The Derivation of Navier-Stokes Equations in Spherical Coordinates

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■ Impressum

This Mathematica-Notebook is part of the book entitled
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■ General Description

This Notebook enables the user to compute the analytic expressions for the components of the velocity Laplacian Dv in the spherical coordinates r, j , q .

The discussion of underlying formulas may be found in Section 5.1 of the above book.

■ User's Guide

■ Step 1

Compile the program cell entitled "TheDefinition of Spherical Coordinates" (see the Section "Program Listing").

■ Step 2

Compile the program cell entitled "The Partial Derivatives $\partial_x\, x_m$, $i, m = 1, 2, 3.$"
(see the Section "Program Listing").

■ Step 3

Compile the program cell entitled "The Expressions for Metric Tensor Components"
(see the Section "Program Listing").

■ Step 4

Compile the program cell entitled "The Continuity Equation"
(see the Section "Program Listing").

■ Step 5

Compile the program cell entitled "The Laplace Operator in the Momentum Equations"
(see the Section "Program Listing").

■ Program Listing

■ The Definition of Spherical Coordinates

The quantities x[1], x[2], x[3] are the Cartesian coordinates;
the quantities k[1], k[2], k[3] are the spherical coordinates.

```
x[1] = k[1] Sin[k[2]] Cos[k[3]];
x[2] = k[1] Sin[k[2]] Sin[k[3]];
x[3] = k[1] Cos[k[2]];
```

■ The Partial Derivatives $\partial_\xi\, x_m$, $i, m = 1, 2, 3.$

```
g = IdentityMatrix[3];
Do[ Do[ g0 = Sum[D[x[m],k[i]]*D[x[m],k[j]],{m, 3}];
g = ReplacePart[g, g0, {i,j}], {i, 3}], {j, 3}]
```

■ The Expressions for Metric Tensor Components

```
Do[Do[gj = g[[i, j]] /. {k[1]  -> r, k[2] -> q, k[3] ->  j};
gj = TrigExpand[TrigReduce[gj]] /. {Cos[q]^2 -> 1 - Sin[q]^2};
 g = ReplacePart[g, Expand[gj], {i, j}], {j, 3}], {i, 3}];
Print["g = ", TraditionalForm[ MatrixForm[g]]];
```

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2(q) \end{pmatrix}$$

■ The Continuity Equation

```
g1 = g[[1, 1]]; g2 = g[[2, 2]]; g3 = g[[3, 3]];
den = PowerExpand[(g1 g2 g3)^(1/2)];
div = 1 / den
  (∂r (PowerExpand[(g2 g3)^(1/2)] v1[r, Q, j]) + D[PowerExpand[(g1 g3)^(1/2)] v2[r, Q, j], q] +
     ∂j (PowerExpand[(g1 g2)^(1/2)] v3[r, Q, j])) ;
  div = Expand[div] // Simplify;
   Print[TraditionalForm[div], " = 0"];
```

$$\frac{1}{r}\left(2\, v_1[r, Q, j] + \cot(q)\, v_2[r, Q, j] + \csc(q)\, v_3^{(0,0,1)}(r, Q, j) + r\, v_1^{(1,0,0)}(r, Q, j)\right) = 0$$

■ The Laplace Operator in the Momentum Equations

```
laplace[k_] :=
 (b[1] = g[[1, 1]]; b[2] = g[[2, 2]];
  b[3] = g[[3, 3]]; i = k + 1;
  If[i > 3, i = 1]; j = 6 - i - k;
  rl1 = {r -> p[1], q -> p[2], j -> p[3]};
b[1] = b[1] /. rl1; b[2] = b[2] /. rl1; b[3] = b[3] /. rl1;
den = PowerExpand[(b[1] b[2] b[3])^(1/2)];
s1 = D[PowerExpand[(b[j] b[k])^(1/2)] * u[p[1], p[2], p[3], i], p[i]];
s2 = D[PowerExpand[(b[i] b[k])^(1/2)] * u[p[1], p[2], p[3], j], p[j]];
s3 = D[PowerExpand[(b[i] b[j])^(1/2)] * u[p[1], p[2], p[3], k], p[k]];
s4 = D[PowerExpand[(b[j] / (b[i] * b[k])) ^ (1 / 2)] *
    D[PowerExpand[b[i]^(1/2)] * u[p[1], p[2], p[3], i], p[k]], p[i]];
s5 = D[PowerExpand[(b[j] / (b[i] * b[k])) ^ (1 / 2)] *
    D[PowerExpand[b[k]^(1/2)] * u[p[1], p[2], p[3], k], p[i]], p[i]];
s6 = D[PowerExpand[(b[i] / (b[j] * b[k])) ^ (1 / 2)] *
    D[PowerExpand[b[k]^(1/2)] * u[p[1], p[2], p[3], k], p[j]], p[j]];
s7 = D[PowerExpand[(b[i] / (b[j] * b[k])) ^ (1 / 2)] *
    D[PowerExpand[b[j]^(1/2)] * u[p[1], p[2], p[3], j], p[k]], p[j]];
          (* invr = {p[1] -> r, p[2] -> q, p[3] -> j}*)
lap = (1 / PowerExpand[b[k]^(1/2)]) * D[(1 / den) * (s1 + s2 + s3) , p[k]] -
    (1 / PowerExpand[(b[i] * b[j])^(1/2)]) * (s4 - s5 - s6 + s7);
   p[1] = r; p[2] = q; p[3] = j;
  lap = Expand[lap] // Simplify;
  lap );
 laplace[1]
 laplace[2]
 laplace[3]
```

$$\frac{1}{r^2}\left(-2\,u[r,q,j,1]-2\,\mathrm{Cot}[q]\,u[r,q,j,2]-2\,\mathrm{Csc}[q]\,u^{(0,0,1,0)}[r,q,j,3]+\mathrm{Csc}[q]^2\,u^{(0,0,2,0)}[r,q,j,1]+\mathrm{Cot}[q]\,u^{(0,1,0,0)}[r,q,j,1]-2\,u^{(0,1,0,0)}[r,q,j,2]+u^{(0,2,0,0)}[r,q,j,1]+2\,r\,u^{(1,0,0,0)}[r,q,j,1]+r^2\,u^{(2,0,0,0)}[r,q,j,1]\right)$$

$$\frac{1}{r^2}\left(-\mathrm{Csc}[q]^2\,u[r,q,j,2]-2\,\mathrm{Cot}[q]\,\mathrm{Csc}[q]\,u^{(0,0,1,0)}[r,q,j,3]+\mathrm{Csc}[q]^2\,u^{(0,0,2,0)}[r,q,j,2]+2\,u^{(0,1,0,0)}[r,q,j,1]+\mathrm{Cot}[q]\,u^{(0,1,0,0)}[r,q,j,2]+u^{(0,2,0,0)}[r,q,j,2]+2\,r\,u^{(1,0,0,0)}[r,q,j,2]+r^2\,u^{(2,0,0,0)}[r,q,j,2]\right)$$

$$\frac{1}{r^2}\left(-\mathrm{Csc}[q]^2\,u[r,q,j,3]+2\,\mathrm{Csc}[q]\,u^{(0,0,1,0)}[r,q,j,1]+2\,\mathrm{Cot}[q]\,\mathrm{Csc}[q]\,u^{(0,0,1,0)}[r,q,j,2]+\mathrm{Csc}[q]^2\,u^{(0,0,2,0)}[r,q,j,3]+\mathrm{Cot}[q]\,u^{(0,1,0,0)}[r,q,j,3]+u^{(0,2,0,0)}[r,q,j,3]+2\,r\,u^{(1,0,0,0)}[r,q,j,3]+r^2\,u^{(2,0,0,0)}[r,q,j,3]\right)$$

■ The Structure of the Output

The following results are obtained with the aid of the above program:

(1) the expression for the metric tensor g in the form of a 3x3 matrix;
(2) the analytic expression for the continuity equation in spherical coordinates;
(3) the expression for the Laplacian operator in the momentum equation for the velocity component v_r;
(4) the expression for the Laplacian operator in the momentum equation for the velocity component v_q;
(5) the expression for the Laplacian operator in the momentum equation for the velocity component v_j.