The Prandtl-Meyer Solution

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■ Impressum

This Mathematica-Notebook is part of the book entitled
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■ General Description

The program prog6-3.nb, which follows below, enables the user to construct the solution in the centered Prandtl-Meyer rarefaction wave arising in a supersonic flow around a dihedral corner for the case where
$|j_0| < |q_+|$.

The formulation of this problem and the discussion of its solution can be found in Section 6.4 of the above mentioned book.

■ User's Guide

■ Step 1

Load and compile the program file beginning with the line

```
q[M1_, M_] := g1*(ArcTan[Sqrt[g2*(M1^2 - 1)]] -
```

(see the Section "Program Listing")

■ Step 2

Specify the input data by entering them in the line (see also Section "Examples of the Input Data" below)

```
Meyer[1.4, 4.0, 1, 10, 1, -1, 1, -0.5, 0.6, 30, 80]
```

Then click in this line and wait for the result of symbolic/numerical computation.
The meaning of the input parameters is explained in the Section "Parameters Used in Program prog6-3.nb".

■ Program Listing

```
  q[M1_, M_] := g1*(ArcTan[Sqrt[g2*(M1^2 - 1)]] -
ArcTan[Sqrt[g2*(M^2 - 1)]]) - ArcTan[Sqrt[M1^2 - 1]] +
ArcTan[Sqrt[M^2 - 1]];

cl2[f_] := Hue[1.2*(Af*f + Bf)];

fij[mm_, mm1_] := ArcTan[Sqrt[mm^2 - 1], 1] - tj + q[mm1, mm];

Meyer[g_, M1r_, M10_, qmax_, surf_, xl_, xr_,
      yr_, h0_, ix_, m1int_] :=
( g2 = (g - 1) / ( g + 1); g1 = 1 / Sqrt[g2];
If[surf == 1, Plot3D[q[m1, m], {m1, 1, M1r}, {m, 1, M1r},
           AspectRatio -> Automatic,
           AxesLabel -> {"M", "M1", "q"}]];
 (* The computation of the Mach number and the
     ratios P/P1  and r/r1  *)
 P = {}; r = {}; Mach = {}; xx = {};
  g2 = (g - 1) / 2; yr1 = yr; xr1 = xr; h1 = h0;
g4 = g / (g - 1); g5 = 1 / (g - 1); z1 = 1 + g2 M10^2;
dq = N[Pi / 180];
  tcr = Floor[N[90*(g1 - 1)]]; tmax = Min[qmax, tcr];
  Do[qj = (1 - j)*dq;
 (* Plot[qj -q[M10, m], {m, 1, M1r}]; *)
a0 = 1; b0 = M1r; isl = Sign[qj - q[M10, a0]];
isr = Sign[qj - q[M10, b0]];
While[N[Abs[a0 - b0]] > 10^-7,
c = (a0 + b0) / 2; isc = Sign[qj - q[M10, c]];
        If[isc == isr, b0 = c, a0 = c]];
AppendTo[Mach, c], {j, tmax}];
 Do[z = 1 + g2 Mach[[i]]^2; pi = N[(z1 / z)^g4];  dd = N[(z1 / z)^g5];
AppendTo[P, pi]; AppendTo[r, dd], {i, tmax}];
Print["q, deg.       M        P/P1         r/r1      a, deg."];
Do[ qj = 1 - j; mj = N[Mach[[j]], 9]; pj = N[P[[j]], 9];
  dd = N[r[[j]], 9]; AppendTo[xx, qj];
   aj = N[(180 / Pi)*ArcTan[1 / Sqrt[Mach[[j]]^2 - 1]], 9];
Print["  ", qj, "    ", mj, "   ", pj,
    "   ", dd, "  ", aj], {j, tmax}];
```

```
(*  --- Graphics --- *)
 lc = MapThread[List, {xx, Mach}];
gr1 = ListPlot[lc, PlotJoined -> True,
          DisplayFunction -> Identity];
  lc = MapThread[List, {xx, P}];
 gr2 = ListPlot[lc, PlotJoined -> True,
           PlotStyle -> {Dashing[{0.02}]},
           DisplayFunction -> Identity];
  lc1 = MapThread[List, {xx, r}];
  gr3 = ListPlot[lc1, PlotStyle -> {PointSize[0.01]},
            DisplayFunction -> Identity];
 Show[gr1, gr3, gr2,  Frame -> True,
          PlotRange -> All,
          DisplayFunction -> $DisplayFunction];
  (* The flowfield in the Prandtl-Meyer wave *)
 If[N[Abs[xr]] < 10^-7, j0 = N[-Pi / 2], j0 = N[ArcTan[yr / xr]]];
 If[xr < 0, j0 = N[j0 - Pi]];
  (* --- The determination of M2 --- *)
 a0 = 1; b0 = M1r; isl = Sign[j0 - q[M10, a0]];
isr = Sign[j0 - q[M10, b0]];
While[N[Abs[a0 - b0]] > 10^-7,
c = (a0 + b0) / 2; isc = Sign[j0 - q[M10, c]];
        If[isc == isr, b0 = c, a0 = c]];
M2 = c; fa2 = N[j0 + ArcSin[1 / M2]]; cf2 = N[Cos[fa2]];
   Print["j0 = ", j0, "; M2 = ", M2, "; a2 = ", fa2];
  p2 = N[(z1 / (1 + g2 * M2^2))^g4];
  xpl = xl; xpr = xr; ypt = N[Abs[xl]]; ypb = yr;
  If[xr < 0.3 * Abs[xl], xpr = 0.5 * Abs[xl]];
  jy = Floor[ix * (ypt - ypb) / (xpr - xpl)];
  hx = (xpr - xpl) / (ix - 1); hy = (ypt - ypb) / (jy - 1);
Print["The color map on the original ", ix, "x", jy, " mesh"];
mij = Table[M10, {i, ix}, {j, jy}];
pij = Table[1, {i, ix}, {j, jy}];
 cta1 = N[Sqrt[M10^2 - 1]];
 Do[ yj = ypb + (j - 0.5) * hy; xwj = N[xr * (yj / yr)];
    xcj = -yj * cf2; xuj = N[yj * cta1];
   Do[ xi = xl + (i - 0.5) * hx;
If[(xi > xcj && yj < 0) || (yj > 0 && xi > xuj),
   tj = ArcTan[xi, yj]; a0 = 1; b0 = M1r;
 isl = Sign[fij[a0, M10]]; isr = Sign[fij[b0, M10]];
 While[N[Abs[a0 - b0]] > 10^-4, c = (a0 + b0) / 2;
```

```
isc = Sign[fij[c, M10]]; If[isc == isr, b0 = c, a0 = c]];
mij[[i, j]] = c; pij[[i, j]] = N[(z1 / (1 + g2 * c^2))^g4]];
If[(xi > xwj && xi < xcj) && (yj < 0),
  mij[[i, j]] = M2; pij[[i, j]] = p2], {i, ix}], {j, jy}];
 Pmax = Max[pij]; Pmin = Min[pij];
 Af = h0 / (Pmin - Pmax); Bf = -Af * Pmax;
Print["Pmin = ", Pmin, ", Pmax = ", Pmax];
(*  Print["P = ", pt];   *)
 gr1 = DensityGraphics[Transpose[pij]];
 gr2 = Show[gr1, ColorFunction -> (cl2[#]&),
         Mesh -> False,
         DisplayFunction -> Identity];

  ynl = -ypb / hy; xn0 = - xpl / hx;  xnr = (xr - xpl) / hx;
obj = Polygon[{{0, ynl}, {xn0, ynl}, {xnr, 0}, {0, 0}, {0, ynl}}];
   wall = Graphics[{RGBColor[0.3, 0.9, 1], obj}];
 gg1 = Show[gr2, wall, AspectRatio -> Automatic,
                      DisplayFunction -> Identity];
 (* -- Making a color map showing the correspondence
       between individual colors and the numerical
       values of P/P1 ----------- *)
dy = N[jy / 10]; dx = dy; mc = {};
  Do[ybj = (j - 1) * dy; ytj = ybj + dy; ycj = ybj + 0.5 dy;
     pcj = Pmin + ((j - 0.5) * dy / jy) * (Pmax - Pmin);
  lis = {{0, ybj}, {0, ytj}, {dx, ytj}, {dx, ybj}, {0, ybj}};
  boxj = Graphics[{cl2[pcj], Polygon[lis]}];
  cont = ListPlot[lis, PlotJoined -> True,
                      DisplayFunction -> Identity];
  txt = Graphics[Text[pcj, {2 dx, ycj}, {-1, 0}]];
  gj = Show[txt, boxj, cont, DisplayFunction -> Identity];
       AppendTo[mc, gj], {j, 10}];
 gg2 = Show[mc, AspectRatio -> Automatic,
                PlotRange -> All,
                DisplayFunction -> Identity];
 Show[GraphicsArray[{gg1, gg2}]];
 (* --- Interpolation on a finer mesh --- *)
tb = {};
 Do[yj = ypb + (j - 1) * hy;
    Do[xi = xl + (i - 1) * hx;
   AppendTo[tb, {xi, yj, pij[[i, j]]}], {i, ix}], {j, jy}];
 Print["Calculation of the interpolating function"];
 vint = Interpolation[tb, InterpolationOrder -> 1];
```

```
      h1n = (xpr - xpl) / (m1int - 1);
      m2int = Floor[m1int * (ypt - ypb) / (xpr - xpl)];
      h2n = (ypt - ypb) / (m2int - 1);
      vn = Table[0.0, {i, m1int}, {j, m2int}];
      (* -- Calculation of the values of P (x,y)/P1 in
            the nodes of a fine mesh ----- *)
     Do[yj = ypb + (j - 1) * h2n;
        Do[xi = xpl + (i - 1) * h1n;
     vn[[i, j]] = vint[xi, yj], {i, m1int}], {j, m2int}];
      gr3 = DensityGraphics[Transpose[vn]];
      gr4 = Show[gr3, ColorFunction -> (cl2[#]&),
                      Mesh -> False,
                      DisplayFunction -> Identity];
      ynl = -ypb / h2n; xn0 = -xpl / h1n;
      xnr = (xr - xpl) / h1n;
      obj = Polygon[{{0, ynl}, {xn0, ynl}, {xnr, 0},
                     {0, 0}, {0, ynl}}];
      wall = Graphics[{RGBColor[0.3, 0.9, 1], obj}];
      gg3 = Show[gr4, wall, AspectRatio -> Automatic,
                            DisplayFunction -> Identity];
      Print["The color map on the fine ", m1int, "x", m2int, " mesh"]
      Show[GraphicsArray[{gg3, gg2}]]; )
```

■ Parameters Used in Program prog6-3.nb

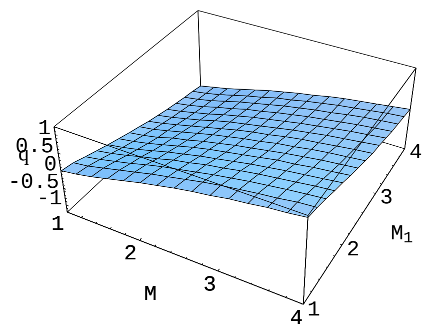
Parameter	Description
g	the ratio of the gas specific heats; g > 0
M1r	the right end of the interval [1,M1r], in which the local value M(x,y) of the Mach number is computed; M1r > 1; M1r > 3 for g = 1.4
M10	the freestream Mach number; M10 >= 1
qmax	the maximum valueof the angle q, in degrees, for making the one-dimensional plots of M(q), $P(q)/P_1$, $r(q)/r_1$, a(q); q is a positive integer, $1 < q < q_+$
surf	if surf = 1 then the surface q = $f(M_1, M)$ is plotted; otherwise this surface is not plotted
xl	the abscissa of the left end (xl,0) of a dihedral corner, xl < 0
xr	the abscissa of the right end of the right facet of the dihedral corner
yr	the ordinate of the right end of the right facet of the dihedral corner , yr < 0
h0	the number4 in the interval 0 < h0 <= 1 to specify the color

	corresponding to the minimum value of the function $P(x, y)/P_1$ in the spatial region under consideration
ix	the number of cells of an uniform rectangular (rough) grid in the interval xl <= x<=xr; ix is a positive integer, ix > 1
m1int	the number of cells of a fine mesh in the interval xl <= x<= xpr; m1int > ix

■ Examples of the Input Data

■ Example 1

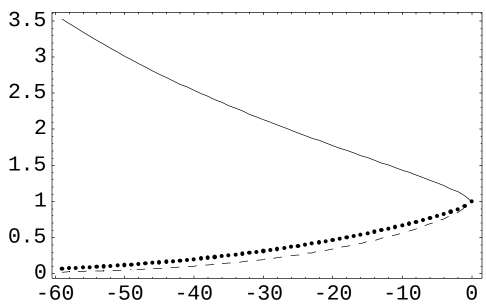
```
Meyer[1.4, 4.0, 1, 60, 1, -1, 1, -0.5, 0.6, 30, 100]
```



q, deg.	M	P/P1	r/r1	a, deg.
0	1.00000009	0.999999896	0.999999925	89.9757717
-1	1.08181283	0.906677246	0.932414639	67.5741964
-2	1.1326026	0.851129804	0.891244779	61.9969031
-3	1.17686996	0.804379084	0.855996478	58.1803101
-4	1.2177352	0.762678786	0.82405905	55.2047574
-5	1.25647327	0.724491496	0.79437173	52.7383063
-6	1.29376224	0.688998854	0.766375805	50.6185967
-7	1.33001676	0.655701263	0.739734253	48.7526432
-8	1.36551365	0.62426639	0.714226041	47.081676
-9	1.4004505	0.594458493	0.689696846	45.5658835
-10	1.43497446	0.566103152	0.666034672	44.1769403
-11	1.46919981	0.539066156	0.643155165	42.8938798
-12	1.50321791	0.513241232	0.620993156	41.7007055
-13	1.53710368	0.488542152	0.599497218	40.5849082

-14	1.57092008	0.464897337	0.57862594	39.5364978
-15	1.6047211	0.442246189	0.558345381	38.5473547
-16	1.63855377	0.420536622	0.538627403	37.6107847
-17	1.67245975	0.399723146	0.519448331	36.7211959
-18	1.70647642	0.379765522	0.500788045	35.8738665
-19	1.74063793	0.360627604	0.482629148	35.0647656
-20	1.77497575	0.342276624	0.46495649	34.2904245
-21	1.80951938	0.324682486	0.44775665	33.5478313
-22	1.84429637	0.307817492	0.431017805	32.8343606
-23	1.87933299	0.291655766	0.414729273	32.1477025
-24	1.91465446	0.276173001	0.398881347	31.4858147
-25	1.95028564	0.261346018	0.383464914	30.8468734
-26	1.98624995	0.247153106	0.36847187	30.2292637
-27	2.022571	0.233573254	0.353894368	29.6315263
-28	2.05927184	0.220586379	0.339725101	29.0523502
-29	2.09637555	0.208173063	0.325957051	28.4905474
-30	2.13390502	0.196314573	0.312583536	27.9450415
-31	2.17188314	0.184992756	0.299598122	27.4148536
-32	2.21033314	0.174189902	0.286994486	26.8990864
-33	2.24927828	0.163888829	0.274766526	26.3969215
-34	2.28874233	0.154072717	0.262908181	25.9076047
-35	2.32874945	0.144725148	0.251413488	25.4304418
-36	2.36932448	0.135830021	0.240276481	24.96479
-37	2.41049227	0.127371699	0.229491384	24.5100608
-38	2.45227894	0.11933474	0.21905228	24.065702
-39	2.49471077	0.111704123	0.208953395	23.6312062
-40	2.53781548	0.104465015	0.199188797	23.2060948
-41	2.58162096	0.0976029996	0.189752705	22.7899284
-42	2.62615672	0.0911038447	0.180639171	22.3822902
-43	2.67145279	0.0849536856	0.171842333	21.9827949
-44	2.71754029	0.0791389419	0.163356299	21.5910817
-45	2.76445213	0.0736462453	0.155175039	21.2068068

-46	2.81222209	0.0684625682	0.147292573	20.8296509
-47	2.86088505	0.0635751985	0.139702931	20.4593154
-48	2.91047803	0.0589716428	0.13240001	20.0955142
-49	2.9610391	0.0546397492	0.125377758	19.7379812
-50	3.01260832	0.0505676313	0.118630052	19.386463
-51	3.06522754	0.0467436995	0.112150755	19.0407203
-52	3.11894056	0.0431566586	0.105933702	18.7005263
-53	3.17379317	0.0397955173	0.099972727	18.3656671
-54	3.22983363	0.0366495658	0.094261616	18.035938
-55	3.28711239	0.033708409	0.0887941733	17.7111459
-56	3.34568235	0.0309619533	0.0835641982	17.3911077
-57	3.4055995	0.0284003915	0.0785654607	17.0756475
-58	3.46692267	0.0260142237	0.0737917423	16.7645984
-59	3.5297139	0.0237942504	0.069236828	16.4578007



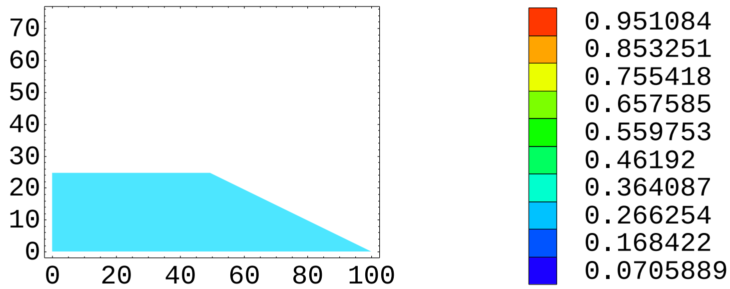
j0 = -0.463648; M2 = 2.00673; a2 = 0.0580166

The color map on the original 30x22 mesh

Pmin = 0.0216725, Pmax = 1

Calculation of the interpolating function

The color map on the fine 100x75 mesh

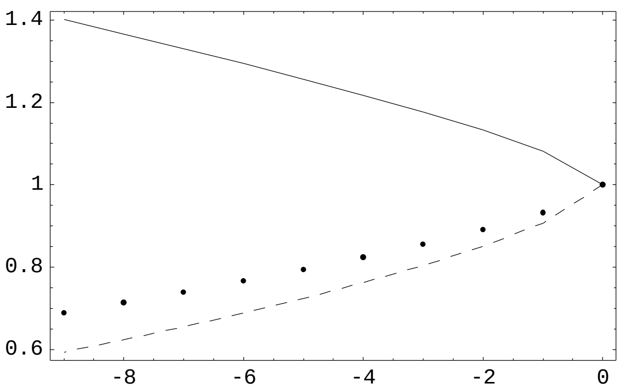


■ Example 2

The dihedral angle magnitude = p/2

```
Meyer[1.4, 8.0, 1, 10, 0, -1, 0.0, -0.5, 0.6, 30, 50]
```

q, deg.	M	P/P1	r/r1	a, deg.
0	1.00000005	0.999999939	0.999999957	89.9814953
-1	1.08181288	0.906677188	0.932414597	67.5741897
-2	1.13260259	0.851129812	0.891244785	61.9969039
-3	1.17687003	0.804379014	0.855996426	58.1803049
-4	1.21773513	0.762678853	0.824059102	55.204762
-5	1.25647319	0.724491576	0.794371792	52.7383112
-6	1.2937622	0.688998888	0.766375832	50.6185987
-7	1.33001668	0.655701337	0.739734312	48.7526472
-8	1.36551367	0.62426637	0.714226025	47.081675
-9	1.40045054	0.594458461	0.68969682	45.565882



j0 = -1.5708; M2 = 6.81904; a2 = -1.42362

The color map on the original 30x30 mesh

Pmin = 0.000539792, Pmax = 1

Calculation of the interpolating function

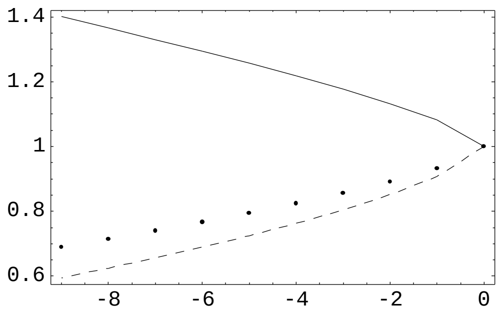
The color map on the fine 50x50 mesh

■ Example 3

The dihedral angle magnitude < p/2

Meyer[**1.4, 10.0, 1, 10, 0, -1, -0.07, -0.5, 0.6, 57, 90**]

q, deg.	M	P/P1	r/r1	a, deg.
0	1.00000007	0.999999922	0.999999944	89.9790177
-1	1.08181281	0.906677271	0.932414657	67.5741993
-2	1.13260258	0.851129828	0.891244797	61.9969053
-3	1.17687003	0.804379014	0.855996426	58.1803049
-4	1.21773509	0.762678898	0.824059136	55.204765
-5	1.25647316	0.724491604	0.794371815	52.738313
-6	1.29376217	0.688998916	0.766375855	50.6186003
-7	1.33001678	0.655701243	0.739734237	48.7526421
-8	1.36551367	0.62426637	0.714226025	47.081675
-9	1.40045052	0.594458474	0.68969683	45.5658826



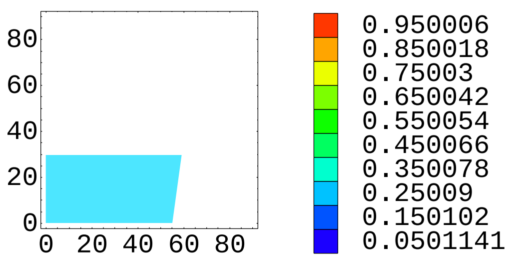
j0 = -1.70989; M2 = 8.6092; a2 = -1.59347

The color map on the original 57x57 mesh

Pmin = 0.000120105, Pmax = 1

Calculation of the interpolating function

The color map on the fine 90x90 mesh



■ The Structure of the Output

The following information is printed out in the process of the work of the program:

1. The surface q = q(M_1,M);

2. The numerical values of the functions M(q), P(q)/ P_1, r(q)/ r_1, a(q) for q = 1,2,...,qmax degrees;

3. The picture showing the curves of the functions listed in item No. 2;

4. The computed numerical values of the angle j_0 (in radians), the Mach number M_2, and the angle a_2 (in radians);

5. The numerical values of min($P(x, y)/P_1$) and max($P(x, y)/P_1$);

6. The color map of the pressure field $P(x, y)/P_1$ on a rough mesh of ix *jy cells along with a colored column giving the correspondence between the individual colors and the values of P/P_1;

7. The color map of the pressure field $P(x, y)/P_1$ on a fine mesh of m1int *m2int cells along with a colored column giving the correspondence between the individual colors and the values of P/P_1;

To resize an individual picture obtained by *Mathematica*, please

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