Metric Tensor Components

S.P. Kiselev, E.V. Vorozhtsov, and V.M. Fomin

■ Impressum

This Mathematica-Notebook is part of the book entitled
S.P. Kiselev, E.V. Vorozhtsov, and V.M. Fomin
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■ General Description

The program prog1-1.nb, which follows below, enables one to compute the analytic expressions for the metric tensor g components in the cylindrical coordinates r, j , z . This program also finds the following analytic expressions:

(a) the squared length element in cylindrical coordinates;
(b) the expressions for the basis vectors e_1, e_2, e_3 of the Cartesian coordinate system in terms of the basis vectors E_1, E_2, and E_3 of the cylindrical coordinate system.

The formulation of this problem and the discussion of its solution can be found in Section 1.1.9 of the above mentioned book, see Problem 1.1.

■ User's Guide

■ Step 1

Compile the program cell entitled "TheDefinition of Cylindrical Coordinates"
(see the Section "Program Listing").

■ Step 2

Compile the program cell entitled "The Partial Derivatives $\partial_x x_m$, $i, m = 1, 2, 3$."
(see the Section "Program Listing").

■ Step 3

Compile the program cell entitled "The Expressions for Metric Tensor Components"
(see the Section "Program Listing").

■ Step 4

Compile the program cell entitled "The Squared Length Element"
(see the Section "Program Listing").

■ Step 5

Compile the program cell entitled "The Expressions for the Basis Vectors"
(see the Section "Program Listing").

■ Program Listing

■ The Definition of Cylindrical Coordinates

The quantities x[1], x[2], x[3] are the Cartesian coordinates;
the quantities k[1], k[2], k[3] are the cylindrical coordinates.

```
x[1] =  k[1] Cos[k[2]]
x[2] = k[1] Sin[k[2]]
x[3] = k[3]
```

```
Cos[k[2]] k[1]

k[1] Sin[k[2]]

k[3]
```

■ The Partial Derivatives $\partial_{\xi} x_m$, $i, m = 1, 2, 3$.

```
g = IdentityMatrix[3];
Do[ Do[ g0 = Sum[D[x[m],k[i]]*D[x[m],k[j]],{m, 3}];
g = ReplacePart[g, g0, {i,j}], {i, 3}], {j, 3}]
```

■ The Expressions for Metric Tensor Components

```
Do [Do[gj = g[[i, j]] /. {k[1] Æ x1, k[2] Æ x2, k[3] Æ x3};
  g = ReplacePart[g, TrigReduce[gj], {i, j}], {j, 3}], {i, 3}];
    Print["g = ", MatrixForm[g]];
```

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_1^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

■ The Squared Length Element

```
dsq = Sum[Sum[g[[i, j]] * dk[i] * dk[j], {i, 3}], {j, 3}];
dsq = dsq /. {k[1] -> r, k[2] -> j, k[3] -> z,
              dk[1] -> dr, dk[2] -> dj, dk[3] -> dz};
Print["ds^2 = ", TraditionalForm[TrigReduce[dsq]]];
```

ds^2 = $dr^2 + dz^2 + dj^2\, x_1^2$

■ The Expressions for the Basis Vectors

```
e = {};
Do[ ei = Sum[D[x[m], k[i]] * B[m], {m, 3}];
   AppendTo[e, ei], {i, 3}];
Do[ei = e[[i]] /. {B[1] -> B⃗1, B[2] -> B⃗2,
   B[3] -> B⃗3, k[1] -> x1, k[2] -> x2, k[3] -> x3};
 Print["e(", i, ") = ", TraditionalForm[ei]], {i, 3}];
```

e(1) = $\cos(x_2)\, \vec{B}_1 + \sin(x_2)\, \vec{B}_2$

e(2) = $\cos(x_2)\, x_1\, \vec{B}_2 - \sin(x_2)\, x_1\, \vec{B}_1$

$$e(3) = \vec{E}_3$$

■ The Structure of the Output

The following results are obtained with the aid of the above program:

(1) the expression for the metric tensor g in the form of a 3x3 matrix;
(2) the analytic expression for the squared length element in the cylindrical coordinates;
(3) the expressions for the basis vectors e_1, e_2, e_3 of the Cartesian coordinate system in terms of the basis vectors E_1, E_2, and E_3 of the cylindrical coordinate system.