r_{22}}\right)^2\right)\Big/\left(3\left(2\left(-1+M_{12}^2-\frac{m_2 r_{11}}{m_1 r_{22}}\right)^3+12\sqrt{3}\,M_{12}\sqrt{\left(1-3M_{12}^2+3M_{12}^4-M_{12}^6+\frac{m_2^3 r_{11}^3}{m_1^3 r_{22}^3}+\right.\right.\right.$$

$$\left.\left.\left.\frac{3m_2^2 r_{11}^2}{m_1^2 r_{22}^2}-\frac{3m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2}+\frac{3m_2 r_{11}}{m_1 r_{22}}+\frac{21m_2 M_{12}^2 r_{11}}{m_1 r_{22}}+\frac{3m_2 M_{12}^4 r_{11}}{m_1 r_{22}}\right)\sqrt{\frac{m_2 r_{11}}{m_1 r_{22}}}+\right.\right.$$

$$\left.\left.\frac{108m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2}-\frac{108m_2 M_{12}^4 r_{11}}{m_1 r_{22}}+\frac{108m_2 M_{12}^2 r_{11}\left(-1+M_{12}^2-\frac{m_2 r_{11}}{m_1 r_{22}}\right)}{m_1 r_{22}}\right)\wedge(1/3)\right)-$$

$$\frac{1}{3\,2^{1/3}}\left(\left(2\left(-1+M_{12}^2-\frac{m_2 r_{11}}{m_1 r_{22}}\right)^3+12\sqrt{3}\,M_{12}\sqrt{\left(1-3M_{12}^2+3M_{12}^4-M_{12}^6+\frac{m_2^3 r_{11}^3}{m_1^3 r_{22}^3}+\frac{3m_2^2 r_{11}^2}{m_1^2 r_{22}^2}-\right.\right.\right.$$

$$\left.\left.\left.\frac{3m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2}+\frac{3m_2 r_{11}}{m_1 r_{22}}+\frac{21m_2 M_{12}^2 r_{11}}{m_1 r_{22}}+\frac{3m_2 M_{12}^4 r_{11}}{m_1 r_{22}}\right)\sqrt{\frac{m_2 r_{11}}{m_1 r_{22}}}+\frac{108m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2}-\right.\right.$$

$$\left.\left.\frac{108m_2 M_{12}^4 r_{11}}{m_1 r_{22}}+\frac{108m_2 M_{12}^2 r_{11}\left(-1+M_{12}^2-\frac{m_2 r_{11}}{m_1 r_{22}}\right)}{m_1 r_{22}}\right)\wedge(1/3)\right)+\frac{m_2 r_{11}}{m_1 r_{22}}\right)-$$

$$\frac{1}{2}\sqrt{\left(1+\frac{1}{3}\left(-1+M_{12}^2-\frac{m_2 r_{11}}{m_1 r_{22}}\right)+\right.}$$

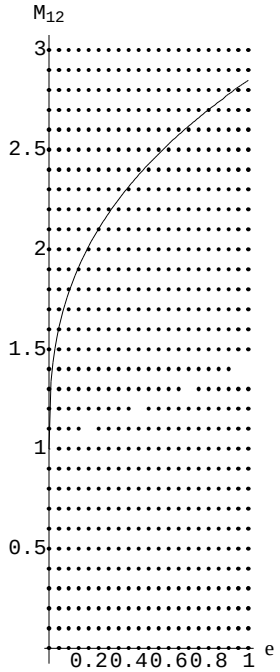
$$\left(2^{1/3}\left(1-M_{12}^2+\frac{m_2 r_{11}}{m_1 r_{22}}\right)^2\right)\Big/\left(3\left(2\left(-1+M_{12}^2-\frac{m_2 r_{11}}{m_1 r_{22}}\right)^3+12\sqrt{3}\,M_{12}\sqrt{\left(1-3M_{12}^2+3M_{12}^4-M_{12}^6+\frac{m_2^3 r_{11}^3}{m_1^3 r_{22}^3}+\right.\right.\right.$$

$$\left.\left.\left.\frac{3m_2^2 r_{11}^2}{m_1^2 r_{22}^2}-\frac{3m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2}+\frac{3m_2 r_{11}}{m_1 r_{22}}+\frac{21m_2 M_{12}^2 r_{11}}{m_1 r_{22}}+\frac{3m_2 M_{12}^4 r_{11}}{m_1 r_{22}}\right)\sqrt{\frac{m_2 r_{11}}{m_1 r_{22}}}+\right.\right.$$

$$\left.\left.\frac{108m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2}-\frac{108m_2 M_{12}^4 r_{11}}{m_1 r_{22}}+\frac{108m_2 M_{12}^2 r_{11}\left(-1+M_{12}^2-\frac{m_2 r_{11}}{m_1 r_{22}}\right)}{m_1 r_{22}}\right)\wedge(1/3)\right)+$$

$$\left.\frac{m_2 r_{11}}{m_1 r_{22}}\right)\Bigg)$$

```
The number of real points = 645
```



Ö Graphics Ö

■ Example 2

Root No. 2

gaspart[2, 1, 1, 1, 3, 21, 31]

lambda =

$$v_2 + c\left(\frac{M_{12}}{2} + \frac{1}{2}\sqrt{\left(1 + M_{12}^2 + \frac{1}{3}\left(1 - M_{12}^2 + \frac{m_2 r_{11}}{m_1 r_{22}}\right) - \left(8M_{12}^3 - 8M_{12}\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right) - \frac{16 m_2 M_{12} r_{11}}{m_1 r_{22}}\right)\right/ \left(4\sqrt{\left(1 + \frac{1}{3}\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right) + \left(2^{1/3}\left(1 - M_{12}^2 + \frac{m_2 r_{11}}{m_1 r_{22}}\right)^2\right)\right/ \left(3\left(2\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right)^3 + 12\sqrt{3}\, M_{12}\sqrt{\left(1 - 3M_{12}^2 + 3M_{12}^4 - M_{12}^6 + \frac{m_2^3 r_{11}^3}{m_1^3 r_{22}^3} + \frac{3 m_2^2 r_{11}^2}{m_1^2 r_{22}^2} - \frac{3 m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2} + \frac{3 m_2 r_{11}}{m_1 r_{22}} + \frac{21 m_2 M_{12}^2 r_{11}}{m_1 r_{22}} + \frac{3 m_2 M_{12}^4 r_{11}}{m_1 r_{22}}\right)\sqrt{\frac{m_2 r_{11}}{m_1 r_{22}}} + \frac{108 m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2} - \frac{108 m_2 M_{12}^4 r_{11}}{m_1 r_{22}} + \frac{108 m_2 M_{12}^2 r_{11}\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right)}{m_1 r_{22}}\right)^{(1/3)}\right) +$$

$$\frac{1}{3\,2^{1/3}}\left(\left(2\left(-1+M_{12}^2-\frac{m_2 r_{11}}{m_1 r_{22}}\right)^3+12\sqrt{3}\,M_{12}\sqrt{\left(1-3\right.\right.\right.$$

$$M_{12}^2+3M_{12}^4-M_{12}^6+\frac{m_2^3 r_{11}^3}{m_1^3 r_{22}^3}+\frac{3m_2^2 r_{11}^2}{m_1^2 r_{22}^2}-\frac{3m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2}+$$

$$\left.\left.\frac{3m_2 r_{11}}{m_1 r_{22}}+\frac{21m_2 M_{12}^2 r_{11}}{m_1 r_{22}}+\frac{3m_2 M_{12}^4 r_{11}}{m_1 r_{22}}\right)\sqrt{\frac{m_2 r_{11}}{m_1 r_{22}}}+\frac{108m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2}-\right.$$

$$\left.\left.\frac{108m_2 M_{12}^4 r_{11}}{m_1 r_{22}}+\frac{108m_2 M_{12}^2 r_{11}\left(-1+M_{12}^2-\frac{m_2 r_{11}}{m_1 r_{22}}\right)}{m_1 r_{22}}\right)\wedge(1/3)\right)+\frac{m_2 r_{11}}{m_1 r_{22}}\right)\Bigg)-$$

$$\left(2^{1/3}\left(1-M_{12}^2+\frac{m_2 r_{11}}{m_1 r_{22}}\right)^2\right)\Big/\left(3\left(2\left(-1+M_{12}^2-\frac{m_2 r_{11}}{m_1 r_{22}}\right)^3+12\sqrt{3}\,M_{12}\sqrt{\left(1-3M_{12}^2+3M_{12}^4-M_{12}^6+\frac{m_2^3 r_{11}^3}{m_1^3 r_{22}^3}+\right.\right.\right.$$

$$\frac{3m_2^2 r_{11}^2}{m_1^2 r_{22}^2}-\frac{3m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2}+\frac{3m_2 r_{11}}{m_1 r_{22}}+\frac{21m_2 M_{12}^2 r_{11}}{m_1 r_{22}}+\frac{3m_2 M_{12}^4 r_{11}}{m_1 r_{22}}\Bigg)\sqrt{\frac{m_2 r_{11}}{m_1 r_{22}}}+$$

$$\left.\left.\frac{108m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2}-\frac{108m_2 M_{12}^4 r_{11}}{m_1 r_{22}}+\frac{108m_2 M_{12}^2 r_{11}\left(-1+M_{12}^2-\frac{m_2 r_{11}}{m_1 r_{22}}\right)}{m_1 r_{22}}\right)\wedge(1/3)\right)-$$

$$\frac{1}{3\,2^{1/3}}\left(\left(2\left(-1+M_{12}^2-\frac{m_2 r_{11}}{m_1 r_{22}}\right)^3+12\sqrt{3}\,M_{12}\sqrt{\left(1-3M_{12}^2+3M_{12}^4-M_{12}^6+\frac{m_2^3 r_{11}^3}{m_1^3 r_{22}^3}+\frac{3m_2^2 r_{11}^2}{m_1^2 r_{22}^2}-\right.\right.\right.$$

$$\frac{3m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2}+\frac{3m_2 r_{11}}{m_1 r_{22}}+\frac{21m_2 M_{12}^2 r_{11}}{m_1 r_{22}}+\frac{3m_2 M_{12}^4 r_{11}}{m_1 r_{22}}\Bigg)\sqrt{\frac{m_2 r_{11}}{m_1 r_{22}}}+\frac{108m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2}-$$

$$\left.\left.\frac{108m_2 M_{12}^4 r_{11}}{m_1 r_{22}}+\frac{108m_2 M_{12}^2 r_{11}\left(-1+M_{12}^2-\frac{m_2 r_{11}}{m_1 r_{22}}\right)}{m_1 r_{22}}\right)\wedge(1/3)\right)+\frac{m_2 r_{11}}{m_1 r_{22}}\right)-$$

$$\frac{1}{2}\sqrt{\left(1+\frac{1}{3}\left(-1+M_{12}^2-\frac{m_2 r_{11}}{m_1 r_{22}}\right)+\right.}$$

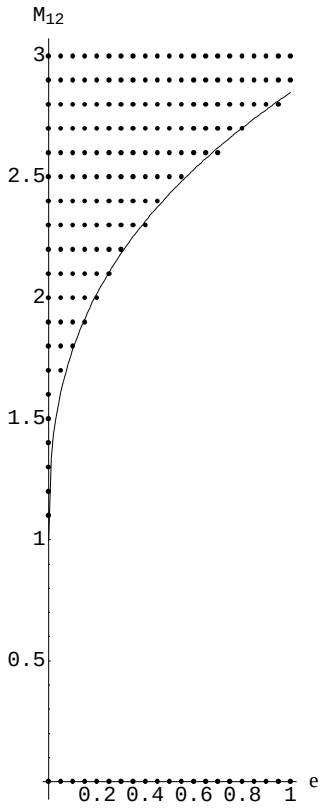
$$\left(2^{1/3}\left(1-M_{12}^2+\frac{m_2 r_{11}}{m_1 r_{22}}\right)^2\right)\Big/\left(3\left(2\left(-1+M_{12}^2-\frac{m_2 r_{11}}{m_1 r_{22}}\right)^3+12\sqrt{3}\,M_{12}\sqrt{\left(1-3M_{12}^2+3M_{12}^4-M_{12}^6+\frac{m_2^3 r_{11}^3}{m_1^3 r_{22}^3}+\right.\right.\right.$$

$$\frac{3m_2^2 r_{11}^2}{m_1^2 r_{22}^2}-\frac{3m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2}+\frac{3m_2 r_{11}}{m_1 r_{22}}+\frac{21m_2 M_{12}^2 r_{11}}{m_1 r_{22}}+\frac{3m_2 M_{12}^4 r_{11}}{m_1 r_{22}}\Bigg)\sqrt{\frac{m_2 r_{11}}{m_1 r_{22}}}+$$

$$\left.\left.\frac{108m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2}-\frac{108m_2 M_{12}^4 r_{11}}{m_1 r_{22}}+\frac{108m_2 M_{12}^2 r_{11}\left(-1+M_{12}^2-\frac{m_2 r_{11}}{m_1 r_{22}}\right)}{m_1 r_{22}}\right)\wedge(1/3)\right)+$$

$$\left.\frac{m_2 r_{11}}{m_1 r_{22}}\right)\Bigg)$$

```
The number of real points = 179
```



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■ Example 3

Root No. 3

gaspart[3, 1, 1, 1, 3, 21, 31]

lambda =

$$v_2 + c\left(\frac{M_{12}}{2} - \frac{1}{2}\sqrt{\left(1 + M_{12}^2 + \frac{1}{3}\left(1 - M_{12}^2 + \frac{m_2 r_{11}}{m_1 r_{22}}\right) + \left(8M_{12}^3 - 8M_{12}\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right) - \frac{16 m_2 M_{12} r_{11}}{m_1 r_{22}}\right)\Big/ \left(4\sqrt{\left(1 + \frac{1}{3}\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right) + \left(2^{1/3}\left(1 - M_{12}^2 + \frac{m_2 r_{11}}{m_1 r_{22}}\right)^2\right)\Big/ \left(3\left(2\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right)^3 + 12\sqrt{3}\, M_{12}\sqrt{\left(1 - 3M_{12}^2 + 3M_{12}^4 - M_{12}^6 + \frac{m_2^3 r_{11}^3}{m_1^3 r_{22}^3} + \frac{3 m_2^2 r_{11}^2}{m_1^2 r_{22}^2} - \frac{3 m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2} + \frac{3 m_2 r_{11}}{m_1 r_{22}} + \frac{21 m_2 M_{12}^2 r_{11}}{m_1 r_{22}} + \frac{3 m_2 M_{12}^4 r_{11}}{m_1 r_{22}}\right)\sqrt{\frac{m_2 r_{11}}{m_1 r_{22}}} + \frac{108 m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2} -\right.\right.\right.\right.\right.\right.\right.\right.$$

$$\frac{108\,m_2\,M_{12}^4\,r_{11}}{m_1\,r_{22}} + \frac{108 m_2\,M_{12}^2\,r_{11}\left(-1+M_{12}^2-\frac{m_2\,r_{11}}{m_1\,r_{22}}\right)}{m_1\,r_{22}}\Bigg)\,\wedge\,(1/3)\Bigg) +$$

$$\frac{1}{3\,2^{1/3}}\Bigg(\Bigg(2\left(-1+M_{12}^2-\frac{m_2\,r_{11}}{m_1\,r_{22}}\right)^3+12\sqrt{3}\,M_{12}\sqrt{\Bigg(1-3$$

$$M_{12}^2+3\,M_{12}^4-M_{12}^6+\frac{m_2^3\,r_{11}^3}{m_1^3\,r_{22}^3}+\frac{3\,m_2^2\,r_{11}^2}{m_1^2\,r_{22}^2}-\frac{3m_2^2\,M_{12}^2\,r_{11}^2}{m_1^2\,r_{22}^2}+$$

$$\frac{3\,m_2\,r_{11}}{m_1\,r_{22}}+\frac{21m_2\,M_{12}^2\,r_{11}}{m_1\,r_{22}}+\frac{3\,m_2\,M_{12}^4\,r_{11}}{m_1\,r_{22}}\Bigg)\sqrt{\frac{m_2\,r_{11}}{m_1\,r_{22}}}+\frac{108m_2^2\,M_{12}^2\,r_{11}^2}{m_1^2\,r_{22}^2}-$$

$$\frac{108\,m_2\,M_{12}^4\,r_{11}}{m_1\,r_{22}}+\frac{108m_2\,M_{12}^2\,r_{11}\left(-1+M_{12}^2-\frac{m_2\,r_{11}}{m_1\,r_{22}}\right)}{m_1\,r_{22}}\Bigg)\,\wedge\,(1/3)\Bigg)+\frac{m_2\,r_{11}}{m_1\,r_{22}}\Bigg)\Bigg)-$$

$$\left(2^{1/3}\left(1-M_{12}^2+\frac{m_2\,r_{11}}{m_1\,r_{22}}\right)^2\right)\Big/\Bigg(3\Bigg(2\left(-1+M_{12}^2-\frac{m_2\,r_{11}}{m_1\,r_{22}}\right)^3+12\sqrt{3}\,M_{12}\sqrt{\Bigg(1-3M_{12}^2+3M_{12}^4-M_{12}^6+\frac{m_2^3\,r_{11}^3}{m_1^3\,r_{22}^3}+$$

$$\frac{3\,m_2^2\,r_{11}^2}{m_1^2\,r_{22}^2}-\frac{3m_2^2\,M_{12}^2\,r_{11}^2}{m_1^2\,r_{22}^2}+\frac{3\,m_2\,r_{11}}{m_1\,r_{22}}+\frac{21m_2\,M_{12}^2\,r_{11}}{m_1\,r_{22}}+\frac{3\,m_2\,M_{12}^4\,r_{11}}{m_1\,r_{22}}\Bigg)\sqrt{\frac{m_2\,r_{11}}{m_1\,r_{22}}}+$$

$$\frac{108\,m_2^2\,M_{12}^2\,r_{11}^2}{m_1^2\,r_{22}^2}-\frac{108m_2\,M_{12}^4\,r_{11}}{m_1\,r_{22}}+\frac{108\,m_2\,M_{12}^2\,r_{11}\left(-1+M_{12}^2-\frac{m_2\,r_{11}}{m_1\,r_{22}}\right)}{m_1\,r_{22}}\Bigg)\,\wedge\,(1/3)\Bigg)-$$

$$\frac{1}{3\,2^{1/3}}\Bigg(\Bigg(2\left(-1+M_{12}^2-\frac{m_2\,r_{11}}{m_1\,r_{22}}\right)^3+12\sqrt{3}\,M_{12}\sqrt{\Bigg(1-3\,M_{12}^2+3\,M_{12}^4-M_{12}^6+\frac{m_2^3\,r_{11}^3}{m_1^3\,r_{22}^3}+\frac{3\,m_2^2\,r_{11}^2}{m_1^2\,r_{22}^2}-$$

$$\frac{3\,m_2^2\,M_{12}^2\,r_{11}^2}{m_1^2\,r_{22}^2}+\frac{3m_2\,r_{11}}{m_1\,r_{22}}+\frac{21\,m_2\,M_{12}^2\,r_{11}}{m_1\,r_{22}}+\frac{3m_2\,M_{12}^4\,r_{11}}{m_1\,r_{22}}\Bigg)\sqrt{\frac{m_2\,r_{11}}{m_1\,r_{22}}}+\frac{108\,m_2^2\,M_{12}^2\,r_{11}^2}{m_1^2\,r_{22}^2}-$$

$$\frac{108\,m_2\,M_{12}^4\,r_{11}}{m_1\,r_{22}}+\frac{108m_2\,M_{12}^2\,r_{11}\left(-1+M_{12}^2-\frac{m_2\,r_{11}}{m_1\,r_{22}}\right)}{m_1\,r_{22}}\Bigg)\,\wedge\,(1/3)\Bigg)+\frac{m_2\,r_{11}}{m_1\,r_{22}}\Bigg)+$$

$$\frac{1}{2}\sqrt{\Bigg(1+\frac{1}{3}\left(-1+M_{12}^2-\frac{m_2\,r_{11}}{m_1\,r_{22}}\right)+$$

$$\left(2^{1/3}\left(1-M_{12}^2+\frac{m_2\,r_{11}}{m_1\,r_{22}}\right)^2\right)\Big/\Bigg(3\Bigg(2\left(-1+M_{12}^2-\frac{m_2\,r_{11}}{m_1\,r_{22}}\right)^3+12\sqrt{3}\,M_{12}\sqrt{\Bigg(1-3M_{12}^2+3M_{12}^4-M_{12}^6+\frac{m_2^3\,r_{11}^3}{m_1^3\,r_{22}^3}+$$

$$\frac{3\,m_2^2\,r_{11}^2}{m_1^2\,r_{22}^2}-\frac{3m_2^2\,M_{12}^2\,r_{11}^2}{m_1^2\,r_{22}^2}+\frac{3\,m_2\,r_{11}}{m_1\,r_{22}}+\frac{21m_2\,M_{12}^2\,r_{11}}{m_1\,r_{22}}+\frac{3\,m_2\,M_{12}^4\,r_{11}}{m_1\,r_{22}}\Bigg)\sqrt{\frac{m_2\,r_{11}}{m_1\,r_{22}}}+$$

$$\frac{108\,m_2^2\,M_{12}^2\,r_{11}^2}{m_1^2\,r_{22}^2}-\frac{108m_2\,M_{12}^4\,r_{11}}{m_1\,r_{22}}+\frac{108\,m_2\,M_{12}^2\,r_{11}\left(-1+M_{12}^2-\frac{m_2\,r_{11}}{m_1\,r_{22}}\right)}{m_1\,r_{22}}\Bigg)\,\wedge\,(1/3)\Bigg)+$$

$$\frac{m_2\,r_{11}}{m_1\,r_{22}}\Bigg)\Bigg)$$

```
The number of real points = 179
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Ö Graphics Ö

■ Example 4

Root No. 4

gaspart[4, 1, 1, 1, 3, 21, 31]

lambda =

$$v_2 + c\left(\frac{M_{12}}{2} + \frac{1}{2}\sqrt{\left(1 + M_{12}^2 + \frac{1}{3}\left(1 - M_{12}^2 + \frac{m_2 r_{11}}{m_1 r_{22}}\right) + \left(8M_{12}^3 - 8M_{12}\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right) - \frac{16 m_2 M_{12} r_{11}}{m_1 r_{22}}\right)\Big/\right.$$

$$\left(4\sqrt{\left(1 + \frac{1}{3}\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right) + \left(2^{1/3}\left(1 - M_{12}^2 + \frac{m_2 r_{11}}{m_1 r_{22}}\right)^2\right)\Big/\right.}$$

$$\left(3\left(2\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right)^3 + 12\sqrt{3}\, M_{12}\sqrt{\left(1 - 3M_{12}^2 + 3M_{12}^4 - \right.\right.\right.$$

$$M_{12}^6 + \frac{m_2^3 r_{11}^3}{m_1^3 r_{22}^3} + \frac{3 m_2^2 r_{11}^2}{m_1^2 r_{22}^2} - \frac{3 m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2} + \frac{3 m_2 r_{11}}{m_1 r_{22}} +$$

$$\left.\left.\frac{21 m_2 M_{12}^2 r_{11}}{m_1 r_{22}} + \frac{3 m_2 M_{12}^4 r_{11}}{m_1 r_{22}}\right)\sqrt{\frac{m_2 r_{11}}{m_1 r_{22}}} + \frac{108 m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2} -\right.$$

$$\left.\left.\frac{108 m_2 M_{12}^4 r_{11}}{m_1 r_{22}} + \frac{108 m_2 M_{12}^2 r_{11}\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right)}{m_1 r_{22}}\right)\wedge(1/3)\right) +$$

$$\frac{1}{3\, 2^{1/3}}\left(\left(\left(2\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right)^3 + 12\sqrt{3}\, M_{12}\sqrt{\left(1 - 3\right.\right.\right.\right.$$

$$M_{12}^2 + 3M_{12}^4 - M_{12}^6 + \frac{m_2^3 r_{11}^3}{m_1^3 r_{22}^3} + \frac{3 m_2^2 r_{11}^2}{m_1^2 r_{22}^2} - \frac{3 m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2} +$$

$$\left.\left.\frac{3 m_2 r_{11}}{m_1 r_{22}} + \frac{21 m_2 M_{12}^2 r_{11}}{m_1 r_{22}} + \frac{3 m_2 M_{12}^4 r_{11}}{m_1 r_{22}}\right)\sqrt{\frac{m_2 r_{11}}{m_1 r_{22}}} + \frac{108 m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2} -\right.$$

$$\left.\left.\frac{108 m_2 M_{12}^4 r_{11}}{m_1 r_{22}} + \frac{108 m_2 M_{12}^2 r_{11}\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right)}{m_1 r_{22}}\right)\wedge(1/3)\right) + \frac{m_2 r_{11}}{m_1 r_{22}}\right) -$$

$$\left(2^{1/3}\left(1 - M_{12}^2 + \frac{m_2 r_{11}}{m_1 r_{22}}\right)^2\right)\Big/\left(3\left(2\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right)^3 + 12\sqrt{3}\, M_{12}\sqrt{\left(1 - 3M_{12}^2 + 3M_{12}^4 - M_{12}^6 + \frac{m_2^3 r_{11}^3}{m_1^3 r_{22}^3} + \right.\right.\right.\right.$$

$$\frac{3 m_2^2 r_{11}^2}{m_1^2 r_{22}^2} - \frac{3 m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2} + \frac{3 m_2 r_{11}}{m_1 r_{22}} + \frac{21 m_2 M_{12}^2 r_{11}}{m_1 r_{22}} + \frac{3 m_2 M_{12}^4 r_{11}}{m_1 r_{22}}\Big)\sqrt{\frac{m_2 r_{11}}{m_1 r_{22}}} +$$

$$\left.\left.\frac{108 m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2} - \frac{108 m_2 M_{12}^4 r_{11}}{m_1 r_{22}} + \frac{108 m_2 M_{12}^2 r_{11}\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right)}{m_1 r_{22}}\right)\wedge(1/3)\right) -$$

$$\frac{1}{3\, 2^{1/3}}\left(\left(\left(2\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right)^3 + 12\sqrt{3}\, M_{12}\sqrt{\left(1 - 3M_{12}^2 + 3M_{12}^4 - M_{12}^6 + \frac{m_2^3 r_{11}^3}{m_1^3 r_{22}^3} + \frac{3 m_2^2 r_{11}^2}{m_1^2 r_{22}^2} - \right.\right.\right.\right.$$

$$\left.\left.\frac{3 m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2} + \frac{3 m_2 r_{11}}{m_1 r_{22}} + \frac{21 m_2 M_{12}^2 r_{11}}{m_1 r_{22}} + \frac{3 m_2 M_{12}^4 r_{11}}{m_1 r_{22}}\right)\sqrt{\frac{m_2 r_{11}}{m_1 r_{22}}} + \frac{108 m_2^2 M_{12}^2 r_{11}^2}{m_1^2 r_{22}^2} -\right.$$

$$\left.\left.\frac{108 m_2 M_{12}^4 r_{11}}{m_1 r_{22}} + \frac{108 m_2 M_{12}^2 r_{11}\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right)}{m_1 r_{22}}\right)\wedge(1/3)\right) + \frac{m_2 r_{11}}{m_1 r_{22}}\right) +$$

$$\frac{1}{2}\sqrt{\left(1 + \frac{1}{3}\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right) +\right.}$$

$$\left(2^{1/3}\left(1 - M_{12}^2 + \frac{m_2 r_{11}}{m_1 r_{22}}\right)^2\right)\Big/\left(3\left(2\left(-1 + M_{12}^2 - \frac{m_2 r_{11}}{m_1 r_{22}}\right)^3 + 12\sqrt{3}\, M_{12}\sqrt{\left(1 - 3M_{12}^2 + 3M_{12}^4 - M_{12}^6 + \frac{m_2^3 r_{11}^3}{m_1^3 r_{22}^3} + \right.\right.\right.\right.$$

$$\frac{3\,m_2^2\,r_{11}^2}{m_1^2\,r_{22}^2} - \frac{3\,m_2^2\,M_{12}^2\,r_{11}^2}{m_1^2\,r_{22}^2} + \frac{3\,m_2\,r_{11}}{m_1\,r_{22}} + \frac{21\,m_2\,M_{12}^2\,r_{11}}{m_1\,r_{22}} + \frac{3\,m_2\,M_{12}^4\,r_{11}}{m_1\,r_{22}}\Bigg)\sqrt{\frac{m_2\,r_{11}}{m_1\,r_{22}}} +$$

$$\frac{108\,m_2^2\,M_{12}^2\,r_{11}^2}{m_1^2\,r_{22}^2} - \frac{108\,m_2\,M_{12}^4\,r_{11}}{m_1\,r_{22}} + \frac{108\,m_2\,M_{12}^2\,r_{11}\left(-1+M_{12}^2-\frac{m_2\,r_{11}}{m_1\,r_{22}}\right)}{m_1\,r_{22}}\Bigg)^{(1/3)}\Bigg) +$$

$$\frac{1}{3\,2^{1/3}}\left(\left(2\left(-1+M_{12}^2-\frac{m_2\,r_{11}}{m_1\,r_{22}}\right)^3 + 12\sqrt{3}\,M_{12}\sqrt{\left(1-3\,M_{12}^2+3\,M_{12}^4-M_{12}^6+\frac{m_2^3\,r_{11}^3}{m_1^3\,r_{22}^3}+\right.\right.\right.$$

$$\frac{3\,m_2^2\,r_{11}^2}{m_1^2\,r_{22}^2} - \frac{3\,m_2^2\,M_{12}^2\,r_{11}^2}{m_1^2\,r_{22}^2} + \frac{3\,m_2\,r_{11}}{m_1\,r_{22}} + \frac{21\,m_2\,M_{12}^2\,r_{11}}{m_1\,r_{22}} + \frac{3\,m_2\,M_{12}^4\,r_{11}}{m_1\,r_{22}}\Bigg)\sqrt{\frac{m_2\,r_{11}}{m_1\,r_{22}}} +$$

$$\frac{108\,m_2^2\,M_{12}^2\,r_{11}^2}{m_1^2\,r_{22}^2} - \frac{108\,m_2\,M_{12}^4\,r_{11}}{m_1\,r_{22}} + \frac{108\,m_2\,M_{12}^2\,r_{11}\left(-1+M_{12}^2-\frac{m_2\,r_{11}}{m_1\,r_{22}}\right)}{m_1\,r_{22}}\Bigg)^{(1/3)}\Bigg) +$$

$$\left.\left.\frac{m_2\,r_{11}}{m_1\,r_{22}}\right)\right)$$

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The number of real points = 646
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■ The Structure of the Output

The following data are printed out in the process of the work of the program:

1. The analytical expression for the root l_{nroot} of equation (7.2.5);

2. The picture showing the region of real roots l_j in the user-specified rectangular region in the (e, M_{12}) plane.