The Christoffel Symbols $(\Gamma^k)_{ij}$ in the Spherical Coordinates

S.P. Kiselev, E.V. Vorozhtsov, and V.M. Fomin

■ Impressum

This Mathematica-Notebook is part of the book entitled
S.P. Kiselev, E.V. Vorozhtsov, and V.M. Fomin
Foundations of Fluid Mechanics with Applications
Problem Solving Using *Mathematica.*
Birkhauser Boston, Basel, 1999.

■ Copyright

■ General Description

This Notebook enables the user to find the analytic expressions for the Christoffel symbols $(G_{ij})^k$ in the spherical coordinates.

A more detailed discussion of the Christoffel symbols and their application may be found in Section 2.1 of the above book.

In the case of other curvilinear coordinates different from the spherical coordinates please specify the expressions for the relations between the Cartesian coordinates x^1, x^2, x^3 and the curvilinear coordinates k[1] = x^1, k[2] = x^2, k[3] = x^3 in the cell entitled "The Definition of Spherical Coordinates".

■ User's Guide

■ Step 1

Compile the program cell entitled "The Definition of Spherical Coordinates" (see the Section "Program Listing").

■ Step 2

Compile the program cell entitled "The Partial Derivatives $\partial_x \, x_m$, $i, m = 1, 2, 3.$".
(see the Section "Program Listing").

■ Step 3

Compile the program cell entitled "The Expressions for Metric Tensor Components "
(see the Section "Program Listing").

■ Step 4

Compile the program cell entitled "The Metric Tensor with Contravariant Components g^{ij} "
(see the Section "Program Listing").

■ Step 5

Compile the program cell entitled "Calculation of the Christoffel Symbols $(G^k)_{ij}$, i,j,k = 1,2,3 "
(see the Section "Program Listing").

■ Program Listing

■ The Definition of Spherical Coordinates

The quantities x[1], x[2], x[3] are the Cartesian coordinates;
the quantities k[1], k[2], k[3] are the spherical coordinates.

```
x[1] =  k[1] Sin[k[2]] Cos[k[3]];
x[2] = k[1] Sin[k[2]] Sin[k[3]];
x[3] = k[1] Cos[k[2]];
```

■ The Partial Derivatives $\partial_\xi \, x_m$, $i, m = 1, 2, 3.$

```
g = IdentityMatrix[3];
Do[ Do[ g0 = Sum[D[x[m], k[i]] * D[x[m], k[j]], {m, 3}];
    g = ReplacePart[g, g0, {i, j}], {i, 3}], {j, 3}];
```

■ The Expressions for Metric Tensor Components

```
Do[ Do[gj = g[[i, j]] /. {k[1]  -> r, k[2] -> q, k[3] ->  j};
gj = TrigExpand[TrigReduce[gj]] /. {Cos[q]^2 -> 1 - Sin[q]^2};
 g = ReplacePart[g, Expand[gj], {i, j}], {j, 3}], {i, 3}];
Print["g = ", TraditionalForm[ MatrixForm[g]]];
```

g = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2(q) \end{pmatrix}$

■ The Metric Tensor with Contravariant Components g^{ij}

```
gc = IdentityMatrix[3];
Do[ Do[ g0 =  g[[i, j]]; aij = 0; If[g0 =!= 0, aij = 1 / g0];
    gc = ReplacePart[gc, aij, {i, j}], {i, 3}], {j, 3}];
  Print["g contravariant = ", TraditionalForm[ MatrixForm[gc]]];
```

g contravariant = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{\csc^2(q)}{r^2} \end{pmatrix}$

■ Calculation of the Christoffel Symbols $(\Gamma^k)_{ij}$, i, j, k = 1, 2, 3

```
Print["The nonzero Christoffel symbols"];
Do[Do[gj = g[[i, j]] /. {r -> y[1], q -> y[2], j -> y[3]};
  g = ReplacePart[g, gj, {i, j}], {j, 3}], {i, 3}];
Do[ Do[ Do[ gk = 1 / 2 * Sum[gc[[k, s]] * (D[g[[i, s]], y[j]] +
      D[g[[j, s]], y[i]] - D[g[[i, j]], y[s]]), {s, 3}];
   gk = gk /. {y[1] -> r, y[2] -> q, y[3] -> j};
If[gk =!= 0, Print["G_{", i, j, "}^", k, " = ",
       TraditionalForm[gk]]], {i, 3}], {j, 3}], {k, 3}];
```

The nonzero Christoffel symbols

G_{22}^1 = $-r$

G_{33}^1 = $-r \sin^2(q)$

G_{21}^2 = $\frac{1}{r}$

G_{12}^2 = $\frac{1}{r}$

G_{33}^2 = $-\cos(q) \sin(q)$

G_{31}^3 = $\frac{1}{r}$

G_{32}^3 = $\cot(q)$

$$G_{13}^3 = \frac{1}{r}$$

$$G_{23}^3 = \cot(q)$$

■ The Structure of the Output

The following results are obtained with the aid of the above program:

(1) the metric tensor g in the form of a 3x3 matrix;
(2) the metric tensor with contravariant components g^{ij} in the form of a 3x3 matrix;
(3) all the nonzero Christoffel symbols $(G^k)_{ij}$, i, j, k = 1,2,3.