Metric Tensor Components

S.P. Kiselev, E.V. Vorozhtsov, and V.M. Fomin

■ Impressum

This Mathematica-Notebook is part of the book entitled
S.P. Kiselev, E.V. Vorozhtsov, and V.M. Fomin
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Problem Solving Using *Mathematica.*
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■ General Description

The program prog1-2.nb, which follows below, enables one to compute the analytic expressions for the metric tensor g components in the spherical coordinates r, j , q . This program also finds the analytic expression for the squared length element in spherical coordinates.

In the case of other curvilinear coordinates different from the spherical coordinates please specify the expressions for the relations between the Cartesian coordinates x^1, x^2, x^3 and the curvilinear coordinates k[1] = x^1, k[2] = x^2, k[3] = x^3 in the cell entitled "The Definition of Spherical Coordinates".

The formulation of this problem and the discussion of its solution can be found in Section 1.1.9 of the above mentioned book, see Problem 1.2.

■ User's Guide

■ **Step 1**

Compile the program cell entitled "TheDefinition of Spherical Coordinates"
(see the Section "Program Listing").

■ Step 2

Compile the program cell entitled "The Partial Derivatives $\partial_x \, x_m$, $i, m = 1, 2, 3$."
(see the Section "Program Listing").

■ Step 3

Compile the program cell entitled "The Expressions for Metric Tensor Components"
(see the Section "Program Listing").

■ Step 4

Compile the program cell entitled "The Squared Length Element"
(see the Section "Program Listing").

■ Program Listing

■ The Definition of Spherical Coordinates

The quantities x[1], x[2], x[3] are the Cartesian coordinates;
the quantities k[1], k[2], k[3] are the cylindrical coordinates.

```
x[1] = k[1] Sin[k[2]] Cos[k[3]]
x[2] = k[1] Sin[k[2]] Sin[k[3]]
x[3] = k[1] Cos[k[2]]
```

```
Cos[k[3]] k[1] Sin[k[2]]

k[1] Sin[k[2]] Sin[k[3]]

Cos[k[2]] k[1]
```

■ The Partial Derivatives $\partial_{\xi} x_m$, $i, m = 1, 2, 3.$

```
g = IdentityMatrix[3];
Do[ Do[ g0 = Sum[D[x[m],k[i]]*D[x[m],k[j]],{m, 3}];
g = ReplacePart[g, g0, {i,j}], {i, 3}], {j, 3}]
```

■ The Expressions for Metric Tensor Components

```
Do[Do[gj = g[[i, j]] /. {k[1] -> r, k[2] -> q, k[3] -> j};
gj = TrigExpand[TrigReduce[gj]] /. {Cos[q]^2 -> 1 - Sin[q]^2};
 g = ReplacePart[g, Expand[gj], {i, j}], {j, 3}], {i, 3}];
Print["g = ", TraditionalForm[ MatrixForm[g]]];
```

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2(q) \end{pmatrix}$$

■ The Squared Length Element

```
dsq = Sum[Sum[g[[i, j]]*dk[i]*dk[j], {i, 3}], {j, 3}];
dsq = dsq /. {k[1] -> r, k[2] -> q, k[3] -> j,
             dk[1] -> dr, dk[2] -> dq, dk[3] -> dj};
Print["ds^2 = ", TraditionalForm[dsq]];
```

ds^2 = $dr^2 + dq^2\, r^2 + dj^2\, r^2 \sin^2(q)$

■ The Structure of the Output

The following results are obtained with the aid of the above program:

(1) the expression for the metric tensor g in the form of a 3x3 matrix;
(2) the analytic expression for the squared length element in the spherical coordinates.