Quasi One-Dimensional Gas flow

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■ Impressum

This Mathematica-Notebook is part of the book entitled
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■ General Description

This Notebook enables the user to find the numerical values of the Mach number M(x), the pressure ratio $P(x)/P_1$, the density ratio $r(x)/r_1$, the gas velocity ratio $v(x)/v_1$, and the temperature ratio $T(x)/T_1$ along the x-coordinate in a one-dimensional isentropic gas motion in the variable section duct. The derivation of the underlying solution formulas may be found in Section 6.1.2 of the above book.

In the present Notebook, a Laval nozzle is taken as the variable section duct. The ordinates of the nozzle wall are computed with the aid of the function fw[x_], see the beginning of the subsequent program listing. A more detailed information about the nozzle geometry is contained in Section 6.1.2 of the above book.

It is assumed in the present notebook that the value of the Mach number M_1 is specified by the user at the nozzle inlet. As was pointed out in Section 6.1.2, the solution of the transcendental equation (6.1.10) for M(x) exists not for any value of M_1. For a variable section duct different from the nozzle used in the present Section, the value of M_1 ensuring the existence of M(x) along the overall channel length is to be determined empirically by performing a series of runs for several guesses M_1. It is not difficult to perform these runs, because the present program is very fast.

The channel wall can also be specified in the form of its (x,y) coordinates as the following two lists:

```
xx = {...};
yw = {...}; ix = Length[xx];
```

In this case, the program lines

```
xx = Table[N[(i - 1) x4 / (ix - 1)], {i, ix}];
yw = Table[N[fw[xx[[i]]]] , {i, ix}];
```

should be replaced with the above two lines (or they can be enclosed in the comment parentheses (* .. *)).
Note that the lists of the numerical values xx and yw should have equal lengths.

If the user wishes to find the dimensional values of P(x), r(x), v(x), and T(x) at any duct point he should specify the values P_1, r_1, v_1, and T_1 at the nozzle inlet. It should be kept in mind that these values are not independent, because they should satisfy the following relations:

$$M_1 = \sqrt{gP_1 / r_1}\,, \quad P_1 = R r_1 T_1,$$

where R is the universal gas constant, R = 8.31 10^3 J/(kmole deg).

■ User's Guide

■ Step 1

Load and compile the program file beginning with the line

```
ClearAll[fw, fm, nozzle];
```

(see the Section "Program Listing")

■ Step 2

Specify the input data by entering them in the line (see also Section "Examples of the Input Data" below)

```
nozzle[0.6, 3.125, 1.0, 0.625, 2.805, 5.005, p / 4, p / 12,
1.4, 0.18, 2.5, 10^ -3, 40, 1]
```

Then click in this line and wait for the result of symbolic/numerical computation.
The meaning of the input parameters is explained in the Section "Parameters Used in Program prog6-1.nb".

■ Program Listing

```
   ClearAll[fw, fm, nozzle];
fw[x_] := (f1 = If[0 £ x && x £ x1, y0 - R0 + √((R0 - x) (R0 + x)) , 0];
f2 = If[x1 £ x && x £ x2, y1 - ct1 (x - x1), f1];
f3 = If[x2 £ x && x £ x3, hc1 + R - √((R - x + x0) (R + x - x0)) , f2];
f4 = If[x3 £ x && x £ x4, y3 + ct2 (x - x3), f3]; f4);

 fm[M_] := N[F / F1 - (b / M) * (1 + g2 M^2)^g3]

nozzle[hc_, cy0_, cR0_, cR_, cx0_, cx4_, thet1_, thet2_,
 g_, M1_, Mmax_, eps_, ix_, reg_] :=
((* -- Specification of the nozzle geometry -- *)
  R0 = cR0 hc; y0 = cy0 hc; x0 = cx0 hc; R = cR hc; hc1 = hc;
 x1 = R0 N[Sin[thet1]]; ct1 = N[Tan[thet1]]; ct2 = N[Tan[thet2]];
  x2 = x0 - R N[Sin[thet1]]; x3 = x0 + R N[Sin[thet2]]; x4 = cx4 hc;
  y1 = y0 - R0 (1 - N[Cos[thet1]]); y3 = hc + R (1 - N[Cos[thet2]]);
  Print["The values of the nozzle geometry", " parameters"];
  Print["hc = ", hc, ", R0 = ", R0, ", R = ", R];
  Print["The abscissa of the critical nozzle section = ", x0];
  Print["thet1 = ", thet1, ", thet2 = ", thet2, ",
   nozzle length = ", x4];
  (* yw is the list for the ordinates of the nozzle wall *)
  xx = Table[N[(i - 1) x4 / (ix - 1)], {i, ix}];
  yw = Table[N[fw[xx[[i]]]] , {i, ix}];
 (* --- The nozzle geometry --- *)
 lc = MapThread[List, {xx, yw}];
  gr1 = ListPlot[lc, PlotJoined -> True,
              PlotStyle -> {Thickness[0.01]},
              DisplayFunction -> Identity];
   gr2 = Graphics[Line[{{0, 0}, {x4, 0}}]];
  Show[gr1, gr2, DisplayFunction -> $DisplayFunction];
  (* -- The computation of the Mach number M (x) -- *)
 g2 = (g - 1) / 2; g3 = (g + 1) / (2 (g - 1)); F1 = yw[[1]];
  Mach = Table[N[M1], {i, ix}];
  b = M1 / ((1 + g2 M1^2)^g3);
```

```
 Print["beta = ", b];
   Do[ F = yw[[i]]; xi = xx[[i]];
    (* Plot[fm[x], {x, 0, 2}]; *)
    (* ff = Table[Sign[fm[0.01 + (j-1)*0.05]], {j,26}];
    Print["x = ", xi, "signj = ", ff]; *)
(* -- Numerical solution of equation
    fm[M] = 0 by the bisection method -- *)
  a0 = 0.01; b0 = Mmax;
   If[xi >= x0 && reg === 1, a0 = 1];
  isl = Sign[fm[a0]]; isr = Sign[fm[b0]];
(* Print["a0=",a0,",b0=",b0,",isl0 = ", isl, ",isr0 = ", isr];*)
  If[isl == isr,
 h = N[(b0 - a0) * 0.1];  mi = N[b0];
    Do[ mi = a0 + (j - 1) h; isr = Sign[fm[mi]];
     If[isl =!= isr, b0 = mi; Break[]], {j, 11}]];
   (* Print["x = ", xi,", b0 = ", b0,", isl =", isl,
                ", isr = ", isr]; *)
If[isl == isr, b0 = a0; xc = M1];
While[N[Abs[a0 - b0]] > eps, xc = N[(a0 + b0) * 0.5];
  nc = Sign[fm[xc]]; If[nc == isr, b0 = xc, a0 = xc]];
    Mach[[i]] = xc, {i, 2, ix}];
   Print["xx = ", xx];
  Print["The Mach number M(x) = ", Mach];
(* The computation of the ratios P/P1,r/r1,T/T1, and v/v1 *)
 P = {}; r = {}; T = {}; v = {};
g4 = g / (g - 1); g5 = 1 / (g - 1); z1 = 1 + g2 M1^2;
Do[z = 1 + g2 Mach[[i]]^2; pi = N[(z1 / z)^g4];
   dd = N[(z1 / z)^g5]; Ti = N[z1 / z];
 vi = N[(Mach[[i]] / M1) Sqrt[z1 / z]];
AppendTo[P, pi]; AppendTo[r, dd]; AppendTo[T, Ti];
AppendTo[v, vi],  {i, ix}];
(*  --- Graphics --- *)
lc = MapThread[List, {xx, Mach}];
gr1 = ListPlot[lc, PlotJoined -> True,
                    DisplayFunction -> Identity];
  lc = MapThread[List, {xx, P}];
 gr2 = ListPlot[lc, PlotJoined -> True,
                    PlotStyle -> {Dashing[{0.02}]},
                     DisplayFunction -> Identity];
  lc1 = MapThread[List, {xx, r}];
  gr3 = ListPlot[lc1, PlotStyle -> {PointSize[0.01]},
                     DisplayFunction -> Identity];
```

```
  lc = MapThread[List, {xx, v}];
 gr4 = ListPlot[lc, PlotJoined -> True,
      PlotStyle -> {Dashing[{0.005, 0.02, 0.02, 0.02}]},
                     Frame -> True,
                     PlotRange -> All];
gr5 = {};
Do[ xj = xx[[j]]; tj = T[[j]];
gj = Graphics[Circle[{xj, tj}, 0.015]];
    AppendTo[gr5, gj], {j, ix}];
Show[gr1,  gr3, gr2, gr5, Frame -> True,
                PlotRange -> All,
                DisplayFunction -> $DisplayFunction];
                )
```

■ Parameters Used in Program prog6-1.nb

Parameter	Description
hc	the nozzle wall height in the critical section of the nozzle, hc > 0;
cy0	the ordinate of the nozzle wall in the inlet section x = 0 related to hc;
cR0	the radius of the curvature of the nozzle wall in its subsonic part related to hc;
cR	the nozzle wall radius of the curvature in the critical section related to hc;
cx0	the abscissa of the critical nozzle section related to hc;
cx4	the nozzle length related to hc, cx4 > cx0;
thet1	the inclination angle of the nozzle wall in the subsonic part of the nozzle; the angle is specified in radians;
thet2	the inclination angle of the nozzle wall in the supersonic part of the nozzle; the angle is specified in radians;
g	the ratio of the gas specific heats; g > 1;
M1	the numerical value of the Mach number at the duct inlet; M1 > 0;
Mmax	the solution of equation (4.1.10) is sought for in the interval [0, Mmax], Mmax > 0. If reg =1, then the value Mmax >1 should be specified;
eps	the user-specified accuracy of the numerical solution of equation (4.1.10) by the bisection method:; 0 < eps < 1;
ix	the number of uniform grid nodes in the interval 0 <= x <= x4, where x4 is the nozzle length along the x axis; ix is a positive integer, ix > 1;
reg	if reg = 0, then the subsonic flow regime is chosen in the diverging part of the Laval nozzle. If reg = 1, then the supersonic flow regime is chosen in the diverging part of the Laval nozzle.

■ Examples of the Input Data

■ Example 1

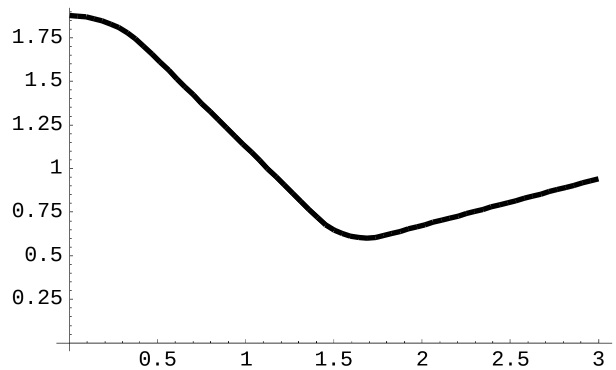
```
nozzle[0.6, 3.125, 1.0, 0.64, 2.805, 5.005, p/4, p/12,
1.4, 0.1891, 2.5, 10^-4, 65, 1]
```

The values of the nozzle geometry parameters

hc = 0.6, R0 = 0.6, R = 0.384

The abscissa of the critical nozzle section = 1.683

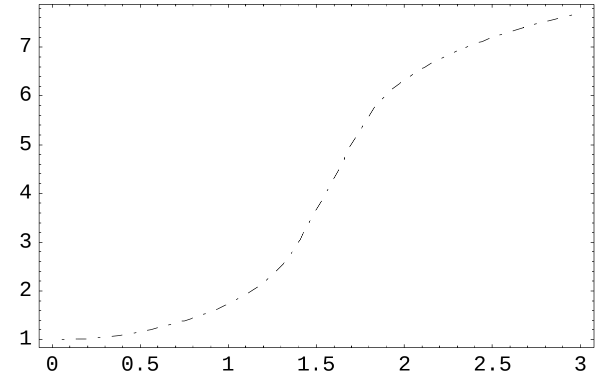
thet1 = $\frac{p}{4}$, thet2 = $\frac{p}{12}$, nozzle length = 3.003

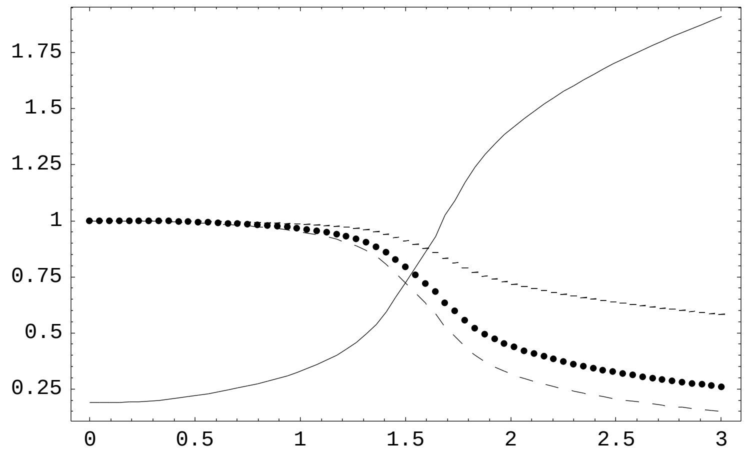


beta = 0.1851

xx = {0, 0.0469219, 0.0938437, 0.140766, 0.187687, 0.234609, 0.281531, 0.328453, 0.375375,
0.422297, 0.469219, 0.516141, 0.563062, 0.609984, 0.656906, 0.703828, 0.75075, 0.797672,
0.844594, 0.891516, 0.938437, 0.985359, 1.03228, 1.0792, 1.12612, 1.17305, 1.21997,
1.26689, 1.31381, 1.36073, 1.40766, 1.45458, 1.5015, 1.54842, 1.59534, 1.64227,
1.68919, 1.73611, 1.78303, 1.82995, 1.87687, 1.9238, 1.97072, 2.01764, 2.06456,
2.11148, 2.15841, 2.20533, 2.25225, 2.29917, 2.34609, 2.39302, 2.43994, 2.48686,
2.53378, 2.5807, 2.62762, 2.67455, 2.72147, 2.76839, 2.81531, 2.86223, 2.90916, 2.95608, 3.003}

The Mach number M(x) = {0.1891, 0.189258, 0.189865, 0.190929, 0.192297, 0.194273, 0.196856,
0.200048, 0.204151, 0.209471, 0.215702, 0.222389, 0.229516, 0.237176, 0.245322, 0.254076,
0.263438, 0.273651, 0.284836, 0.296873, 0.310125, 0.324594, 0.340764, 0.358636, 0.378819,
0.401677, 0.427817, 0.458455, 0.494687, 0.538973, 0.595509, 0.660616, 0.72627, 0.793292, 0.861865,
0.931531, 1.02573, 1.09109, 1.1713, 1.24088, 1.29562, 1.3425, 1.38425, 1.42215, 1.45712,
1.48972, 1.52048, 1.54959, 1.57724, 1.60361, 1.62888, 1.65323, 1.67667, 1.69937,
1.72134, 1.74258, 1.76309, 1.78323, 1.80264, 1.82169, 1.84018, 1.85812, 1.87589, 1.8931, 1.90994}





■ Example 2

The same nozzle as in Example 1, but the value reg = 0 (subsonic flow regime) is input:

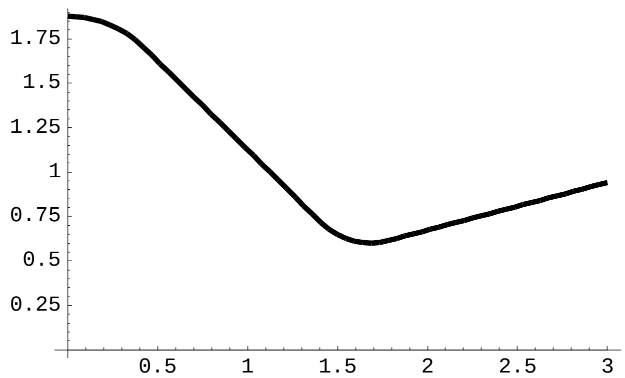
`nozzle[0.6, 3.125, 1.0, 0.64, 2.805, 5.005, p/4, p/12,`
`1.4, 0.1891, 2.5, 10^-3, 65, 0]`

```
The values of the nozzle geometry parameters

hc = 0.6, R0 = 0.6, R = 0.384

The abscissa of the critical nozzle section = 1.683
```

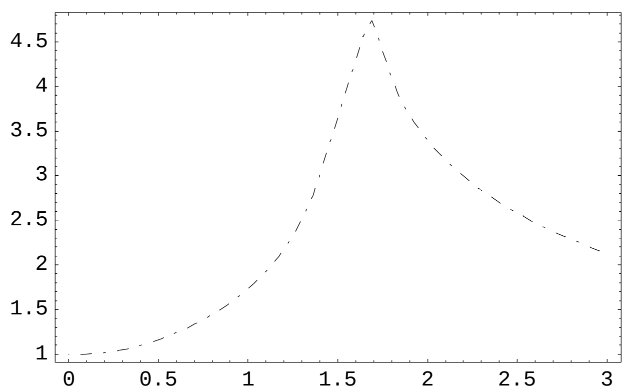
thet1 = $\frac{P}{4}$, thet2 = $\frac{P}{12}$,   nozzle length = 3.003

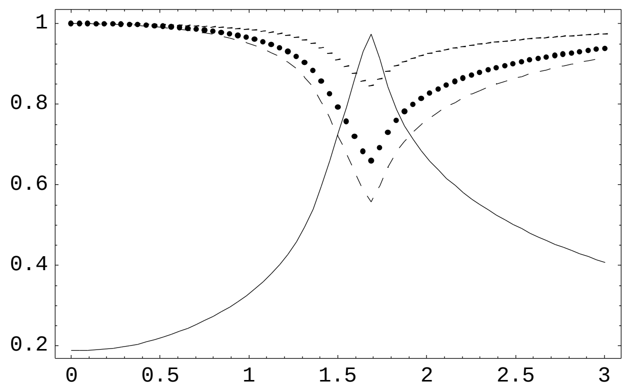


beta = 0.1851

xx = {0, 0.0469219, 0.0938437, 0.140766, 0.187687, 0.234609, 0.281531, 0.328453, 0.375375, 0.422297, 0.469219, 0.516141, 0.563062, 0.609984, 0.656906, 0.703828, 0.75075, 0.797672, 0.844594, 0.891516, 0.938437, 0.985359, 1.03228, 1.0792, 1.12612, 1.17305, 1.21997, 1.26689, 1.31381, 1.36073, 1.40766, 1.45458, 1.5015, 1.54842, 1.59534, 1.64227, 1.68919, 1.73611, 1.78303, 1.82995, 1.87687, 1.9238, 1.97072, 2.01764, 2.06456, 2.11148, 2.15841, 2.20533, 2.25225, 2.29917, 2.34609, 2.39302, 2.43994, 2.48686, 2.53378, 2.5807, 2.62762, 2.67455, 2.72147, 2.76839, 2.81531, 2.86223, 2.90916, 2.95608, 3.003}

The Mach number M(x) = {0.1891, 0.189333, 0.189333, 0.190549, 0.191765, 0.194197, 0.196628, 0.200276, 0.203923, 0.210002, 0.216082, 0.222161, 0.228848, 0.236629, 0.24441, 0.254137, 0.263863, 0.27359, 0.285262, 0.296934, 0.310551, 0.324168, 0.341676, 0.359184, 0.378637, 0.40198, 0.42727, 0.458395, 0.495355, 0.538882, 0.595782, 0.659978, 0.725632, 0.792988, 0.861074, 0.931105, 0.973902, 0.913598, 0.843566, 0.787152, 0.746058, 0.712501, 0.68478, 0.658519, 0.636634, 0.616208, 0.5987, 0.581192, 0.565144, 0.550554, 0.537423, 0.524292, 0.51262, 0.501191, 0.491465, 0.479793, 0.470066, 0.462285, 0.452559, 0.444777, 0.436996, 0.429215, 0.421434, 0.413652, 0.407816}





■ The Structure of the Output

The following information is obtained as a result of the work of the above program:

1. The values of the nozzle geometry parameters.
2. The Figure showing the nozzle wall.
3. The numerical values of the grid nodes coordinates on the x-axis (the list xx) and the numerical values of the Mach number M(x) (the list Mach).
4. The Figure showing the curve $v(x)/ v_1$ (the velocity ratio) by a dash-dot line.
5. The Figure showing the following curves:

______ M(x) the Mach number
\- - - - - - $P(x)/ P_1$ the pressure ratio
. . . . . . $r(x)/ r_1$ the density ratio
o o o o o $T(x)/ T_1$ the temperature ratio

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