

# Lectures on Homotopy Theory

The links below are to pdf files, which comprise my lecture notes for a first course on Homotopy Theory. I last gave this course at the University of Western Ontario during the Winter term of 2018.

The course material is widely applicable, in fields including Topology, Geometry, Number Theory, Mathematical Physics, and some forms of data analysis.

This collection of files is the basic source material for the course, and this page is an outline of the course contents. In practice, some of this is elective - I usually don't get much beyond proving the Hurewicz Theorem in classroom lectures. Also, despite the titles, each of the files covers much more material than one can usually present in a single lecture.

More detail on topics covered here can be found in the Goerss-Jardine book *Simplicial Homotopy Theory*, which appears in the References.

It would be quite helpful for a student to have a background in basic Algebraic Topology and/or Homological Algebra prior to working through this course.

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## References

## Lectures on Homotopy Theory

[http://uwo.ca/math/faculty/jardine/courses/homth/homotopy\\_theory.html](http://uwo.ca/math/faculty/jardine/courses/homth/homotopy_theory.html)

### Basic References

- [1] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*. Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition
- [2] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999
- [3] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998

# Lecture 01: Homological algebra

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## 1 Chain complexes

$R$  = commutative ring with 1 (eg.  $\mathbb{Z}$ , a field  $k$ )

### **$R$ -modules: basic definitions and facts**

- $f : M \rightarrow N$  an  $R$ -module homomorphism:

The *kernel*  $\ker(f)$  of  $f$  is defined by

$$\ker(f) = \{\text{all } x \in M \text{ such that } f(x) = 0\}.$$

$\ker(f) \subset M$  is a submodule.

The *image*  $\text{im}(f) \subset N$  of  $f$  is defined by

$$\text{im}(f) = \{f(x) \mid x \in M\}.$$

The *cokernel*  $\text{cok}(f)$  of  $f$  is the quotient

$$\text{cok}(f) = N / \text{im}(f).$$

- A sequence

$$M \xrightarrow{f} M' \xrightarrow{g} M''$$

is *exact* if  $\ker(g) = \operatorname{im}(f)$ . Equivalently,  $g \cdot f = 0$  and  $\operatorname{im}(f) \subset \ker(g)$  is surjective.

The sequence  $M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_n$  is *exact* if  $\ker = \operatorname{im}$  everywhere.

**Examples:** 1) The sequence

$$0 \rightarrow \ker(f) \rightarrow M \xrightarrow{f} N \rightarrow \operatorname{cok}(f) \rightarrow 0$$

is exact.

2) The sequence

$$0 \rightarrow M \xrightarrow{f} N$$

is exact if and only if  $f$  is a monomorphism (monic, injective)

3) The sequence

$$M \xrightarrow{f} N \rightarrow 0$$

is exact if and only if  $f$  is an epimorphism (epi, surjective).

**Lemma 1.1** (Snake Lemma). *Given a commutative diagram of  $R$ -module homomorphisms*

$$\begin{array}{ccccccc} & & A_1 & \longrightarrow & A_2 & \xrightarrow{p} & A_3 \longrightarrow 0 \\ & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 \\ 0 & \longrightarrow & B_1 & \xrightarrow{i} & B_2 & \longrightarrow & B_3 \end{array}$$

*in which the horizontal sequences are exact. There is an induced exact sequence*

$$\ker(f_1) \rightarrow \ker(f_2) \rightarrow \ker(f_3) \xrightarrow{\partial} \operatorname{cok}(f_1) \rightarrow \operatorname{cok}(f_2) \rightarrow \operatorname{cok}(f_3).$$

$\partial(y) = [z]$  for  $y \in \ker(f_3)$ , where  $y = p(x)$ , and  $f_2(x) = i(z)$ .

**Lemma 1.2** ( $(3 \times 3)$ -Lemma). *Given a commutative diagram of  $R$ -module maps*

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & B_1 & \xrightarrow{f} & B_2 & \xrightarrow{g} & B_3 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

*With exact columns.*

- 1) If either the top two or bottom two rows are exact, then so is the third.
- 2) If the top and bottom rows are exact, and  $g \cdot f = 0$ , then the middle row is exact.

**Lemma 1.3 (5-Lemma).** *Given a commutative diagram of  $R$ -module homomorphisms*

$$\begin{array}{ccccccccc}
 A_1 & \xrightarrow{f_1} & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \xrightarrow{g_1} & A_5 \\
 \downarrow h_1 & & \downarrow h_2 & & \downarrow h_3 & & \downarrow h_4 & & \downarrow h_5 \\
 B_1 & \xrightarrow{f_2} & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \xrightarrow{g_2} & B_5
 \end{array}$$

*with exact rows, such that  $h_1, h_2, h_4, h_5$  are isomorphisms. Then  $h_3$  is an isomorphism.*

The Snake Lemma is proved with an element chase.  
The  $(3 \times 3)$ -Lemma and 5-Lemma are consequences.

e.g. Prove the 5-Lemma with the induced diagram

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \text{cok}(f_1) & \longrightarrow & A_3 & \longrightarrow & \text{ker}(g_1) & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow h_3 & & \downarrow \cong & & \\
 0 & \longrightarrow & \text{cok}(f_2) & \longrightarrow & B_3 & \longrightarrow & \text{ker}(g_2) & \longrightarrow & 0
 \end{array}$$



## Chain complexes

A *chain complex*  $C$  in  $R$ -modules is a sequence of  $R$ -module homomorphisms

$$\dots \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} C_{-1} \xrightarrow{\partial} \dots$$

such that  $\partial^2 = 0$  (or that  $\text{im}(\partial) \subset \ker(\partial)$ ) everywhere.  $C_n$  is the module of  $n$ -chains.

A *morphism*  $f : C \rightarrow D$  of chain complexes consists of  $R$ -module maps  $f_n : C_n \rightarrow D_n$ ,  $n \in \mathbb{Z}$  such that there are comm. diagrams

$$\begin{array}{ccc} C_n & \xrightarrow{f_n} & D_n \\ \partial \downarrow & & \downarrow \partial \\ C_{n-1} & \xrightarrow{f_{n-1}} & D_{n-1} \end{array}$$

The chain complexes and their morphisms form a category, denoted by  $Ch(R)$ .

- If  $C$  is a chain complex such that  $C_n = 0$  for  $n < 0$ , then  $C$  is an *ordinary* chain complex. We usually drop all the 0 objects, and write

$$\rightarrow C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$$

$Ch_+(R)$  is the full subcategory of ordinary chain complexes in  $Ch(R)$ .

- Chain complexes indexed by the integers are often called *unbounded* complexes.

*Slogan:* Ordinary chain complexes are spaces, and unbounded complexes are spectra.

- Chain complexes of the form

$$\cdots \rightarrow 0 \rightarrow C_0 \rightarrow C_{-1} \rightarrow \cdots$$

are *cochain complexes*, written (classically) as

$$C^0 \rightarrow C^1 \rightarrow C^2 \rightarrow \cdots$$

Both notations are in common (confusing) use.

Morphisms of chain complexes have kernels and cokernels, defined degreewise.

A sequence of chain complex morphisms

$$C \rightarrow D \rightarrow E$$

is *exact* if all sequences of morphisms

$$C_n \rightarrow D_n \rightarrow E_n$$

are exact.

## Homology

Given a chain complex  $C$  :

$$\cdots \rightarrow C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \rightarrow \cdots$$

Write

$$Z_n = Z_n(C) = \ker(\partial : C_n \rightarrow C_{n-1}), \text{ (} n\text{-cycles), and}$$
$$B_n = B_n(C) = \operatorname{im}(\partial : C_{n+1} \rightarrow C_n) \text{ (} n\text{-boundaries).}$$

$$\partial^2 = 0, \text{ so } B_n(C) \subset Z_n(C).$$

The  $n^{\text{th}}$  homology group  $H_n(C)$  of  $C$  is defined by

$$H_n(C) = Z_n(C) / B_n(C).$$

A chain map  $f : C \rightarrow D$  induces  $R$ -module maps

$$f_* : H_n(C) \rightarrow H_n(D), \quad n \in \mathbb{Z}.$$

$f : C \rightarrow D$  is a *homology isomorphism* (resp. *quasi-isomorphism*, *acyclic map*, *weak equivalence*) if all induced maps  $f_* : H_n(C) \rightarrow H_n(D)$ ,  $n \in \mathbb{Z}$  are isomorphisms.

A complex  $C$  is *acyclic* if the map  $0 \rightarrow C$  is a homology isomorphism, or if  $H_n(C) \cong 0$  for all  $n$ , or if the sequence

$$\cdots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} C_{-1} \xrightarrow{\partial} \cdots$$

is exact.

**Lemma 1.4.** *A short exact sequence*

$$0 \rightarrow C \rightarrow D \rightarrow E \rightarrow 0$$

*induces a natural long exact sequence*

$$\dots \xrightarrow{\partial} H_n(C) \rightarrow H_n(D) \rightarrow H_n(E) \xrightarrow{\partial} H_{n-1}(C) \rightarrow \dots$$

*Proof.* The short exact sequence induces comparisons of exact sequences

$$\begin{array}{ccccccc} C_n/B_n(C) & \longrightarrow & D_n/B_n(D) & \longrightarrow & E_n/B_n(E) & \longrightarrow & 0 \\ \downarrow \partial_* & & \downarrow \partial_* & & \downarrow \partial_* & & \\ 0 \longrightarrow & Z_{n-1}(C) & \longrightarrow & Z_{n-1}(D) & \longrightarrow & Z_{n-1}(E) & \end{array}$$

Use the natural exact sequence

$$0 \rightarrow H_n(C) \rightarrow C_n/B_n(C) \xrightarrow{\partial_*} Z_{n-1}(C) \rightarrow H_{n-1}(C) \rightarrow 0$$

Apply the Snake Lemma.  $\square$

## 2 Ordinary chain complexes

A map  $f : C \rightarrow D$  in  $Ch_+(R)$  is a

- *weak equivalence* if  $f$  is a homology isomorphism,
- *fibration* if  $f : C_n \rightarrow D_n$  is surjective for  $n > 0$ ,
- *cofibration* if  $f$  has the left lifting property (LLP) with respect to all morphisms of  $Ch_+(R)$  which

are simultaneously fibrations and weak equivalences.

A *trivial fibration* is a map which is both a fibration and a weak equivalence. A *trivial cofibration* is both a cofibration and a weak equivalence.

$f$  has the *left lifting property* with respect to all trivial fibrations (ie.  $f$  is a cofibration) if given any solid arrow commutative diagram

$$\begin{array}{ccc} C & \longrightarrow & X \\ f \downarrow & \nearrow & \downarrow p \\ D & \longrightarrow & Y \end{array}$$

in  $Ch_+(R)$  with  $p$  a trivial fibration, then the dotted arrow exists making the diagram commute.

Special chain complexes and chain maps:

- $R(n)$  [=  $R[-n]$  in “shift notation”] consists of a copy of the free  $R$ -module  $R$ , concentrated in degree  $n$ :

$$\cdots \rightarrow 0 \rightarrow 0 \rightarrow \overset{n}{R} \rightarrow 0 \rightarrow 0 \rightarrow \cdots$$

There is a natural  $R$ -module isomorphism

$$\text{hom}_{Ch_+(R)}(R(n), C) \cong Z_n(C).$$

- $R\langle n+1 \rangle$  is the complex

$$\cdots \rightarrow 0 \rightarrow \overset{n+1}{R} \xrightarrow{1} \overset{n}{R} \rightarrow 0 \rightarrow \cdots$$

- There is a natural  $R$ -module isomorphism

$$\text{hom}_{Ch_+(R)}(R\langle n+1 \rangle, C) \cong C_{n+1}.$$

- There is a chain  $\alpha : R(n) \rightarrow R\langle n+1 \rangle$

$$\begin{array}{ccccccc} \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & R \longrightarrow 0 \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow 1 \\ \dots & \longrightarrow & 0 & \longrightarrow & R & \xrightarrow{1} & R \longrightarrow 0 \longrightarrow \dots \end{array}$$

$\alpha$  classifies the cycle  $1 \in R\langle n+1 \rangle_n$ .

**Lemma 2.1.** *Suppose that  $p : A \rightarrow B$  is a fibration and that  $i : K \rightarrow A$  is the inclusion of the kernel of  $p$ . Then there is a long exact sequence*

$$\begin{aligned} \dots &\xrightarrow{p_*} H_{n+1}(B) \xrightarrow{\partial} H_n(K) \xrightarrow{i_*} H_n(A) \xrightarrow{p_*} H_n(B) \xrightarrow{\partial} \dots \\ &\dots \xrightarrow{\partial} H_0(K) \xrightarrow{i_*} H_0(A) \xrightarrow{p_*} H_0(B). \end{aligned}$$

*Proof.*  $j : \text{im}(p) \subset B$ , and write  $\pi : A \rightarrow \text{im}(p)$  for the induced epimorphism. Then  $H_n(\text{im}(p)) = H_n(B)$  for  $n > 0$ , and there is a diagram

$$\begin{array}{ccc} H_0(A) & \xrightarrow{p_*} & H_0(B) \\ & \searrow \pi_* & \nearrow i_* \\ & H_0(\text{im}(p)) & \end{array}$$

in which  $\pi_*$  is an epimorphism and  $i_*$  is a monomorphism (exercise). The long exact sequence is con-

structed from the long exact sequence in homology for the short exact sequence

$$0 \rightarrow K \xrightarrow{i} A \xrightarrow{\pi} \text{im}(p) \rightarrow 0,$$

with the monic  $i_* : H_0(\text{im}(p)) \rightarrow H_0(B)$ .  $\square$

**Lemma 2.2.**  $p : A \rightarrow B$  is a fibration if and only if  $p$  has the RLP wrt. all maps  $0 \rightarrow R\langle n+1 \rangle$ ,  $n \geq 0$ .

*Proof.* The lift exists in all solid arrow diagrams

$$\begin{array}{ccc} 0 & \longrightarrow & A \\ \downarrow & \nearrow & \downarrow p \\ R\langle n+1 \rangle & \longrightarrow & B \end{array}$$

for  $n \geq 0$ .  $\square$

**Corollary 2.3.**  $0 \rightarrow R\langle n+1 \rangle$  is a cofibration for all  $n \geq 0$ .

*Proof.* This map has the LLP wrt all fibrations, hence wrt all trivial fibrations.  $\square$

**Lemma 2.4.** The map  $0 \rightarrow R(n)$  is a cofibration.

*Proof.* The trivial fibration  $p : A \rightarrow B$  induces an epimorphism  $Z_n(A) \rightarrow Z_n(B)$  for all  $n \geq 0$ :

$$\begin{array}{ccccccccc} A_{n+1} & \twoheadrightarrow & B_n(A) & \longrightarrow & Z_n(A) & \longrightarrow & H_n(A) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \cong & & \\ B_{n+1} & \twoheadrightarrow & B_n(B) & \longrightarrow & Z_n(B) & \longrightarrow & H_n(B) & \longrightarrow & 0 \end{array}$$

$\square$

A chain complex  $A$  is *cofibrant* if the map  $0 \rightarrow A$  is a cofibration.

eg.  $R\langle n+1 \rangle$  and  $R(n)$  are cofibrant.

All chain complexes  $C$  are *fibrant*, because all chain maps  $C \rightarrow 0$  are fibrations.

**Proposition 2.5.**  $p : A \rightarrow B$  is a trivial fibration and if and only if

- 1)  $p : A_0 \rightarrow B_0$  is a surjection, and
- 2)  $p$  has the RLP wrt all  $\alpha : R(n) \rightarrow R\langle n+1 \rangle$ .

**Corollary 2.6.**  $\alpha : R(n) \rightarrow R\langle n+1 \rangle$  is a cofibration.

*Proof of Proposition 2.5.* 1) Suppose that  $p : A \rightarrow B$  is a trivial fibration with kernel  $K$ .

Use Snake Lemma with the comparison

$$\begin{array}{ccccccc} A_1 & \xrightarrow{\partial} & A_0 & \longrightarrow & H_0(A) & \longrightarrow & 0 \\ p \downarrow & & p \downarrow & & \downarrow \cong & & \\ B_1 & \xrightarrow{\partial} & B_0 & \longrightarrow & H_0(B) & \longrightarrow & 0 \end{array}$$

to show that  $p : A_0 \rightarrow B_0$  is surjective.

Suppose given a diagram

$$\begin{array}{ccc} R(n) & \xrightarrow{x} & A \\ \alpha \downarrow & & \downarrow p \\ R\langle n+1 \rangle & \xrightarrow{y} & B \end{array}$$



Choose  $z \in A_{n+1}$  such that  $p(z) = y$ . Then  $x - \partial(z)$  is a cycle of  $K$ , and  $K$  is acyclic (exercise) so there is a  $v \in K_{n+1}$  such that  $\partial(v) = x - \partial(z)$ .  $\partial(z + v) = x$  and  $p(z + v) = p(v) = y$ , so  $v + z$  is the desired lift.

2) Suppose that  $p : A_0 \rightarrow B_0$  is surjective and that  $p$  has the right lifting property with respect to all  $R(n) \rightarrow R\langle n+1 \rangle$ .

The solutions of the lifting problems

$$\begin{array}{ccc} R(n) & \xrightarrow{0} & A \\ \downarrow & \nearrow & \downarrow p \\ R\langle n+1 \rangle & \xrightarrow{x} & B \end{array}$$

show that  $p$  is surjective on all cycles, while the solutions of the lifting problems

$$\begin{array}{ccc} R(n) & \xrightarrow{x} & A \\ \downarrow & \nearrow & \downarrow p \\ R\langle n+1 \rangle & \xrightarrow{y} & B \end{array}$$

show that  $p$  induces a monomorphism in all homology groups. It follows that  $p$  is a weak equivalence.

We have the diagram

$$\begin{array}{ccccccc}
Z_{n+1}(A) & \longrightarrow & A_{n+1} & \xrightarrow{\partial} & Z_n(A) & \longrightarrow & H_n(A) \longrightarrow 0 \\
\downarrow p & & \downarrow p & & \downarrow p & & p \downarrow \cong \\
Z_{n+1}(B) & \longrightarrow & B_{n+1} & \xrightarrow{\partial} & Z_n(B) & \longrightarrow & H_n(B) \longrightarrow 0
\end{array}$$

Then  $p : B_n(A) \rightarrow B_n(B)$  is epi, so  $p : A_{n+1} \rightarrow B_{n+1}$  is epi, for all  $n \geq 0$ .  $\square$

**Proposition 2.7.** *Every chain map  $f : C \rightarrow D$  has two factorizations*

$$\begin{array}{ccc}
& E & \\
i \nearrow & & \searrow p \\
C & \xrightarrow{f} & D \\
j \searrow & & \nearrow q \\
& F &
\end{array}$$

where

- 1)  $p$  is a fibration.  $i$  is a monomorphism, a weak equivalence and has the LLP wrt all fibrations.
- 2)  $q$  is a trivial fibration and  $j$  is a monomorphism and a cofibration.

*Proof.* 1) Form the factorization

$$\begin{array}{ccc}
 & C \oplus (\bigoplus_{x \in D_{n+1}, n \geq 0} R\langle n+1 \rangle) & \\
 i \nearrow & & \searrow p \\
 C & \xrightarrow{f} & D
 \end{array}$$

$p$  is the sum of  $f$  and all classifying maps for chains  $x$  in all non-zero degrees. It is therefore surjective in non-zero degrees, hence a fibration.

$i$  is the inclusion of a direct summand with acyclic cokernel, and is thus a monomorphism and a weak equivalence.  $i$  is a direct sum of maps which have the LLP wrt all fibrations, and thus has the same lifting property.

2) Recall that  $A \rightarrow B$  is a trivial fibration if and only if it has the RLP wrt all cofibrations  $R(n) \rightarrow R\langle n+1 \rangle$ ,  $n \geq -1$ .

Notation:  $R(-1) \rightarrow R\langle 0 \rangle$  is the map  $0 \rightarrow R(0)$ .

Consider the set of all diagrams

$$\begin{array}{ccc}
 D: & R(n_D) & \xrightarrow{\alpha_D} C \\
 & \downarrow & \downarrow f=q_0 \\
 & R\langle n_D+1 \rangle & \xrightarrow{\beta_D} D
 \end{array}$$

and form the pushout

$$\begin{array}{ccc}
 \bigoplus_D R(n_D) & \xrightarrow{(\alpha_D)} & C_0 \\
 \downarrow & & \downarrow j_1 \\
 \bigoplus_D R\langle n_D + 1 \rangle & \xrightarrow{(\theta_D)} & C_1 \\
 & \searrow (\beta_D) & \swarrow q_0 \\
 & & D
 \end{array}$$

where  $C = C_0$ . Then  $j_1$  is a monomorphism and a cofibration, because the collection of all such maps is closed under direct sum and pushout.

Every lifting problem  $D$  as above is solved in  $C_1$ :

$$\begin{array}{ccccc}
 R(n_D) & \xrightarrow{\alpha_D} & C_0 & \xrightarrow{j_1} & C_1 \\
 \downarrow & & & \nearrow \theta_D & \downarrow q_1 \\
 R\langle n_D + 1 \rangle & & & & D \\
 & \xrightarrow{\beta_D} & & & 
 \end{array}$$

commutes.

Repeat this process inductively for the maps  $q_i$  to produce a string of factorizations

$$\begin{array}{ccccccc}
 C_0 & \xrightarrow{j_1} & C_1 & \xrightarrow{j_2} & C_2 & \xrightarrow{j_3} & \dots \\
 q_0 \downarrow & q_1 \swarrow & & \nearrow q_2 & & & \\
 & & D & & & & 
 \end{array}$$

Let  $F = \varinjlim C_i$ . Then  $f$  has a factorization

$$\begin{array}{ccc}
 C & \xrightarrow{j} & F \\
 & \searrow f & \downarrow q \\
 & & D
 \end{array}$$

Then  $j$  is a cofibration and a monomorphism, because all  $j_k$  have these properties and the family of such maps is closed under (infinite) composition.

Finally, given a diagram

$$\begin{array}{ccc} R(n) & \xrightarrow{\alpha} & F \\ \downarrow & & \downarrow q \\ R\langle n+1 \rangle & \xrightarrow{\beta} & D \end{array}$$

The map  $\alpha$  factors through some finite stage of the filtered colimit defining  $F$ , so that  $\alpha$  is a composite

$$R(n) \xrightarrow{\alpha'} C_k \rightarrow F$$

for some  $k$ . The lifting problem

$$\begin{array}{ccc} R(n) & \xrightarrow{\alpha'} & C_k \\ \downarrow & & \downarrow q_k \\ R\langle n+1 \rangle & \xrightarrow{\beta} & D \end{array}$$

is solved in  $C_{k+1}$ , hence in  $F$ . □

**Remark:** This proof is a *small object argument*.

The  $R(n)$  are *small* (or compact):  $\text{hom}(R(n), \text{ })$  commutes with filtered colimits.

**Corollary 2.8.** *1) Every cofibration is a monomorphism.*

*2) Suppose that  $j : C \rightarrow D$  is a cofibration and a weak equivalence. Then  $j$  has the LLP wrt all fibrations.*

*Proof.* 2) The map  $j$  has a factorization

$$\begin{array}{ccc} C & \xrightarrow{i} & F \\ & \searrow j & \downarrow p \\ & & D \end{array}$$

where  $i$  has the left lifting property with respect to all fibrations and is a weak equivalence, and  $p$  is a fibration. Then  $p$  is a trivial fibration, so the lifting exists in the diagram

$$\begin{array}{ccc} C & \xrightarrow{i} & F \\ j \downarrow & \nearrow & \downarrow p \\ D & \xrightarrow{1} & D \end{array}$$

since  $j$  is a cofibration. Then  $j$  is a retract of a map (namely  $i$ ) which has the LLP wrt all fibrations, and so  $j$  has the same property.

1) is an exercise. □

## Resolutions

Suppose that  $P$  is a chain complex. Proposition 2.7 says that  $0 \rightarrow P$  has a factorization

$$\begin{array}{ccc} 0 & \xrightarrow{j} & F \\ & \searrow & \downarrow q \\ & & P \end{array}$$

where  $j$  is a cofibration (so that  $F$  is cofibrant) and  $q$  is a trivial fibration, hence a weak equivalence.

The proof of Proposition 2.7 implies that each  $R$ -module  $F_n$  is free, so  $F$  is a *free resolution* of  $P$ .

If the complex  $P$  is cofibrant, then the lift exists in

$$\begin{array}{ccc} 0 & \longrightarrow & F \\ \downarrow & \nearrow & \downarrow q \\ P & \xrightarrow{1} & P \end{array}$$

All modules  $P_n$  are direct summands of free modules and are therefore projective.

This observation has a converse:

**Lemma 2.9.** *A chain complex  $P$  is cofibrant if and only if all modules  $P_n$  are projective.*

*Proof.* Suppose that  $P$  is a complex of projectives, and  $p : A \rightarrow B$  is a trivial fibration.

Then  $p : A_n \rightarrow B_n$  is surjective for all  $n \geq 0$  and has acyclic kernel  $i : K \rightarrow A$ .

Suppose given a lifting problem

$$\begin{array}{ccc} 0 & \longrightarrow & A \\ \downarrow & \nearrow \theta & \downarrow p \\ P & \xrightarrow{f} & B \end{array}$$

There is a map  $\theta_0 : P_0 \rightarrow A_0$  which lifts  $f_0$ :

$$\begin{array}{ccc} & & A_0 \\ & \nearrow \theta_0 & \downarrow p_0 \\ P_0 & \xrightarrow{f_0} & B_0 \end{array}$$

Suppose given a lift up to degree  $n$ , ie. homomorphisms  $\theta_i : P_i \rightarrow A_i$  for  $i \leq n$  such that  $p_i \theta_i = f_i$  for  $i \leq n$  and  $\partial \theta_i = \theta_{i-1} \partial$  for  $1 \leq i \leq n$

There is a map  $\theta'_{n+1} : P_{n+1} \rightarrow A_{n+1}$  such that  $p_{n+1} \theta'_{n+1} = f_{n+1}$ .

Then

$$p_n(\partial \theta'_{n+1} - \theta_n \partial) = \partial p_{n+1} \theta'_{n+1} - f_n \partial = \partial f_{n+1} - f_n \partial = 0$$

so there is a  $v : P_{n+1} \rightarrow K_n$  such that

$$i_n v = \partial \theta'_{n+1} - \theta_n \partial.$$

Also

$$\partial(\partial \theta'_{n+1} - \theta_n \partial) = 0$$



and  $K$  is acyclic, so there is a  $w : P_{n+1} \rightarrow K_{n+1}$  such that

$$i_n \partial w = \partial \theta'_{n+1} - \theta_n \partial.$$

Then

$$\partial(\theta'_{n+1} - i_{n+1} w) = \theta_n \partial$$

and

$$p_{n+1}(\theta'_{n+1} - i_{n+1} w) = p_{n+1} \theta'_{n+1} = f_{n+1}.$$

□

**Remarks:**

- 1) Every chain complex  $C$  has a *cofibrant model*, i.e. a weak equivalence  $p : P \rightarrow C$  with  $P$  cofibrant (aka. complex of projectives).
- 2)  $M =$  an  $R$ -module. A cofibrant model  $P \rightarrow M(0)$  is a projective resolution of  $M$  in the usual sense.
- 3) Cofibrant models  $P \rightarrow C$  are also (commonly) constructed with Eilenberg-Cartan resolutions.

### 3 Closed model categories

A *closed model category* is a category  $\mathcal{M}$  equipped with three classes of maps, namely weak equivalences, fibrations and cofibrations, such that the following conditions are satisfied:

**CM1** The category  $\mathcal{M}$  has all finite limits and colimits.

**CM2** Given a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ & \searrow h \quad \swarrow f & \\ & Z & \end{array}$$

of morphisms in  $\mathcal{M}$ , if any two of  $f, g$  and  $h$  are weak equivalences, then so is the third.

**CM3** The classes of cofibrations, fibrations and weak equivalences are closed under retraction.

**CM4** Given a commutative solid arrow diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & \nearrow & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

such that  $i$  is a cofibration and  $p$  is a fibration. Then the lift exists making the diagram commute if either  $i$  or  $p$  is a weak equivalence.

**CM5** Every morphism  $f : X \rightarrow Y$  has factorizations

$$\begin{array}{ccc}
 & Z & \\
 i \nearrow & & \searrow p \\
 X & \xrightarrow{f} & Y \\
 j \searrow & & \nearrow q \\
 & W &
 \end{array}$$

where  $p$  is a fibration and  $i$  is a trivial cofibration, and  $q$  is a trivial fibration and  $j$  is a cofibration.

**Theorem 3.1.** *With the definition of weak equivalence, fibration and cofibration given above,  $Ch_+(R)$  satisfies the axioms for a closed model category.*

*Proof.* **CM1**, **CM2** and **CM3** are exercises. **CM5** is Proposition 2.7, and **CM4** is Corollary 2.8.  $\square$

**Exercise:** A map  $f : C \rightarrow D$  of  $Ch(R)$  (unbounded chain complexes) is a *weak equivalence* if it is a homology isomorphism.

$f$  is a *fibration* if all maps  $f : C_n \rightarrow D_n$ ,  $n \in \mathbb{Z}$  are surjective.

A map of is a *cofibration* if and only if it has the left lifting property with respect to all trivial fibrations.

Show that, with these definitions,  $Ch(R)$  has the structure of a closed model category.

## Lecture 02: Spaces

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### 4 Spaces and homotopy groups

#### Some definitions

**CGWH** is the category of compactly generated weak Hausdorff spaces.

A space  $X$  is *compactly generated* if a subset  $Z$  is closed if and only if  $Z \cap K$  is closed for all maps  $K \rightarrow X$  with  $K$  compact.

A compactly generated space  $X$  is *weakly Hausdorff* if and only if the image of the diagonal  $\Delta : X \rightarrow X \times X$  is closed in  $X \times X$ , where the product is in the category of compactly generated spaces.

**CGWH** is the “convenient category” for homotopy theory, because it’s cartesian closed, as well as complete and cocomplete.

The product  $X \times Y$  in **CGWH** has the underlying point set that you expect, but it’s topologized as

a colimit of all products  $C \times D$  where  $C \rightarrow X$  and  $D \rightarrow Y$  are maps such that  $C$  and  $D$  are compact.

If  $X$  and  $Y$  are compact, this definition doesn't affect the topology on  $X \times Y$ .

All  $CW$ -complexes (spaces inductively built from cells) are members of **CGWH**.

See the preprint

N.P. Strickland. The category of CGWH spaces. Preprint, Sheffield <http://www.neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf>, 2009

## Examples that we care about

The *topological standard  $n$ -simplex* is the space  $|\Delta^n|$  defined by

$$|\Delta^n| = \{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1, t_i \geq 0 \}.$$

$|\Delta^0|$  is a point,  $|\Delta^1|$  is a copy of the unit interval,  $|\Delta^2|$  is a triangle, etc.

$\mathbf{n} = \{0, 1, \dots, n\}$ ,  $n \geq 0$ , with the obvious poset structure — this is the *finite ordinal number  $\mathbf{n}$* .

The finite ordinal numbers  $\mathbf{n}$ ,  $n \geq 0$ , form a category  $\Delta$ , whose morphisms are the order-preserving functions (aka. poset morphisms)  $\theta : \mathbf{m} \rightarrow \mathbf{n}$ .

The monomorphisms  $d^i : \mathbf{n} - \mathbf{1} \rightarrow \mathbf{n}$  have the form

$$d^i(j) = \begin{cases} j & \text{if } j < i, \text{ and} \\ j+1 & \text{if } j \geq i. \end{cases}$$

with  $0 \leq i \leq n$ . The map  $d^i$  misses the element  $i \in \mathbf{n}$ .

$s^j : \mathbf{n} + \mathbf{1} \rightarrow \mathbf{n}$ ,  $0 \leq j \leq n$ , is the unique poset epimorphism such that  $s^j(j) = s^j(j+1) = j$ .

The  $s^j$ ,  $0 \leq j \leq n$  form a complete list of epimorphisms  $\mathbf{n} + \mathbf{1} \rightarrow \mathbf{n}$  in  $\Delta$ .

### The singular set

There is a functor

$$|\Delta| : \Delta \rightarrow \mathbf{CGWH}$$

with  $\mathbf{n} \mapsto |\Delta^n|$ . The morphism  $\theta : \mathbf{m} \rightarrow \mathbf{n}$  induces the continuous map  $\theta_* : |\Delta^m| \rightarrow |\Delta^n|$ , with

$$\theta_*(t_0, \dots, t_m) = (s_0, \dots, s_n),$$

and

$$s_i = \sum_{j \in \theta^{-1}(i)} t_j.$$

An *n-simplex* of a space  $X$  is a continuous map  $\sigma : |\Delta^n| \rightarrow X$ .

The  $i^{th}$  face  $d_i(\sigma)$  of the  $n$ -simplex  $\sigma$  is the composite

$$|\Delta^{n-1}| \xrightarrow{d^i} |\Delta^n| \xrightarrow{\sigma} X.$$

The vertex  $v_j$  of  $\sigma$  is the composite

$$|\Delta^0| \xrightarrow{j} |\Delta^n| \xrightarrow{\sigma} X,$$

(an element of  $X$ ), where  $j : \mathbf{0} \rightarrow \mathbf{n}$  is defined by  $j(0) = j \in \mathbf{n}$ .

$v_j$  is the vertex opposite the face  $d_j(\sigma)$ .

**Example:** Suppose  $\sigma : |\Delta^2| \rightarrow X$  is a 2-simplex. Here's the picture:

$$\begin{array}{ccc} v_0 & \xrightarrow{d_2(\sigma)} & v_1 \\ & \searrow d_1(\sigma) & \downarrow d_0\sigma \\ & & v_2 \end{array} \qquad \begin{array}{ccc} 0 & \xrightarrow{[0,1]} & 1 \\ & \searrow [0,2] & \downarrow [1,2] \\ & & 2 \end{array}$$

Some language:

$$S(X)_n = \text{hom}(|\Delta^n|, X)$$

is the set of  $n$ -simplices of  $X$ .

An ordinal number map  $\theta : \mathbf{m} \rightarrow \mathbf{n}$  induces a function  $\theta^* : S_n(X) \rightarrow S_m(X)$  by precomposition with  $\theta : |\Delta^m| \rightarrow |\Delta^n|$ .

The composite

$$|\Delta^m| \xrightarrow{\theta} |\Delta^n| \xrightarrow{\sigma} X$$

is  $\theta^*(\sigma) \in S_m(X)$ .

The simplices and precompositions define a (contravariant) functor

$$S(X) : \Delta^{op} \rightarrow \mathbf{Set}$$

taking values in sets.  $S(X)$  is a *simplicial set*, called the *singular set* for the space  $X$ .

### Path components

A *path* in  $X$  is a 1-simplex  $\omega : |\Delta^1| \rightarrow X$  of  $X$ , while a vertex is an element  $x : |\Delta^0| \rightarrow X$ .

A path has a natural orientation:

$$x = d_1(\omega) \xrightarrow{\omega} d_0(\omega) = y$$

reflects

$$d^1(0) = 0 \rightarrow 1 = d^0(0)$$

in  $\mathbf{1}$ .

The set of *path components*  $\pi_0|X|$  of  $X$  is defined by a coequalizer

$$S(X)_1 \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} S(X)_0 \longrightarrow \pi_0|X|$$

in the set category.



## Fundamental groupoid

Suppose that the paths  $\omega, \omega' : |\Delta^1| \rightarrow X$  start at  $x$  and end at  $y$  in the sense that  $d_1(\omega) = d_1(\omega') = x$  and  $d_0(\omega) = d_0(\omega') = y$ .

Alternate notation:

$$\partial(\omega) = \partial(\omega') = (y, x).$$

Say that  $\omega$  is *homotopic to  $\omega'$  rel. end points* if there is a commutative diagram

$$\begin{array}{ccc} (|\Delta^0| \sqcup |\Delta^0|) \times I & \xrightarrow{pr} & (|\Delta^0| \sqcup |\Delta^0|) \\ (d^1, d^0) \times I \downarrow & & \downarrow (x, y) \\ |\Delta^1| \times I & \xrightarrow{h} & X \\ (d^1, d^0) \uparrow & \nearrow (\omega, \omega') & \\ |\Delta^1| \sqcup |\Delta^1| & & \end{array}$$

Here,  $I = [0, 1]$ , or some homeomorphic copy of it, like  $|\Delta^1|$ .

The map  $h$  is a *homotopy* from  $\omega$  to  $\omega'$  (note the direction). One represents  $h$  by the following picture:

$$\begin{array}{ccc} x & \xrightarrow{\omega} & y \\ x \downarrow & \Downarrow & \downarrow y \\ x & \xrightarrow{\omega'} & y \end{array}$$

Homotopy of paths rel end points in a space  $X$  is an equivalence relation (exercise), and the set of

homotopy classes of paths rel end points from  $x$  to  $y$  is denoted by  $\pi(X)(x, y)$ . This is the set of morphisms from  $x$  to  $y$  in the *fundamental groupoid*  $\pi(X)$  of the space  $X$ .

There's a law of composition for  $\pi(X)$ , but we need more notation to describe it.

### Nice little spaces

1)  $|\partial\Delta^n|$  is the *topological boundary* of  $|\Delta^n|$ : it is the union of the faces  $d^i : |\Delta^{n-1}| \rightarrow |\Delta^n|$ . Any two such faces intersect in a lower dimensional face  $|\Delta^{n-2}|$ , and there is a coequalizer picture

$$\bigsqcup_{i < j, 0 \leq i, j \leq n} |\Delta^{n-2}| \rightrightarrows \bigsqcup_{0 \leq i \leq n} |\Delta^{n-1}| \longrightarrow |\partial\Delta^n|$$

in spaces, which is defined by the identities  $d^j d^i = d^i d^{j-1}$  for  $i < j$ .

2)  $|\Lambda_k^n| \subset |\partial\Delta^n|$  is obtained by throwing away the  $k^{th}$  face  $d^k : |\Delta^{n-1}| \rightarrow |\Delta^n|$ . There is a coequalizer

$$\bigsqcup_{i < j, i, j \neq k} |\Delta^{n-2}| \rightrightarrows \bigsqcup_{0 \leq i \leq n, i \neq k} |\Delta^{n-1}| \longrightarrow |\Lambda_k^n|$$

defined by the identities  $d^j d^i = d^i d^{j-1}$  for  $i < j$ .  $|\Lambda_k^n|$  is the  $k^{th}$  *horn* of  $|\Delta^n|$ .

The inclusion  $i : |\Lambda_k^n| \subset |\Delta^n|$  is a *strong deformation retraction*.

There is a map  $r : |\Delta^n| \rightarrow |\Lambda_k^n|$  (projection along the normal to the missing simplex) such that  $r \cdot i = 1$  and  $i \cdot r$  is homotopic to the identity on  $|\Delta^n| \text{ rel } |\Lambda_k^n|$ .

It follows that the dotted arrow exists, making the diagram commute in all solid arrow pictures

$$\begin{array}{ccc} |\Lambda_k^n| & \xrightarrow{\alpha} & X \\ i \downarrow & \nearrow & \downarrow \\ |\Delta^n| & \longrightarrow & * \end{array}$$

The lift is given by the composite  $\alpha \cdot r$ , and  $*$  is the one-point space (aka  $|\Delta^0|$ ), which is terminal in **CGWH**.

Here's some other inclusions which admit strong deformation retractions (secretly made up of instances of inclusions of horns in simplices):

- $(|\Delta^n| \times \{\varepsilon\}) \cup (|\partial\Delta^n| \times I) \subset |\Delta^n| \times I$  where  $\varepsilon = 0, 1$ ,
- $(|\Delta^n| \times \{0, 1\}) \cup (|\Lambda_k^n| \times I) \subset |\Delta^n| \times I$ .

$I = [0, 1]$  is the unit interval, and  $I \cong |\Delta^1|$ .

Any space  $X$  has the right lifting property for these inclusions, as for the maps  $|\Lambda_k^n| \subset |\Delta^n|$ .

## Composition law

A map  $|\Lambda_1^2| \rightarrow X$  is a string of paths  $x \xrightarrow{\omega} y \xrightarrow{\gamma} z$  in  $X$ , and there is an extension

$$\begin{array}{ccc} |\Lambda_1^2| & \xrightarrow{(\gamma, ?, \omega)} & X \\ \downarrow & \nearrow \sigma & \\ |\Delta^2| & & \end{array}$$

The face  $d_1\sigma$  represents a well defined element  $[d_1\sigma]$  of  $\pi(X)(x, z)$  which is independent of the classes of  $\omega$  and  $\gamma$ , by an argument involving an extension

$$\begin{array}{ccc} (|\Delta^2| \times \{0, 1\}) \cup (|\Lambda_1^2| \times I) & \longrightarrow & X \\ \downarrow & \nearrow & \\ |\Delta^2| \times I & & \end{array}$$

[This is a prototypical “prismatic” argument. Filling in the labels on the diagram is an exercise.]

We therefore have the composition law

$$[\gamma] * [\omega] = [d_1(\sigma)]$$

defined for the fundamental groupoid  $\pi(X)$ .

## Associativity

Suppose given a string of paths

$$x_0 \xrightarrow{\omega_1} x_1 \xrightarrow{\omega_2} x_2 \xrightarrow{\omega_3} x_3$$

in  $X$ . There is a corresponding string of paths

$$0 \xrightarrow{[0,1]} 1 \xrightarrow{[1,2]} 2 \xrightarrow{[2,3]} 3$$

which defines a subspace  $P \subset |\Delta^3|$ , while the string of paths in  $X$  can be represented as a map  $\omega : P \rightarrow X$ . The map  $\omega : P \rightarrow X$  extends to a map  $\sigma : |\Delta^3| \rightarrow X$ , in the sense that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\omega} & X \\ \downarrow & \nearrow \sigma & \\ |\Delta^3| & & \end{array}$$

commutes.

[Fill in  $[0, 1, 2]$ ,  $[1, 2, 3]$ ,  $[0, 1, 3]$ , then  $[0, 1, 2, 3]$ .]

The image of the path  $[0, 3]$  in  $|\Delta^3|$  represents both  $[\omega_3] * ([\omega_2] * [\omega_1])$  and  $([\omega_3] * [\omega_2]) * [\omega_1]$ , so the composition law in  $\pi(X)$  is associative.

## Identities

Write  $x$  for the *constant path*

$$|\Delta^1| \xrightarrow{s^0} |\Delta^0| \xrightarrow{x} X$$

at an element  $x$  of  $X$ .

Suppose that  $\omega : x \rightarrow y$  is a path of  $X$ . Then

$$\partial s_0(\omega) = (\omega, \omega, x) \text{ and } \partial s_1(\omega) = (y, \omega, \omega).$$

The constant paths are 2-sided identities for the composition law.

## Inverses

Again, suppose that  $\omega : x \rightarrow y$  is a path of  $X$ . Then there are extensions

$$\begin{array}{ccc} |\Lambda_0^2| & \xrightarrow{(? , x, \omega)} & X \\ \downarrow & \nearrow \text{dotted} & \\ |\Delta^2| & & \end{array} \quad \text{and} \quad \begin{array}{ccc} |\Lambda_2^2| & \xrightarrow{(\omega, y, ?)} & X \\ \downarrow & \nearrow \text{dotted} & \\ |\Delta^2| & & \end{array}$$

so that the composition law on  $\pi(X)$  is invertible.

We have therefore shown that the fundamental groupoid  $\pi(X)$  of a space  $X$  is a groupoid.

## Fundamental groups

The *fundamental group*  $\pi_1(X, x)$  of  $X$  based at the element  $x$  is the set of homomorphisms (isomorphisms)  $\pi(X)(x, x)$  from  $x$  to itself in  $\pi(X)$ .

Explicitly,  $\pi_1(X, x)$  is the group of homotopy classes of loops  $x \rightarrow x$  rel end points in  $X$ , with composition law defined by extensions

$$\begin{array}{ccc} |\Lambda_1^2| & \xrightarrow{(\omega_2, ?, \omega_1)} & X \\ \downarrow & \nearrow \text{dotted} & \\ |\Delta^2| & & \end{array}$$

with identity defined by the constant path at  $x$ .

## Higher homotopy groups

Suppose that  $x$  is a vertex (aka. element) of  $X$ . The members of  $\pi_n(X, x)$  are homotopy classes

$$[(|\Delta^n|, |\partial\Delta^n|), (X, x)]$$

of simplices with boundary mapping to  $x$ , rel boundary. These classes are represented by diagrams

$$\begin{array}{ccc} |\partial\Delta^n| & \xrightarrow{x} & X \\ \downarrow & \nearrow \alpha & \\ |\Delta^n| & & \end{array}$$

which one tends to refer to by the name of the simplex, in this case  $\alpha$ .

Here's a cheat: one can show inductively (or by an explicit homeomorphism of pairs) that the set

$$[(|\Delta^n|, |\partial\Delta^n|), (X, x)]$$

is in bijective correspondence with the set

$$[(I^{\times n}, \partial I^{\times n}), (X, *)].$$

One starts the induction by using using extensions

$$\begin{array}{ccc} & & (|\Delta^n| \times \{0, 1\}) \cup (|\Lambda_0^n| \times I) \xrightarrow{((\alpha, x), x)} X \\ & & \downarrow \\ |\Delta^{n-1}| \times I & \xrightarrow{d^0 \times I} & |\Delta^n| \times I \end{array}$$

to show that there is a bijection

$$[(|\Delta^n|, |\partial\Delta^n|), (X, x)] \cong [(|\Delta^{n-1}| \times I, \partial(|\Delta^{n-1}| \times I)), (X, x)].$$

Homotopy classes of maps  $(I^{\times n}, \partial I^{\times n}) \rightarrow (X, x)$  can be composed in multiple directions, potentially giving  $n$  different group structures according to the description given above (recall that  $I = |\Delta^1|$ ).

These multiplications have a common identity, namely the constant cell at  $x$ , and they satisfy *interchange laws*

$$(a_1 *_i a_2) *_j (b_1 *_i b_2) = (a_1 *_j b_1) *_i (a_2 *_j b_2).$$

The multiplications therefore coincide and are abelian if  $n \geq 2$  (exercise).

The interchange laws follow from the existence of solutions to lifting problems

$$\begin{array}{ccc} |\Lambda_1^2| \times |\Lambda_1^2| & \longrightarrow & X \\ \downarrow & \nearrow & \\ |\Delta^2| \times |\Delta^2| & & \end{array}$$

(exercise again).



## Homotopy equivalences, weak equivalences

The construction of  $\pi_0(X)$ ,  $\pi(X)$  and all  $\pi_n(X, x)$  are functorial: every map  $f : X \rightarrow Y$  induces

- $f_* : \pi_0(X) \rightarrow \pi_0(Y)$  (function between sets),
- $f_* : \pi(X) \rightarrow \pi(Y)$  (functor between groupoids),
- $f_* : \pi_n(X, x) \rightarrow \pi_n(Y, f(x))$ ,  $n \geq 1$ ,  $x \in X$  (group homomorphisms).

A) A map  $f : X \rightarrow Y$  is said to be a *homotopy equivalence* if there is a map  $g : Y \rightarrow X$  such that  $g \cdot f \simeq 1_X$  (homotopic to the identity on  $X$ ) and  $f \cdot g \simeq 1_Y$ .

B)  $f : X \rightarrow Y$  is a *weak equivalence* if

- 1)  $f_* : \pi_0(X) \rightarrow \pi_0(Y)$  is a bijection, and
- 2)  $f_* : \pi_n(X, x) \rightarrow \pi_n(Y, f(x))$  is an isomorphism for all  $n \geq 1$  and all  $x \in X$ .

## Exercises

- 1) Show that every homotopy equivalence is a weak equivalence.
- 2) Show that every weak equivalence  $f : X \rightarrow Y$  induces an equivalence of groupoids  $f_* : \pi(X) \rightarrow \pi(Y)$ .

## Group objects

Write  $X_0$  for the set of points underlying  $X$  (the vertices of  $S(X)$ ). Every base point  $x \in X_0$  has an associated homotopy group  $\pi_n(X, x)$ , and we can collect all such homotopy groups together to define a function

$$\pi_n(X) = \bigsqcup_{x \in X_0} \pi_n(X, x) \rightarrow \bigsqcup_{x \in X_0} * = X_0.$$

The function  $\pi_n(X) \rightarrow X_0$  defines a group object over the set  $X_0$  for  $n \geq 1$  which is abelian if  $n \geq 2$ .

**Fact:** A map  $f : X \rightarrow Y$  is a weak equivalence if and only if

- 1) the function  $f_* : \pi_0(X) \rightarrow \pi_0(Y)$  is a bijection, and
- 2) the induced diagrams

$$\begin{array}{ccc} \pi_n(X) & \xrightarrow{f_*} & \pi_n(Y) \\ \downarrow & & \downarrow \\ X_0 & \xrightarrow{f_*} & Y_0 \end{array}$$

are pullbacks (in **Set**) for  $n \geq 1$ .

## 5 Serre fibrations and the model structure for CGWH

A map  $p : X \rightarrow Y$  is said to be a *Serre fibration* if it has the RLP wrt all  $|\Lambda_k^n| \subset |\Delta^n|$ ,  $n \geq 1$ .

All spaces  $X$  are fibrant: the map  $X \rightarrow *$  is a Serre fibration.

Main formal properties of Serre fibrations:

**Lemma 5.1.** *A map  $p : X \rightarrow Y$  is a Serre fibration and a weak equivalence if and only if it has the right lifting property with respect to all inclusions  $|\partial\Delta^n| \subset |\Delta^n|$ ,  $n \geq 0$ .*

Here,  $|\partial\Delta^0| = \emptyset$ .

**Lemma 5.2.** *Suppose that  $p : X \rightarrow Y$  is a Serre fibration, and that  $F = p^{-1}(y)$  is the fibre over an element  $y \in Y$ . Then we have the following:*

1) *For each  $x \in F$  there is a sequence of pointed sets*

$$\begin{aligned} \dots \pi_n(F, x) &\xrightarrow{i_*} \pi_n(X, x) \xrightarrow{p_*} \pi_n(Y, y) \xrightarrow{\partial} \pi_{n-1}(F, x) \rightarrow \dots \\ \dots \pi_1(Y, y) &\xrightarrow{\partial} \pi_0(F) \xrightarrow{i_*} \pi_0(X) \xrightarrow{p_*} \pi_0(Y) \end{aligned}$$

*which is exact in the sense that  $\ker = \text{im}$  everywhere.*

2) *There is a group action*

$$* : \pi_1(Y, y) \times \pi_0(F) \rightarrow \pi_0(F)$$

*such that  $\partial([\alpha]) = [\alpha] * [x]$ , and such that  $i_*[z] = i_*[w]$  if and only if there is an element  $[\beta] \in \pi_1(Y, y)$  such that  $[\beta] * [z] = [w]$ .*

The boundary map

$$\partial : \pi_n(Y, p(x)) \rightarrow \pi_{n-1}(F, x)$$

is defined by  $\partial([\alpha]) = [d_0\theta]$ , where  $\theta$  is a choice of lifting in the following diagram

$$\begin{array}{ccc} |\Lambda_0^n| & \xrightarrow{x} & X \\ \downarrow & \nearrow \theta & \downarrow p \\ |\Delta^n| & \xrightarrow{\alpha} & Y \end{array}$$

Lemma 5.1 is needed for the following result, while Lemma 5.2 is needed for almost all calculations of homotopy groups.

The proof of Lemma 5.1 is sketched below, and the proof of Lemma 5.2 is an exercise.

A map  $i : A \rightarrow B$  is said to be a *cofibration* if it has the LLP wrt all trivial Serre fibrations.

Lemma 5.1 implies that all inclusions  $|\partial\Delta^n| \subset |\Delta^n|$  are cofibrations. All CW-complexes (spaces built inductively by attaching cells) are *cofibrant*.

**Theorem 5.3.** *The weak equivalences, Serre fibrations and cofibrations as defined above give **CGWH** the structure of a closed model category.*

*Proof.*  $p : X \rightarrow Y$  is a Serre fibration if and only if it has the RLP wrt all  $|\Lambda_k^n| \subset |\Delta^n|$ .

$p$  is a trivial Serre fibration if and only if it has the RLP wrt all  $|\partial\Delta^n| \subset |\Delta^n|$  by Lemma 5.1.

All inclusions  $|\Lambda_k^n| \subset |\Delta^n|$  are strong deformation retractions, as are all of their pushouts.

Pushouts of monomorphisms are monomorphisms.

A small object argument (which depends on an observation of J.H.C. Whitehead that a compact subset of a  $CW$ -complex meets only finitely many cells) shows that every continuous map  $f : X \rightarrow Y$  has factorizations

$$\begin{array}{ccc} & Z & \\ i \nearrow & & \searrow p \\ X & \xrightarrow{f} & Y \\ j \searrow & & \nearrow q \\ & W & \end{array}$$

such that  $i$  is a trivial cofibration which has the LLP wrt all fibrations and  $p$  is a Serre fibration,

and  $j$  is a cofibration and a monomorphism and  $q$  is a trivial Serre fibration. This gives **CM5**.

Suppose that  $j : A \rightarrow B$  is a trivial cofibration.  $j$  has a factorization

$$\begin{array}{ccc} A & \xrightarrow{i} & C \\ & \searrow j & \downarrow p \\ & & B \end{array}$$

such that  $i$  is a trivial cofibration which has the LLP wrt all Serre fibrations, and  $p$  is a Serre fibration. Then  $p$  is a trivial Serre fibration, so the lift exists in the diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & C \\ j \downarrow & \nearrow \alpha & \downarrow p \\ B & \xrightarrow{1} & B \end{array}$$

Then  $j$  is a retract of  $i$ , so  $j$  has the LLP wrt all Serre fibrations.

For **CM4**, suppose given a diagram (lifting problem)

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & \nearrow & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

where  $i$  is a cofibration and  $p$  is a Serre fibration. The lift exists if  $p$  is trivial (definition of cofibration), and we just showed that every trivial cofibration has the LLP wrt all Serre fibrations.

The other model axioms are exercises.  $\square$

We need the following for the proof of Lemma 5.1:

**Lemma 5.4.** *A map  $\alpha : (\Delta^n, \partial\Delta^n) \rightarrow (X, x)$  represents the identity element of  $\pi_n(X, x)$  if and only if the lifting problem*

$$\begin{array}{ccc} |\partial\Delta^{n+1}| & \xrightarrow{(\alpha, x, \dots, x)} & X \\ \downarrow & \nearrow & \\ |\Delta^{n+1}| & & \end{array}$$

*can be solved.*

*Proof.* Exercise.  $\square$

*Proof of Lemma 5.1.* 1) Suppose  $p : X \rightarrow Y$  is a trivial Serre fibration, and suppose given a lifting problem

$$\begin{array}{ccc} |\partial\Delta^n| & \xrightarrow{\alpha} & X \\ \downarrow & \nearrow & \downarrow p \\ |\Delta^n| & \xrightarrow{\beta} & Y \end{array}$$

Suppose  $x = \alpha(0)$ . There is a homotopy of diagrams

$$\begin{array}{ccc} |\partial\Delta^n| \times I & \longrightarrow & X \\ \downarrow & & \downarrow \\ |\Delta^n| \times I & \longrightarrow & Y \end{array}$$

from the original diagram to one of the form

$$\begin{array}{ccc} |\partial\Delta^n| & \xrightarrow{(\alpha_0, x, \dots, x)} & X \\ \downarrow & \nearrow & \downarrow p \\ |\Delta^n| & \xrightarrow{\beta'} & Y \end{array}$$

so the two lifting problems are equivalent.

$p_*([\alpha_0]) = 0$  so  $[\alpha_0] = 0 \in \pi_{n-1}(X, x)$ , and it follows from a second homotopy of diagrams that the original lifting problem is equivalent to one of the form

$$\begin{array}{ccc} |\partial\Delta^n| & \xrightarrow{x} & X \\ \downarrow & \nearrow & \downarrow p \\ |\Delta^n| & \xrightarrow{\beta''} & Y \end{array}$$

Since  $p_* : \pi_n(X, x) \rightarrow \pi_n(Y, p(x))$  is surjective,  $\beta''$  lifts up to homotopy rel  $|\partial\Delta^n|$  to a simplex of  $X$ , so that this last diagram is homotopic to a diagram for which the lifting problem is solved.

2) Suppose  $p : X \rightarrow Y$  has the RLP wrt all  $|\partial\Delta^n| \subset |\Delta^n|$ .

Then  $p$  has the right lifting property with respect to all  $|\Lambda_k^n| \subset |\Delta^n|$  (exercise), so  $p$  is a Serre fibration.



Suppose that  $[\alpha] \in \pi_n(X, x)$  such that  $p_*([\alpha]) = 0$ .  
Then there is a commutative diagram

$$\begin{array}{ccc} |\partial\Delta^{n+1}| & \xrightarrow{(\alpha, x, \dots, x)} & X \\ \downarrow & \nearrow & \downarrow p \\ |\Delta^{n+1}| & \xrightarrow{\beta} & Y \end{array}$$

and the existence of the indicated lift implies that  $[\alpha] = 0 \in \pi_n(X, x)$ . Thus,  $p_*$  is a monomorphism.

The existence of liftings

$$\begin{array}{ccc} |\partial\Delta^n| & \xrightarrow{x} & X \\ \downarrow & \nearrow \theta & \downarrow p \\ |\Delta^n| & \xrightarrow{\beta} & Y \end{array}$$

means that  $p_*$  is surjective:  $p_*([\theta]) = [\beta]$ . □

## Lecture 03: Homotopical algebra

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### 6 Example: Chain homotopy

$C$  = an ordinary chain complex. We have two constructions:

1)  $C^I$  is the complex with

$$C_n^I = C_n \oplus C_n \oplus C_{n+1}$$

for  $n > 0$ , and with

$$C_0^I = \{(x, y, z) \in C_0 \oplus C_0 \oplus C_1 \mid (x - y) + \partial(z) = 0\}.$$

The boundary map  $\partial : C_n^I \rightarrow C_{n-1}^I$  is defined by

$$\partial(x, y, z) = (\partial(x), \partial(y), (-1)^n(x - y) + \partial(z)).$$

2)  $\tilde{C}$  is the chain complex with

$$\tilde{C}_n = C_n \oplus C_{n+1}$$

for  $n > 0$  and

$$\tilde{C}_0 = \{(x, z) \in C_0 \oplus C_1 \mid x + \partial(z) = 0\}.$$

The boundary  $\partial : \tilde{C}_n \rightarrow \tilde{C}_{n-1}$  of  $\tilde{C}$  is defined by

$$\partial(x, z) = (\partial(x), (-1)^n x + \partial(z)).$$

**Lemma 6.1.** *The complex  $\tilde{C}$  is acyclic.*

*Proof.* If  $\partial(x, z) = 0$  then  $\partial(x) = 0$  and  $\partial(z) = (-1)^{n+1}x$ . It follows that

$$\partial((-1)^{n+1}z, 0) = (x, z)$$

if  $(x, z)$  is a cycle, so  $(x, z)$  is a boundary.  $\square$

There is a pullback diagram

$$\begin{array}{ccc} C^I & \xrightarrow{\alpha} & \tilde{C} \\ p \downarrow & & \downarrow p' \\ C \oplus C & \xrightarrow{\beta} & C \end{array}$$

in which  $p$  and  $p'$  are projections defined in each degree by  $p(x, y, z) = (x, y)$  and  $p'(x, z) = x$ . The map  $\alpha$  is defined by  $\alpha(x, y, z) = (x - y, z)$ , while  $\beta(x, y) = x - y$ .

$p'$  is a fibration, and fibrations are closed under pullback, so  $p$  is also a fibration. The maps  $\alpha$  and  $\beta$  are surjective in all degrees, and the diagram above expands to a comparison

$$\begin{array}{ccccccc} & & C^I & \xrightarrow{\alpha} & \tilde{C} & \longrightarrow & 0 \\ & \nearrow s & \downarrow p & & \downarrow p' & & \\ 0 \longrightarrow & C & \xrightarrow{\Delta} & C \oplus C & \xrightarrow{\beta} & C & \longrightarrow 0 \end{array}$$

where  $\Delta$  is the diagonal map.

Lemma 6.1 and a long exact sequence argument imply that the map  $s$  is a weak equivalence.

We have a functorial diagram

$$\begin{array}{ccc} & & C^I \\ & \nearrow s & \downarrow p \\ C & \xrightarrow{\Delta} & C \oplus C \end{array} \quad (1)$$

in which  $p$  is a fibration and  $s$  is a weak equivalence. This is a *path object*.

A commutative diagram of chain maps

$$\begin{array}{ccc} & & C^I \\ & \nearrow h & \downarrow p \\ D & \xrightarrow{(f,g)} & C \oplus C \end{array} \quad (2)$$

is a *right homotopy* between the chain maps  $f, g : D \rightarrow C$

The map  $h$ , if it exists, is defined by

$$h(x) = (f(x), g(x), s(x))$$

for a collection of  $R$ -module maps  $s : D_n \rightarrow C_{n+1}$ .

The fact that  $h$  is a chain map forces

$$s(\partial(x)) = (-1)^n(f(x) - g(x)) + \partial(s(x))$$

for  $x \in D_n$ . Thus

$$(-1)^n s(\partial(x)) = (f(x) - g(x)) + \partial((-1)^n s(x)),$$

so

$$(-1)^n s(\partial(x)) + \partial((-1)^{n+1} s(x)) = f(x) - g(x).$$

The maps  $x \mapsto (-1)^{n+1} s(x)$ ,  $x \in D_n$ , arising from the right homotopy  $h$  define a *chain homotopy* between the chain maps  $f$  and  $g$ .

All chain homotopies arise in this way.

**Exercise:** Show that there is a functorial diagram of the form (1) for unbounded chain complexes  $C$ , such that the corresponding right homotopies (2) define chain homotopies between maps  $f, g : D \rightarrow C$  of unbounded chain complexes.

## 7 Homotopical algebra

A *closed model category* is a category  $\mathcal{M}$  equipped with weak equivalences, fibrations and cofibrations, such that the following hold:

**CM1** The category  $\mathcal{M}$  has all (finite) limits and colimits.

**CM2** Given a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ & \searrow h & \swarrow f \\ & Z & \end{array}$$

in  $\mathcal{M}$ , if any two of  $f, g$  and  $h$  are weak equivalences, then so is the third.

**CM3** The classes of cofibrations, fibrations and weak equivalences are closed under retraction.

**CM4** Given a commutative solid arrow diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & \nearrow & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

such that  $i$  is a cofibration and  $p$  is a fibration. Then the lift exists if either  $i$  or  $p$  is a weak equivalence.

**CM5** Every morphism  $f : X \rightarrow Y$  has factorizations

$$\begin{array}{ccccc} & & Z & & \\ & i \nearrow & & \searrow p & \\ X & \xrightarrow{f} & Y & & \\ & j \searrow & & \nearrow q & \\ & & W & & \end{array}$$

where  $p$  is a fibration and  $i$  is a trivial cofibration, and  $q$  is a trivial fibration and  $j$  is a cofibration.

Here's the meaning of the word “closed”:

**Lemma 7.1.** *1)  $i : A \rightarrow B$  is a cofibration if and only if it has the LLP wrt all trivial fibrations.*

*2)  $i : A \rightarrow B$  is a trivial cofibration if and only if it has the LLP wrt all fibrations.*

*3)  $p : X \rightarrow Y$  is a fibration if and only if it has the RLP wrt all trivial cofibrations.*

*4)  $p$  is a trivial fibration if and only if it has the RLP wrt all cofibrations.*

*Proof.* I'll prove statement 2). The rest are similar.

If  $i$  is a trivial cofibration, then it has the LLP wrt all fibrations by **CM4**.

Suppose  $i$  has the LLP wrt all fibrations.  $i$  has a factorization

$$\begin{array}{ccc} A & \xrightarrow{j} & X \\ & \searrow i & \downarrow p \\ & & B \end{array}$$

where  $j$  is a trivial cofibration and  $p$  is a fibration. Then the lifting exists in the diagram

$$\begin{array}{ccc} A & \xrightarrow{j} & X \\ i \downarrow & \nearrow & \downarrow p \\ B & \xrightarrow{1} & B \end{array}$$

Then  $i$  is a retract of  $j$  and is therefore a trivial cofibration by **CM3**.  $\square$

**Corollary 7.2.** *1) The classes of cofibrations and trivial cofibrations are closed under compositions and pushout. Any isomorphism is a trivial cofibration.*

*2) The classes of fibrations and trivial fibrations are closed under composition and pullback. Any isomorphism is a trivial fibration.*

**Remark:** Lemma 7.1 implies that, in order to describe a closed model structure, one needs only specify the weak equivalences and either the cofibrations or fibrations.

We saw this in the descriptions of the model structures for the chain complex categories and for spaces.

## Homotopies

1) A *path object* for  $Y \in \mathcal{M}$  is a commutative diagram

$$\begin{array}{ccc} & & Y^I \\ & \nearrow s & \downarrow p \\ Y & \xrightarrow{\Delta} & Y \times Y \end{array}$$



such that  $\Delta$  is the diagonal map,  $s$  is a weak equivalence and  $p$  is a fibration.

- 2) A *right homotopy* between maps  $f, g : X \rightarrow Y$  is a commutative diagram

$$\begin{array}{ccccc} & & Y^I & & \\ & \nearrow h & \downarrow p & \nwarrow s & \\ X & \xrightarrow{(f,g)} & Y \times Y & \xleftarrow{\Delta} & Y \end{array}$$

where  $p$  is the fibration for *some* (displayed) path object for  $Y$ .

$f$  is *right homotopic* to  $g$  if such a right homotopy exists. Write  $f \sim_r g$ .

**Examples:** 1) Path objects abound in nature, since the diagonal map  $\Delta : Y \rightarrow Y \times Y$  factorizes as a fibration following a trivial cofibration, by **CM5**.

2) Chain homotopy is a type of right homotopy in both  $Ch_+(R)$  and  $Ch(R)$ .

3) For ordinary spaces  $X$ , there is a space  $X^I$ , whose elements are the paths  $I \rightarrow X$  in  $X$ . Restricting to the two ends of the paths defines a map  $d : X^I \rightarrow X \times X$ , which is a Serre fibration (exercise). There is a constant path map  $s : X \rightarrow X^I$ , and a commu-

tative diagram

$$\begin{array}{ccc} & & X^I \\ & \nearrow s & \downarrow d \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

The composite  $X^I \xrightarrow{d} X \times X \xrightarrow{pr_L} X$  is a trivial fibration (exercise), so  $s$  is a weak equivalence.

The traditional path space defines a path object construction. Right homotopies  $X \rightarrow Y^I$  are traditional homotopies  $X \times I \rightarrow Y$  by adjointness.

Here's the dual cluster of definitions:

- 1) A *cylinder object* for an object  $X \in \mathcal{M}$  is a commutative diagram

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{\nabla} & X \\ \downarrow i & \nearrow \sigma & \\ X \otimes I & & \end{array}$$

where  $\nabla$  is the “fold” map,  $i$  is a cofibration and  $\sigma$  is a weak equivalence.

- 2) A *left homotopy* between maps  $f, g : X \rightarrow Y$  is a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{\nabla} & X \sqcup X & \xrightarrow{(f,g)} & Y \\ & \searrow \sigma & \downarrow i & \nearrow h & \\ & & X \otimes I & & \end{array}$$

where  $i$  is the cofibration appearing in *some* cylinder object for  $X$ .

Say  $f$  is *left homotopic* to  $g$  if such a left homotopy exists. Write  $f \sim_l g$ .

**Examples:** 1) Suppose  $X$  is a  $CW$ -complex and  $I$  is the unit interval. The standard picture

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{\nabla} & X \\ i \downarrow & \nearrow pr & \\ X \times I & & \end{array}$$

is a cylinder object for  $X$ . The space  $X \times I$  is obtained from  $X \sqcup X$  by attaching cells, so  $i$  is a cofibration.

2) There are lots of cylinder objects: the map  $\nabla : X \sqcup X \rightarrow X$  has a factorization as a cofibration followed by a trivial fibration, by **CM5**.

## Duality

Here is what I mean by “dual”:

**Lemma 7.3.**  $\mathcal{M} =$  a closed model category.

Say a morphism  $f^{op} : Y \rightarrow X$  of the opposite category  $\mathcal{M}^{op}$  is a fibration (resp. cofibration, weak equivalence) if and only if the corresponding map  $f : X \rightarrow Y$  is a cofibration (resp. fibration, weak equivalence) of  $\mathcal{M}$ .

Then with these definitions,  $\mathcal{M}^{op}$  satisfies the axioms for a closed model category.

*Proof.* Exercise. □

Reversing the arrows in a cylinder object gives a path object, and vice versa. All homotopical facts about a model category  $\mathcal{M}$  have equivalent dual assertions in  $\mathcal{M}^{op}$ .

**Examples:** In Lemma 7.1, statement 3) is the dual of statement 1), and statement 4) is the dual of statement 2).

**Lemma 7.4.** *Right homotopy of maps  $X \rightarrow Y$  is an equivalence relation if  $Y$  is fibrant.*

The dual of Lemma 7.4 is the following:

**Lemma 7.5.** *Left homotopy of maps  $X \rightarrow Y$  is an equivalence relation if  $X$  is cofibrant.*

*Proof.* Lemma 7.5 is equivalent to Lemma 7.4 in  $\mathcal{M}^{op}$ . □

*Proof of Lemma 7.4.* If  $Y$  is fibrant then any projection  $X \times Y \rightarrow X$  is a fibration (exercise).

Thus, if

$$\begin{array}{ccc} & & Y^I \\ & \nearrow s & \downarrow (p_0, p_1) \\ Y & \xrightarrow{\Delta} & Y \times Y \end{array}$$

is a path object for a fibrant object  $Y$ , then the maps  $p_0$  and  $p_1$  are trivial fibrations.

Suppose given right homotopies

$$\begin{array}{ccc} & Y^I & \\ h_1 \nearrow & \downarrow (p_0, p_1) & \\ X & \xrightarrow{(f_1, f_2)} & Y \times Y \end{array} \quad \text{and} \quad \begin{array}{ccc} & Y^J & \\ h_2 \nearrow & \downarrow (q_0, q_1) & \\ X & \xrightarrow{(f_2, f_3)} & Y \times Y \end{array}$$

Form the pullback

$$\begin{array}{ccc} Y^I \times_Y Y^J & \xrightarrow{p_*} & Y^J \\ q_* \downarrow & & \downarrow q_0 \\ Y^I & \xrightarrow{p_1} & Y \end{array}$$

The diagram

$$\begin{array}{ccc} Y^I \times_Y Y^J & \xrightarrow{p_*} & Y^J \\ (q_*, q_1 p_*) \downarrow & & \downarrow (q_0, q_1) \\ Y^I \times Y & \xrightarrow{p_1 \times 1} & Y \times Y \end{array}$$

is a pullback and  $p_0 \times 1 : Y^I \times Y \rightarrow Y \times Y$  is a fibration, so the composite

$$Y^I \times_Y Y^J \xrightarrow{(p_0 q_*, q_1 p_*)} Y \times Y$$

is a fibration. The weak equivalences  $s, s'$  from the respective path objects determine a commutative

diagram

$$\begin{array}{ccc}
 & Y^I \times_Y Y^J & \\
 (s,s') \nearrow & \downarrow (p_0 q_*, q_1 p_*) & \\
 Y & \xrightarrow{\Delta} Y \times Y & 
 \end{array}$$

and the map  $(s, s')$  is a weak equivalence since  $p_0 q_*$  is a trivial fibration.

The homotopies  $h, h'$  therefore determine a right homotopy

$$\begin{array}{ccc}
 & Y^I \times_Y Y^J & \\
 (h,h') \nearrow & \downarrow (p_0 q_*, q_1 p_*) & \\
 X & \xrightarrow{(f_1, f_3)} Y \times Y & 
 \end{array}$$

It follows that the right homotopy relation is transitive.

Right homotopy is symmetric, since the twist isomorphism  $Y \times Y \xrightarrow{\cong} Y \times Y$  is a fibration.

Right homotopy is reflexive, since the morphism  $s$  in a path object is a right homotopy from the identity to itself.  $\square$

Here's the result that ties the homotopical room together:

**Lemma 7.6.** *1) Suppose  $Y$  is fibrant and  $X \otimes I$  is a fixed choice of cylinder object for an object  $X$ . Suppose  $f, g : X \rightarrow Y$  are right homotopic. Then there is a left homotopy*

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{(f,g)} & Y \\ i \downarrow & \nearrow h & \\ X \otimes I & & \end{array}$$

*2) Suppose  $X$  is cofibrant and  $Y^I$  is a fixed choice of path object for an object  $Y$ . Suppose  $f, g : X \rightarrow Y$  are left homotopic. Then there is a right homotopy*

$$\begin{array}{ccc} & & Y^I \\ & \nearrow h & \downarrow p \\ X & \xrightarrow{(f,g)} & Y \times Y \end{array}$$

*Proof.* Statement 2) is the dual of statement 1). We'll prove statement 1).

Suppose

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{\nabla} & X \\ i \downarrow & \nearrow \sigma & \\ X \otimes I & & \end{array} \quad \text{and} \quad \begin{array}{ccc} & & Y^I \\ & \nearrow s & \downarrow (p_0, p_1) \\ Y & \xrightarrow{\Delta} & Y \times Y \end{array}$$

are the fixed choice of cylinder and the path object involved in the right homotopy  $f \sim_r g$ , respectively, and let  $h : X \rightarrow Y^I$  be the right homotopy. Form the diagram

$$\begin{array}{ccccc} X \sqcup X & \xrightarrow{(sf, h)} & Y^I & \xrightarrow{p_1} & Y \\ \downarrow i & \nearrow \theta & \downarrow p_0 & & \\ X \otimes I & \xrightarrow{f\sigma} & Y & & \end{array}$$

The lift  $\theta$  exists because  $p_0$  is a trivial fibration since  $Y$  is fibrant (exercise). The composite  $p_1 \theta$  is the desired left homotopy.  $\square$

**Corollary 7.7.** *Suppose  $f, g : X \rightarrow Y$  are morphisms of  $\mathcal{M}$ , where  $X$  is cofibrant and  $Y$  is fibrant. Suppose*

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{\nabla} & X \\ \downarrow i & \nearrow \sigma & \\ X \otimes I & & \end{array} \quad \text{and} \quad \begin{array}{ccc} & & Y^I \\ & \nearrow s & \downarrow p \\ Y & \xrightarrow{\Delta} & Y \times Y \end{array}$$

*are fixed choices of cylinder and path objects for  $X$  and  $Y$  respectively. Then the following are equivalent:*

- $f$  is left homotopic to  $g$ .
- There is a right homotopy  $h : X \rightarrow Y^I$  from  $f$  to  $g$ .



- $f$  is right homotopic to  $g$ .
- There is a left homotopy  $H : X \otimes I \rightarrow Y$  from  $f$  to  $g$ .

Thus, if  $X$  is cofibrant and  $Y$  is fibrant, all notions of homotopy of maps  $X \rightarrow Y$  collapse to the same thing.

Write  $f \sim g$  to say that  $f$  is homotopic to  $g$  (by whatever means) in this case.

Here's the first big application:

**Theorem 7.8** (Whitehead Theorem). *Suppose  $f : X \rightarrow Y$  is a weak equivalence, and the objects  $X$  and  $Y$  are both fibrant and cofibrant. Then  $f$  is a homotopy equivalence.*

*Proof.* We can assume that  $f$  is a trivial fibration: every weak equivalence is a composite of a trivial fibration with a trivial cofibration, and the trivial cofibration case is dual.

$Y$  is cofibrant, so the lifting exists in the diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \downarrow & \nearrow j & \downarrow f \\ Y & \xrightarrow{1} & Y \end{array}$$

Suppose

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{\nabla} & X \\ i \downarrow & \nearrow \sigma & \\ X \otimes I & & \end{array}$$

is a cylinder object for  $X$ , and then form the diagram

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{(jf, 1)} & X \\ i \downarrow & \nearrow h & \downarrow f \\ X \otimes I & \xrightarrow{f\sigma} & Y \end{array}$$

The indicated lift (and required homotopy) exists because  $f$  is a trivial fibration.  $\square$

**Examples:** 1) (traditional Whitehead Theorem) Every weak equivalence  $f : X \rightarrow Y$  between  $CW$ -complexes is a homotopy equivalence.

2) Every weak equivalence  $f : C \rightarrow D$  in  $Ch_+(R)$  between complexes of projective  $R$ -modules is a chain homotopy equivalence.

3) Any two projective resolutions  $p : P \rightarrow M(0)$ ,  $q : Q \rightarrow M(0)$  of a module  $M$  are chain homotopy equivalent.

The maps  $p$  and  $q$  are trivial fibrations, and both  $P$  and  $Q$  are cofibrant chain complexes, so the lift  $\theta$

exists in the diagram

$$\begin{array}{ccc} & & Q \\ & \nearrow \theta & \downarrow q \\ P & \xrightarrow{p} & M(0) \end{array}$$

The map  $\theta$  is a weak equivalence of cofibrant complexes, hence a chain homotopy equivalence.

3 bis)  $f : M \rightarrow N$  a homomorphism of modules.  
 $p : P \rightarrow M(0)$ ,  $q : Q \rightarrow N(0)$  projective resolutions.

The lift exists in the diagram

$$\begin{array}{ccc} 0 & \xrightarrow{\quad} & Q \\ & \searrow f_1 & \downarrow q \\ P & \xrightarrow{p} M(0) \xrightarrow{f} & N(0) \end{array}$$

since  $P$  is cofibrant and  $q$  is a trivial fibration, so  $f$  lifts to a chain complex map  $f_1$ .

If  $f$  also lifts to some other chain complex map  $f_2 : P \rightarrow Q$ , there is a commutative diagram

$$\begin{array}{ccc} P \oplus P & \xrightarrow{(f_1, f_2)} & Q \\ \downarrow i & \searrow h & \downarrow q \\ P \otimes I & \xrightarrow{\sigma} P \xrightarrow{p} M(0) \xrightarrow{f} & N(0) \end{array}$$

for some (any) choice of cylinder  $P \otimes I$ .

Then  $f_1 \simeq_l f_2$ , so  $f_1$  and  $f_2$  are *chain homotopic* since  $P$  is cofibrant and  $Q$  is fibrant.

4)  $X =$  a space. There is a trivial fibration  $p : U \rightarrow X$  such that  $U$  is a *CW* complex (exercise).

Suppose  $Y$  is a cofibrant space. Then  $Y$  is a retract of a *CW*-complex (exercise).

Suppose  $f : X \rightarrow Y$  and choose trivial fibrations  $p : U \rightarrow X$  and  $q : V \rightarrow Y$  such that  $U$  and  $V$  are *CW*-complexes. Then there is a map  $f' : U \rightarrow V$  which lifts  $f$  in the sense that the diagram

$$\begin{array}{ccc} U & \xrightarrow{f'} & V \\ p \downarrow & & \downarrow q \\ X & \xrightarrow{f} & Y \end{array}$$

commutes, and any two such maps are “naively” homotopic (exercise).

## 8 The homotopy category

For all  $X \in \mathcal{M}$  find maps

$$X \xleftarrow{p_X} QX \xrightarrow{j_X} RQX$$

such that

- $p_X$  is a trivial fibration and  $QX$  is cofibrant, and  $j_X$  is a trivial cofibration and  $RQX$  is fibrant (and cofibrant),
- $QX = X$  and  $p_X = 1_X$  if  $X$  is cofibrant, and  $RQX = QX$  and  $j_X = 1_{QX}$  if  $QX$  is fibrant.

Every map  $f : X \rightarrow Y$  determines a diagram

$$\begin{array}{ccccc} X & \xleftarrow{p_X} & QX & \xrightarrow{j_X} & RQX \\ f \downarrow & & \downarrow f_1 & & \downarrow f_2 \\ Y & \xleftarrow{p_Y} & QY & \xrightarrow{j_Y} & RQY \end{array}$$

since  $QX$  is cofibrant and  $RQY$  is fibrant.

**Lemma 8.1.** *The map  $f_2$  is uniquely determined up to homotopy.*

*Proof.* Suppose  $f'_1$  and  $f'_2$  are different choices for  $f_1$  and  $f_2$  respectively.

There is a diagram

$$\begin{array}{ccc}
 QX \sqcup QX & \xrightarrow{(f_1, f'_1)} & QY \\
 i \downarrow & \nearrow & \downarrow p_Y \\
 QX \otimes I & \xrightarrow{\sigma} QX \xrightarrow{f p_X} & Y
 \end{array}$$

for any choice of cylinder  $QX \otimes I$  for  $QX$ , so  $f_1$  and  $f'_1$  are left homotopic.

The maps  $j_Y f_1$  and  $j_Y f'_1$  are left homotopic, hence right homotopic because  $QX$  is cofibrant and  $RQY$  is fibrant. Thus, there is a right homotopy

$$\begin{array}{ccc}
 & & RQY^I \\
 & \nearrow h & \downarrow p \\
 QX & \xrightarrow{(j_Y f_1, j_Y f'_1)} & RQY \times RQY
 \end{array}$$

for some (actually any) path object  $RQY^I$ . Form the diagram

$$\begin{array}{ccc}
 QX & \xrightarrow{h} & RQY^I \\
 j_X \downarrow & \nearrow H & \downarrow p \\
 RQX & \xrightarrow{(f_2, f'_2)} & RQY \times RQY
 \end{array}$$

Then  $f_2$  and  $f'_2$  are homotopic. □

$\pi(\mathcal{M})_{cf}$  is the category whose objects are the cofibrant-fibrant objects of  $\mathcal{M}$ , and whose morphisms are homotopy classes of maps.

Lemma 8.1 implies that there is a well-defined functor

$$\mathcal{M} \rightarrow \pi(\mathcal{M})_{cf}$$

defined by  $X \mapsto RQX$  and  $f \mapsto [RQ(f)]$ , where

$$RQ(f) = f_2.$$

The *homotopy category*  $\text{Ho}(\mathcal{M})$  of  $\mathcal{M}$  has the same objects as  $\mathcal{M}$ , and has

$$\text{hom}_{\text{Ho}(\mathcal{M})}(X, Y) = \text{hom}_{\pi(\mathcal{M})_{cf}}(RQX, RQY).$$

There is a functor

$$\gamma : \mathcal{M} \rightarrow \text{Ho}(\mathcal{M})$$

that is the identity on objects, and sends  $f : X \rightarrow Y$  to the homotopy class  $[RQ(f)]$ .

$\gamma$  takes weak equivalences to isomorphisms in  $\text{Ho}(\mathcal{M})$ , by the Whitehead Theorem (Theorem 7.8).

**Lemma 8.2.** *Suppose  $f : RQX \rightarrow RQY$  represents a morphism  $[f] : X \rightarrow Y$  of  $\text{Ho}(\mathcal{M})$ . Then there is a commutative diagram*

$$\begin{array}{ccccc} X & \xleftarrow{\gamma(p_X)} & QX & \xrightarrow{\gamma(j_X)} & RQX \\ [f] \downarrow & & \downarrow [f] & & \downarrow \gamma(f) \\ Y & \xleftarrow{\gamma(p_Y)} & QY & \xrightarrow{\gamma(j_Y)} & RQY \end{array}$$

in  $\text{Ho}(\mathcal{M})$ .

*Proof.* The maps  $\gamma(p_X)$  and  $\gamma(j_X)$  are isomorphisms defined by the class  $[1_{RQX}]$  in  $\pi(\mathcal{M})_{cf}$ .  $\square$

**Theorem 8.3.** *Suppose  $\mathcal{M}$  is a closed model category, and  $F : \mathcal{M} \rightarrow D$  takes weak equivalences to isomorphisms.*

*There is a unique functor  $F_* : \text{Ho}(\mathcal{M}) \rightarrow D$  such that the diagram of functors*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\gamma} & \text{Ho}(\mathcal{M}) \\ & \searrow F & \downarrow F_* \\ & & D \end{array}$$

*commutes.*

*Proof.* This result is a corollary of Lemma 8.2.  $\square$

**Remarks:** 1)  $\text{Ho}(\mathcal{M})$  is a model for the category  $\mathcal{M}[WE]^{-1}$  obtained from  $\mathcal{M}$  by formally inverting all weak equivalences.

2)  $\gamma : \mathcal{M} \rightarrow \text{Ho}(\mathcal{M})$  induces a fully faithful functor  $\gamma_* : \pi(\mathcal{M}_{cf}) \rightarrow \text{Ho}(\mathcal{M})$ . Every object of  $\text{Ho}(\mathcal{M})$  is isomorphic to a (cofibrant fibrant) object in the image of  $\gamma_*$ .

It follows that the functor  $\gamma_*$  is an equivalence of categories.

This last observation specializes to well known phenomena:



- The homotopy category of **CGWH** is equivalent to the category of  $CW$ -complexes and ordinary homotopy classes of maps between them.
- The derived category of  $Ch_+(R)$  is equivalent to the category of chain complexes of projectives and chain homotopy classes of maps between them.

One final thing: the functor  $\gamma : \mathcal{M} \rightarrow \mathrm{Ho}(\mathcal{M})$  reflects weak equivalences:

**Proposition 8.4.** *Suppose that  $\mathcal{M}$  is a closed model category, and that  $f : X \rightarrow Y$  is a morphism such that  $\gamma(f)$  is an isomorphism in  $\mathrm{Ho}(\mathcal{M})$ . Then  $f$  is a weak equivalence of  $\mathcal{M}$ .*

For the proof, it is enough to suppose that both  $X$  and  $Y$  are fibrant and cofibrant and that  $f$  is a fibration with a homotopy inverse  $g : Y \rightarrow X$ . Then the idea is to show that  $f$  is a weak equivalence.

This claim is a triviality in almost all cases of interest, but it is a bit tricky to prove in full generality. This result appears as Proposition II.1.14 in [1].

## References

- [1] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*. Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition.

## Lecture 04: Simplicial sets

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### 9 Simplicial sets

A **simplicial set** is a functor

$$X : \Delta^{op} \rightarrow \mathbf{Set},$$

ie. a contravariant set-valued functor defined on the ordinal number category  $\Delta$ .

One usually writes  $\mathbf{n} \mapsto X_n$ .

$X_n$  is the set of  $n$ -**simplices** of  $X$ .

A **simplicial map**  $f : X \rightarrow Y$  is a natural transformation of such functors.

The simplicial sets and simplicial maps form the category of simplicial sets, denoted by  $s\mathbf{Set}$  — one also sees the notation  $\mathbf{S}$  for this category.

If  $\mathcal{A}$  is some category, then a **simplicial object** in  $\mathcal{A}$  is a functor

$$A : \Delta^{op} \rightarrow \mathcal{A}.$$

Maps between simplicial objects are natural transformations.

The simplicial objects in  $\mathcal{A}$  and their morphisms form a category  $s\mathcal{A}$ .

**Examples:** 1)  $s\mathbf{Gr}$  = simplicial groups.

2)  $s\mathbf{Ab}$  = simplicial abelian groups.

3)  $s(R - \mathbf{Mod})$  = simplicial  $R$ -modules.

4)  $s(s\mathbf{Set}) = s^2\mathbf{Set}$  is the category of **bisimplicial sets**.

Simplicial objects are everywhere.

**Examples of simplicial sets:**

1) We've already met the *singular set*  $S(X)$  for a topological space  $X$ , in Section 4.

$S(X)$  is defined by the *cosimplicial space* (covariant functor)  $\mathbf{n} \mapsto |\Delta^n|$ , by

$$S(X)_n = \text{hom}(|\Delta^n|, X).$$

$\theta : \mathbf{m} \rightarrow \mathbf{n}$  defines a function

$$S(X)_n = \text{hom}(|\Delta^n|, X) \xrightarrow{\theta^*} \text{hom}(|\Delta^m|, X) = S(X)_m$$

by precomposition with the map  $\theta : |\Delta^m| \rightarrow |\Delta^n|$ .

The assignment  $X \mapsto S(X)$  defines the **singular functor**

$$S : \mathbf{CGWH} \rightarrow s\mathbf{Set}.$$

2) The ordinal number  $\mathbf{n}$  represents a contravariant functor

$$\Delta^n = \text{hom}_\Delta(, \mathbf{n}) : \Delta^{op} \rightarrow \mathbf{Set},$$

called the **standard  $n$ -simplex**.

$$\iota_n := 1_{\mathbf{n}} \in \text{hom}_\Delta(\mathbf{n}, \mathbf{n}).$$

The  $n$ -simplex  $\iota_n$  is the **classifying  $n$ -simplex**.

The Yoneda Lemma implies that there is a natural bijection

$$\text{hom}_{s\mathbf{Set}}(\Delta^n, Y) \cong Y_n$$

defined by sending the map  $\sigma : \Delta^n \rightarrow Y$  to the element  $\sigma(\iota_n) \in Y_n$ .

A map  $\Delta^n \rightarrow Y$  is an  *$n$ -simplex of  $Y$* .

Every ordinal number morphism  $\theta : \mathbf{m} \rightarrow \mathbf{n}$  induces a simplicial set map

$$\theta : \Delta^m \rightarrow \Delta^n,$$

defined by composition.

We have a covariant functor

$$\Delta : \Delta \rightarrow s\mathbf{Set}$$

with  $\mathbf{n} \mapsto \Delta^n$ . This is a *cosimplicial object* in  $s\mathbf{Set}$ .

If  $\sigma : \Delta^n \rightarrow X$  is a simplex of  $X$ , the  $i^{th}$  **face**  $d_i(\sigma)$  is the composite

$$\Delta^{n-1} \xrightarrow{d^i} \Delta^n \xrightarrow{\sigma} X,$$

The  $j^{th}$  **degeneracy**  $s_j(\sigma)$  is the composite

$$\Delta^{n+1} \xrightarrow{s^j} \Delta^n \xrightarrow{\sigma} X.$$

3)  $\partial\Delta^n$  is the subobject of  $\Delta^n$  which is generated by the  $(n-1)$ -simplices  $d^i$ ,  $0 \leq i \leq n$ .

$\Lambda_k^n$  is the subobject of  $\partial\Delta^n$  which is generated by the simplices  $d^i$ ,  $i \neq k$ .

$\partial\Delta^n$  is the **boundary** of  $\Delta^n$ , and  $\Lambda_k^n$  is the  $k^{th}$  **horn**.

The faces  $d^i : \Delta^{n-1} \rightarrow \Delta^n$  determine a covering

$$\bigsqcup_{i=0}^n \Delta^{n-1} \rightarrow \partial\Delta^n,$$

and for each  $i < j$  there are pullback diagrams

$$\begin{array}{ccc} \Delta^{n-2} & \xrightarrow{d^{j-1}} & \Delta^{n-1} \\ d^i \downarrow & & \downarrow d^i \\ \Delta^{n-1} & \xrightarrow{d^j} & \Delta^n \end{array}$$

(Excercise!). It follows that there is a coequalizer

$$\bigsqcup_{i < j, 0 \leq i, j \leq n} \Delta^{n-2} \rightrightarrows \bigsqcup_{0 \leq i \leq n} \Delta^{n-1} \longrightarrow \partial\Delta^n$$

in  $s\mathbf{Set}$ .

Similarly, there is a coequalizer

$$\bigsqcup_{i < j, i, j \neq k} \Delta^{n-2} \rightrightarrows \bigsqcup_{0 \leq i \leq n, i \neq k} \Delta^{n-1} \longrightarrow \Lambda_k^n.$$

4) Suppose the category  $C$  is **small**, ie. the morphisms  $\text{Mor}(C)$  (and objects  $\text{Ob}(C)$ ) form a set.

Examples include all finite ordinal numbers  $\mathbf{n}$  (because they are posets), all monoids (small categories having one object), and all groups.

There is a simplicial set  $BC$  with  $n$ -simplices

$$BC_n = \text{hom}(\mathbf{n}, C),$$

ie. the functors  $\mathbf{n} \rightarrow C$ .

The simplicial structure on  $BC$  is defined by precomposition with ordinal number maps: if  $\theta : \mathbf{m} \rightarrow \mathbf{n}$  is an ordinal number map (aka. functor) and  $\sigma : \mathbf{n} \rightarrow C$  is an  $n$ -simplex, then  $\theta^*(\sigma)$  is the composite functor

$$\mathbf{m} \xrightarrow{\theta} \mathbf{n} \xrightarrow{\sigma} C.$$

The object  $BC$  is called the **classifying space** or **nerve** of  $C$  (the notation  $NC$  is also common).

If  $G$  is a (discrete) group,  $BG$  “is” the standard classifying space for  $G$  in **CGWH**, which classifies principal  $G$ -bundles.

**NB:**  $B\mathbf{n} = \Delta^n$ .

5) Suppose  $I$  is a small category, and  $X : I \rightarrow \mathbf{Set}$  is a set-valued functor (aka. a diagram in sets).

The **translation category** (“category of elements”)  $E_I(X)$  has objects given by all pairs  $(i, x)$  with  $x \in X(i)$ .

A morphism  $\alpha : (i, x) \rightarrow (j, y)$  is a morphism  $\alpha : i \rightarrow j$  of  $I$  such that  $\alpha_*(x) = y$ .

The simplicial set  $B(E_I X)$  is the **homotopy colimit** for the functor  $X$ . One often writes

$$\underline{\mathrm{holim}}_I X = B(E_I X).$$

Here’s a different description of the nerve  $BI$ :

$$BI = \underline{\mathrm{holim}}_I *.$$

$BI$  is the homotopy colimit of the (constant) functor  $I \rightarrow \mathbf{Set}$  which associates the one-point set  $*$  to every object of  $I$ .

There is a functor

$$E_I X \rightarrow I,$$

defined by the assignment  $(i, x) \mapsto i$ .

This functor induces a simplicial set map

$$\pi : B(E_I X) = \underline{\mathrm{holim}}_I X \rightarrow BI.$$

A functor  $\mathbf{n} \rightarrow C$  is specified by a string of arrows

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} a_n$$

in  $C$ , for then all composites of these arrows are uniquely determined.

The functors  $\mathbf{n} \rightarrow E_I X$  can be identified with strings

$$(i_0, x_0) \xrightarrow{\alpha_1} (i_1, x_1) \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} (i_n, x_n).$$

Such a string is specified by the underlying string  $i_0 \rightarrow \dots \rightarrow i_n$  in the index category  $Y$  and  $x_0 \in X(i_0)$ .

It follows that there is an identification

$$(\underline{\text{holim}}_I X)_n = B(E_I X)_n = \bigsqcup_{i_0 \rightarrow \dots \rightarrow i_n} X(i_0).$$

The construction is functorial with respect to natural transformations in diagrams  $X$ .

A diagram  $X : I \rightarrow s\mathbf{Set}$  in simplicial sets (a simplicial object in set-valued functors) determines a simplicial category  $m \mapsto E_I(X_m)$  and a corresponding bisimplicial set with  $(n, m)$  simplices

$$B(E_I X)_m = \bigsqcup_{i_0 \rightarrow \dots \rightarrow i_n} X(i_0)_m.$$

The **diagonal**  $d(Y)$  of a bisimplicial set  $Y$  is the simplicial set with  $n$ -simplices  $Y_{n,n}$ . Equivalently,



$d(Y)$  is the composite functor

$$\Delta^{op} \xrightarrow{\Delta} \Delta^{op} \times \Delta^{op} \xrightarrow{Y} \mathbf{Set}$$

where  $\Delta$  is the diagonal functor.

The diagonal  $dB(E_I X)$  of the bisimplicial set  $B(E_I X)$  is the **homotopy colimit**  $\varinjlim_I X$  of the functor  $X : I \rightarrow s\mathbf{Set}$ .

There is a natural simplicial set map

$$\pi : \varinjlim_I X \rightarrow BI.$$

6) Suppose  $X$  and  $Y$  are simplicial sets. The **function complex**

$$\mathbf{hom}(X, Y)$$

has  $n$ -simplices

$$\mathbf{hom}(X, Y)_n = \mathbf{hom}(X \times \Delta^n, Y).$$

If  $\theta : \mathbf{m} \rightarrow \mathbf{n}$  is an ordinal number map and  $f : X \times \Delta^n \rightarrow Y$  is an  $n$ -simplex of  $\mathbf{hom}(X, Y)$ , then  $\theta^*(f)$  is the composite

$$X \times \Delta^m \xrightarrow{1 \times \theta} X \times \Delta^n \xrightarrow{f} Y.$$

There is a natural simplicial set map

$$ev : X \times \mathbf{hom}(X, Y) \rightarrow Y$$

defined by

$$(x, f : X \times \Delta^n \rightarrow Y) \mapsto f(x, \mathbf{l}_n).$$

Suppose  $K$  is a simplicial set.

The function

$$ev_* : \text{hom}(K, \mathbf{hom}(X, Y)) \rightarrow \text{hom}(X \times K, Y),$$

is defined by sending  $g : K \rightarrow \mathbf{hom}(X, Y)$  to the composite

$$X \times K \xrightarrow{1 \times g} X \times \mathbf{hom}(X, Y) \xrightarrow{ev} Y.$$

The function  $ev_*$  is a *bijection*, with inverse that takes  $f : X \times K \rightarrow Y$  to the morphism  $f_* : K \rightarrow \mathbf{hom}(X, Y)$ , where  $f_*(y)$  is the composite

$$X \times \Delta^n \xrightarrow{1 \times y} X \times K \xrightarrow{f} Y.$$

The natural bijection

$$\text{hom}(X \times K, Y) \cong \text{hom}(K, \mathbf{hom}(X, Y))$$

is called the **exponential law**.

$s\mathbf{Set}$  is a *cartesian closed category*.

The function complexes also give  $s\mathbf{Set}$  the structure of a *category enriched in simplicial sets*.

## 10 The simplex category and realization

Suppose  $X$  is a simplicial set.

The **simplex category**  $\Delta/X$  has for objects all simplices  $\Delta^n \rightarrow X$ .

Its morphisms are the *incidence relations* between the simplices, meaning all commutative diagrams

$$\begin{array}{ccc} \Delta^m & & \\ \theta \downarrow & \searrow \tau & \\ \Delta^n & \nearrow \sigma & X \end{array} \quad (1)$$

$\Delta/X$  is a type of *slice category*. It is denoted by  $\Delta \downarrow X$  in [2]. See also [6].

In the broader context of homotopy theories associated to a test category (long story — see [4]) one says that the simplex category is a *cell category*.

**Exercise:** Show that a simplicial set  $X$  is a colimit of its simplices, ie. the simplices  $\Delta^n \rightarrow X$  define a simplicial set map

$$\varinjlim_{\Delta^n \rightarrow X} \Delta^n \rightarrow X,$$

which is an isomorphism.

There is a space  $|X|$ , called the **realization** of the simplicial set  $X$ , which is defined by

$$|X| = \varinjlim_{\Delta^n \rightarrow X} |\Delta^n|.$$

Here  $|\Delta^n|$  is the topological standard  $n$ -simplex, as described in Section 4.

$|X|$  is the colimit of the functor  $\Delta/X \rightarrow \mathbf{CGWH}$  which takes the morphism (1) to the map

$$|\Delta^m| \xrightarrow{\theta} |\Delta^n|.$$

The assignment  $X \mapsto |X|$  defines a functor

$$|| : s\mathbf{Set} \rightarrow \mathbf{CGWH},$$

called the **realization functor**.

**Lemma 10.1.** *The realization functor is left adjoint to the singular functor  $S : \mathbf{CGWH} \rightarrow s\mathbf{Set}$ .*

*Proof.* A simplicial set  $X$  is a colimit of its simplices. Thus, for a simplicial set  $X$  and a space  $Y$ ,

there are natural isomorphisms

$$\begin{aligned}
\text{hom}(X, S(Y)) &\cong \text{hom}\left(\varinjlim_{\Delta^n \rightarrow X} \Delta^n, S(Y)\right) \\
&\cong \varprojlim_{\Delta^n \rightarrow X} \text{hom}(\Delta^n, S(Y)) \\
&\cong \varprojlim_{\Delta^n \rightarrow X} \text{hom}(|\Delta^n|, Y) \\
&\cong \text{hom}\left(\varinjlim_{\Delta^n \rightarrow X} |\Delta^n|, Y\right) \\
&= \text{hom}(|X|, Y).
\end{aligned}$$

□

**Remark:** Kan introduced the concept of adjoint functors to describe the relation between the realization and singular functors.

**Examples:**

- 1)  $|\Delta^n| = |\Delta^n|$ , since the simplex category  $\Delta/\Delta^n$  has a terminal object, namely  $1 : \Delta^n \rightarrow \Delta^n$ .
- 2)  $|\partial\Delta^n| = |\partial\Delta^n|$  and  $|\Lambda_k^n| = |\Lambda_k^n|$ , since the realization functor is a left adjoint and therefore preserves coequalizers and coproducts.

The  $n^{\text{th}}$  **skeleton**  $\text{sk}_n X$  of a simplicial set  $X$  is the subobject generated by the simplices  $X_i$ ,  $0 \leq i \leq n$ . The ascending sequence of subcomplexes

$$\text{sk}_0 X \subset \text{sk}_1 X \subset \text{sk}_2 X \subset \dots$$

defines a filtration of  $X$ , and there are pushout diagrams

$$\begin{array}{ccc} \bigsqcup_{x \in NX_n} \partial \Delta^n & \longrightarrow & \mathrm{sk}_{n-1} X \\ \downarrow & & \downarrow \\ \bigsqcup_{x \in NX_n} \Delta^n & \longrightarrow & \mathrm{sk}_n X \end{array} \quad (2)$$

$NX_n$  is the set of non-degenerate  $n$ -simplices of  $X$ .

$\sigma \in X_n$  is **non-degenerate** if it is not of the form  $s_j(y)$  for some  $(n-1)$ -simplex  $y$  and some  $j$ .

**Exercise:** Show that the diagram (2) is indeed a pushout.

For this, it's helpful to know that the functor  $X \mapsto \mathrm{sk}_n X$  is left adjoint to truncation up to level  $n$ .

For *that*, you should know that every simplex  $x$  of a simplicial set  $X$  has a unique representation  $x = s^*(y)$  where  $s : \mathbf{n} \twoheadrightarrow \mathbf{k}$  is an ordinal number epi and  $y \in X_k$  is non-degenerate.

**Corollary 10.2.** *The realization  $|X|$  of a simplicial set  $X$  is a CW-complex.*

Every monomorphism  $A \rightarrow B$  of simplicial sets induces a cofibration  $|A| \rightarrow |B|$  of spaces. ie.  $|B|$  is constructed from  $|A|$  by attaching cells.

**Lemma 10.3.** *The realization functor preserves finite limits.*

*Proof.* There are isomorphisms

$$\begin{aligned}
|X \times Y| &\cong \left| \varinjlim_{\Delta^n \rightarrow X, \Delta^m \rightarrow Y} \Delta^n \times \Delta^m \right| \\
&\cong \varinjlim_{\Delta^n \rightarrow X, \Delta^m \rightarrow Y} |\Delta^n \times \Delta^m| \\
&\cong \varinjlim_{\Delta^n \rightarrow X, \Delta^m \rightarrow Y} |\Delta^n| \times |\Delta^m| \\
&\cong |X| \times |Y|
\end{aligned}$$

One shows that the canonical maps

$$|\Delta^n \times \Delta^m| \rightarrow |\Delta^n| \times |\Delta^m|$$

are isomorphisms with an argument involving shuffles — see [1, p.52].

If  $\sigma, \tau : \Delta^n \rightarrow Y$  are simplices such that

$$|\sigma| = |\tau| : |\Delta^n| \rightarrow |Y|,$$

then  $\sigma = \tau$  (exercise).

Suppose  $f, g : X \rightarrow Y$  are simplicial set maps, and  $x \in |X|$  is an element such that  $f_*(x) = g_*(x)$ .

If  $\sigma$  is the “carrier” of  $x$  (ie. non-degenerate simplex of  $X$  such that  $x$  is interior to the cell defined by  $\sigma$ ), then  $f_*(y) = g_*(y)$  for all  $y$  in the interior of

$|\sigma|$  (by transforming by a suitable automorphism of the cosimplicial space  $|\Delta|$  — see [1, p.51]).

But then

$$|f\sigma| = |g\sigma| : |\Delta^n| \rightarrow |Y|,$$

so  $f\sigma = g\sigma$  and  $x \in |E|$ , where  $E$  is the equalizer of  $f$  and  $g$  in  $s\mathbf{Set}$ .  $\square$

## 11 Model structure for simplicial sets

A map  $f : X \rightarrow Y$  of simplicial sets is a **weak equivalence** if  $f_* : |X| \rightarrow |Y|$  is a weak equivalence of **CGWH**.

A map  $i : A \rightarrow B$  of simplicial sets is a **cofibration** if and only if it is a monomorphism, ie. all functions  $i : A_n \rightarrow B_n$  are injective.

A simplicial set map  $p : X \rightarrow Y$  is a **fibration** if it has the RLP wrt all trivial cofibrations.

**Remark:** There is a natural commutative diagram

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{\nabla} & X \\ (i_0, i_1) \downarrow & \nearrow pr & \\ X \times \Delta^1 & & \end{array} \quad (3)$$

for simplicial sets  $X$ .  $(i_0, i_1)$  is the cofibration

$$1_X \times i : X \times \partial\Delta^1 \rightarrow X \times \Delta^1$$



induced by the inclusion  $i : \partial\Delta^1 \subset \Delta^1$ . The two inclusions  $i_\varepsilon$  of the end points of the cylinder are weak equivalences, as is  $pr : X \times \Delta^1 \rightarrow X$ .

The diagram (3) is a natural cylinder object for the model structure on simplicial sets (see Theorem 11.6). Left homotopy with respect to this cylinder is classical **simplicial homotopy**.

**Lemma 11.1.** *A map  $p : X \rightarrow Y$  is a trivial fibration if and only if it has the RLP wrt all inclusions  $\partial\Delta^n \subset \Delta^n$ ,  $n \geq 0$ .*

*Proof.* 1) Suppose  $p$  has the lifting property.

Then  $p$  has the RLP wrt all cofibrations (exercise: induct through relative skeleta), so the lifting  $s$  exists in the diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \downarrow & \nearrow s & \downarrow p \\ Y & \xrightarrow{1_Y} & Y \end{array}$$

since all simplicial sets are cofibrant.

The lifting  $h$  exists in the diagram

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{(sp, 1)} & X \\ \downarrow i & \nearrow h & \downarrow p \\ X \times \Delta^1 & \xrightarrow{p \cdot pr} & Y \end{array}$$

so the map  $p_* : |X| \rightarrow |Y|$  is a homotopy equivalence, hence a weak equivalence.

2) Suppose  $p$  is a trivial fibration and choose a factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & U \\ & \searrow p & \downarrow q \\ & & Y \end{array}$$

such that  $j$  is a cofibration and  $q$  has the RLP wrt all maps  $\partial\Delta^n \subset \Delta^n$  (such things exist by a small object argument).

$q$  is a weak equivalence by part 1), so  $j$  is a trivial cofibration and the lift  $r$  exists in the diagram

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ j \downarrow & \nearrow r & \downarrow p \\ U & \xrightarrow{q} & Y \end{array}$$

Then  $p$  is a retract of  $q$ , and has the RLP. □

Say that a simplicial set  $A$  is **countable** if it has countably many non-degenerate simplices.

A simplicial set  $K$  is **finite** if it has only finitely many non-degenerate simplices, eg.  $\Delta^n$ ,  $\partial\Delta^n$ ,  $\Lambda_k^n$ .

**Fact:** If  $X$  is countable (resp. finite), then all subcomplexes of  $X$  are countable (resp. finite).

The following result is proved with simplicial approximation techniques:

**Lemma 11.2.** *Suppose that  $X$  has countably many non-degenerate simplices.*

*Then  $\pi_0|X|$  and all homotopy groups  $\pi_n(|X|, x)$  are countable.*

*Proof.* Suppose  $x$  is a vertex of  $X$ , identified with  $x \in |X|$ .

A continuous map

$$(|\Delta^k|, |\partial\Delta^k|) \rightarrow (|X|, x)$$

is homotopic, rel boundary, to the realization of a simplicial set map

$$(\text{sd}^N \Delta^k, \text{sd}^N \partial\Delta^k) \rightarrow (X, x),$$

by simplicial approximation [3].

The (iterated) subdivisions  $\text{sd}^M \Delta^k$  are finite complexes, and there are only countably many maps  $\text{sd}^M \Delta^k \rightarrow X$  for  $M \geq 0$ .  $\square$

Here's a consequence:

**Lemma 11.3** (Bounded cofibration lemma). *Suppose given cofibrations*

$$\begin{array}{ccc} & X & \\ & \downarrow i & \\ A & \longrightarrow & Y \end{array}$$

*where  $i$  is trivial and  $A$  is countable.*

*Then there is a countable  $B \subset Y$  with  $A \subset B$ , such that the map  $B \cap X \rightarrow B$  is a trivial cofibration.*

*Proof.* Write  $B_0 = A$  and consider the map

$$B_0 \cap X \rightarrow B_0.$$

The homotopy groups of  $|B_0|$  and  $|B_0 \cap X|$  are countable, by Lemma 11.2.

$Y$  is a union of its countable subcomplexes.

Suppose that

$$\alpha, \beta : (|\Delta^n|, |\partial\Delta^n|) \rightarrow (|B_0 \cap X|, x)$$

become homotopic in  $|B_0|$  hence in  $|X|$ .

The map defining the homotopy in  $|X|$  is compact (ie. defined on a  $CW$ -complex with finitely many cells), so there is a countable  $B' \subset Y$  with  $B_0 \subset B'$  such that the homotopy lives in  $|B' \cap X|$ .

The image in  $|Y|$  of any morphism

$$\gamma: (|\Delta^n|, |\partial\Delta^n|) \rightarrow (|B_0|, x)$$

lifts to  $|X|$  up to homotopy, and that homotopy lives in  $|B''|$  for some countable subcomplex  $B'' \subset Y$  with  $B_0 \subset B''$ .

It follows that there is a countable subcomplex  $B_1 \subset Y$  with  $B_0 \subset B_1$  such that any two elements

$$[\alpha], [\beta] \in \pi_n(|B_0 \cap X|, x)$$

which map to the same element in  $\pi_n(|B_0|, x)$  must also map to the same element of  $\pi_n(|B_1 \cap X|, x)$ , and every element

$$[\gamma] \in \pi_n(|B_0|, x)$$

lifts to an element of  $\pi_n(|B_1 \cap X|, x)$ , and this for all  $n \geq 0$  and all (countably many) vertices  $x$ .

Repeat the construction inductively, to form a countable collection

$$A = B_0 \subset B_1 \subset B_2 \subset \dots$$

of subcomplexes of  $Y$ .

Then  $B = \bigcup B_i$  is a countable subcomplex of  $Y$ , and the map  $B \cap X \rightarrow B$  is a weak equivalence.  $\square$

Say that a cofibration  $A \rightarrow B$  is **countable** if  $B$  is countable.

**Lemma 11.4.** *Every simplicial set map  $f : X \rightarrow Y$  has a factorization*

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ & \searrow f & \downarrow q \\ & & Y \end{array}$$

*such that  $q$  has the RLP wrt all countable trivial cofibrations, and  $i$  is constructed from countable trivial cofibrations by pushout and composition.*

The proof of Lemma 11.4 is an example of a *transfinite small object argument*.

Lang's *Algebra* [5] has a quick introduction to cardinal arithmetic.

*Proof.* Choose an uncountable cardinal number  $\kappa$ , interpreted as the (totally ordered) poset of ordinal numbers  $s < \kappa$ .

Construct a system of factorizations

$$\begin{array}{ccc} X & \xrightarrow{i_s} & Z_s \\ & \searrow f & \downarrow q_s \\ & & Y \end{array} \tag{4}$$

of  $f$  with  $j_s$  a trivial cofibration as follows:

- given factorization of the form (4) consider all diagrams

$$D : \begin{array}{ccc} A_D & \longrightarrow & Z_s \\ i_D \downarrow & & \downarrow q_s \\ B_D & \longrightarrow & Y \end{array}$$

such that  $i_D$  is a countable trivial cofibration, and form the pushout

$$\begin{array}{ccc} \bigsqcup_D A_D & \longrightarrow & Z_s \\ \downarrow & & \downarrow j_s \\ \bigsqcup_D B_D & \longrightarrow & Z_{s+1} \end{array}$$

Then the map  $j_s$  is a trivial cofibration, and the diagrams together induce a map  $q_{s+1} : Z_{s+1} \rightarrow Y$ . Let  $i_{s+1} = j_s i_s$ .

- if  $\gamma < \kappa$  is a limit ordinal, let  $Z_\gamma = \varinjlim_{t < \gamma} Z_t$ .

Now let  $Z = \varinjlim_{s < \kappa} Z_s$  with induced factorization

$$X \xrightarrow{j} Z \xrightarrow{q} Y$$

Suppose given a lifting problem

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & Z \\ j \downarrow & & \downarrow q \\ B & \longrightarrow & Y \end{array}$$

with  $j : A \rightarrow B$  a countable trivial cofibration. Then  $\alpha(A)$  is a countable subcomplex of  $X$ , so  $\alpha(A) \subset$

$Z_s$  for some  $s < \kappa$ , for otherwise  $\alpha(A)$  has too many elements.

The lifting problem is solved in  $Z_{s+1}$ .  $\square$

**Remark:** The map  $j : X \rightarrow Z$  is in the saturation of the set of countable trivial cofibrations.

The **saturation** of a set of cofibrations  $I$  is the smallest class of cofibrations containing  $I$  which is closed under pushout, coproducts, (long) compositions and retraction.

If a map  $p$  has the RLP wrt all maps of  $I$  then it has the RLP wrt all maps in the saturation of  $I$ . (exercise)

Classes of cofibrations which are defined by a left lifting property with respect to some family of maps are saturated in this sense. (exercise)

**Lemma 11.5.** *A map  $q : X \rightarrow Y$  is a fibration if and only if it has the RLP wrt (the set of) all countable trivial cofibrations.*

We use a recurring trick for the proof of this result. It amounts to verifying a “solution set condition”.



*Proof.* 1) Suppose given a diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ j \downarrow & & \downarrow f \\ B & \longrightarrow & Y \end{array}$$

where  $j$  is a cofibration,  $B$  is countable and  $f$  is a weak equivalence.

Lemma 11.1 says that  $f$  has a factorization  $f = q \cdot i$ , where  $i$  is a trivial cofibration and  $q$  has the RLP wrt all cofibrations.

The lift exists in the diagram

$$\begin{array}{ccccc} A & \longrightarrow & X & & \\ \downarrow j & & \downarrow i & & \\ & \nearrow \theta & Z & & \\ & & \downarrow q & & \\ B & \longrightarrow & Y & & \end{array}$$

$\theta(B)$  is countable, so there is a countable subcomplex  $D \subset Z$  with  $\theta(B) \subset D$  such that the map  $D \cap X \rightarrow D$  is a trivial cofibration.

We have a factorization

$$\begin{array}{ccccc} A & \longrightarrow & D \cap X & \longrightarrow & X \\ j \downarrow & & \downarrow & & \downarrow f \\ B & \longrightarrow & D & \longrightarrow & Y \end{array}$$

of the original diagram through a countable trivial cofibration.

2) Suppose that  $i : C \rightarrow D$  is a trivial cofibration.

Then  $i$  has a factorization

$$\begin{array}{ccc} C & \xrightarrow{j} & E \\ & \searrow i & \downarrow p \\ & & D \end{array}$$

such that  $p$  has the RLP wrt all countable trivial cofibrations, and  $j$  is built from countable trivial cofibrations by pushout and composition. Then  $j$  is a weak equivalence, so  $p$  is a weak equivalence.

Part 1) implies that  $p$  has the RLP wrt all countable cofibrations, and hence wrt all cofibrations.

The lift therefore exists in the diagram

$$\begin{array}{ccc} C & \xrightarrow{j} & E \\ i \downarrow & \theta \nearrow & \downarrow p \\ D & \xrightarrow{1_D} & D \end{array}$$

so  $i$  is a retract of  $j$ .

Thus, if  $q : Z \rightarrow W$  has the RLP wrt all countable trivial cofibrations, then it has the RLP wrt all trivial cofibrations.  $\square$

**Exercise:** Find a different, simpler proof for Lemma 11.5. Hint: use Zorn's lemma.

**Theorem 11.6.** *With the definitions of weak equivalence, cofibration and fibration given above the category  $s\mathbf{Set}$  of simplicial sets satisfies the axioms for a closed model category.*

*Proof.* The axioms **CM1**, **CM2** and **CM3** are easy to verify.

Every map  $f : X \rightarrow Y$  has a factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & W \\ & \searrow f & \downarrow q \\ & & Y \end{array}$$

such that  $j$  is a cofibration and  $q$  is a trivial fibration — this follows from Lemma 11.1 and a standard small object argument. The other half of the factorization axiom **CM5** is a consequence of Lemma 11.4 and Lemma 11.5.

**CM4** also follows from Lemma 11.1. □

**Remark:** In the adjoint pair of functors

$$| | : s\mathbf{Set} \rightleftarrows \mathbf{CGWH} : S$$

the realization functor (the left adjoint part) preserves cofibrations and trivial cofibrations. It's an immediate consequence that the singular functor  $S$  preserves fibrations and trivial fibrations.

Adjunctions like this between closed model category are called **Quillen adjunctions** or **Quillen pairs**. We'll see later on, and this is a huge result, that these functors form a Quillen equivalence.

**Remark:** We defined the weak equivalences of simplicial sets to be those maps whose realizations are weak equivalences of spaces. In this way, the model structure for  $s\mathbf{Set}$ , as it is described here, is *induced* from the model structure for  $\mathbf{CGWH}$  via the realization functor  $|\cdot|$ .

Alternatively, one says that the model structure on simplicial sets is obtained from that on spaces by *transfer*.

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## Lecture 05: Fibrations, geometric realization

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### 12 Kan fibrations

A map  $p : X \rightarrow Y$  is a **Kan fibration** if it has the RLP wrt all inclusions  $\Lambda_k^n \subset \Delta^n$ .

**Example:** A fibration of simplicial sets, (Section 11), is a Kan fibration, since  $|\Lambda_k^n| \rightarrow |\Delta^n|$  is a weak equivalence.

The converse statement is also true: every Kan fibration is a fibration. This is Theorem 13.5 below.

Say that  $X$  is a **Kan complex** if the map  $X \rightarrow *$  is a Kan fibration.

**Exercise:** Suppose  $C$  is a small category. Show that the nerve  $BC$  is a Kan complex if and only if  $C$  is a groupoid.

**Example:** The ordinal number posets  $\mathbf{n}$  are not groupoids if  $n \geq 1$ , so the simplices  $\Delta^n = B\mathbf{n}$  are *not* Kan complexes.

The saturation of the set of cofibrations  $\Lambda_k^n \subset \Delta^n$  is normally called the class of **anodyne extensions**.

This is the class of cofibrations which has the LLP wrt all Kan fibrations.

**Lemma 12.1.** *The following sets of cofibrations have the same saturations:*

- $\mathbf{A}_1 = \text{all maps } \Lambda_k^n \subset \Delta^n,$
- $\mathbf{A}_2 = \text{all inclusions}$

$$(\Delta^1 \times \partial\Delta^n) \cup (\{\varepsilon\} \times \Delta^n) \subset \Delta^1 \times \Delta^n, \varepsilon = 0, 1.$$

*Proof.* 1) The saturation of  $\mathbf{A}_2$  includes all maps

$$(\Delta^1 \times K) \cup (\{\varepsilon\} \times L) \subset \Delta^1 \times L, \varepsilon = 0, 1.$$

induced by inclusions  $K \subset L$ , since  $L$  is built from  $K$  by attaching cells.

The functor  $r_k : \mathbf{n} \times \mathbf{1} \rightarrow \mathbf{n}$  specified by the picture

$$\begin{array}{ccccccc} 0 & \longrightarrow & 1 & \longrightarrow & \dots & \longrightarrow & k & \longrightarrow & k & \longrightarrow & \dots & \longrightarrow & k \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\ 0 & \longrightarrow & 1 & \longrightarrow & \dots & \longrightarrow & k & \longrightarrow & k+1 & \longrightarrow & \dots & \longrightarrow & n \end{array}$$

and the functor  $i : \mathbf{n} \rightarrow \mathbf{n} \times \mathbf{1}$  defined by  $i(j) = (j, 1)$  together determine a retraction diagram

$$\begin{array}{ccccc} \Lambda_k^n & \longrightarrow & (\Lambda_k^n \times \Delta^1) \cup (\Delta^n \times \{0\}) & \longrightarrow & \Lambda_k^n \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & \Delta^n \times \Delta^1 & \longrightarrow & \Delta^n \end{array}$$

(NB:  $\Delta^n \times \{0\}$  is mapped into  $\Lambda_k^n$ ) so  $\Lambda_k^n \subset \Delta^n$  is in the saturation of the family  $\mathbf{A}_2$  if  $k < n$ .

The map  $\Lambda_k^n \subset \Delta^n$  is a retraction of

$$(\Lambda_k^n \times \Delta^1) \cup (\Delta^n \times \{1\}) \subset \Delta^n \times \Delta^1$$

if  $k > 0$ . Thus, the saturation of  $\mathbf{A}_1$  is contained in the saturation of  $\mathbf{A}_2$ .

2) The non-degenerate  $(n+1)$ -simplices of  $h_i : \Delta^n \times \Delta^1$  are functors  $\mathbf{n} + \mathbf{1} \rightarrow \mathbf{n} \times \mathbf{1}$  defined by the pictures

$$\begin{array}{ccccccc} (0,0) & \longrightarrow & (1,0) & \longrightarrow & \dots & \longrightarrow & (i,0) \\ & & & & & & \downarrow \\ & & & & & & (i,1) \longrightarrow \dots \longrightarrow (i,n) \end{array}$$

Let  $(\Delta^n \times \Delta^1)^{(i)}$  be the subcomplex of  $\Delta^n \times \Delta^1$  generated by  $\partial \Delta^n \times \Delta^1$  and the simplices  $h_0, \dots, h_i$ . Let

$$(\Delta^n \times \Delta^1)^{(-1)} = (\partial \Delta^n \times \Delta^1) \cup (\Delta^n \times \{0\}).$$

Then  $(\Delta^n \times \Delta^1)^{(n)} = \Delta^n \times \Delta^1$ , and there are pushouts

$$\begin{array}{ccc} \Lambda_{i+2}^{n+1} & \longrightarrow & (\Delta^n \times \Delta^1)^{(i)} \\ \downarrow & & \downarrow \\ \Delta^{n+1} & \longrightarrow & (\Delta^n \times \Delta^1)^{(i+1)} \end{array}$$

It follows that the members of  $\mathbf{A}_2$  are in the saturation of the set  $\mathbf{A}_1$ .  $\square$

**Lemma 12.2.** *Suppose  $i : K \rightarrow L$  is an anodyne extension and  $j : A \rightarrow B$  is a cofibration.*

*Then the inclusion*

$$(K \times B) \cup (L \times A) \subset L \times B$$

*is anodyne.*

*Proof.* The class of cofibrations  $K' \rightarrow L'$  such that

$$(K' \times B) \cup (L' \times A) \subset L' \times B$$

is anodyne is saturated, and includes all cofibrations

$$(\Delta^1 \times \partial\Delta^n) \cup (\{\varepsilon\} \times \Delta^n) \subset \Delta^1 \times \Delta^n, \quad \varepsilon = 0, 1,$$

by rebracketing (see [2, I.4.6]). □

**Corollary 12.3.** *The cofibrations*

$$(\Lambda_k^n \times \Delta^m) \cup (\Delta^n \times \partial\Delta^m) \subset \Delta^n \times \Delta^m$$

*are anodyne.*



Here's something else that Lemma 12.2 buys you:

**Corollary 12.4.** *Suppose  $p : X \rightarrow Y$  is a Kan fibration and  $j : A \rightarrow B$  is a cofibration.*

*Then the map*

$$\mathbf{hom}(B, X) \xrightarrow{(j^*, p_*)} \mathbf{hom}(A, X) \times_{\mathbf{hom}(A, Y)} \mathbf{hom}(B, Y)$$

*is a Kan fibration.*

*If either  $p$  is a trivial fibration or  $j$  is anodyne, then the map  $(j^*, p_*)$  is a trivial fibration.*

*Proof.* Solutions of the lifting problem

$$\begin{array}{ccc} K & \longrightarrow & \mathbf{hom}(B, X) \\ i \downarrow & \nearrow & \downarrow (j^*, p_*) \\ L & \longrightarrow & \mathbf{hom}(A, X) \times_{\mathbf{hom}(A, Y)} \mathbf{hom}(B, Y) \end{array}$$

are equivalent to solutions of the lifting problem

$$\begin{array}{ccc} (L \times A) \cup (K \times B) & \longrightarrow & X \\ (i, j)_* \downarrow & \nearrow & \downarrow p \\ L \times B & \longrightarrow & Y \end{array}$$

by the exponential law. The map  $(i, j)_*$  is anodyne if either  $i$  or  $j$  is anodyne, by Lemma 12.2.  $\square$

**Corollary 12.5.** *The function complex  $\mathbf{hom}(X, Y)$  is a Kan complex if  $Y$  is a Kan complex.*

The proof of Corollary 12.5 is an exercise.

**Lemma 12.6.** *Simplicial homotopy of maps*

$$X \rightarrow Y$$

is an equivalence relation if  $Y$  is a Kan complex.

*Proof.* It's enough to show that simplicial homotopy classes of vertices  $\Delta^0 \rightarrow Z$  is an equivalence relation if  $Z$  is a Kan complex, since  $\mathbf{hom}(X, Y)$  is a Kan complex.

The paths

$$x \xrightarrow{\omega_2} y \xrightarrow{\omega_0} z$$

define a map  $(\omega_0, \cdot, \omega_2) : \Lambda_1^2 \rightarrow Z$  which extends to a 2-simplex  $\sigma : \Delta^2 \rightarrow Z$ . The 1-simplex  $d_1\sigma$  is a path  $x \rightarrow z$ . Thus, the path relation is transitive.

Suppose  $\omega_2 : x \rightarrow y$  is a path in a Kan complex  $Z$ . Let  $x : x \rightarrow x$  denote the constant path (degenerate 1-simplex) at  $x$ . Then there is a diagram

$$\begin{array}{ccc} \Lambda_0^2 & \xrightarrow{(\cdot, x, \omega_2)} & Z \\ \downarrow & \nearrow \theta & \\ \Delta^2 & & \end{array}$$

so there is a path  $d_0\theta : y \rightarrow x$ . The path relation is therefore symmetric.

The constant path  $\Delta^1 \xrightarrow{s^0} \Delta^0 \xrightarrow{x} X$  is a path from  $x$  to  $x$ , so the relation is reflexive.  $\square$

## Path components

Write  $\pi_0(Z)$  for the path components, aka. simplicial homotopy classes of vertices  $\Delta^0 \rightarrow Z$  for a Kan complex of  $Z$ .

The argument for Lemma 12.6 implies that there is a coequalizer

$$Z_1 \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} Z_0 \longrightarrow \pi_0(Z)$$

in **Set**.

More generally, the set  $\pi_0 X$  of **path components** is defined for an arbitrary simplicial set  $X$  by the coequalizer

$$X_1 \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} X_0 \longrightarrow \pi_0(X)$$

**Exercise:** Show that there is a natural bijection

$$\pi_0(X) \cong \pi_0(|X|)$$

for simplicial sets  $X$ .

## Combinatorial homotopy groups

Suppose  $Y$  is a Kan complex, and  $x \in Y_0$  is a vertex.

The map  $i^*$  in the pullback diagram

$$\begin{array}{ccc} F_x & \longrightarrow & \mathbf{hom}(\Delta^n, Y) \\ \downarrow & & \downarrow i^* \\ \Delta^0 & \xrightarrow{x} & \mathbf{hom}(\partial\Delta^n, Y) \end{array}$$

is a Kan fibration by Corollary 12.4.

The vertices of the Kan complex  $F_x$  are diagrams

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{x} & Y \\ i \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

or simplices  $\alpha : \Delta^n \rightarrow Y$  which restrict to the trivial map  $\partial\Delta^n \rightarrow \Delta^0 \xrightarrow{x} Y$  on the boundary.

The path components  $\pi_0(F_x)$  are the simplicial homotopy classes of maps

$$(\Delta^n, \partial\Delta^n) \rightarrow (Y, x)$$

rel  $\partial\Delta^n$ .

$\pi_n^s(Y, x)$  denotes this set of simplicial homotopy classes.

The set  $\pi_n^s(Y, x)$  has the structure of a group for  $n \geq 1$ , and this group is abelian if  $n \geq 2$ . These are

the *simplicial homotopy groups* of a Kan complex.

The multiplication is specified for  $[\alpha], [\beta] \in \pi_n^s(Y, x)$  by

$$[\alpha] * [\beta] = [d_n \sigma],$$

where  $\sigma : \Delta^{n+1} \rightarrow Y$  is a lifting

$$\begin{array}{ccc} \Lambda_n^{n+1} & \xrightarrow{(x, \dots, x, \alpha, ?, \beta)} & Y \\ \downarrow & \nearrow \sigma & \\ \Delta^{n+1} & & \end{array}$$

Equivalently,  $\pi_n^s(Y, x)$  can be identified with homotopy classes of maps

$$((\Delta^1)^{\times n}, \partial((\Delta^1)^{\times n})) \rightarrow (Y, x)$$

by the same (prismatic) argument as the corresponding result for topological spaces (Section 5).

The group  $\pi_2^s(Y, x)$  is the group of automorphisms of the constant loop  $x \rightarrow x$  in the combinatorial fundamental groupoid  $\pi^s(\Omega(Y))$  for the loop object  $\Omega(Y)$  at  $x$ . The loop “space”  $\Omega(Y)$  is defined by the pullback diagram

$$\begin{array}{ccc} \Omega(Y) & \longrightarrow & \mathbf{hom}(\Delta^1, Y) \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{x} & \mathbf{hom}(\partial \Delta^1, Y) \end{array}$$

The **combinatorial fundamental groupoid**  $\pi^s(Z)$  is defined for a Kan complex  $Z$  by analogy with the definition of the fundamental groupoid of a space. Exercise: Construct  $\pi^s(Z)$ .

This group multiplication is defined by one of the two directions implicit in the maps

$$((\Delta^1)^{\times 2}, \partial(\Delta^1)^{\times 2}) \rightarrow (Y, x).$$

The second multiplication coincides with this one (and has the same identity), since the inclusions

$$\Lambda_1^2 \times \Lambda_1^2 \rightarrow \Delta^2 \times \Delta^2$$

are anodyne (exercise). It follows that  $\pi_2^s(Y, x)$  is an abelian group.

The group laws for all  $\pi_n^s(Y, x), n \geq 2$ , are constructed similarly, and are abelian.  $\pi_n^s(Y, x)$  is an automorphism group of the combinatorial fundamental groupoid  $\pi^s \Omega^{n-1}(Y)$  of the iterated loop space  $\Omega^{n-1}(Y)$  (at  $x$ ).

## Long exact sequence

Suppose  $p : X \rightarrow Y$  is a Kan fibration such that  $Y$  (hence  $X$ ) is a Kan complex.

Define the **fibre**  $F$  over a vertex  $y \in Y$  by the pull-back diagram

$$\begin{array}{ccc} F & \xrightarrow{i} & X \\ \downarrow & & \downarrow p \\ \Delta^0 & \xrightarrow{y} & Y \end{array}$$

Suppose  $x$  is a vertex of  $F$ . There is a boundary homomorphism

$$\partial : \pi_{n+1}^s(Y, y) \rightarrow \pi_n^s(F, x)$$

which is defined for  $[\alpha] \in \pi_{n+1}^s(Y, y)$  by setting  $\partial([\alpha]) = [d_0\theta]$ , where  $\theta$  is a choice of lifting making the diagram

$$\begin{array}{ccc} \Lambda_0^{n+1} & \xrightarrow{x} & X \\ \downarrow & \nearrow \theta & \downarrow p \\ \Delta^{n+1} & \xrightarrow{\alpha} & Y \end{array}$$

commute.

The same arguments as for Lemma 5.2 apply, giving

**Lemma 12.7.**  $p : X \rightarrow Y$  is a Kan fibration such that  $Y$  is a Kan complex, and  $F$  is the fibre over a vertex  $y \in Y$ .

1) For each vertex  $x \in F$  there is a sequence of pointed sets

$$\begin{aligned} \dots \pi_n^s(F, x) &\xrightarrow{i_*} \pi_n^s(X, x) \xrightarrow{p_*} \pi_n^s(Y, p(x)) \xrightarrow{\partial} \pi_{n-1}^s(F, x) \rightarrow \dots \\ \dots \pi_1^s(Y, p(x)) &\xrightarrow{\partial} \pi_0(F) \xrightarrow{i_*} \pi_0(X) \xrightarrow{p_*} \pi_0(Y) \end{aligned}$$

which is exact in the sense that  $\ker = \text{im}$  everywhere.

2) There is a group action

$$* : \pi_1^s(Y, p(x)) \times \pi_0(F) \rightarrow \pi_0(F)$$

such that  $\partial([\alpha]) = [\alpha] * [x]$ , and  $i_*[z] = i_*[w]$  iff there is  $[\beta] \in \pi_1(Y, p(x))$  st  $[\beta] * [z] = [w]$ .

Here's a combinatorial analogue of Lemma 5.1:

**Lemma 12.8.**  $p : X \rightarrow Y$  is a Kan fibration and  $Y$  is a Kan complex. Suppose  $p$  induces a bijection  $\pi_0(X) \cong \pi_0(Y)$ , and isomorphisms  $\pi_n^s(X, x) \cong \pi_n^s(Y, p(x))$  for all  $n \geq 1$  and all vertices  $x$  of  $X$ . Then  $p$  is a trivial fibration of  $s\mathbf{Set}$ .

*Proof.* Show that  $p$  has the right lifting property with respect to all inclusions  $\partial\Delta^n \subset \Delta^n$ ,  $n \geq 0$ . The argument is the same as for Lemma 5.1.  $\square$



A **combinatorial weak equivalence** is a map  $f : X \rightarrow Y$  of *Kan complexes* that induces an isomorphism in all possible simplicial homotopy groups, ie.  $f$  induces a bijection and isomorphisms

$$\pi_n^s(X, x) \cong \pi_n^s(Y, f(x)), \quad x \in X_0, n \geq 1.$$

Equivalently,  $f$  induces a bijection

$$\pi_0(X) \cong \pi_0(Y)$$

and all diagrams

$$\begin{array}{ccc} \pi_n^s(X) & \longrightarrow & \pi_n^s(Y) \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

are pullbacks of sets. Here,

$$\pi_n^s(X) := \bigsqcup_{x \in X_0} \pi_n^s(X, x).$$

By Lemma 12.8, a map  $p$  that is a Kan fibration and a combinatorial weak equivalence between Kan complexes must also be a trivial fibration.

### 13 Simplicial sets and spaces

Here's a major theorem, due to Quillen:

**Theorem 13.1.** *The realization of a Kan fibration is a Serre fibration.*

*Proof.* This will only be a brief sketch — the details can be found, for example, [2, I.10].

The idea is to use the theory of minimal fibrations to show that every Kan fibration  $p : X \rightarrow Y$  has a factorization

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ & \searrow p & \downarrow q \\ & & Y \end{array}$$

where  $g$  is a trivial fibration (ie. has the right lifting property with respect to all  $\partial\Delta^n \subset \Delta^n$ ) and  $q$  is a minimal Kan fibration.

Garbriel and Zisman show [1], [2] that the realization of a minimal fibration  $q : Z \rightarrow Y$  is a Serre fibration: the idea is that every pullback  $q^{-1}(\sigma)$  of a simplex  $\sigma : \Delta^n \rightarrow Y$  is isomorphic over  $\Delta^n$  to a simplicial set  $F \times \Delta^n$ , where  $F$  is a fibre over some vertex  $\Delta^n$ , and it follows that the realization of  $q$  is locally a projection, hence a Serre fibration.

The trivial fibration  $g$  sits in a diagram

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ (1_X, g) \downarrow & \nearrow & \downarrow g \\ X \times Z & \xrightarrow{pr} & Z \end{array}$$

and is therefore a retract of a projection.  $\square$

A Kan fibration  $p : X \rightarrow Y$  is said to be **minimal** if, given simplices  $\alpha, \beta : \Delta^n \rightarrow Y$  (with  $\partial(\alpha) = \partial(\beta)$  and  $p(\alpha) = p(\beta)$ ), then the existence of a diagram

$$\begin{array}{ccc} \partial\Delta^n \times \Delta^1 & \xrightarrow{pr} & \partial\Delta^n \\ i \times 1 \downarrow & & \downarrow \\ \Delta^n \times \Delta^1 & \xrightarrow{h} & X \\ pr \downarrow & & \downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

(fibrewise homotopy rel boundary) forces  $\alpha = \beta$ .

Every Kan fibration has a minimal Kan fibration as a strong fibrewise deformation retract, and every fibrewise weak equivalence of minimal fibrations is an isomorphism. See [2, I.10].

The *Milnor Theorem* is a consequence of Quillen's theorem:

**Theorem 13.2** (Milnor). *Suppose that  $Y$  is a Kan complex and  $\eta : Y \rightarrow S(|Y|)$  is the adjunction homomorphism.*

*Then  $\eta$  is a combinatorial weak equivalence.*

We need the path-loop fibre sequence for the proof of Theorem 13.2.

If  $Y$  is a Kan complex, then the map  $\partial\Delta^1 \subset \Delta^1$  induces a Kan fibration

$$\mathbf{hom}(\Delta^1, Y) \xrightarrow{(p_0, p_1)} Y \times Y \cong \mathbf{hom}(\partial\Delta^1, Y),$$

and the induced maps  $p_0, p_1$  are trivial fibrations, by Corollary 12.4.

Take a vertex  $x \in Y$ , and form the pullback

$$\begin{array}{ccc} P_x Y & \xrightarrow{i} & \mathbf{hom}(\Delta^1, Y) \\ p_{0*} \downarrow & & \downarrow p_0 \\ \Delta^0 & \xrightarrow{x} & Y \end{array}$$

The map  $p_{0*}$  is a trivial fibration, so  $P_x Y$  is contractible.

There is a pullback

$$\begin{array}{ccc} P_x Y & \xrightarrow{i} & \mathbf{hom}(\Delta^1, Y) \\ (p_{0*}, p_1 i) \downarrow & & \downarrow (p_0, p_1) \\ \Delta^0 \times Y & \xrightarrow{(x, 1_Y)} & Y \times Y \end{array}$$

so  $\pi = p_1 i : P_x Y \rightarrow Y$  is a Kan fibration. The **loop space**  $\Omega Y$  is the fibre of  $\pi$  over  $x \in Y$ .

We have the Kan fibre sequence

$$\Omega Y \rightarrow P_x Y \xrightarrow{\pi} Y$$

This is the **path-loop fibre sequence** for the Kan complex  $Y$ .

$P_x Y$  is the **path space** at  $x$ .

*Proof of Theorem 13.2.* The map  $\eta : Y \rightarrow S(|Y|)$  induces a bijection  $\pi_0(Y) \cong \pi_0(S(|Y|))$ .

The maps

$$S(|\Omega Y|) \rightarrow S(|P_x Y|) \rightarrow S(|Y|)$$

form a Kan fibre sequence by Theorem 13.1 and the exactness of the realization functor (Lemma 10.1).

The Kan complex  $S(|P_x Y|)$  is contractible.

There is a commutative diagram of functions

$$\begin{array}{ccc} \pi_1^s(Y, x) & \xrightarrow{\eta_*} & \pi_1^s(S(|Y|), x) \\ \partial \downarrow \cong & & \cong \downarrow \partial \\ \pi_0(\Omega Y) & \xrightarrow[\eta_*]{\cong} & \pi_0(S(|\Omega Y|)) \end{array}$$

so  $\eta_* : \pi_1^s(Y, x) \rightarrow \pi_1^s(S(|Y|), x)$  is an isomorphism.

Inductively, all maps  $\pi_n^s(Y, x) \rightarrow \pi_n^s(S(|Y|), x)$  are isomorphisms, for all vertices  $x$  of  $Y$ .  $\square$

**Corollary 13.3.** *There are natural isomorphisms*

$$\pi_n^s(Y, x) \cong \pi_n(|Y|, x)$$

*at all vertices  $x$  for all Kan complexes  $Y$ .*

*Proof.* The adjunction isomorphism

$$[(\Delta^n, \partial\Delta^n), (S(X), x)] \cong [(|\Delta^n|, |\partial\Delta^n|), (X, x)]$$

gives an isomorphism

$$\pi_n^s(S(X), x) \cong \pi_n(X, x)$$

for each space  $X$ . □

**Lemma 13.4.** *Suppose  $p : X \rightarrow Y$  is a Kan fibration and a weak equivalence. Then  $p$  is a trivial fibration.*

*Proof.* The class of maps which are both Kan fibrations and weak equivalences is stable under pullback.

In effect, given a pullback diagram

$$\begin{array}{ccc} Z \times_Y X & \longrightarrow & X \\ p_* \downarrow & & \downarrow p \\ Z & \longrightarrow & Y \end{array}$$

the realization  $|p|$  is a trivial Serre fibration by Theorem 13.1, so  $|p_*|$  is also a trivial Serre fibration, since realization preserves pullbacks.

It is enough to show (by a lifting argument) that, if  $p : X \rightarrow \Delta^n$  is a Kan fibration and a weak equivalence, then  $p$  is a trivial fibration.

As in the proof of Theorem 13.1,  $p$  has a factorization

$$\begin{array}{ccc} X & \xrightarrow{g} & F \times \Delta^n \\ & \searrow p & \downarrow pr \\ & & \Delta^n \end{array}$$

where  $g$  is a trivial fibration and the projection  $pr$  is minimal.

$pr$  is a weak equivalence, so all homotopy groups of the space  $|F|$  vanish, and Theorem 13.2 (Milnor Theorem) implies that all simplicial homotopy groups of  $F$  vanish.

By Lemma 12.8, all lifting problems

$$\begin{array}{ccc} \partial\Delta^m & \longrightarrow & F \times \Delta^n \\ \downarrow & \nearrow \text{dotted} & \downarrow pr \\ \Delta^m & \longrightarrow & \Delta^n \end{array}$$

have solutions. □

**Theorem 13.5.** *Every Kan fibration is a fibration.*

*Proof.* Suppose  $i : A \rightarrow B$  is a trivial cofibration. Then  $i$  has a factorization

$$\begin{array}{ccc} A & \xrightarrow{j} & Z \\ & \searrow i & \downarrow p \\ & & B \end{array}$$

such that  $j$  is an anodyne extension and  $p$  is a Kan fibration.

Then  $j$  is a weak equivalence, so  $p$  is a weak equivalence, and is a trivial fibration by Lemma 13.4.

The lifting exists in the diagram

$$\begin{array}{ccc} A & \xrightarrow{j} & Z \\ i \downarrow & \nearrow & \downarrow p \\ B & \xrightarrow{1_B} & B \end{array}$$

so  $i$  is a retract of an anodyne extension and is therefore an anodyne extension.

Thus, every Kan fibration has the right lifting property with respect to all trivial cofibrations.  $\square$

**Remark:** The approach to constructing the model structure for simplicial sets that is given here is non-standard.



Normally, as in [2], one decrees at the outset that the Kan fibrations are the fibrations, and the weak equivalences and cofibrations are as defined here.

The model structure is produced much more quickly in these notes (as in [3]), at the expense of knowing that the Kan fibrations are the fibrations until the very end.

### Replacing maps by fibrations

Suppose  $f : X \rightarrow Y$  is a map of Kan complexes.

Form the pullback diagram

$$\begin{array}{ccc} X \times_Y \mathbf{hom}(\Delta^1, Y) & \xrightarrow{f_*} & \mathbf{hom}(\Delta^1, Y) \xrightarrow{p_1} Y \\ p_{0*} \downarrow & & \downarrow p_0 \\ X & \xrightarrow{f} & Y \end{array}$$

where  $p_0$  and  $p_1$  are the trivial fibrations arising from the standard path object

$$\begin{array}{ccc} & \mathbf{hom}(\Delta^1, Y) & \\ & \downarrow (p_0, p_1) & \\ Y & \xrightarrow{\Delta} Y \times Y & \end{array} \quad \begin{array}{c} \nearrow s \\ \end{array}$$

for the Kan complex  $Y$ .

**Remark:** The right homotopy relation associated to this path object is classical simplicial homotopy.

There is a pullback diagram

$$\begin{array}{ccc} X \times_Y \mathbf{hom}(\Delta^1, Y) & \xrightarrow{f_*} & \mathbf{hom}(\Delta^1, Y) \\ (p_{0*}, p_{1*}f_*) \downarrow & & \downarrow (p_0, p_1) \\ X \times Y & \xrightarrow{f \times 1_Y} & Y \times Y \end{array}$$

and  $X$  is fibrant, so  $\pi := p_{1*}f_*$  is a fibration.

$p_{0*}$  is a trivial fibration. The map  $sf$  defines a section  $s_*$  of  $p_{0*}$ , so  $s_*$  is a weak equivalence.

Finally,  $\pi s_* = p_{1*}sf = f$ .

Thus, every map  $f : X \rightarrow Y$  between Kan complexes has a functorial factorization

$$\begin{array}{ccc} X & \xrightarrow{s_*} & X \times_Y \mathbf{hom}(\Delta^1, Y) \\ & \searrow f & \downarrow \pi \\ & & Y \end{array} \quad (1)$$

such that  $\pi$  is a fibration and  $s_*$  is a section of a trivial fibration.

**Remark:** This construction is an abstraction of the classical replacement of a map by a fibration, and works for the subcategory of fibrant objects in an arbitrary simplicial model category.

The dual of this construction is the mapping cylinder, which replaces a map by a cofibration up to weak equivalence (exercise).

## Simplicial sets and spaces

**Theorem 13.6.** *The adjunction maps  $\eta : X \rightarrow S(|X|)$  and  $\varepsilon : |S(Y)| \rightarrow Y$  are weak equivalences, for all simplicial sets  $X$  and spaces  $Y$ , respectively.*

*Proof.* Every combinatorial weak equivalence  $f : X \rightarrow Y$  between Kan complexes is a weak equivalence.

In effect, every map which is a fibration and a combinatorial weak equivalence is a weak equivalence by Lemma 12.8, and then one finishes by replacing the map  $f$  with a fibration as above.

The adjunction map  $\eta : X \rightarrow S(|X|)$  is a weak equivalence if  $X$  is fibrant (Theorem 13.2).

Choose a fibrant model for an arbitrary simplicial set  $X$ , ie. a weak equivalence  $j : X \rightarrow Z$  such that  $Z$  is fibrant.

Then in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta} & S(|X|) \\ \simeq \downarrow & & \downarrow \simeq \\ Z & \xrightarrow[\eta]{\simeq} & S(|Z|) \end{array}$$

the indicated maps are weak equivalences, so  $\eta : X \rightarrow S(|X|)$  is a weak equivalence too.

Suppose  $Y$  is a space. In the triangle identity

$$\begin{array}{ccc} S(Y) & \xrightarrow{\eta} & S(|S(Y)|) \\ & \searrow 1 & \downarrow s(\varepsilon) \\ & & S(Y) \end{array}$$

$S(\varepsilon)$  is a weak equiv. of Kan complexes, so  $\varepsilon : |S(Y)| \rightarrow Y$  is a weak equiv. of spaces.  $\square$

The realization and singular functor adjunction

$$| | : s\mathbf{Set} \rightleftarrows \mathbf{CGWH} : S$$

is a classic example of a Quillen equivalence. In particular we have the following:

**Corollary 13.7.** *The realization and singular functors induce an adjoint equivalence*

$$| | : \mathrm{Ho}(s\mathbf{Set}) \rightleftarrows \mathrm{Ho}(\mathbf{CGWH}) : S.$$

The final result of this section gives the closed “simplicial” model structure for the  $s\mathbf{Set}$ .

**Lemma 13.8.** *Suppose  $p : X \rightarrow Y$  is a fibration and  $i : A \rightarrow B$  is a cofibration.*

*Then the induced map*

$$\mathbf{hom}(B, X) \xrightarrow{(i^*, p_*)} \mathbf{hom}(A, X) \times_{\mathbf{hom}(A, Y)} \mathbf{hom}(B, X) \quad (2)$$

is a fibration. This map is a trivial fibration if either  $i$  or  $p$  is a weak equivalence.

*Proof.* If  $j : K \rightarrow L$  is a cofibration, then the induced map

$$(B \times K) \cup_{(A \times K)} (A \times L) \rightarrow B \times L \quad (3)$$

is a cofibration, which is a weak equivalence if either  $i$  or  $j$  is a weak equivalence (exercise).

Use an adjunction argument to show that the map (2) has the RLP wrt  $j : K \rightarrow L$  if and only if the map  $p : X \rightarrow Y$  has the RLP wrt the map (3).  $\square$

Roughly speaking (see [2] for a full definition), a **closed simplicial model category** is a closed model category  $\mathcal{M}$  together with an internal function space construction with exponential law such that the following holds:

**SM7:** Suppose  $p : X \rightarrow Y$  is a fibration and  $i : A \rightarrow B$  is a cofibration. Then the map

$$\mathbf{hom}(B, X) \xrightarrow{(i^*, p_*)} \mathbf{hom}(A, X) \times_{\mathbf{hom}(A, Y)} \mathbf{hom}(B, X)$$

is a fibration, which is trivial if either  $i$  or  $p$  is a weak equivalence.

**Second example:** The category **CGWH** has a closed simplicial model category structure, with the usual

mapping space construction. The statement **SM7** follows from the observation that two cofibrations  $i : A \rightarrow B$  and  $j : C \rightarrow D$  induce a cofibration

$$(B \times C) \cup_{(A \times C)} (A \times D) \rightarrow B \times D,$$

which is trivial if either  $i$  or  $j$  is trivial (exercise).

## References

- [1] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.
- [2] P. G. Goerss and J. F. Jardine. *Simplicial Homotopy Theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.
- [3] J. F. Jardine. Simplicial approximation. *Theory Appl. Categ.*, 12:No. 2, 34–72 (electronic), 2004.

## Lecture 06: Simplicial groups, simplicial modules

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### 14 Simplicial groups

A **simplicial group** is a functor  $G : \Delta^{op} \rightarrow \mathbf{Grp}$ .

A morphism of simplicial groups is a natural transformation of such functors.

The category of simplicial groups is denoted by  $s\mathbf{Gr}$ .

We use the same notation for a simplicial group  $G$  and its underlying simplicial set.

**Lemma 14.1** (Moore). *Every simplicial group is a Kan complex.*

The proof of Lemma 14.1 involves the classical

simplicial identities. Here's the full list:

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i \text{ if } i < j \\
d_i s_j &= \begin{cases} s_{j-1} d_i & \text{if } i < j \\ 1 & \text{if } i = j, j+1 \\ s_j d_{i-1} & \text{if } i > j+1 \end{cases} \\
s_i s_j &= s_{j+1} s_i \text{ if } i \leq j.
\end{aligned}$$

*Proof.* Suppose

$$(x_0, \dots, x_{k-1}, x_{\ell-1}, \dots, x_n)$$

( $\ell \geq k+2$ ) is a family of  $(n-1)$ -simplices of  $G$  such that  $d_i x_j = d_{j-1} x_i$  for  $i < j$ .

Suppose there is an  $n$ -simplex  $y \in G$  such that  $d_i(y) = x_i$  for  $i \leq k-1$  and  $i \geq \ell$ .

Then  $d_i x_{\ell-1} = d_i d_{\ell-1}(y)$  for  $i \leq k-1$  and  $i \geq \ell-1$ , and

$$d_i(s_{\ell-2}(x_{\ell-1} d_{\ell-1}(y^{-1})))y = x_i$$

for  $i \leq k-1$  and  $i \geq \ell-1$ .

*Alternatively,* suppose  $S \subset \mathbf{n}$  and  $|S| \leq n$ .

Write  $\Delta^n \langle S \rangle$  for the subcomplex of  $\partial \Delta^n$  which is generated by the faces  $d_i \iota_n$  for  $i \in S$ .

Write

$$G_{\langle S \rangle} := \text{hom}(\Delta^n \langle S \rangle, G).$$



Restriction to faces determines a group homomorphism  $d : G_n \rightarrow G_{\langle S \rangle}$ .

We show that  $d$  is surjective, by induction on  $|S|$ .

There is a  $j \in S$  such that either  $j - 1$  or  $j + 1$  is not a member of  $S$ , since  $|S| \leq n$ .

Pick such a  $j$ , and suppose  $\theta : \Delta^n \langle S \rangle \rightarrow G$  is a simplicial set map such that  $\theta_i = \theta(d_i \iota_n) = e$  for  $i \neq j$ . Then there is a simplex  $y \in G_n$  such that  $d_j(y) = \theta$ .

For this, set  $y = s_j \theta_j$  if  $j + 1 \notin S$  or  $y = s_{j-1} \theta_j$  if  $j - 1 \notin S$ .

Now suppose  $\sigma : \Delta^n \langle S \rangle \rightarrow G$  is a simplicial set map, and let  $\sigma^{(j)}$  denote the composite

$$\Delta^n \langle S - \{j\} \rangle \subset \Delta^n \langle S \rangle \xrightarrow{\sigma} G.$$

Inductively, there is a  $y \in G_n$  such that  $d(y) = \sigma^{(j)}$ , or such that  $d_i y = \sigma_i$  for  $i \neq j$ . Let  $y_S$  be the restriction of  $y$  to  $\Delta^n \langle S \rangle$ .

The product  $\sigma \cdot y_S^{-1}$  is a map such that  $(\sigma \cdot y_S^{-1})_i = e$  for  $i \neq j$ . Thus, there is a  $\theta \in G_n$  such that  $d(\theta) = \sigma \cdot y_S^{-1}$ .

Then  $d(\theta \cdot y) = \sigma$ . □

The following result will be useful:

**Lemma 14.2.** 1) Suppose that  $S \subset \mathbf{n}$  such that  $|S| \leq n$ . Then the inclusion  $\Delta^n \langle S \rangle \subset \Delta^n$  is anodyne.  
 2) If  $T \subset S$ , and  $T \neq \emptyset$ , then  $\Delta^n \langle T \rangle \subset \Delta^n \langle S \rangle$  is anodyne.

*Proof.* For 1), we argue by induction on  $n$ .

Suppose that  $k$  is the largest element of  $S$ . There is a pushout diagram

$$\begin{array}{ccc} \Delta^{n-1} \langle S - \{k\} \rangle & \xrightarrow{d^{k-1}} & \Delta^n \langle S - \{k\} \rangle \\ \downarrow & & \downarrow \\ \Delta^{n-1} & \xrightarrow{d^k} & \Delta^n \langle S \rangle \end{array} \quad (1)$$

By adding  $(n-1)$ -simplices to  $\Delta^n \langle S \rangle$ , one finds a  $k \in \mathbf{n}$  such that the maps in the string

$$\Delta^n \langle S \rangle \subset \Lambda_k^n \subset \Delta^n$$

are anodyne. □

Write

$$N_n(G) = \cap_{i < n} \ker(d_i : G_n \rightarrow G_{n-1}).$$

The simplicial identities imply that the face map  $d_n$  induces a homomorphism

$$d_n : N_n(G) \rightarrow N_{n-1}(G).$$

In effect, if  $i < n-1$ , then  $i < n$  and

$$d_i d_n(x) = d_{n-1} d_i(x) = e$$

for  $x \in N_n(G)$ .

The image of  $d_n : N_n(G) \rightarrow N_{n-1}(G)$  is normal in  $G_n$ , since

$$d_n((s_{n-1}x)y(s_{n-1}x)^{-1}) = xd_n(y)x^{-1}.$$

for  $y \in N_{n+1}(G)$  and  $x \in G_n$ .

**Lemma 14.3.** 1) *There are isomorphisms*

$$\frac{\ker(d_n : N_n(G) \rightarrow N_{n-1}(G))}{\operatorname{im}(d_{n+1} : N_{n+1}(G) \rightarrow N_n(G))} \xrightarrow{\cong} \pi_n(G, e)$$

*for all  $n \geq 0$ .*

2) *The homotopy groups  $\pi_n(G, e)$  are abelian for  $n \geq 1$ .*

3) *There are isomorphisms*

$$\pi_n(G, x) \cong \pi_n(G, e)$$

*for any  $x \in G_0$ .*

*Proof.* The group multiplication on  $G$  induces a multiplication on  $\pi_n(G, e)$  which has identity represented by  $e \in G$  and satisfies an interchange law with the standard multiplication on the simplicial homotopy group  $\pi_n(G, e)$ .

Thus, the two group structures on  $\pi_n(G, e)$  coincide and are abelian for  $n \geq 1$ .

Multiplication by the vertex  $x$  defines a group homomorphism

$$\pi_n(G, e) \rightarrow \pi_n(G, x),$$

with inverse defined by multiplication by  $x^{-1}$ .  $\square$

**Corollary 14.4.** *A map  $f : G \rightarrow H$  of simplicial groups is a weak equivalence if and only if it induces isomorphisms*

$$\begin{aligned} \pi_0(G) &\cong \pi_0(H), \text{ and} \\ \pi_n(G, e) &\cong \pi_n(H, e), \quad n \geq 1. \end{aligned}$$

**Lemma 14.5.** *Suppose  $p : G \rightarrow H$  is a simplicial group homomorphism such that  $p : G_i \rightarrow H_i$  is a surjective group homomorphism for  $i \leq n$ .*

*Then  $p$  has the RLP wrt all morphisms  $\Lambda_k^m \subset \Delta^m$  for  $m \leq n$ .*

*Proof.* Suppose given a commutative diagram

$$\begin{array}{ccc} \Lambda_k^m & \xrightarrow{\alpha} & G \\ \downarrow & & \downarrow p \\ \Delta^m & \xrightarrow{\beta} & H \end{array}$$

and let  $K$  be the kernel of  $p$ .

Since  $m \leq n$  there is a simplex  $\theta : \Delta^m \rightarrow G$  such that  $p\theta = \beta$ . Then  $p\theta|_{\Lambda_k^m} = p\alpha$ , and there is a

simplex  $\gamma : \Delta^m \rightarrow K$  such that the diagram

$$\begin{array}{ccc} \Lambda_k^m & \xrightarrow{\alpha(\theta|_{\Lambda_k^m})^{-1}} & K \\ \downarrow & \nearrow \gamma & \\ \Delta^m & & \end{array}$$

commutes, since  $K$  is a Kan complex (Lemma 14.1). Then  $(\gamma\theta)|_{\Lambda_k^m} = \alpha$  and  $p(\gamma\theta) = \beta$ .  $\square$

**Lemma 14.6.** *Suppose  $p : G \rightarrow H$  is a simplicial group homomorphism such that the induced homomorphisms  $N_i(G) \rightarrow N_i(H)$  are surjective for  $i \leq n$ .*

*Then  $p$  is surjective up to level  $n$ .*

*Proof.* Suppose  $\beta : \Delta^n \rightarrow H$  is an  $n$ -simplex, and suppose that  $p$  is surjective up to level  $n - 1$ .

$p$  is surjective up to level  $n - 1$  and is a fibration up to level  $n - 1$  by Lemma 14.5.

It follows from the proof of Lemma 14.2, ie. the pushouts (1), that  $p$  has the RLP wrt to the inclusion  $\Delta^{n-1} \subset \Delta^n \langle S \rangle$  defined by the inclusion of the minimal simplex of  $S$ .

Thus, there is map  $\alpha : \Lambda_n^n \rightarrow G$  such that the fol-

lowing commutes

$$\begin{array}{ccc} \Lambda_n^n & \xrightarrow{\alpha} & G \\ \downarrow & & \downarrow p \\ \Delta^n & \xrightarrow{\beta} & H \end{array}$$

Choose a simplex  $\theta : \Delta^n \rightarrow G$  which extends  $\alpha$ . Then  $(\beta p(\theta)^{-1})|_{\Lambda_n^n} = e$  so there is an  $n$ -simplex  $\gamma \in N_n(G)$  such that  $p(\gamma) = \beta p(\theta)^{-1}$ .

But then  $\beta = p(\gamma\theta)$ . □

**Lemma 14.7.** *The following are equivalent for a simplicial group homomorphism  $p : G \rightarrow H$ :*

- 1) *The map  $p$  is a fibration.*
- 2) *The induced map  $p_* : N_n(G) \rightarrow N_n(H)$  is surjective for  $n \geq 1$ .*

*Proof.* We will show that 2) implies 1). The other implication is an exercise.

Consider the diagram

$$\begin{array}{ccc} G & \xrightarrow{p} & H \\ \downarrow & & \downarrow \\ K(\pi_0 G, 0) & \xrightarrow{p_*} & K(\pi_0 H, 0) \end{array}$$

where  $K(X, 0)$  denotes the constant simplicial set on a set  $X$ .

Example:  $K(\pi_0 G, 0)$  is the constant simplicial group on the group  $\pi_0(G)$ .

Every map  $K(X, 0) \rightarrow K(Y, 0)$  induced by a function  $X \rightarrow Y$  is a fibration (exercise), so that the map  $p_*$  is a fibration, and the map

$$K(\pi_0 G, 0) \times_{K(\pi_0 H, 0)} H \rightarrow H$$

is a fibration.

The functor  $G \mapsto N_n(G)$  preserves pullbacks, and the map

$$p' : G \rightarrow K(\pi_0 G, 0) \times_{K(\pi_0 H, 0)} H$$

is surjective in degree 0 (exercise).

Then  $p'$  induces surjections

$$N_n(G) \rightarrow N_n(K(\pi_0 G, 0) \times_{K(\pi_0 H, 0)} H)$$

for  $n \geq 0$ , and is a fibration by Lemmas 14.5 and 14.6. □

Here are some definitions:

- A homomorphism  $p : G \rightarrow H$  of simplicial groups is said to be a **fibration** if the underlying map of simplicial sets is a fibration.
- The homomorphism  $f : A \rightarrow B$  in  $s\mathbf{Gr}$  is a **weak equivalence** if the underlying map of simplicial sets is a weak equivalence.

- A **cofibration** of  $s\mathbf{Gr}$  is a map which has the left lifting property with respect to all trivial fibrations.

The forgetful functor  $U : s\mathbf{Gr} \rightarrow s\mathbf{Set}$  has a left adjoint  $X \mapsto G(X)$  which is defined by the free group functor in all degrees.

A map  $G \rightarrow H$  is a fibration (respectively weak equivalence) of  $s\mathbf{Gr}$  iff  $U(G) \rightarrow U(H)$  is a fibration (resp. weak equivalence) of simplicial sets.

If  $i : A \rightarrow B$  is a cofibration of simplicial sets, then the map  $i_* : G(A) \rightarrow G(B)$  of simplicial groups is a cofibration.

Suppose  $G$  and  $H$  are simplicial groups and that  $K$  is a simplicial set.

The simplicial group  $G \otimes K$  has

$$(G \otimes K)_n = *_{x \in K_n} G_n$$

(generalized free product, or coproduct in  $\mathbf{Gr}$ ).

The **function complex**  $\mathbf{hom}(G, H)$  for simplicial groups  $G, H$  is defined by

$$\mathbf{hom}(G, H)_n = \{G \otimes \Delta^n \rightarrow H\}.$$

There is a natural bijection

$$\mathbf{hom}(G \otimes K, H) \cong \mathbf{hom}(K, \mathbf{hom}(G, H)).$$



There is a simplicial group  $H^K$  defined as a simplicial set by

$$H^K = \mathbf{hom}(K, H),$$

with the group structure induced from  $H$ . There is an exponential law

$$\mathbf{hom}(G \otimes K, H) \cong \mathbf{hom}(G, H^K).$$

**Proposition 14.8.** *With the definitions of fibration, weak equivalence and cofibration given above the category  $s\mathbf{Gr}$  satisfies the axioms for a closed simplicial model category.*

*Proof.* The proof is exercise. A map  $p : G \rightarrow H$  is a fibration (respectively trivial fibration) if and only if it has the RLP wrt all maps  $G(\Lambda_k^n) \rightarrow G(\Delta^n)$  (respectively with respect to all  $G(\partial\Delta^n) \rightarrow G(\Delta^n)$ ), so a standard small object argument proves the factorization axiom, subject to proving Lemma 14.9 below.

(We need the Lemma to show that the maps  $G(A) \rightarrow G(B)$  induced by trivial cofibrations  $A \rightarrow B$  push out to trivial cofibrations).

The axiom **SM7** reduces to the assertion that if  $p : G \rightarrow H$  is a fibration and  $i : K \rightarrow L$  is an in-

clusion of simplicial sets, then the induced homomorphism

$$G^L \rightarrow G^K \times_{H^K} H^L$$

is a fibration which is trivial if either  $i$  or  $p$  is trivial. For this, one uses the natural isomorphism

$$G(X) \otimes K \cong G(X \times K)$$

and the simplicial model axiom for simplicial sets.  $\square$

**Lemma 14.9.** *Suppose  $i : A \rightarrow B$  is a trivial cofibration of simplicial sets. Then the induced map  $i_* : G(A) \rightarrow G(B)$  is a strong deformation retraction of simplicial groups.*

*Proof.* All simplicial groups are fibrant, so the lift  $\sigma$  exists in the diagram

$$\begin{array}{ccc} G(A) & \xrightarrow{1} & G(A) \\ i_* \downarrow & \nearrow \sigma & \downarrow \\ G(B) & \longrightarrow & e \end{array}$$

The lift  $h$  also exists in the diagram

$$\begin{array}{ccc} G(A) & \xrightarrow{si_*} & G(B)^{\Delta^1} \\ i_* \downarrow & \nearrow h & \downarrow (p_0, p_1) \\ G(B) & \xrightarrow{(i_*\sigma, 1)} & G(B) \times G(B) \end{array}$$

and  $h$  is the required homotopy.  $\square$

**Corollary 14.10.** *The free group functor  $G : s\mathbf{Set} \rightarrow s\mathbf{Gr}$  preserves weak equivalences.*

The proof of Corollary 14.10 uses the **mapping cylinder** construction. Let  $f : X \rightarrow Y$  be a map of simplicial sets, and form the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i_0 \downarrow & & \downarrow i_{0*} \\ X & \xrightarrow{i_1} X \times \Delta^1 \xrightarrow{f_*} (X \times \Delta^1) \cup_X Y \end{array}$$

Let  $j = f_* i_1$ , and observe that this map is a cofibration since  $X$  is cofibrant. The map  $pr : X \times \Delta^1 \rightarrow X$  induces a map  $pr_* : (X \times \Delta^1) \cup_X Y \rightarrow Y$  such that  $pr_* i_{0*} = 1_Y$  and one sees that the diagram

$$\begin{array}{ccc} X & \xrightarrow{j} & (X \times \Delta^1) \cup_X Y \\ & \searrow f & \downarrow pr_* \\ & & Y \end{array} \quad (2)$$

commutes. In other words, any simplicial set map  $f : X \rightarrow Y$  has a (natural) factorization as above such that  $j$  is a cofibration and  $pr_*$  has a section which is a trivial cofibration.

**Remark:** A functor  $s\mathbf{Set} \rightarrow \mathcal{M}$  taking values in a model category which takes trivial cofibrations to weak equivalences must preserve weak equivalences.

A similar statement holds for functors defined on any category of cofibrant objects and taking values in  $\mathcal{M}$ .

**Remark:** The construction of (2) is an abstraction of the classical replacement of the map  $f$  by a cofibration. It is dual to the replacement of a map by a fibration in a category of fibrant objects displayed in (1 — see p. 21) of Section 13.

**Remark:** We have used the forgetful-free group functor adjunction to induce a model structure on  $s\mathbf{Gr}$  from that on simplicial sets, in such a way that the functors

$$G : s\mathbf{Set} \rightleftarrows s\mathbf{Gr} : U$$

form a Quillen adjunction.

## 15 Simplicial modules

$s(R - \mathbf{Mod})$  is the category of simplicial  $R$ -modules, where  $R$  is some unitary ring.

The forgetful functor  $U : s(R - \mathbf{Mod}) \rightarrow s\mathbf{Set}$  has a left adjoint

$$R : s\mathbf{Set} \rightarrow s(R - \mathbf{Mod}).$$

$R(X)_n$  is the free  $R$ -module on the set  $X_n$  for  $n \geq 0$ .

$s(R - \mathbf{Mod})$  has a closed model structure which is induced from simplicial sets by the forgetful-free abelian group functor adjoint pair, in the same way that the category  $s\mathbf{Gr}$  of simplicial groups acquires its model structure.

A morphism  $f : A \rightarrow B$  of simplicial  $R$ -modules is a **weak equivalence** (respectively **fibration**) if the underlying morphism of simplicial sets is a weak equivalence (respectively fibration).

A **cofibration** of simplicial  $R$ -modules is a map which has the LLP wrt all trivial fibrations.

Examples of cofibrations of  $s(R - \mathbf{Mod})$  include all maps  $R(A) \rightarrow R(B)$  induced by cofibrations of simplicial sets.

Suppose  $A$  and  $B$  are simplicial groups and that  $K$  is a simplicial set. Then there is a simplicial abelian group  $A \otimes K$  with

$$(A \otimes K)_n = \bigoplus_{x \in K_n} A_n \cong A_n \otimes R(K)_n.$$

The **function complex**  $\mathbf{hom}(A, B)$  for simplicial abelian groups  $A, B$  is defined by

$$\mathbf{hom}(A, B)_n = \{A \otimes \Delta^n \rightarrow B\}.$$

Then there is a natural bijection

$$\mathrm{hom}(A \otimes K, B) \cong \mathrm{hom}(K, \mathbf{hom}(A, B)).$$

There is a simplicial module  $B^K$  defined as a simplicial set by

$$B^K = \mathbf{hom}(K, B),$$

with  $R$ -module structure induced from  $B$ .

There is an exponential law

$$\mathrm{hom}(A \otimes K, B) \cong \mathrm{hom}(A, B^K).$$

**Proposition 15.1.** *With the definitions of fibration, weak equivalence and cofibration given above the category  $s(R - \mathbf{Mod})$  satisfies the axioms for a closed simplicial model category.*

*Proof.* The proof is by analogy with the corresponding result for simplicial groups (Prop. 14.8).  $\square$

The proof of Proposition 15.1 also uses the following analog of Lemma 14.9, in the same way:

**Lemma 15.2.** *Suppose  $i : A \rightarrow B$  is a trivial cofibration of simplicial sets. Then the induced map  $i_* : R(A) \rightarrow R(B)$  is a strong deformation retraction of simplicial  $R$ -modules.*

**Corollary 15.3.** *The free  $R$ -module functor*

$$R : s\mathbf{Set} \rightarrow s(R - \mathbf{Mod})$$

*preserves weak equivalences.*

Once again, the adjoint functors

$$R : s\mathbf{Set} \rightleftarrows s(R - \mathbf{Mod}) : U$$

form a Quillen adjunction.

**Example:**  $R = \mathbb{Z}$ : The category  $s(\mathbb{Z} - \mathbf{Mod})$  is the category of simplicial abelian groups, also denoted by  $s\mathbf{Ab}$ .

The adjunction homomorphism  $\eta : X \rightarrow U\mathbb{Z}(X)$  for this case is usually written as

$$h : X \rightarrow \mathbb{Z}(X)$$

and is called the **Hurewicz homomorphism**. More on this later.

Simplicial  $R$ -modules are simplicial groups, so we know a few things:

- For a simplicial  $R$ -module  $A$  the modules  $N_n A = \bigcap_{i < n} \ker(d_i)$  and the morphisms

$$N_n A \xrightarrow{(-1)^n d_n} N_{n-1} A$$

form an ordinary chain complex, called the **normalized chain complex** of  $A$ . The assignment

$A \mapsto NA$  defines a functor

$$N : s(R - \mathbf{Mod}) \rightarrow Ch_+(R).$$

- There is a natural isomorphism

$$\pi_n(A, 0) \cong H_n(NA),$$

and a map  $f : A \rightarrow B$  is a weak equivalence if and only if the induced chain map  $NA \rightarrow NB$  is a homology isomorphism (Corollary 14.4).

- A map  $p : A \rightarrow B$  is a fibration of  $s(R - \mathbf{mod})$  if and only if the induced map  $p_* : NA \rightarrow NB$  is a fibration of  $Ch_+(R)$  (Lemma 14.7).

This precise relationship between simplicial modules and chain complexes is not an accident.

The **Moore complex**  $M(A)$  for a simplicial module  $A$  has  $n$ -chains given by  $M(A)_n = A_n$  and boundary

$$\partial = \sum_{i=0}^n (-1)^i d_i : A_n \rightarrow A_{n-1}.$$

The fact that  $\partial^2 = 0$  is an exercise involving the simplicial identities  $d_i d_j = d_{j-1} d_i$ ,  $i < j$ .

The construction is functorial:

$$M : s(R - \mathbf{Mod}) \rightarrow Ch_+(R).$$



The Moore chains functor is *not* the normalized chains functor, but the inclusions  $N_n A \subset A_n$  determine a natural chain map

$$N(A) \subset M(A).$$

**Example:** If  $Y$  is a space, the  $n^{th}$  **singular homology module**  $H_n(Y, R)$  with coefficients in  $R$  is defined by

$$H_n(Y, R) = H_n M(R(S(Y))).$$

If  $N$  is any  $R$ -module, then

$$H_n(Y, N) = H_n(M(R(S(Y)) \otimes_R N))$$

defines the  $n^{th}$  **singular homology module** of  $Y$  with coefficients in  $N$ .

The subobject  $D(A)_n \subset M(A)_n$  is defined by

$$D(A)_n = \langle s_j(y) \mid 0 \leq j \leq n-1, y \in A_{n-1} \rangle.$$

$D(A)_n$  is the submodule generated by degenerate simplices.

The Moore chains boundary  $\partial$  restricts to a boundary map  $\partial : D(A)_n \rightarrow D(A)_{n-1}$  (exercise), and the inclusions  $D(A)_n \subset A_n$  form a natural chain map

$$D(A) \subset M(A).$$

Here's what you need to know:

**Theorem 15.4.** 1) *The composite chain map*

$$N(A) \subset M(A) \rightarrow M(A)/D(A)$$

*is a natural isomorphism.*

2) *The inclusion  $N(A) \subset M(A)$  is a natural chain homotopy equivalence.*

*Proof.* There is a subcomplex  $N_j(A) \subset M(A)$  with  $N_j A_n = N A_n$  if  $n \leq j+1$  and

$$N_j A_n = \cap_{i=0}^j \ker(d_i) \text{ if } n \geq j+2.$$

$D_j(A_n) :=$  the submodule of  $A_n$  generated by all  $s_i(x)$  with  $i \leq j$ .

1) We show that the composite

$$\phi : N_j(A_n) \rightarrow A_n \rightarrow A_n/D_j(A_n)$$

is an isomorphism for all  $j < n$ , by induction on  $j$ .

There is a commutative diagram

$$\begin{array}{ccccccc} N_{j-1}A_{n-1} & \xrightarrow{s_j} & N_{j-1}A_n & \xleftarrow{i} & N_jA_n & & \\ \cong \downarrow \phi & & \cong \downarrow \phi & & \downarrow \phi & & \\ 0 \longrightarrow & A_{n-1}/D_{j-1}A_{n-1} & \xrightarrow{s_j} & A_n/D_{j-1}A_n & \longrightarrow & A_n/D_jA_n & \longrightarrow 0 \end{array}$$

in which the bottom sequence is exact and  $i$  is the obvious inclusion.

If  $[x] \in A_n/D_jA_n$  for  $x \in N_{j-1}A_n$ , then  $[x - s_jd_jx] = [x]$  and  $x - s_jd_jx \in N_jA_n$ , so  $\phi : N_jA_n \rightarrow A_n/d_jA_n$  is surjective.

If  $\phi(x) = 0$  for  $x \in N_jA_n$  then  $x = s_j(y)$  for some  $y \in N_{j-1}A_{n-1}$ . But  $d_jx = 0$  so  $0 = d_js_jy = y$ .

For 2), we have  $N_{j+1}A \subset N_jA$  and

$$NA = \bigcap_{j \geq 0} N_jA$$

in finitely many stages in each degree.

We show that  $i : N_{j+1}A \subset N_jA$  is a chain homotopy equivalence (this is cheating a bit, but is easily fixed — see [2, p.149]).

There are chain maps  $f : N_jA \rightarrow N_{j+1}A$  defined by

$$f(x) = \begin{cases} x - s_{j+1}d_{j+1}(x) & \text{if } n \geq j+2, \\ x & \text{if } n \leq j+1. \end{cases}$$

Write  $t = (-1)^j s_{j+1} : N_jA_n \rightarrow N_{j+1}A_{n+1}$  if  $n \geq j+1$  and set  $t = 0$  otherwise. Then  $f(i(x)) = x$  and

$$1 - i \cdot f = \partial t + t \partial.$$

□

Suppose  $A$  is a simplicial  $R$ -module. Every monomorphism  $d : \mathbf{m} \rightarrow \mathbf{n}$  induces a homomorphism  $d^* : NA_n \rightarrow NA_m$ , and  $d^* = 0$  unless  $d = d^n$ .

Suppose  $C$  is a chain complex. Associate the module  $C_n$  to the ordinal number  $\mathbf{n}$ , and associate to each ordinal number monomorphism  $d$  the morphism  $d^* : C_n \rightarrow C_m$ , where

$$d^* = \begin{cases} 0 & \text{if } d \neq d^n, \\ (-1)^n \partial : C_n \rightarrow C_{n-1} & \text{if } d = d^n. \end{cases}$$

Define

$$\Gamma(C)_n = \bigoplus_{s:\mathbf{n} \rightarrow \mathbf{k}} C_k.$$

The ordinal number map  $\theta : \mathbf{m} \rightarrow \mathbf{n}$  induces an  $R$ -module homomorphism

$$\theta^* : \Gamma(C)_n \rightarrow \Gamma(C)_m$$

which is defined on the summand corresponding to the epi  $s : \mathbf{n} \twoheadrightarrow \mathbf{k}$  by the composite

$$C_k \xrightarrow{d^*} C_r \xrightarrow{in_t} \bigoplus_{\mathbf{m} \rightarrow \mathbf{r}} C_r,$$

where the ordinal number maps

$$\mathbf{m} \xrightarrow{t} \mathbf{r} \xrightarrow{d} \mathbf{k}$$

give the epi-monic factorization of the composite

$$\mathbf{m} \xrightarrow{\theta} \mathbf{n} \xrightarrow{s} \mathbf{k}.$$

and  $d^*$  is induced by  $d$  according to the prescription above.

The assignment  $C \mapsto \Gamma(C)$  defines a functor

$$\Gamma : Ch_+(R) \rightarrow s(R - \mathbf{Mod}).$$

**Theorem 15.5** (Dold-Kan). *The functor  $\Gamma$  is an inverse up to natural isomorphism for the normalized chains functor  $N$ .*

The equivalence of categories defined by the functors  $N$  and  $\Gamma$  is the **Dold-Kan correspondence**.

*Proof.* One can show that

$$D(\Gamma(C))_n = \bigoplus_{s:\mathbf{n} \twoheadrightarrow \mathbf{k}, k \leq n-1} C_k,$$

so there is a natural isomorphism

$$C \cong M(\Gamma(C))/D(\Gamma(C)) \cong N(\Gamma(C))$$

There is a natural homomorphism of simplicial modules

$$\Psi : \Gamma(NA) \rightarrow A,$$

which in degree  $n$  is the homomorphism

$$\bigoplus_{s:\mathbf{n} \twoheadrightarrow \mathbf{k}} NA_k \rightarrow A_n$$

defined on the summand corresponding to  $s : \mathbf{n} \twoheadrightarrow \mathbf{k}$  by the composite

$$NA_k \subset A_k \xrightarrow{s^*} A_n.$$

Collapsing  $\Psi$  by degeneracies gives the canonical isomorphism  $NA \cong A/D(A)$ , so the map

$$N(\Psi) : N(\Gamma(NA)) \rightarrow NA$$

is an isomorphism of chain complexes.

It follows from Lemma 14.6 that the natural map  $\Psi$  is surjective in all degrees.

The functor  $A \mapsto NA$  is exact: it is left exact from the definition, and it preserves epimorphisms by Lemma 14.7.

It follows that the normalized chains functor reflects isomorphisms.

To see this, suppose  $f : A \rightarrow B$  is a simplicial module map and that the sequence

$$0 \rightarrow K \rightarrow A \xrightarrow{f} B \rightarrow C \rightarrow 0$$

is exact. Suppose also that  $Nf$  is an isomorphism. Then the sequence of chain complex maps

$$0 \rightarrow NK \rightarrow NA \xrightarrow{Nf} NB \rightarrow NC \rightarrow 0$$

is exact, so that  $NK = NC = 0$ . But then  $K = C = 0$  since  $\Psi$  is a natural epimorphism, so that  $f$  is an isomorphism.

Finally,  $N\Psi$  is an isomorphism, so that  $\Psi$  is an isomorphism.  $\square$

## 16 Eilenberg-Mac Lane spaces

Under the Dold-Kan correspondence

$$\Gamma : Ch_+(R) \rightleftarrows s(R - \mathbf{Mod}) : N$$

a map  $f : A \rightarrow B$  of simplicial modules is a weak equivalence (respectively fibration, cofibration) if and only if the induced map  $f_* : NA \rightarrow NB$  is a weak equivalence (resp. fibration, cofibration) of  $Ch_+(R)$ .

There are natural isomorphisms

$$\pi_n(|A|, 0) \cong \pi_n^s(A, 0) \cong H_n(N(A)) \cong H_n(M(A)).$$

for simplicial modules  $A$ .

Suppose that  $C$  is a chain complex.

Take  $n \geq 0$ . Write  $C[-n]$  for the **shifted** chain complex with

$$C[-n]_k = \begin{cases} C_{k-n} & k \geq n, \\ 0 & k < n. \end{cases}$$

There is a natural short exact sequence of chain complexes

$$0 \rightarrow C \rightarrow \widetilde{C[-1]} \rightarrow C[-1] \rightarrow 0.$$

In general (see Section 6),  $\tilde{D}$  is the acyclic complex with  $\tilde{D}_n = D_n \oplus D_{n+1}$  for  $n > 0$ ,

$$\tilde{D}_0 = \{(x, z) \in D_0 \oplus D_1 \mid x + \partial(z) = 0\},$$

and with boundary map defined by

$$\partial(x, z) = (\partial(x), (-1)^n x + \partial(z))$$

for  $(x, z) \in \tilde{D}_n$ .

For a simplicial module  $A$ , the objects  $\Gamma(\widetilde{NA[-1]})$  and  $\Gamma(NA[-1])$  have special names, due to Eilenberg and Mac Lane:

$$\overline{W}(A) := \Gamma(NA[-1]),$$

and

$$W(A) := \Gamma(\widetilde{NA[-1]}).$$

There is a natural short exact (hence fibre) sequence of simplicial modules

$$0 \rightarrow A \rightarrow W(A) \rightarrow \overline{W}(A) \rightarrow 0,$$

(exercise) and there are isomorphisms

$$\pi_n(A) \cong \pi_{n+1}(\overline{W}(A)).$$

The object  $\overline{W}(A)$  is a natural delooping of the simplicial module  $A$ , usually thought of as either a *suspension* or a *classifying space* for  $A$ .



Suppose  $B$  is an  $R$ -module, and write  $B(0)$  for the chain complex concentrated in degree 0, which consists of  $B$  in that degree and 0 elsewhere.

Then  $B(n) = B(0)[-n]$  is the chain complex with  $B$  in degree  $n$ . Write

$$K(B, n) = \Gamma(B(n)).$$

There are natural isomorphisms

$$\pi_j K(B, n) \cong H_j(B(n)) \cong \begin{cases} B & j = n \\ 0 & j \neq n. \end{cases}$$

The object  $K(B, n)$  (or  $|K(B, n)|$ ) is an **Eilenberg-Mac Lane space of type  $(B, n)$** .

This is a standard method of constructing these spaces, together with the natural fibre sequences

$$K(B, n) \rightarrow W(K(B, n)) \rightarrow K(B, n+1)$$

for modules (or abelian groups)  $B$ . These fibre sequences are short exact sequences of simplicial modules.

### Non-abelian groups

The non-abelian world is different. Here's an exercise:

**Exercise:** Show that a functor  $f : G \rightarrow H$  between groupoids induces a fibration  $BG \rightarrow BH$  if and

only if  $f$  has the **path lifting property** in the sense that all lifting problems

$$\begin{array}{ccc} * & \longrightarrow & G \\ 0 \downarrow & \nearrow & \downarrow f \\ \mathbf{1} & \longrightarrow & H \end{array}$$

can be solved.

Suppose  $G$  is a group, identified with a groupoid with one object  $*$ , and recall that the slice category  $*/G$  has as objects all group elements (morphisms)  $* \xrightarrow{g} *$ , and as morphisms all commutative diagrams

$$\begin{array}{ccc} & & * \\ & \nearrow g & \downarrow k \\ * & & \\ & \searrow h & \downarrow \\ & & * \end{array}$$

The canonical functor  $\pi : */G \rightarrow G$  sends the morphism above to the morphism  $k$  of  $G$ .

The functor  $\pi$  has the path lifting property, and the fibre over the vertex  $*$  of the fibration  $\pi : B(*/G) \rightarrow BG$  is a copy of  $K(G, 0)$ .

One usually writes

$$EG = B(*/G).$$

This is a contractible space, since it has an initial object  $e$  and the unique maps  $\gamma_g : e \rightarrow g$  define a contracting homotopy  $*/G \times \mathbf{1} \rightarrow */G$ .

The Kan complex  $BG$  is connected, since it has only one vertex. The long exact sequence in homotopy groups associated to the fibre sequence

$$K(G, 0) \rightarrow EG \xrightarrow{\pi} BG$$

can be used to show that  $\pi_n^s(BG)$  is trivial for  $n \neq 1$ , and that the boundary map

$$\pi_1^s(BG) \xrightarrow{\partial} G = \pi_0(K(G, 0))$$

is a bijection.

For this, there is a surjective homomorphism

$$G \rightarrow \pi_1^s(BG),$$

defined by taking  $g$  to the homotopy group element  $[g]$  represented by the simplex  $* \xrightarrow{g} *$ . One shows that the composite

$$G \rightarrow \pi_1^s(BG) \xrightarrow{\partial} G \tag{3}$$

is the identity on  $G$ , so that the homomorphism  $G \rightarrow \pi_1^s(BG)$  is a bijection.

To see that the composite (3) is the identity, observe that there is a commutative diagram

$$\begin{array}{ccc} \Lambda_0^1 & \xrightarrow{e} & EG \\ \downarrow & \nearrow \gamma_g & \downarrow \pi \\ \Delta^1 & \xrightarrow{g} & BG \end{array}$$

Then  $\partial([g]) = d_0(\gamma_g) = g$ .

The classifying space  $BG$  for a group  $G$  is an Eilenberg-Mac lane space  $K(G, 1)$ . This is a standard model.

### Some facts about groupoids

Suppose that  $H$  is a connected groupoid. This means that, for any two objects  $x, y \in H$  there is a morphism (isomorphism)  $\omega : x \rightarrow y$ .

Fix an object  $x$  of  $H$  and chose isomorphisms  $\gamma_y : y \rightarrow x$  for all objects of  $H$ , such that  $\gamma_x = 1_x$ . There is an inclusion functor

$$i : H_x = H(x, x) \subset H.$$

We define a functor  $r : H \rightarrow H_x$  by conjugation with the maps  $\gamma_y$ : if  $\alpha : y \rightarrow z$  is a morphism of  $H$ , then  $r(\alpha) = \gamma_z^{-1} \alpha \gamma_x$ , so that the diagrams

$$\begin{array}{ccc} x & \xrightarrow{\gamma_y} & y \\ r(\alpha) \downarrow & & \downarrow \alpha \\ x & \xrightarrow{\gamma_z} & z \end{array}$$

commute.

The functor  $r$  is uniquely determined by the isomorphisms  $\gamma_y$ , and the composite

$$H_x \xrightarrow{i} H \xrightarrow{r} H_x$$

is the identity.

The maps  $\gamma_y$  define a natural transformation

$$\gamma: i \cdot r \rightarrow 1_H.$$

We have shown that the inclusion  $BH_x \rightarrow BH$  is a homotopy equivalence, even a strong deformation retraction.

It follows that, for arbitrary small groupoids  $H$ , there is a homotopy equivalence

$$BH \simeq \bigsqcup_{[x] \in \pi_0(H)} BH(x, x). \quad (4)$$

Thus, a groupoid  $H$  has no higher homotopy groups in the sense that  $\pi_k(BH, x) = 0$  for  $k \geq 2$  and all objects  $x$ , since the same is true of classifying spaces of groups.

### **Example: Group actions**

Suppose that  $G \times F \rightarrow F$  is the action of a group  $G$  on a set  $F$ .

Recall that the corresponding translation groupoid  $E_GF$  has objects  $x \in F$  and morphisms  $x \rightarrow g \cdot x$ .

The space  $B(E_GF) = EG \times_G F$  is the Borel construction for the action of  $G$  on  $F$ .

The group of automorphisms  $x \rightarrow x$  in  $E_GF$  can be identified with the subgroup  $G_x \subset G$  that stabilizes

$x$ . If  $\alpha : x \rightarrow y$  is a morphism of  $E_G F$ , then  $G_x$  is conjugate to  $G_y$  as subgroups of  $G$  (exercise).

There is a bijection

$$\pi_0(EG \times_G F) \cong F/G,$$

and the identification (4) translates to a homotopy equivalence

$$EG \times_G F \simeq \bigsqcup_{[x] \in F/G} BG_x. \quad (5)$$

Then  $EG \times_G F$  is contractible if and only if

- 1)  $G$  acts transitively on  $F$ , ie.  $F/G \cong *$ , and
- 2) the stabilizer subgroups  $G_x$  (fundamental groups) are trivial for all  $x \in F$ .

One usually summarizes conditions 1) and 2) by saying that  $G$  acts simply transitively on  $F$ , or that  $G$  acts principally on  $F$ .

In ordinary set theory, this means precisely that there is a  $G$ -equivariant isomorphism  $G \xrightarrow{\cong} F$ .

In the topos world, where  $G \times F \rightarrow F$  is the action of a sheaf of groups  $G$  on a sheaf  $F$ , the assertion that the Borel construction  $EG \times_G F$  is (locally) contractible is equivalent to the assertion that  $F$  is a  $G$ -torsor.

The canonical groupoid morphism  $E_GF \rightarrow G$  has the path lifting property, and hence induces a Kan fibration

$$\pi : EG \times_G F \rightarrow BG$$

with fibre  $F$ .

The use of this fibration  $\pi$ , in number theory, geometry and topology, is to derive calculations of homology invariants of  $BG$  from calculations of the corresponding invariants of the spaces  $BG_x$  associated to stabilizers, usually via spectral sequence calculations.

The Borel construction made its first appearance in the Borel seminar on transformation groups at IAS in 1958-59 [1].

If the action  $G \times F \rightarrow F$  is simple in the sense that all stabilizer groups  $G_x$  are trivial, then all orbits are copies of  $G$  up to equivariant isomorphism, and the canonical map

$$EG \times_G F \rightarrow F/G$$

is a weak equivalence.

It is a consequence of Quillen's Theorem 23.4 below that if  $G \times X \rightarrow G$  is an action of  $G$  on a simplicial set  $X$ , then  $X$  is the homotopy fibre of the canonical map  $EG \times_G X \rightarrow BG$ .

It follows that, if the action  $G \times X \rightarrow X$  is simple in all degrees and the simplicial set  $X$  is contractible, then the maps

$$\begin{array}{c} EG \times_G X \xrightarrow{\cong} X/G \\ \pi \downarrow \simeq \\ BG \end{array}$$

are weak equivalences, so that  $BG$  is weakly equivalent to  $X/G$ . This is a well known classical result.

## References

- [1] Armand Borel. *Seminar on transformation groups*. With contributions by G. Bredon, E. E. Floyd, D. Montgomery, R. Palais. Annals of Mathematics Studies, No. 46. Princeton University Press, Princeton, N.J., 1960.
- [2] P. G. Goerss and J. F. Jardine. *Simplicial Homotopy Theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.



## Lecture 07: Properness, diagrams of spaces

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### 17 Proper model structures

$\mathbf{M}$  is a fixed closed model category, for a while.

Here's a basic principle:

**Lemma 17.1.** *Suppose  $f : X \rightarrow Y$  is a morphism of  $\mathbf{M}$ , with both  $X$  and  $Y$  cofibrant.*

*Then  $f$  has a factorization*

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ & \searrow f & \downarrow u \\ & & Y \end{array}$$

*such that  $i$  is a cofibration, and  $u$  is a weak equivalence which is left inverse to a trivial cofibration  $j : Y \rightarrow Z$ .*

*Proof.* The construction is an abstraction of the classical mapping cylinder. It is dual to the replacement of a map between fibrant objects by a fibration (Section 13).  $\square$

**Lemma 17.2.** *Suppose given a pushout diagram*

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ i \downarrow & & \downarrow \\ C & \xrightarrow{u_*} & D \end{array}$$

*in  $\mathbf{M}$  with all objects cofibrant,  $i$  a cofibration and  $u$  a weak equivalence.*

*Then  $u_*$  is a weak equivalence.*

To put it a different way, in the category of cofibrant objects in a model category  $\mathbf{M}$ , the class of weak equivalences is closed under pushout along cofibrations.

*Proof.* By Lemma 17.1, and since trivial cofibrations are closed under pushout, it suffices to assume that there is a trivial cofibration  $j : B \rightarrow A$  with  $uj = 1_B$ .

Form the diagram

$$\begin{array}{ccccccc} B & \xrightarrow{j} & A & \xrightarrow{u} & B & & \\ j \downarrow & & \downarrow & & \downarrow & \searrow & \\ A & & & & & & \\ i \downarrow & & j_* \downarrow & & i \downarrow & & \\ C & \xrightarrow{\tilde{j}} & \tilde{D} & \xrightarrow{\tilde{u}} & B_* & \xrightarrow{f_*} & D \\ & & \downarrow f & & \downarrow f & & \\ & & C & \xrightarrow{u_*} & D & & \end{array}$$

$1_C$

in which the two back squares are pushouts.

$j$  is a trivial cofibration so  $\tilde{j}$  is a trivial cofibration, and so  $\tilde{u}$  is a weak equivalence (since  $\tilde{u}\tilde{j}$  is an isomorphism).  $f$  is a weak equivalence, so it suffices to show that the map  $f_*$  is a weak equivalence.

$f_*$  is a map between cofibrant objects of the model category  $B/\mathbf{M}$  which is obtained by pushing out the map  $j_* \xrightarrow{f} i$  of  $A/\mathbf{M}$  along  $u$ .

The pushout functor takes trivial cofibrations of slice categories to trivial cofibrations, thus preserves weak equivalences between cofibrant objects.  $\square$

**Remark:** For the last proof, you need to know (exercise) that if  $\mathbf{M}$  is a model category and  $A$  is an object of  $\mathbf{M}$ , then the slice category  $A/\mathbf{M}$  has a model structure for which a morphism

$$\begin{array}{ccc} & A & \\ \swarrow & & \searrow \\ B & \xrightarrow{f} & C \end{array}$$

is a weak equivalence (respectively cofibration, fibration) if and only if the map  $f : B \rightarrow C$  is a weak equivalence (respectively cofibration, fibration) of  $\mathbf{M}$ .

The dual structure on the slice category  $\mathbf{M}/A$  has a similar description.

Here is the dual of Lemma 17.2:

**Lemma 17.3.** *Suppose given a pullback diagram*

$$\begin{array}{ccc} W & \xrightarrow{u_*} & X \\ \downarrow & & \downarrow p \\ Z & \xrightarrow{u} & Y \end{array}$$

*in  $\mathbf{M}$ , with all objects fibrant,  $p$  a fibration and  $u$  a weak equivalence.*

*Then  $u_*$  is a weak equivalence.*

Thus, in the category of fibrant objects in  $\mathbf{M}$  the class of weak equivalences is closed under pullback along fibrations.

**Definition 17.4.** A model category  $\mathbf{M}$  is

- 1) *right proper* if the class of weak equivalences is closed under pullback along fibrations,
- 2) *left proper* if the class of weak equivalences is closed under pushout along cofibrations,
- 3) *proper* if it is both right and left proper.

**Examples:** 1) The category  $s\mathbf{Set}$  is proper.

All simplicial sets are cofibrant, so  $s\mathbf{Set}$  is left proper.

Given a pullback

$$\begin{array}{ccc} W & \xrightarrow{u_*} & X \\ \downarrow & & \downarrow p \\ Z & \xrightarrow{u} & Y \end{array}$$

in  $s\mathbf{Set}$  with  $p$  a fibration and  $u$  a weak equivalence, the induced diagram

$$\begin{array}{ccc} |W| & \xrightarrow{|u_*|} & |X| \\ \downarrow & & \downarrow |p| \\ |Z| & \xrightarrow{|u|} & |Y| \end{array}$$

of spaces is a pullback (realization is exact) in which  $|p|$  is a Serre fibration (Quillen's theorem: Theorem 13.1) and  $|u|$  is a weak equivalence. All spaces are fibrant, so  $|u_*|$  is a weak equivalence by Lemma 17.3, and so  $u_*$  is a weak equivalence of  $s\mathbf{Set}$ .

2) All spaces are fibrant, so  $\mathbf{CGWH}$  is right proper by Lemma 17.3. This category is also left proper by (non-abelian) *excision*, and the fact that  $s\mathbf{Set}$  is left proper.

The excision statement is the following:

**Lemma 17.5.** *Suppose the open subsets  $U_1, U_2$  cover a space  $Y$ .*

*Then the induced map*

$$S(U_1) \cup_{S(U_1 \cap U_2)} S(U_2) \rightarrow S(Y)$$

*is a weak equivalence of simplicial sets.*

Lemma 17.5 can be proved with simplicial approximation techniques [3].

3) The categories of simplicial groups and simplicial modules are right proper. The category of simplicial modules is also left proper (exercise).

4) There is a model structure on  $s\mathbf{Set}$  for which the cofibrations are the monomorphisms, and the weak equivalences are those maps  $X \rightarrow Y$  which induce rational homology isomorphisms

$$H_*(X, \mathbb{Q}) \cong H_*(Y, \mathbb{Q})$$

(this is the rational homology local model structure — it is one of the objects of study of rational homotopy theory).

There is a pullback square

$$\begin{array}{ccc} K(\mathbb{Q}/\mathbb{Z}, 0) & \xrightarrow{u_*} & P \\ \downarrow & & \downarrow p \\ K(\mathbb{Z}, 1) & \xrightarrow{u} & K(\mathbb{Q}, 1) \end{array}$$

where  $p$  is a fibration,  $P$  is contractible, and  $u$  is induced by the inclusion  $\mathbb{Z} \subset \mathbb{Q}$ . The map  $u$  is a rational homology isomorphism since  $\mathbb{Q}/\mathbb{Z}$  consists of torsion groups, while  $u_*$  is not.

Here's the **glueing lemma**:

**Lemma 17.6.** *Suppose given a commutative cube*

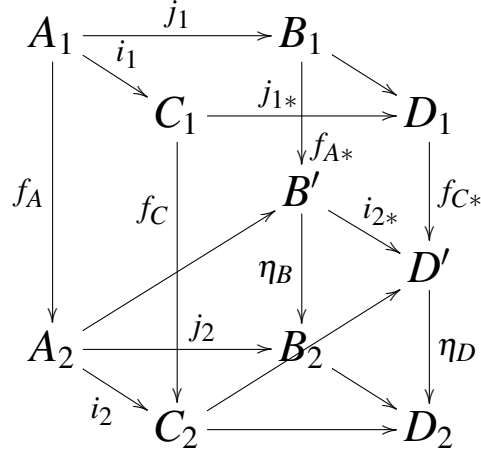
$$\begin{array}{ccccc} A_1 & \xrightarrow{j_1} & B_1 & & \\ & \searrow i_1 & \downarrow & \searrow & \\ & C_1 & \xrightarrow{\quad} & D_1 & \\ f_A \downarrow & \swarrow f_C & \downarrow f_B & \downarrow f_D & \\ A_2 & \xrightarrow{\quad} & B_2 & & \\ & \searrow i_2 & \downarrow j_2 & \searrow & \\ & C_2 & \xrightarrow{\quad} & D_2 & \end{array}$$

*in which all objects are cofibrant,  $i_1$  and  $i_2$  are cofibrations, the top and bottom faces are pushouts, and the maps  $f_A$ ,  $f_B$  and  $f_C$  are weak equivalences.*

*Then  $f_D$  is a weak equivalence.*

*Proof.* By Lemma 17.2, it suffices to assume that the maps  $j_1$  and  $j_2$  are cofibrations.

Form the diagram



in which  $f_{A*}$  is the pushout of  $f_A$  along  $j_1$  and  $f_{C*}$  is the pushout of  $f_C$  along  $j_{1*}$ .

All squares in the prism are pushouts,  $i_{2*}$  is a cofibration, and  $\eta_B$  is a weak equivalence. It follows from Lemma 17.2 that  $\eta_D$  is a weak equivalence.

$f_{C*}$  is also a weak equivalence, so  $f_D$  is a weak equivalence.  $\square$

**Remarks:**

- 1) Lemma 17.6 has a dual, which is usually called the **coglueing lemma**.
- 2) The statement of Lemma 17.6 holds in any left proper model category, by the same argument, while its dual holds in any right proper model category.



## 18 Homotopy cartesian diagrams

Here's the cogluing lemma for right proper model categories:

**Lemma 18.1.** *Suppose  $\mathbf{M}$  is right proper model. Suppose given a diagram*

$$\begin{array}{ccccc} X_1 & \longrightarrow & Y_1 & \xleftarrow{p_1} & Z_1 \\ \simeq \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ X_2 & \longrightarrow & Y_2 & \xleftarrow{p_2} & Z_2 \end{array}$$

*for which the vertical maps are weak equivalences and the maps  $p_1, p_2$  are fibrations.*

*Then the map*

$$X_1 \times_{Y_1} Z_1 \rightarrow X_2 \times_{Y_2} Z_2$$

*is a weak equivalence.*

The model category  $\mathbf{M}$  will be right proper throughout this section.

A commutative diagram

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Z & \xrightarrow{g} & Y \end{array} \tag{1}$$

in  $\mathbf{M}$  is **homotopy cartesian** if  $f$  has a factoriza-

tion

$$\begin{array}{ccc} X & \xrightarrow{\theta} & U \\ & \searrow f & \downarrow p \\ & & Y \end{array} \quad (2)$$

such that  $p$  is a fibration and  $\theta$  is a weak equivalence, and such that the induced map

$$W \xrightarrow{\theta_*} Z \times_Y U$$

is a weak equivalence.

**Slogan 1:** The choice of factorization of  $f$  doesn't matter.

**Lemma 18.2.** *Suppose given a second factorization*

$$\begin{array}{ccc} X & \xrightarrow{\theta'} & U' \\ & \searrow f & \downarrow p' \\ & & Y \end{array}$$

*of the map  $f$  in the commutative square (1). with  $\theta'$  a weak equivalence and  $p'$  a fibration. Then the map*

$$W \xrightarrow{\theta_*} Z \times_Y U$$

*is a weak equivalence if and only if the map*

$$W \xrightarrow{\theta'_*} Z \times_Y U'$$

*is a weak equivalence.*

*Proof.* It suffices to assume that the maps  $\theta$  and  $\theta'$  are trivial cofibrations. To see this, factorize  $\theta$  as

$$\begin{array}{ccc} X & \xrightarrow{i} & V \\ & \searrow \theta & \downarrow \pi \\ & & U \end{array}$$

where  $\pi$  is a trivial fibration and  $i$  is a trivial cofibration. Then in the diagram

$$\begin{array}{ccc} W & \xrightarrow{i_*} & Z \times_Y V \\ & \searrow \theta_* & \downarrow \pi_* \\ & & Z \times_Y U \end{array}$$

the map  $\pi_*$  is a trivial fibration, so  $\theta_*$  is a weak equivalence if and only if  $i_*$  is a weak equivalence.

Now suppose  $\theta$  and  $\theta'$  are trivial cofibrations. Then the lifting  $s$  exists in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\theta} & U \\ \theta' \downarrow & \nearrow s & \downarrow p \\ U' & \xrightarrow{p'} & Y \end{array}$$

and the induced map  $s_*$  in the diagram

$$\begin{array}{ccc} W & \xrightarrow{\theta'_*} & Z \times_Y U' \\ & \searrow \theta_* & \downarrow s_* \\ & & Z \times_Y U \end{array}$$

is a weak equivalence by Lemma 18.1. Thus,  $\theta_*$  is a weak equivalence if and only if  $\theta'_*$  is a weak equivalence.  $\square$

**Slogan 2:** It doesn't matter whether you factorize  $f$  or  $g$ .

**Lemma 18.3.** *Suppose*

$$\begin{array}{ccc} Z & \xrightarrow{\gamma} & V \\ & \searrow g & \downarrow q \\ & & Y \end{array}$$

*is a factorization of the map  $g$  in the diagram (1) with  $q$  a fibration and  $\gamma$  a weak equivalence, and  $f = p \cdot \theta$  with  $p$  a fibration and  $\theta$  a weak equivalence as in (2). Then the map  $\theta_* : W \rightarrow Z \times_Y U$  is a weak equivalence if and only if the map  $\gamma_* : W \rightarrow V \times_Y X$  is a weak equivalence.*

*Proof.* There is a commutative square

$$\begin{array}{ccc} W & \xrightarrow{\theta_*} & Z \times_Y U \\ \gamma_* \downarrow & & \simeq \downarrow \gamma_* \\ V \times_Y X & \xrightarrow[\theta_*]{\simeq} & V \times_Y U \end{array}$$

The indicated maps are weak equivalences since they are pull backs of weak equivalences along fibrations.  $\square$

The following is a rephrasing of the homotopy coglueing lemma for a right proper model category  $\mathbf{M}$ :

**Lemma 18.4.** *Suppose given a commutative cube*

$$\begin{array}{ccccc}
 W_1 & \xrightarrow{\quad} & X_1 & & \\
 \downarrow f_W & \searrow & \downarrow & \searrow & \\
 & Z_1 & \xrightarrow{\quad} & Y_1 & \\
 & \downarrow f_Z & \downarrow f_X & \downarrow f_Y & \\
 W_2 & \xrightarrow{\quad} & X_2 & & \\
 & \searrow & \downarrow & \searrow & \\
 & Z_2 & \xrightarrow{\quad} & Y_2 & 
 \end{array}$$

*in a right proper model category  $\mathbf{M}$  such that the top and bottom faces are homotopy cartesian, and the vertical maps  $f_Z$ ,  $f_X$  and  $f_Y$  are weak equivalences.*

*Then  $f_W$  is a weak equivalence.*

This result follows from the dual of Lemma 17.6.

Homotopy cartesian diagrams behave much like pullback diagrams:

**Lemma 18.5.** *Suppose  $\mathbf{M}$  is right proper.*

*1) Suppose given a commutative diagram*

$$\begin{array}{ccc}
 X_1 & \xrightarrow{\alpha} & X_2 \\
 \downarrow & & \downarrow \\
 Y_1 & \xrightarrow{\beta} & Y_2
 \end{array}$$

in  $\mathbf{M}$  such that the maps  $\alpha$  and  $\beta$  are weak equivalences. Then this diagram is homotopy cartesian.

2) Suppose given a commutative diagram

$$\begin{array}{ccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 \\ \downarrow & & \downarrow & & \downarrow \\ Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 \end{array}$$

$\mathbf{I}$                        $\mathbf{II}$

Then

- a) if the squares  $\mathbf{I}$  and  $\mathbf{II}$  are homotopy cartesian, then the composite square  $\mathbf{I} + \mathbf{II}$  is homotopy cartesian,
- b) if  $\mathbf{I} + \mathbf{II}$  and  $\mathbf{II}$  is homotopy cartesian then  $\mathbf{I}$  is homotopy cartesian.

*Proof.* The proof is an (important) exercise. □

A **homotopy fibre sequence** (or just **fibre sequence**) is a homotopy cartesian diagram

$$\begin{array}{ccc} F & \longrightarrow & X \\ \downarrow & & \downarrow f \\ P & \longrightarrow & Y \end{array}$$

in which  $P$  is contractible (ie. weakly equivalent to the terminal object).  $F$  is a **homotopy fibre** of the map  $f$ .

**Remark:** All concepts and results of this section have duals in left proper model categories, where one has homotopy cocartesian diagrams, homotopy cofibre sequences, and homotopy cofibres.

## 19 Diagrams of spaces

Suppose  $I$  is a small category.  $s\mathbf{Set}^I$  denotes the category of functors  $I \rightarrow s\mathbf{Set}$  and their natural transformations.  $s\mathbf{Set}^I$  is a **diagram category**.

Some people say that it is the category of *simplicial presheaves* on the category  $I$ .

$s\mathbf{Set}^I$  is the category of simplicial sheaves for the chaotic topology on  $I$  (which means no topology at all).

A map (natural transformation)  $f : X \rightarrow Y$  of  $I$ -diagrams is a **weak equivalence** (sometimes called a *sectionwise weak equivalence* or *pointwise weak equivalence*) if all maps  $f : X(i) \rightarrow Y(i)$ ,  $i \in I$ , are weak equivalences of simplicial sets.

There are many model structures on the diagram category  $s\mathbf{Set}^I$  for which the weak equivalences are as described, but I will single out two of them:

- The **projective structure**: The fibrations are defined sectionwise: a *projective fibration* is a map  $p : X \rightarrow Y$  for which consists of Kan fibrations  $f : X(i) \rightarrow Y(i)$ ,  $i \in I$ , in sections. A *projective cofibration* is a map which has the left lifting property with respect to all trivial projective fibrations.
- The **injective structure**: The cofibrations are defined sectionwise. A *cofibration* of  $I$ -diagrams is a monomorphism of  $s\mathbf{Set}^I$ , and an *injective fibration* is a map which has the right lifting property with respect to all trivial cofibrations.

The projective structure was introduced by Bousfield and Kan [1], and is easy to construct.

The *i*-sections functor  $X \mapsto X(i)$  has a left adjoint  $L_i$  with

$$L_i(K) = \text{hom}(i, \ ) \times K$$

for simplicial sets  $K$ .

A map  $p : X \rightarrow Y$  of  $s\mathbf{Set}^I$  is a projective fibration (respectively projective trivial fibration) if and only if it has the right lifting property with respect to the set of all maps  $L_i(\Lambda_k^n) \rightarrow L_i(\Delta^n)$  (repectively with respect to the set of maps  $L_i(\partial\Delta^m) \rightarrow L_i(\Delta^m)$ ).



The factorization axiom is proved by standard small object arguments, **CM4** is proved by the usual tricks, and the rest of the axioms are easily verified.

Note that we have specified generating sets for the trivial projective cofibrations and the projective cofibrations.

To summarize:

**Lemma 19.1.** *The sectionwise weak equivalences, projective fibrations and projective cofibrations give the diagram category  $s\mathbf{Set}^I$  the structure of a proper closed simplicial model category. This model structure is cofibrantly generated.*

Heller [2] is credited with the introduction of the injective structure on  $s\mathbf{Set}^I$ . It is also a special case of the model structure for simplicial sheaves which first appeared in Joyal's seminal letter to Grothendieck [4].

The injective structure is a little trickier to derive. Pick an infinite cardinal  $\alpha > |\mathbf{Mor}(I)|$ . Then one must prove a bounded cofibration condition:

**Lemma 19.2.** *Given a trivial cofibration  $X \rightarrow Y$  and an  $\alpha$ -bounded subobject  $A \subset Y$  there is an  $\alpha$ -bounded  $B$  with  $A \subset B \subset Y$  such that  $B \cap X \rightarrow B$  is a trivial cofibration.*

An  $I$ -diagram  $A$  is  **$\alpha$ -bounded** if  $|A(i)| < \alpha$  for all  $i \in I$ , and a cofibration  $A \rightarrow B$  is  $\alpha$ -bounded if  $B$  is  $\alpha$ -bounded.

It follows (see the proof of Lemma 11.5 (Lecture 04)) that a map  $p : X \rightarrow Y$  of  $s\mathbf{Set}$  is an injective fibration (respectively trivial injective fibration) if and only if it has the right lifting property with respect to all  $\alpha$ -bounded trivial cofibrations (respectively with respect to all  $\alpha$ -bounded cofibrations).

The factorization axiom **CM5** for the injective structure follows from a transfinite small object argument — see the proof of Lemma 11.4. The lifting axiom **CM4** also follows, while the remaining axioms **CM1** — **CM3** are easy to show.

We have “proved”:

**Theorem 19.3.** *The sectionwise weak equivalences, cofibrations and injective fibrations give the category  $s\mathbf{Set}^I$  the structure of a proper closed simplicial model category. This model structure is cofibrantly generated.*

For  $I$ -diagrams  $X$  and  $Y$ , write  $\mathbf{hom}(X, Y)$  for the simplicial set whose set of  $n$ -simplices is the collection of maps  $X \times \Delta^n \rightarrow Y$  (here  $\Delta^n$  is identified

with a constant  $I$ -diagram). For a simplicial set  $K$  and  $I$ -diagram  $X$ , the  $I$ -diagram  $X^K$  is specified at objects  $i \in I$  by

$$X^K(i) = \mathbf{hom}(K, X(i)).$$

There is also an  $I$ -diagram  $X \otimes K := X \times K$  given by

$$(X \times K)(i) = X(i) \times K.$$

If  $i : A \rightarrow B$  is a cofibration (respectively projective cofibration) and  $j : K \rightarrow L$  is a cofibration of simplicial sets, then the map

$$(i, j) : (B \times K) \cup (A \times L) \subset B \times L$$

is a cofibration (respectively projective cofibration) which is trivial if either  $i$  or  $j$  is trivial. The only issue with this is in showing that  $(i, j)$  is projective if  $i$  is projective, but it's true for generators  $L_k(A') \rightarrow L_k(B')$ , so it's true.

Finally, every projective cofibration is a cofibration, and every injective fibration is a projective fibration.

It follows that weak equivalences are stable under pullback along injective fibrations, and weak equivalences are stable under pushout along projective cofibrations, by properness for simplicial sets.

A model category  $\mathbf{M}$  is **cofibrantly generated** if there is a set  $I$  of trivial cofibrations and a set  $J$  of cofibrations such that a map  $p$  is a fibration (respectively trivial fibration) if and only if it has the right lifting property with respect to all members of  $I$  (respectively  $J$ ).

**Exercise:** Fill in the blanks in the proofs of Lemma 19.1 and Theorem 19.3.

## 20 Homotopy limits and colimits

The constant functor  $\Gamma : s\mathbf{Set} \rightarrow s\mathbf{Set}^I$  has both a right and left adjoint, given by limit and colimit, respectively.

Specifically,

$$\Gamma(X)(i) = X.$$

and all maps  $i \rightarrow j$  of  $I$  are sent to  $1_X$ .

$\Gamma$  preserves weak equivalences and cofibrations, and takes fibrations to projective fibrations.

The colimit functor

$$\varinjlim : s\mathbf{Set}^I \rightarrow s\mathbf{Set}$$

therefore takes projective cofibrations to cofibrations and takes trivial projective cofibrations to trivial cofibrations.

## Homotopy colimits

The adjunction

$$\varinjlim : s\mathbf{Set}^I \rightleftarrows s\mathbf{Set} : \Gamma$$

forms a Quillen adjunction for the projective structure on  $s\mathbf{Set}^I$ .

The **homotopy left derived functor**  $L\varinjlim$  is defined by

$$L\varinjlim(X) = \varinjlim Y,$$

where  $Y \rightarrow X$  is a weak equivalence with  $Y$  projective cofibrant.

$Y$  is a *projective cofibrant replacement* (or *projective cofibrant resolution*, or *projective cofibrant model*) of  $X$ .

The functor  $X \mapsto \varinjlim X$  takes trivial projective cofibrations to trivial cofibrations, hence takes weak equivalences between projective cofibrant objects to weak equivalences.

The homotopy type of  $L\varinjlim(X)$  is independent of the choice of projective cofibrant resolution for  $X$ .

The object  $L\varinjlim(X)$  has another name: it's called the **homotopy colimit** for the diagram  $X$ , and one

writes

$$\underline{\mathrm{holim}} X = L\underline{\mathrm{lim}}(X) = \underline{\mathrm{lim}} Y,$$

where  $Y \rightarrow X$  is a projective cofibrant model.

### Examples:

1) Consider all diagrams

$$B \leftarrow A \rightarrow C$$

of simplicial sets. This diagram is projective cofibrant if and only if all displayed morphisms are cofibrations (exercise). Every diagram

$$Z \xleftarrow{f} X \xrightarrow{g} Y$$

has a resolution by a diagram of cofibrations.

Thus, to form the homotopy pushout of  $f$  and  $g$ , replace  $f$  and  $g$  by cofibrations  $i$  and  $j$ , as in

$$\begin{array}{ccccc} & B & & C & \\ & \swarrow j & & \nearrow i & \\ \simeq \downarrow & & X & & \downarrow \simeq \\ Z & \xleftarrow{f} & & \xrightarrow{g} & Y \end{array}$$

and then the homotopy pushout is  $B \cup_X C$ .

By (left) properness, you only need to replace one of  $f$  or  $g$ : there are weak equivalences

$$B \cup_X Y \xleftarrow{\simeq} B \cup_X C \xrightarrow{\simeq} Z \cup_X C.$$

Thus, any homotopy co-cartesian diagram constructs the homotopy pushout.

2) All discrete diagrams are projective cofibrant, so homotopy coproducts and coproducts coincide.

3) Consider all countable diagrams

$$X : X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} X_3 \xrightarrow{\alpha_3} \dots$$

Such a diagram is projective cofibrant if and only if all  $\alpha_i$  are cofibrations.

If the comparison

$$\begin{array}{ccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & \dots \\ \simeq \downarrow & & \downarrow \simeq & & \downarrow \simeq & & \\ X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & \dots \end{array}$$

is a projective cofibrant resolution of  $X$ , then the induced map

$$\varinjlim_n A_n \rightarrow \varinjlim_n X_n$$

is a weak equivalence by comparing homotopy groups.

It follows that the canonical map

$$\varinjlim X = \varinjlim A \rightarrow \varinjlim X$$

is a weak equivalence.

## Homotopy limits

The inverse limit functor

$$\varprojlim : s\mathbf{Set}^I \rightarrow s\mathbf{Set}$$

takes injective fibrations to fibrations and takes trivial injective fibrations to trivial fibrations.

The adjunction

$$\Gamma : s\mathbf{Set} \rightleftarrows s\mathbf{Set}^I : \varprojlim$$

forms a Quillen adjunction for the injective structure on  $s\mathbf{Set}^I$ .

The **homotopy right derived functor**  $R\varprojlim$  is defined by

$$R\varprojlim(X) = \varprojlim Z$$

where  $\alpha : X \rightarrow Z$  is an injective fibrant model for  $X$  (ie.  $\alpha$  is a sectionwise weak equivalence with  $Z$  injective fibrant).

The functor  $Z \mapsto \varprojlim Z$  takes trivial injective fibrations to weak equivalences, and therefore takes weak equivalences between injective fibrant objects  $Z$  to weak equivalences.

The homotopy type of  $R\varprojlim(X)$  is independent the choice of injective fibrant model for  $X$ .



The object  $R\varprojlim(X)$  is the **homotopy inverse limit** of the diagram  $X$ , and one writes

$$\mathbf{holim} X = R\varprojlim(X) = \varprojlim Z$$

where  $X \rightarrow Z$  is an injective fibrant model for  $X$ .

**Examples:**

1) A diagram

$$X \xrightarrow{p} Y \xleftarrow{q} Z$$

of simplicial sets is injective fibrant if and only if  $Y$  is fibrant and  $p$  and  $q$  are fibrations.

Suppose given a diagram

$$X_1 \xrightarrow{f} X_2 \xleftarrow{g} X_3$$

and form an injective fibrant model

$$\begin{array}{ccccc} X_1 & \xrightarrow{f} & X_2 & \xleftarrow{g} & X_3 \\ j_1 \downarrow & & \downarrow j_2 & & \downarrow j_3 \\ Z_1 & \xrightarrow{p} & Z_2 & \xleftarrow{q} & Z_3 \end{array}$$

by choosing a fibrant model  $j_2$  and then factorizing both  $j_2 f$  and  $j_2 g$  as a trivial cofibration followed by a fibration.

Factorize  $g$  as  $g = \pi \cdot j$  where  $j$  is a trivial cofibration and  $\pi$  is a fibration. There is a lifting

$$\begin{array}{ccc} X_3 & \xrightarrow{j_3} & Z_3 \\ j \downarrow & \nearrow & \downarrow q \\ X'_3 & \xrightarrow{j_2 \pi} & Z_2 \end{array}$$

There is a comparison diagram

$$\begin{array}{ccccc} X_1 & \xrightarrow{f} & X_2 & \xleftarrow{\pi} & X'_3 \\ j_1 \downarrow & & \downarrow j_2 & & \downarrow \\ Z_1 & \xrightarrow{p} & Z_2 & \xleftarrow{q} & Z_3 \end{array}$$

in which the vertical maps are weak equivalence and  $\pi$  and  $q$  are fibrations. The induced map

$$X_1 \times_{X_2} X'_3 \rightarrow Z_1 \times_{Z_2} Z_3$$

is a weak equivalence by coglueing (Lemma 18.4).

Every homotopy cartesian diagram of simplicial sets computes the homotopy pullback.

2) A discrete diagram  $\{X_i\}$  in  $s\mathbf{Set}$  is injective fibrant if and only if all objects  $X_i$  are fibrant. The homotopy product of a diagram  $\{Y_i\}$  is constructed by taking fibrant replacements  $Y_i \rightarrow X_i$  for all  $i$ , and then forming the product  $\prod_i X_i$ .

This construction is serious: consider the simplicial sets  $A_n$ ,  $n \geq 1$ , where  $A_n$  is the string of  $n$  copies of  $\Delta^1$

$$0 \rightarrow 1 \rightarrow 2 \rightarrow \cdots \rightarrow n$$

glued end to end.

Each  $A_n$  is weakly equivalent to a point so their homotopy product is contractible, but  $\prod_{n \geq 1} A_n$  is not path connected.

3) A countable diagram (aka. a “tower”)

$$X : X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots$$

is injective fibrant if and only if  $X_1$  is fibrant and all morphisms in the tower are fibrations.

The long exact sequences associated to the fibrations in the tower entangle to define a spectral sequence (the Bousfield-Kan spectral sequence [1]) which computes the homotopy groups of  $\varprojlim X_n$ , at least in good cases.

## References

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## Lecture 08: Bisimplicial sets, homotopy limits and colimits

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### 21 Bisimplicial sets

A **bisimplicial set**  $X$  is a simplicial object

$$X : \Delta^{op} \rightarrow s\mathbf{Set}$$

in simplicial sets, or equivalently a functor

$$X : \Delta^{op} \times \Delta^{op} \rightarrow \mathbf{Set}.$$

I write

$$X_{m,n} = X(\mathbf{m}, \mathbf{n})$$

for the set of **bisimplices** in bidgree  $(m, n)$  and

$$X_m = X_{m,*}$$

for the **vertical simplicial set** in horiz. degree  $m$ .

Morphisms  $X \rightarrow Y$  of bisimplicial sets are natural transformations.

$s^2\mathbf{Set}$  is the category of bisimplicial sets.

**Examples:**

1)  $\Delta^{p,q}$  is the contravariant representable functor

$$\Delta^{p,q} = \text{hom}(\_, (\mathbf{p}, \mathbf{q}))$$

on  $\Delta \times \Delta$ .

$$\Delta_m^{p,q} = \bigsqcup_{\mathbf{m} \rightarrow \mathbf{p}} \Delta^q.$$

The maps  $\Delta^{p,q} \rightarrow X$  classify bisimplices in  $X_{p,q}$ .

The *bisimplex category*  $(\Delta \times \Delta)/X$  has the bisimplices of  $X$  as objects, with morphisms the incidence relations

$$\begin{array}{ccc} \Delta^{p,q} & & \\ \downarrow & \searrow & \\ \Delta^{r,s} & \nearrow & X \end{array}$$

2) Suppose  $K$  and  $L$  are simplicial sets.

The bisimplicial set  $K \tilde{\times} L$  has bisimplices

$$(K \tilde{\times} L)_{p,q} = K_p \times L_q.$$

The object  $K \tilde{\times} L$  is the *external product* of  $K$  and  $L$ .

There is a natural isomorphism

$$\Delta^{p,q} \cong \Delta^p \tilde{\times} \Delta^q.$$

3) Suppose  $I$  is a small category and that  $X : I \rightarrow s\mathbf{Set}$  is an  $I$ -diagram in simplicial sets.

Recall (Lecture 04) that there is a bisimplicial set  $\underline{\mathrm{holim}}_I X$  (“the” *homotopy colimit*) with vertical simplicial sets

$$\bigsqcup_{i_0 \rightarrow \cdots \rightarrow i_n} X(i_0)$$

in horizontal degrees  $n$ .

The transformation  $X \rightarrow *$  induces a bisimplicial set map

$$\pi : \bigsqcup_{i_0 \rightarrow \cdots \rightarrow i_n} X(i_0) \rightarrow \bigsqcup_{i_0 \rightarrow \cdots \rightarrow i_n} * = BI_n,$$

where the set  $BI_n$  has been identified with the discrete simplicial set  $K(BI_n, 0)$  in each horizontal degree.

**Example:** Suppose that  $G$  is a group, and that  $X$  is a simplicial set with a  $G$ -action  $G \times X \rightarrow X$ . If  $G$  is identified with a one-object groupoid, then the  $G$ -action defines a functor  $X : G \rightarrow s\mathbf{Set}$  which sends the single object of  $G$  to  $X$ .

The corresponding bisimplicial set has vertical simplicial sets of the form

$$\bigsqcup_{* \xrightarrow{g_1} * \xrightarrow{g_2} * \cdots \xrightarrow{g_n} *} X \cong G^{\times n} \times X,$$

which is a model in bisimplicial sets for the Borel construction  $EG \times_G X$ .

Applying the diagonal functor (see below) gives the Borel construction in simplicial sets.

Every simplicial set  $X$  determines bisimplicial sets which are constant in each vertical degree or each horizontal degree. We write  $X$  for the *constant* bisimplicial set determined by  $X$  either horizontally or vertically.

From this point of view, the canonical map  $\pi$  is a map of bisimplicial sets

$$\pi : \varinjlim_I X \rightarrow BI.$$

The *diagonal* simplicial set  $d(X)$  for bisimplicial set  $X$  has simplices

$$d(X)_n = X_{n,n}$$

with simplicial structure maps

$$(\theta, \theta)^* : X_{n,n} \rightarrow X_{m,m}$$

for ordinal number maps  $\theta : \mathbf{m} \rightarrow \mathbf{n}$ .

This construction defines a functor

$$d : s^2\mathbf{Set} \rightarrow s\mathbf{Set}.$$

Recall that  $X_n$  denotes the vertical simplicial set in horizontal degree  $n$  for a bisimplicial set  $X$ . The maps

$$\begin{array}{ccc} X_n \times \Delta^m & \xrightarrow{1 \times \theta} & X_n \times \Delta^n \\ \theta^* \times 1 \downarrow & & \\ X_m \times \Delta^m & & \end{array}$$

associated to the ordinal number maps  $\theta : \mathbf{m} \rightarrow \mathbf{n}$  determine morphisms

$$\bigsqcup_{\theta : \mathbf{m} \rightarrow \mathbf{n}} X_n \times \Delta^m \rightrightarrows \bigsqcup_{n \geq 0} X_n \times \Delta^n. \quad (1)$$

There are simplicial set maps

$$\gamma_n : X_n \times \Delta^n \rightarrow d(X)$$

defined on  $r$ -simplices by

$$\gamma_n(x, \tau : \mathbf{r} \rightarrow \mathbf{n}) = \tau^*(x) \in X_{r,r}.$$

The maps in (1) above and the morphisms  $\gamma_n$ ,  $n \geq 0$  together determine a diagram

$$\bigsqcup_{\theta : \mathbf{m} \rightarrow \mathbf{n}} X_n \times \Delta^m \rightrightarrows \bigsqcup_{n \geq 0} X_n \times \Delta^n \xrightarrow{\gamma} d(X). \quad (2)$$

**Exercise:** Show that the diagram (2) is a coequalizer in simplicial sets.

**Example:** There are natural isomorphisms

$$d(K \tilde{\times} L) \cong K \times L.$$



In particular, there are isomorphisms

$$d(\Delta^{p,q}) \cong \Delta^p \times \Delta^q.$$

The diagonal simplicial set  $d(X)$  has a filtration by subobjects  $d(X)^{(n)}$ ,  $n \geq 0$ , where

$$d(X)^{(n)} = \text{image of } \bigsqcup_{p \leq n} X_p \times \Delta^p \text{ in } d(X).$$

The (horizontal) degenerate part of the vertical simplicial set  $X_{n+1}$  is filtered by subobjects

$$s_{[r]}X_n = \bigcup_{0 \leq i \leq r} s_i(X_n) \subset X_{n+1}$$

where  $r \leq n$ . There are natural pushout diagrams of cofibrations

$$\begin{array}{ccc} s_{[r]}X_{n-1} & \xrightarrow{s_{r+1}} & s_{[r]}X_n \\ \downarrow & & \downarrow \\ X_n & \xrightarrow{s_{r+1}} & s_{[r+1]}X_n \end{array} \quad (3)$$

and

$$\begin{array}{ccc} (s_{[n]}X_n \times \Delta^{n+1}) \cup (X_{n+1} \times \partial\Delta^{n+1}) & \longrightarrow & d(X)^{(n)} \\ \downarrow & & \downarrow \\ X_{n+1} \times \Delta^{n+1} & \longrightarrow & d(X)^{(n+1)} \end{array} \quad (4)$$

in which all vertical maps are cofibrations.

The natural filtration  $\{d(X)^{(n)}\}$  of  $d(X)$  and the natural pushout diagrams (3) and (4) are used with glueing lemma arguments to show the following:

**Lemma 21.1.** *Suppose  $f : X \rightarrow Y$  is a map of bisimplicial sets such that all maps  $X_n \rightarrow Y_n$ ,  $n \geq 0$ , of vertical simplicial sets are weak equivalences.*

*Then the induced map  $d(X) \rightarrow d(Y)$  is a weak equivalence of diagonal simplicial sets.*

**Example:** Suppose that  $G \times X \rightarrow X$  is an action of a group  $G$  on a simplicial set  $X$ . The bisimplicial set

$$\begin{array}{c} \bigsqcup \\ * \xrightarrow{g_1} * \xrightarrow{g_2} * \dots \xrightarrow{g_n} * \end{array} \quad X \cong G^{\times n} \times X$$

has horizontal path components  $X/G$ , and the map to path components defines a simplicial set map

$$\pi : EG \times_G X \rightarrow X/G,$$

which is natural in  $G$ -sets  $X$ .

If the action  $G \times X \rightarrow X$  is free, then the path components the simplicial sets  $EG \times_G X_n$  are isomorphic to copies of the contractible space  $EG = EG \times_G G$ . It follows that the map  $\pi$  is a weak equivalence in this case.

If the action  $G \times X \rightarrow X$  is free and  $X$  is contractible, then we have weak equivalences

$$\begin{array}{c} EG \times_G X \xrightarrow[\simeq]{\pi} X/G \\ p \downarrow \simeq \\ BG \end{array}$$

### Model structures

There are multiple closed model structures for bisimplicial sets. Here are three of them:

- 1) The **projective structure**, for which a map  $X \rightarrow Y$  of bisimplicial sets is a weak equivalence (respectively projective fibration) if all maps  $X_n \rightarrow Y_n$  are weak equivalences (respectively fibrations) of simplicial sets. The cofibrations for this structure are called the projective cofibrations.
- 2) The **injective structure**, for which  $X \rightarrow Y$  is a weak equivalence (respectively cofibration) if all maps  $X_n \rightarrow Y_n$  are weak equivalences (respectively cofibrations) of simplicial sets. The fibrations for this theory are called the injective fibrations.
- 3) There is a **diagonal model structure** on  $s^2\mathbf{Set}$  for which a map  $X \rightarrow Y$  is a weak equivalence if it is a *diagonal weak equivalence* ie. that the map  $d(X) \rightarrow d(Y)$  of simplicial sets is a weak equivalence, and the cofibrations are the monomorphisms

of bisimplicial sets as in 2).

The existence of the diagonal structure is originally due to Joyal and Tierney, but they did not publish the result. A proof appears in [3].

The projective structure is a special case of the projective structure for  $I$ -diagrams of simplicial sets of Lemma 19.1 (Lecture 07) — it is called the Bousfield-Kan structure in [2, IV.3.1].

The injective structure is similarly a special case of the injective structure for  $I$ -diagrams, of Theorem 19.3.

The injective structure is also an instance of the Reedy structure for simplicial objects in a model category [2, IV.3.2, VII.2].

The weak equivalences for both the projective and injective structures are called *level equivalences*.

Lemma 21.1 says that every level equivalence is a diagonal equivalence.

The diagonal functor  $X \mapsto d(X)$  is left adjoint to a “singular functor”  $X \mapsto d_*(X)$ , where

$$d_*(X)_{p,q} = \text{hom}(\Delta^p \times \Delta^q, X).$$

One can show, by verifying a (countable) bounded cofibration condition, that a bisimplicial set map

$p : X \rightarrow Y$  is a fibration for the diagonal model structure if and only if it has the right lifting property with respect to all trivial cofibrations  $A \rightarrow B$  which are countable in the sense that all sets of bisimplices  $B_{p,q}$  are countable.

The bounded cofibration condition is a somewhat tough exercise to prove — one uses the fact that the diagonal functor has a left adjoint as well as a right adjoint.

## 22 Homotopy colimits and limits (revisited)

Suppose  $X : I \rightarrow s\mathbf{Set}$  is an  $I$ -diagram which takes values in Kan complexes.

Following [1], one writes

$$\mathop{\mathrm{holim}}\limits_I X = \mathbf{hom}(B(I/?), X),$$

where the function complex is standard, and  $B(I/?)$  is the functor  $i \mapsto B(I/i)$ .

Suppose  $Y$  is a simplicial set, and  $X$  is still our prototypical  $I$ -diagram.

### Homotopy colimits

The assignment  $i \mapsto \mathbf{hom}(X(i), Y)$  defines an  $I^{op}$ -diagram

$$\mathbf{hom}(X, Y) : I^{op} \rightarrow s\mathbf{Set}.$$

There is a natural isomorphism of function spaces

$$\mathbf{hom}(\underline{\mathrm{holim}}_I X, Y) \cong \underline{\mathrm{holim}}_{I^{op}} \mathbf{hom}(X, Y),$$

where  $\underline{\mathrm{holim}}_I X$  is defined by the coequalizer

$$\bigsqcup_{\alpha: i \rightarrow j \text{ in } I} B(j/I) \times X(i) \rightrightarrows \bigsqcup_{i \in \mathrm{Ob}(I)} B(i/I) \times X(i) \rightarrow \underline{\mathrm{holim}}_I X.$$

By looking at maps

$$\underline{\mathrm{holim}}_I X \rightarrow Y,$$

one shows (exercise) that  $\underline{\mathrm{holim}}_I X$  is the diagonal of the bisimplicial set, with vertical  $n$ -simplices

$$\bigsqcup_{i_0 \rightarrow \cdots \rightarrow i_n} X(i_0),$$

up to isomorphism.

This is the (standard) description of the homotopy colimit of  $X$  that was introduced in Section 9.

This definition of homotopy colimit coincides up to equivalence with the “colimit of projective cofibrant model” description of Section 20.

Here is the key to comparing the two:

**Lemma 22.1.** *Suppose  $X : I \rightarrow s\mathbf{Set}$  is a projective cofibrant  $I$ -diagram. Then the canonical map*

$$\underline{\mathrm{holim}}_I X \rightarrow \varinjlim_I X$$

*is a weak equivalence.*

*Proof.*  $\varinjlim_I X_m$  is the set of path components of the simplicial set

$$\bigsqcup_{i_0 \rightarrow \cdots \rightarrow i_n} X(i_0)_m,$$

so  $\varinjlim_I X$  can be identified with the simplicial set of horizontal path components of the bisimplicial set  $\operatorname{holim}_I X$ .

The space  $B(i/I)$  is contractible since the category  $i/I$  has an initial object. Thus, every projection

$$B(i/I) \times K \rightarrow K$$

is a weak equivalence.

The simplicial set  $B(i/I) \times K$  is the homotopy colimit of the  $I$  diagram  $\operatorname{hom}(i, \_) \times K$  and the projection is isomorphic to the map

$$\operatorname{holim}_I (\operatorname{hom}(i, \_) \times K) \rightarrow \varinjlim_I (\operatorname{hom}(i, \_) \times K)$$

Thus, all diagrams  $\operatorname{hom}(i, \_) \times K$  are members of the class of  $I$ -diagrams  $X$  for which the map

$$\operatorname{holim}_I X \rightarrow \varinjlim_I X \tag{5}$$

is a weak equivalence.

Suppose given a pushout diagram

$$\begin{array}{ccc} \mathrm{hom}(i, \ ) \times K & \longrightarrow & X \\ 1 \times j \downarrow & & \downarrow \\ \mathrm{hom}(i, \ ) \times L & \longrightarrow & Y \end{array}$$

of  $I$ -diagrams, where  $j$  is a cofibration. Suppose also that the map (5) is a weak equivalence. Then the induced map

$$\underline{\mathrm{holim}}_I Y \rightarrow \underline{\mathrm{lim}}_I Y$$

is a weak equivalence.

For this, the induced diagram

$$\begin{array}{ccc} \underline{\mathrm{lim}}_I (\mathrm{hom}(i, \ ) \times K) & \longrightarrow & \underline{\mathrm{lim}}_I X \\ \downarrow & & \downarrow \\ \underline{\mathrm{lim}}_I (\mathrm{hom}(i, \ ) \times L) & \longrightarrow & \underline{\mathrm{lim}}_I Y \end{array}$$

is a pushout, and one uses the glueing lemma to see the desired weak equivalence.

Suppose given a diagram of cofibrations of  $I$ -diagrams

$$X_0 \rightarrow X_1 \rightarrow \dots$$

such that all maps

$$\underline{\mathrm{holim}}_I X_s \rightarrow \underline{\mathrm{lim}}_I X_s$$



are weak equivalences. Then the map

$$\underline{\mathrm{holim}}_I(\underline{\lim}_s X_s) \rightarrow \underline{\lim}_I(\underline{\lim}_s X_s)$$

is a weak equivalence.

In effect, the colimit and homotopy colimit functors commute, and filtered colimits preserve weak equivalences in  $s\mathbf{Set}$ .

A small object argument shows that, for every  $I$ -diagram  $Y$ , there is a trivial projective fibration  $p : X \rightarrow Y$  such that  $X$  is projective cofibrant and the map (5) is a weak equivalence.

If  $Y$  is projective cofibrant, then  $Y$  is a retract of the covering  $X$ , so the map

$$\underline{\mathrm{holim}}_I Y \rightarrow \underline{\lim}_I Y$$

is a weak equivalence. □

**Corollary 22.2.** *Suppose  $X : I \rightarrow s\mathbf{Set}$  is an  $I$ -diagram of simplicial sets, and let  $\pi : U \rightarrow X$  be a projective cofibrant model of  $X$ . Then there are weak equivalences*

$$\underline{\mathrm{holim}}_I X \xleftarrow[\pi_*]{\simeq} \underline{\mathrm{holim}}_I U \xrightarrow[\simeq]{\simeq} \underline{\lim}_I U.$$

*Proof.* Generally, if  $f : X \rightarrow Y$  is a weak equivalence of  $I$ -diagrams, then the induced maps

$$\bigsqcup_{i_0 \rightarrow \cdots \rightarrow i_n} X(i_0) \rightarrow \bigsqcup_{i_0 \rightarrow \cdots \rightarrow i_n} Y(i_0)$$

is a weak equivalence of simplicial sets for each vertical degree  $n$ , and it follows from Lemma 21.1 that the induced map

$$\underline{\mathrm{holim}}_I X \rightarrow \underline{\mathrm{holim}}_I Y$$

is a weak equivalence.

It follows that the map

$$\underline{\mathrm{holim}}_I X \xleftarrow{\pi_*} \underline{\mathrm{holim}}_I U$$

is a weak equivalence, and Lemma 22.1 shows that

$$\underline{\mathrm{holim}}_I U \rightarrow \varinjlim_I U$$

is a weak equivalence. □

## Homotopy limits

Each slice category  $I/i$  has a terminal object, so  $B(I/i)$  is contractible, and the map

$$B(I/?) \rightarrow *$$

of  $I$ -diagrams is a weak equivalence.

If  $Z$  is an injective fibrant  $I$ -diagram, then the induced map

$$\varprojlim_I Z \cong \mathbf{hom}(*, Z) \rightarrow \mathbf{hom}(B(I/?), Z) =: \varprojlim_I Z$$

is a weak equivalence.

Here's the interesting thing to prove:

**Proposition 22.3.** *Suppose  $p : X \rightarrow Y$  is a projective fibration (resp. trivial projective fibration). Then*

$$p_* : \varprojlim_I X \rightarrow \varprojlim_I Y$$

*is a fibration (resp. trivial fibration) of  $s\mathbf{Set}$ .*

There are a few concepts involved in the proof of Proposition 22.3.

1) Every  $I$ -diagram  $Y$  has an associated cosimplicial space (aka.  $\Delta$ -diagram in simplicial sets)  $\prod^* Y$  with

$$\prod^n Y = \prod^* Y(\mathbf{n}) = \prod_{i_0 \rightarrow \dots \rightarrow i_n} Y(i_n),$$

and with cosimplicial structure map  $\theta_* : \prod^m Y \rightarrow \prod^n Y$  defined for an ordinal number map  $\theta : \mathbf{m} \rightarrow \mathbf{n}$

defined by the picture

$$\begin{array}{ccc} \prod_{\gamma: j_0 \rightarrow \dots \rightarrow j_m} Y(j_m) & \xrightarrow{\theta_*} & \prod_{\sigma: i_0 \rightarrow \dots \rightarrow i_n} Y(i_n) \\ \downarrow \text{pr}_{\theta^*(\sigma)} & & \downarrow \text{pr}_{\sigma} \\ Y(i_{\theta(m)}) & \longrightarrow & Y(i_n) \end{array}$$

in which the bottom horizontal map is induced by the morphism  $i_{\theta(m)} \rightarrow i_n$  of  $I$ .

2) There is a cosimplicial space  $\Delta$  consisting of the standard  $n$ -simplices and the maps between them, and there is a natural bijection

$$\text{hom}(\Delta, \prod^* Y) \cong \text{hom}(B(I/?), Y)$$

This bijection induces a natural isomorphism of simplicial sets

$$\mathbf{hom}(\Delta, \prod^* Y) \cong \mathbf{hom}(B(I/?), Y) = \varprojlim_I Y.$$

Bousfield and Kan call this isomorphism “cosimplicial replacement of diagrams” in [1].

3) We also use the “matching spaces”  $M^n Z$  for a cosimplicial space  $Z$ . Explicitly,

$$M^n Z \subset \prod_{i=0}^n Z^n$$

is the set of  $(n+1)$ -tuples  $(z_0, \dots, z_n)$  such that  $s^j z_i = s^i z_{j+1}$  for  $i \leq j$ .

There is a natural simplicial set map

$$s : Z^{n+1} \rightarrow M^n Z$$

defined by  $s(z) = (s^0 z, s^1 z, \dots, s^n z)$ .

**Lemma 22.4.** *Suppose  $X$  is an  $I$ -diagram of sets. Then the map*

$$s : \prod^{n+1} X = \prod_{\sigma: i_0 \rightarrow \dots \rightarrow i_{n+1}} X(i_{n+1}) \rightarrow M^n \prod^* X$$

*factors through a bijection*

$$\prod_{\sigma: i_0 \rightarrow \dots \rightarrow i_{n+1} \in D(BI)_{n+1}} X(i_{n+1}) \xrightarrow{\cong} M^n \prod^* X,$$

where  $D(BI)_{n+1}$  is the set of degenerate simplices in  $BI_{n+1}$ .

*Proof.* Write  $X = \bigsqcup_{i \in \text{Ob}(I)} X(i)$ , and let  $\pi : X \rightarrow \text{Ob}(I)$  be the canonical map.

An element  $\alpha$  of  $\prod^m X$  is a commutative diagram

$$\begin{array}{ccc} BI_m & \xrightarrow{\alpha} & X \\ & \searrow v_m & \swarrow \pi \\ & \text{Ob}(I) & \end{array}$$

where  $v_m$  is induced by the inclusion  $\{m\} \subset \mathbf{m}$  of the vertex  $m$ .

If  $s : \mathbf{m} \rightarrow \mathbf{n}$  is an ordinal number epimorphism

then the diagram

$$\begin{array}{ccc}
 BI_n & \xrightarrow{s_*(\alpha)} & X \\
 & \searrow s^* \quad \nearrow \alpha & \\
 & BI_m & \\
 & \searrow v_n \quad \nearrow \pi & \\
 & \downarrow v_m & \\
 & Ob(I) & 
 \end{array}$$

commutes.

The degeneracies  $s_i : BI_n \rightarrow BI_{n+1}$  take values in  $DBI_{n+1}$  and the simplicial identities  $s_i s_j = s_{j+1} s_i$ ,  $i \leq j$  determine a coequalizer

$$\bigsqcup_{i \leq j} BI_{n-1} \rightrightarrows \bigsqcup_{i=0}^n BI_n \rightarrow DBI_{n+1}.$$

Write  $p_1, p_2$  for the maps defining the coequalizer.

An element of  $M_n \prod^* X$  is a map

$$\begin{array}{ccc}
 \bigsqcup_{i=0}^n BI_n & \xrightarrow{f} & X \\
 & \searrow (v_n) \quad \nearrow \pi & \\
 & Ob(I) & 
 \end{array}$$

fibred over  $Ob(I)$ , such that  $f \cdot p_1 = f \cdot p_2$ . It follows that  $f$  factors uniquely through a function  $DBI_{n+1} \rightarrow X$ , fibred over  $Ob(I)$ .  $\square$

*Proof of Proposition 22.3.* By an adjointness argument and cosimplicial replacement of diagrams, showing that the map  $\varprojlim_I X \rightarrow \varprojlim_I Y$  has the RLP wrt an inclusion  $i : K \subset L$  of simplicial sets amounts to solving a lifting problem

$$\begin{array}{ccc} \Delta \times K & \longrightarrow & \prod^* X \\ 1 \times i \downarrow & \nearrow & \downarrow \\ \Delta \times L & \longrightarrow & \prod^* Y \end{array}$$

in cosimplicial spaces.

One solves such lifting problems inductively in cosimplicial degrees by solving lifting problems

$$\begin{array}{ccc} (L \times \partial \Delta^{n+1}) \cup (K \times \Delta^{n+1}) & \longrightarrow & \prod^{n+1} X \\ \downarrow & \nearrow & \downarrow (p,s) \\ L \times \Delta^{n+1} & \longrightarrow & \prod^{n+1} Y \times_{M^n \prod^* Y} M^n \prod^* X \end{array}$$

By Lemma 22.4, solving this lifting problem amounts to solving lifting problems

$$\begin{array}{ccc} (L \times \partial \Delta^{n+1}) \cup (K \times \Delta^{n+1}) & \longrightarrow & X(i_{n+1}) \\ \downarrow & \nearrow & \downarrow p \\ L \times \Delta^{n+1} & \longrightarrow & Y(i_{n+1}) \end{array}$$

one for each non-degenerate simplex  $\sigma : i_0 \rightarrow \cdots \rightarrow i_{n+1}$  of  $BI_{n+1}$ . This can be done if either  $K \subset L$  is anodyne or if  $p$  is trivial, since  $p$  is a projective fibration.  $\square$

**Corollary 22.5.** *Suppose  $X$  is a projective fibrant  $I$ -diagram and that  $X \rightarrow Z$  is an injective fibrant model of  $X$ . Then there are weak equivalences*

$$\varprojlim_I X \xrightarrow{\sim} \varprojlim_I Z \xleftarrow{\sim} \varprojlim_I Z.$$

**Example:** Every bisimplicial set  $X$  is a functor

$$X : \Delta^{op} \rightarrow s\mathbf{Set}.$$

The homotopy colimit  $\varinjlim_{\Delta^{op}} X$  is defined by the coend (ie. colimit of all diagrams)

$$\begin{array}{c} B(\mathbf{m}/\Delta^{op}) \times X_n \xrightarrow{1 \times \theta^*} B(\mathbf{m}/\Delta^{op}) \times X_m \\ \theta^* \times 1 \downarrow \\ B(\mathbf{n}/\Delta^{op}) \times X_n \end{array}$$

and therefore by the coend

$$\begin{array}{c} B(\Delta/\mathbf{m}) \times X_n \xrightarrow{1 \times \theta^*} B(\Delta/\mathbf{m}) \times X_m \\ \theta \times 1 \downarrow \\ B(\Delta/\mathbf{n}) \times X_n \end{array}$$

There is a natural map of cosimplicial categories

$$h : \Delta/\mathbf{n} \rightarrow \mathbf{n}$$

(the “last vertex map”) which takes an object  $\alpha : \mathbf{k} \rightarrow \mathbf{n}$  to  $\alpha(k) \in \mathbf{n}$ .

This map induces a morphism of coends

$$B(\Delta/\mathbf{n}) \times X_n \xrightarrow{h \times 1} \Delta^n \times X_n,$$



and therefore induces a natural map

$$h_* : \underline{\mathrm{holim}}_{\Delta^{op}} X \rightarrow d(X).$$

**Claim:** This map  $h_*$  is a weak equivalence.

Both functors involved in  $h$  preserve levelwise weak equivalences in  $X$ , so we can assume that  $X$  is projective cofibrant. If  $Y$  is a Kan complex, then the induced map

$$\mathbf{hom}(d(X), Y) \rightarrow \mathbf{hom}(\underline{\mathrm{holim}}_{\Delta^{op}} X, Y)$$

can be identified up to isomorphism with the map

$$\mathbf{hom}(X, \mathbf{hom}(\Delta, Y)) \rightarrow \mathbf{hom}(X, \mathbf{hom}(B(\Delta/?), Y)). \quad (6)$$

The map

$$\mathbf{hom}(\Delta, Y) \rightarrow \mathbf{hom}(B(\Delta/?), Y)$$

is a weak equivalence of projective fibrant simplicial spaces, so the map in (6) is a weak equivalence since  $X$  is projective cofibrant.

This is true for all Kan complexes  $Y$ , so  $h_*$  is a weak equivalence as claimed.

**Example:** Suppose  $Y$  is an injective fibrant cosimplicial space. Then the weak equivalence  $h$  induces a weak equivalence

$$\mathbf{hom}(\Delta, Y) \xrightarrow{h_*} \mathbf{hom}(B(\Delta/?), Y) = \underline{\mathrm{holim}}_{\Delta} Y.$$

This is also true if the cosimplicial space  $Y$  is Bousfield-Kan fibrant [1] in the sense that all maps

$$s : Y^{n+1} \rightarrow M_n Y$$

are fibrations — see [1, X.4] or [2]. Every injective fibrant cosimplicial space is fibrant in this sense.

Following [1], the space  $\mathbf{hom}(\Delta, Y)$  is usually denoted by  $\mathrm{Tot}(Y)$ .

### 23 Applications, Quillen's Theorem B

Suppose  $p : X \rightarrow Y$  is a map of simplicial sets, and choose pullbacks

$$\begin{array}{ccc} p^{-1}(\sigma) & \longrightarrow & X \\ \downarrow & & \downarrow p \\ \Delta^n & \xrightarrow{\sigma} & Y \end{array}$$

for all simplices  $\sigma : \Delta^n \rightarrow Y$  of the base  $Y$ .

A morphism  $\alpha : \sigma \rightarrow \tau$  in  $\Delta/Y$  of  $Y$  induces a simplicial set map  $p^{-1}(\sigma) \rightarrow p^{-1}(\tau)$ , and we have a functor

$$p^{-1} : \Delta/Y \rightarrow s\mathbf{Set}.$$

The maps  $p^{-1}(\sigma) \rightarrow X$  induce maps of simplicial

sets

$$\omega : \bigsqcup_{\sigma_0 \rightarrow \dots \rightarrow \sigma_n} p^{-1}(\sigma_0) \rightarrow X$$

or rather a morphism of bisimplicial sets

$$\omega : \varinjlim_{\sigma : \Delta^n \rightarrow Y} p^{-1}(\sigma) \rightarrow X.$$

**Lemma 23.1.** *The bisimplicial set map*

$$\omega : \varinjlim_{\sigma : \Delta^n \rightarrow Y} p^{-1}(\sigma) \rightarrow X$$

*is a diagonal weak equivalence.*

*Proof.* The simplicial set  $Y$  is a colimit of its simplices in the sense that the canonical map

$$\varinjlim_{\Delta^n \rightarrow Y} \Delta^n \rightarrow Y$$

is an isomorphism. The pullback functor is exact, so the canonical map

$$\varinjlim_{\Delta^n \rightarrow Y} p^{-1}(\sigma) \rightarrow X$$

is an isomorphism.

Take  $\tau \in X_m$ . Then fibre  $\omega^{-1}(\tau)$  over  $\tau$  for the simplicial set map

$$\omega : \bigsqcup_{\sigma_0 \rightarrow \dots \rightarrow \sigma_n} p^{-1}(\sigma_0)_m \rightarrow X_m$$

is the nerve of a category  $C_\tau$  whose objects consist of pairs  $(\sigma, y)$ , where  $\sigma : \Delta^n \rightarrow Y$  is a simplex of  $Y$  and  $y \in p^{-1}(\sigma)_m$  such that  $y \mapsto \tau$  under the map  $p^{-1}(\sigma) \rightarrow X$ .

A morphism  $(\sigma, y) \rightarrow (\gamma, z)$  of  $C_\tau$  is a map  $\sigma \rightarrow \gamma$  of the simplex category  $\Delta/Y$  such that  $y \mapsto z$  under the map  $p^{-1}(\sigma) \rightarrow p^{-1}(\gamma)$ .

There is an element  $x_\tau \in p^{-1}(p(\tau))$  such that  $x_\tau \mapsto \tau \in X$  and  $x_\tau \mapsto \iota_m \in \Delta^m$ . The element  $(p(\tau), x_\tau)$  is initial in  $C_\tau$  (exercise), and this is true for all  $\tau \in X_m$ , so the map  $\omega$  is a weak equivalence in each vertical degree  $m$ .

Finish the proof by using Lemma 21.1.  $\square$

Here's a first consequence, originally due to Kan and Thurston [4]:

**Corollary 23.2.** *There are natural weak equivalences*

$$B(\Delta/X) \xleftarrow{\simeq} \varinjlim_{\Delta^n \rightarrow X} \Delta^n \xrightarrow{\simeq} X$$

for each simplicial set  $X$ .

*Proof.* The map

$$\varinjlim_{\Delta^n \rightarrow X} \Delta^n \rightarrow B(\Delta/X)$$

is induced by the weak equivalence of diagrams  $\Delta^n \rightarrow *$  on the simplex category.

The other map is a weak equivalence, by Lemma 23.1 applied to the identity map  $X \rightarrow X$ .  $\square$

Suppose  $f : C \rightarrow D$  is a functor between small categories, and consider the pullback squares of functors

$$\begin{array}{ccc} f/d & \longrightarrow & C \\ \downarrow & & \downarrow \\ D/d & \longrightarrow & D \end{array}$$

for  $d \in \text{Ob}(D)$ .

Here,  $f/d$  is the category whose objects are pairs  $(c, \alpha)$  where  $c \in \text{Ob}(C)$  and  $\alpha : f(c) \rightarrow d$  is a morphism of  $D$ .

A morphism  $\gamma : (c, \alpha) \rightarrow (c', \beta)$  is a morphism  $\gamma : c \rightarrow c'$  of  $C$  such that the diagram

$$\begin{array}{ccc} f(c) & \xrightarrow{\alpha} & d \\ f(\gamma) \downarrow & & \nearrow \beta \\ f(c') & & \end{array}$$

commutes in  $D$ .

Any morphism  $d \rightarrow d'$  of  $D$  induces a functor  $f/d \rightarrow f/d'$ , and there is a  $D$ -diagram in simplicial sets  $d \mapsto B(f/d)$ .

The forgetful functors  $f/d \rightarrow C$  (with  $(c, \alpha) \mapsto c$ ) define a map of bisimplicial sets

$$\omega : \bigsqcup_{d_0 \rightarrow \cdots \rightarrow d_n} B(f/d_0) \rightarrow BC.$$

Then we have the following categorical analogue of Lemma 23.1:

**Lemma 23.3** (Quillen [5]). *The map  $\omega$  induces a weak equivalence of diagonal simplicial sets.*

*Proof.* The homotopy colimit in the statement of the Lemma is the bisimplicial set with  $(n, m)$ -bisimplices consisting of pairs

$$(c_0 \rightarrow \cdots \rightarrow c_m, f(c_m) \rightarrow d_0 \rightarrow \cdots \rightarrow d_n)$$

of strings of arrows in  $C$  and  $D$ , respectively.

The fibre of  $\omega$  over the  $m$ -simplex  $c_0 \rightarrow \cdots \rightarrow c_m$  is the nerve  $B(f(c_m)/D)$ , which is contractible.

This is true for all elements of  $BC_m$  so  $\omega$  is a weak equivalence in each vertical degree  $m$ , and is therefore a diagonal weak equivalence.  $\square$

Now here's what we're really after:

**Theorem 23.4** (Quillen). *Suppose  $X : I \rightarrow s\mathbf{Set}$  is a diagram such that each map  $i \rightarrow j$  of  $I$  induces a weak equivalence  $X(i) \rightarrow X(j)$ .*

*Then all pullback diagrams*

$$\begin{array}{ccc} X(i) & \longrightarrow & \operatorname{holim}_I X \\ \downarrow & & \downarrow \pi \\ \Delta^0 & \xrightarrow{i} & BI \end{array}$$

*are homotopy cartesian.*

Functors  $X : I \rightarrow s\mathbf{Set}$  which take all morphisms of  $I$  to weak equivalences of simplicial sets are **diagrams of equivalences**.

If  $f : I \rightarrow J$  is a functor between small categories and  $X : J \rightarrow s\mathbf{Set}$  is a  $J$ -diagram of simplicial sets, then the diagram

$$\begin{array}{ccc} \operatorname{holim}_I Xf & \longrightarrow & \operatorname{holim}_J X \\ \pi \downarrow & & \downarrow \pi \\ BI & \xrightarrow{f_*} & BJ \end{array}$$

is a pullback (exercise).

In particular, the diagram in the statement of the Theorem is a pullback.

*Proof.* There are two tricks in this proof:

- Factor the map  $i : \Delta^0 \rightarrow BI$  as the composite

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{i} & BI \\ & \searrow j & \nearrow p \\ & U & \end{array}$$

such that  $p$  is a fibration and  $j$  is a trivial cofibration, and show that the induced map  $X(i) \rightarrow U \times_{BI} \underline{\operatorname{holim}}_I X$  is a weak equivalence.

- Use the fact that pullback along a simplicial set map is exact (so it preserves all colimits and monomorphisms), to reduce to showing that every composite  $\Lambda_k^n \subset \Delta^n \rightarrow BI$  induces a weak equivalence

$$\Lambda_k^n \times_{BI} \underline{\operatorname{holim}}_I X \rightarrow \Delta^n \times_{BI} \underline{\operatorname{holim}}_I X.$$

To finish off, the map  $\Delta^n \rightarrow BI$  is induced by a functor  $\sigma : \mathbf{n} \rightarrow I$ , so there is an isomorphism

$$\underline{\operatorname{holim}}_{\mathbf{n}} X\sigma \cong \Delta^n \times_{BI} \underline{\operatorname{holim}}_I X.$$

The composite functor  $X\sigma$  is a diagram of equivalences, and so the initial object  $0 \in \mathbf{n}$  determines a natural transformation

$$X\sigma(0) \rightarrow X\sigma$$

of  $\mathbf{n}$ -diagrams defined on a constant diagram which is a weak equivalence of diagrams.



The induced weak equivalence

$$B\mathbf{n} \times X(\sigma(0)) \cong \underline{\mathrm{holim}}_{\mathbf{n}} X(\sigma(0)) \rightarrow \underline{\mathrm{holim}}_{\mathbf{n}} X\sigma$$

pulls back to a weak equivalence

$$\Lambda_k^n \times X(\sigma(0)) \cong \Lambda_k^n \times_{B\mathbf{n}} \underline{\mathrm{holim}}_{\mathbf{n}} X(\sigma(0)) \rightarrow \Lambda_k^n \times_{B\mathbf{n}} \underline{\mathrm{holim}}_{\mathbf{n}} X\sigma.$$

It follows that there is a commutative diagram

$$\begin{array}{ccc} \Lambda_k^n \times X(\sigma(0)) & \xrightarrow{\simeq} & \Delta^n \times X(\sigma(0)) \\ \simeq \downarrow & & \downarrow \simeq \\ \Lambda_k^n \times_{BI} \underline{\mathrm{holim}}_I X & \longrightarrow & \Delta^n \times_{BI} \underline{\mathrm{holim}}_I X. \end{array}$$

so the bottom horizontal map is a weak equivalence.  $\square$

It's hard to overstate the importance of Theorem 23.4.

The conditions for the Theorem are always satisfied, for example, by diagrams defined on groupoids. In particular, if  $G$  is a group and  $X$  is a space carrying a  $G$ -action, then there is a fibre sequence

$$X \rightarrow EG \times_G X \rightarrow BG$$

defined by the Borel construction, aka. the homotopy colimit for the action of  $G$  on  $X$ .

Theorem 23.4 first appeared as a lemma in the proof of Quillen's "Theorem B" in [5].

Theorem B is the homotopy-theoretic starting point for Quillen's description of higher algebraic  $K$ -theory:

**Theorem 23.5** (Quillen). *Suppose  $f : C \rightarrow D$  is a functor between small categories such that all morphisms  $d \rightarrow d'$  of  $D$  induce weak equivalences  $B(f/d) \rightarrow B(f/d')$ .*

*Then all diagrams*

$$\begin{array}{ccc} B(f/d) & \longrightarrow & BC \\ \downarrow & & \downarrow f_* \\ B(D/d) & \longrightarrow & BD \end{array}$$

*of simplicial set maps are homotopy cartesian.*

*Proof.* Form the diagram

$$\begin{array}{ccccc} B(f/d) & \longrightarrow & \varinjlim_{d \in D} B(f/d) & \xrightarrow{\simeq} & BC \\ \downarrow & & \text{I} \downarrow & & \text{II} \downarrow \\ B(D/d) & \longrightarrow & \varinjlim_{d \in D} B(D/d) & \xrightarrow{\simeq} & BD \\ \simeq \downarrow & & \text{III} \downarrow & & \downarrow \simeq \\ \Delta^0 & \xrightarrow{d} & BD & & \end{array}$$

The indicated horizontal maps are weak equivalences by Lemma 23.3, while the indicated vertical maps are weak equivalences since the spaces  $B(D/d)$  are contractible.

Theorem 23.4 says that the composite diagram  $\mathbf{I} + \mathbf{III}$  is homotopy cartesian, so Lemma 18.5 (Lecture 07) implies that  $\mathbf{I}$  is homotopy cartesian. It follows, again from Lemma 18.5, that the composite  $\mathbf{I} + \mathbf{II}$  is homotopy cartesian.  $\square$

## References

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# Lecture 09: Bisimplicial abelian groups

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## 24 Derived functors

### Homology

Suppose  $A : I \rightarrow \mathbf{Ab}$  is a diagram of abelian groups, defined on a small category  $I$ .

There is a simplicial abelian group  $E_I A$ , with

$$E_I A_n = \bigoplus_{\sigma: i_0 \rightarrow \dots \rightarrow i_n} A(i_0)$$

and with simplicial structure maps  $\theta^*$  defined for  $\theta : \mathbf{m} \rightarrow \mathbf{n}$  by the commutative diagrams

$$\begin{array}{ccc} A(i_0) & \xrightarrow{\alpha_*} & A(i_{\theta(0)}) \\ \text{\scriptsize $in_\sigma$} \downarrow & & \downarrow \text{\scriptsize $in_{\theta^*(\sigma)}$} \\ \bigoplus_{\sigma: i_0 \rightarrow \dots \rightarrow i_n} A(i_0) & \xrightarrow{\theta^*} & \bigoplus_{\gamma: j_0 \rightarrow \dots \rightarrow j_m} A(j_0) \end{array}$$

where  $\alpha : i_0 \rightarrow i_{\theta(0)}$  is the morphism of  $I$  defined by  $\theta$ .

The simplicial abelian group  $E_I A$  defines the homotopy colimit within simplicial abelian groups.

Specifically, every diagram  $B : I \rightarrow s\mathbf{Ab}$  of simplicial abelian groups determines a bisimplicial abelian group  $E_I B$  with horizontal objects

$$E_I B_n = \bigoplus_{\sigma: i_0 \rightarrow \dots \rightarrow i_n} B(i_0).$$

There is a **projective model structure** on  $s\mathbf{Ab}^I$ , for which  $f : A \rightarrow B$  is a weak equivalence (respectively fibration) if and only if each map  $f : A_i \rightarrow B_i$  is a weak equivalence (respectively fibration) of simplicial abelian groups (exercise).

**Lemma 24.1.** *The canonical map*

$$E_I B \rightarrow \varinjlim_I B,$$

*induces a weak equivalence of simplicial abelian groups*

$$\pi : d(E_I B) \rightarrow \varinjlim_I B$$

*if  $B$  is projective cofibrant.*

*Proof.* The generating projective cofibrations are induced from the generating projective cofibrations

$$j \times 1 : K \times \text{hom}(i, \_) \rightarrow L \times \text{hom}(i, \_)$$

of  $I$ -diagrams of simplicial sets by applying the free abelian group functor.

There is an isomorphism

$$E_I \mathbb{Z}(X) \cong \mathbb{Z}(\varinjlim_I X)$$

for all  $I$ -diagrams of simplicial sets  $X$ .

The map

$$E_I \mathbb{Z}(\text{hom}(i, \_) \times K) \rightarrow \varinjlim_I \mathbb{Z}(\text{hom}(i, \_) \times K)$$

is the result of applying the free abelian group functor to a diagonal weak equivalence of bisimplicial sets.

Every pointwise weak equivalence of  $I$ -diagrams  $A \rightarrow B$  induces a diagonal weak equivalence

$$d(E_I A) \rightarrow d(E_I B).$$

This is a consequence of Lemma 24.2. □

**Lemma 24.2.** *Every level weak equivalence  $A \rightarrow B$  of bisimplicial abelian groups induces a weak equivalence  $d(A) \rightarrow d(B)$ .*

Lemma 24.2 follows from Lemma 21.1 (the bisimplicial sets result).

**Corollary 24.3.** *Suppose the level weak equivalence  $p : A \rightarrow B$  is a projective cofibrant replacement of an  $I$ -diagram  $B$ .*

*Then there are weak equivalences*

$$d(E_I B) \xleftarrow{\simeq} d(E_I A) \xrightarrow{\simeq} \varinjlim_I A.$$

**Example:** Suppose  $A : I \rightarrow \mathbf{Ab}$  is a diagram of abelian groups. Write  $\mathbf{Ab}^I$  for the category of such  $I$ -diagrams and natural transformations.

$\mathbf{Ab}^I$  has a set of projective generators, ie. all functors  $\mathbb{Z}(\text{hom}(i, \_))$  obtained by applying the free abelian group functor to the functors  $\text{hom}(i, \_)$ ,  $i \in \text{Ob}(I)$ .

It follows that every  $I$ -diagram  $A : I \rightarrow \mathbf{Ab}$  of abelian groups has a projective resolution

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0.$$

The  $I$ -diagram  $\Gamma(P_*)$  of simplicial abelian groups is projective cofibrant (exercise), so there are weak equivalences of simplicial abelian groups

$$E_I A = d(E_I A) \xleftarrow{\simeq} d(E_I \Gamma(P_*)) \xrightarrow{\simeq} \varinjlim_I \Gamma(P_*) \cong \Gamma(\varinjlim_I P_*).$$

Thus, there are isomorphisms

$$\pi_k(E_I A) \cong \pi_k(\Gamma(\varinjlim_I P_*)) \cong H_k(\varinjlim_I P_*).$$

We have proved the following:

**Lemma 24.4.** *There are natural isomorphisms*

$$\pi_k(E_I A) \cong L(\varinjlim_I)_k(A)$$

*for all  $I$ -diagrams of abelian groups  $A$ .*



In other words, the homotopy (or homology) groups of  $E_I A$  coincide with the left derived functors of the colimit functor in abelian groups.

**Remark:** Exactly the same script works for diagrams of simplicial modules over an arbitrary commutative unitary ring  $R$ .

**Example:** Suppose  $G$  is a group, and let  $R(G)$  be the corresponding group-algebra over  $R$ . An  $R(G)$ -module, or simply a  $G$ -module in  $R - \mathbf{Mod}$ , is a diagram

$$M : G \rightarrow R - \mathbf{Mod},$$

and the higher derived functors of  $\varinjlim_G$  for  $M$  are the group homology groups  $H_n(G, M)$ , as defined classically.

In effect, one can show that there is an isomorphism of simplicial  $R$ -modules

$$\begin{aligned} L(\varinjlim_G)_k(M) &= H_k(E_G M) \\ &\cong H_k(R(EG) \otimes_G M) = H_k(G, M). \end{aligned}$$

Here,  $EG = B(* / G)$  is the standard contractible cover of  $BG$  so  $R(EG) \rightarrow R$  is a free  $G$ -resolution of the trivial  $G$ -module  $R$ .

$R(EG) \otimes_G M$  is the **Borel construction** for the  $G$ -module  $M$ .

The  $R$ -module  $\varinjlim_G M$  is the module of **coinvariants** of the  $G$ -module  $M$ , and it is common to write

$$M/G = \varinjlim_G M.$$

**Example:** Suppose that  $A : \Delta^{op} \rightarrow s\mathbf{Ab}$  is a bisimplicial abelian group.

The colimit  $\varinjlim_{\mathbf{n}} A_n$  is the coequalizer

$$A_1 \rightrightarrows A_0 \rightarrow \pi_0 A = \varinjlim_n A_n$$

of the face maps  $d_0, d_1 : A_1 \rightarrow A_0$ .

The bisimplicial set  $\Delta^n \tilde{\times} K$  has (horizontal) colimit

$$\pi_0 \Delta^n \times K \cong K.$$

It follows that the map

$$\mathbf{Z}(\Delta^n \tilde{\times} K) \rightarrow \varinjlim_p \mathbf{Z}(\Delta_p^n \times K)$$

is a levelwise equivalence (in vertical degrees) of bisimplicial abelian groups. This implies that the bisimplicial abelian group map

$$A \rightarrow \pi_0 A = \varinjlim_n A_n$$

is a weak equivalence in all vertical degrees for all projective cofibrant objects  $A$ , and therefore in-

duces a diagonal weak equivalence

$$d(A) \xrightarrow{\simeq} \varinjlim_n A_n$$

for all such objects  $A$ .

It follows that if  $A \rightarrow B$  is a projective cofibrant resolution of a bisimplicial abelian group  $B$ , then there are weak equivalences

$$d(B) \xleftarrow{\simeq} d(A) \xrightarrow{\simeq} \varinjlim_n A_n,$$

and so the diagonal  $d(B)$  is naturally equivalent to the homotopy colimit of the simplicial object  $A$ .

## Cohomology

There is a cohomological version of the theory presented so far in this section. A little more technology is involved.

- 1) The category  $\mathbf{Ab}^I$  of  $I$ -diagrams of abelian groups has enough injectives.
- 2) If  $A$  is an  $I$ -diagram of abelian groups, then there is an isomorphism of cochain complexes

$$\mathrm{hom}(B(I/?), A) \cong \prod^* A.$$

- 3) The functor  $\mathrm{hom}(\_, J)$  is exact if  $J$  is injective (exercise), and thus takes weak equivalences  $X \rightarrow$

$Y$  of  $I$ -diagrams of simplicial sets to cohomology isomorphisms  $\text{hom}(Y, J) \rightarrow \text{hom}(X, J)$ .

The canonical map  $B(I/?) \rightarrow *$  is a weak equivalence of  $I$ -diagrams, so the morphism

$$\text{hom}(*, J) \rightarrow \text{hom}(B(I/?), J)$$

is a cohomology isomorphism if  $J$  is injective. Thus, there are isomorphisms

$$H^k \prod^* J \cong \begin{cases} \varprojlim_I J & \text{if } k = 0, \text{ and} \\ 0 & \text{if } k > 0. \end{cases}$$

4) More generally, there are isomorphisms

$$H^k \prod^* A \cong R(\varprojlim_I)^k A =: \varprojlim_I^k A$$

for  $k \geq 0$  and for all  $I$ -diagrams  $A$ .

In effect,  $A$  has an injective resolution  $A \rightarrow J^*$  and both (cohomological) spectral sequences for the bicomplex  $\prod^* J^*$  collapse.

5) If  $A$  is an  $I$ -diagram of abelian groups, then there is an isomorphism

$$[* , K(A, n)] \cong \varprojlim_I^n (A), \quad (1)$$

where  $[ , ]$  denotes morphisms in the homotopy category of  $I$ -diagrams of simplicial sets.

The best argument that I know of for the isomorphism (1) appears in [4] (also [5]).

The theory of higher right derived functors of inverse limit is a type of sheaf cohomology theory.

6) There are isomorphisms

$$\begin{aligned}
\pi_0 \varprojlim_I K(A, n) &\cong \pi_0 \varprojlim_I Z \\
&\cong \pi_0 \varprojlim_I Z \\
&\cong [* , Z] \\
&\cong [* , K(A, n)] \\
&\cong \varprojlim_I^n A,
\end{aligned}$$

where  $K(A, n) \rightarrow Z$  is an injective fibrant model of  $K(A, n)$ .

The object  $K(A, n)$  is a de-looping of  $K(A, n - 1)$ , so there are isomorphisms

$$\pi_k \varprojlim_I K(A, n) \cong \begin{cases} \varprojlim_I^{n-k} A & \text{if } 0 \leq k \leq n, \text{ and} \\ 0 & \text{if } k > n \end{cases}$$

## 25 Spectral sequences for a bicomplex

This section contains a very basic introduction to spectral sequences.

We shall only explicitly discuss the spectral sequences in homology which are associated to a bicomplex.

These spectral sequences, their cohomological analogs (used at the end of Section 24), and the Bousfield-Kan spectral sequence for a tower of fibrations [1], are the most common prototypes for spectral sequences that one meets in nature.

Most of the material of this section in Mac Lane's "Homology" [6]. There are many other sources.

A **bicomplex**  $C$  consists of an array of abelian groups  $C_{p,q}$ ,  $p, q \geq 0$  and morphisms

$$\partial_v : C_{p,q} \rightarrow C_{p,q-1} \quad \text{and} \quad \partial_h : C_{p,q} \rightarrow C_{p-1,q},$$

such that

$$\begin{aligned} \partial_v^2 = \partial_h^2 = 0 \quad \text{and} \\ \partial_v \partial_h + \partial_h \partial_v = 0. \end{aligned}$$

A **morphism**  $f : C \rightarrow D$  of bicomplexes consists of morphisms  $f : C_{p,q} \rightarrow D_{p,q}$  that respect the differentials.

Write  $Ch_+^2$  for the corresponding category.

There is a functor

$$\text{Tot} : Ch_+^2 \rightarrow Ch_+$$

taking values in ordinary chain complexes with

$$\text{Tot}(C)_n = \bigoplus_{p+q=n} C_{p,q}$$

and with differential  $\partial : \text{Tot}(C)_n \rightarrow \text{Tot}(C)_{n-1}$  defined on the summand  $C_{p,q}$  by

$$\partial(x) = \partial_v(x) + \partial_h(x).$$

Every bicomplex  $C$  has two filtrations, horizontal and vertical.

The  $p^{\text{th}}$  stage  $F_p C$  of the **horizontal filtration** has

$$F_p C_{r,s} = \begin{cases} C_{r,s} & \text{if } r \leq p, \text{ and} \\ 0 & \text{if } r > p. \end{cases}$$

Then

$$0 = F_{-1}C \subset F_0C \subset F_1C \subset \dots$$

and

$$\bigcup_p F_p(C) = C$$

The functor  $C \mapsto \text{Tot}(C)$  is exact, so this filtration on  $C$  induces a filtration on  $\text{Tot}(C)$ .

One filters  $\text{Tot}(C)_n$  in finitely many stages:

$$0 = F_1 \text{Tot}(C)_n \subset F_0 \text{Tot}(C)_n \subset \dots \subset F_n \text{Tot}(C)_n = \text{Tot}(C)_n.$$

Generally, the long exact sequences in homology associated to the exact sequences

$$0 \rightarrow F_{p-1}C \xrightarrow{i} F_pC \xrightarrow{p} F_pC/F_{p-1}C \rightarrow 0$$

arising from a filtration  $\{F_p C\}$  on a chain complex  $C$  fit together to define a **spectral sequence** for the filtered complex.

This spectral sequence arises from the “ladder diagram”

$$\begin{array}{ccccc}
 \vdots & & \vdots & & \\
 \downarrow & & \downarrow & & \\
 H_{p+q}(F_{p-2}) & & H_{p+q-1}(F_{p-3}) & & \\
 \downarrow i_* & & \downarrow i_* & & \\
 H_{p+q}(F_{p-1}) & & H_{p+q-1}(F_{p-2}) \longrightarrow H_{p+q-1}(F_{p-2}/F_{p-3}) & & \\
 \downarrow i_* & & \downarrow i_* & & \\
 H_{p+q}(F_p) \xrightarrow{p_*} H_{p+q}(F_p/F_{p-1}) \xrightarrow{\partial} H_{p+q-1}(F_{p-1}) & & & & \\
 \downarrow i_* & & \downarrow i_* & & \\
 H_{p+q}(F_{p+1}) & & H_{p+q-1}(F_p) \xrightarrow{p_*} H_{p+q-1}(F_p/F_{p-1}) & & \\
 \downarrow i_* & & \downarrow i_* & & \\
 \vdots & & \vdots & & \\
 \downarrow & & \downarrow & & \\
 H_{p+q}(C) & & H_{p+q-1}(C) & & 
 \end{array}$$

Set

$$Z_r^{p,q} = \{x \in H_{p+q}(F_p/F_{p-1}) \mid \partial(x) \in \text{im}(i_*^{r-1})\}$$

and

$$B_r^{p,q} = p_*(\ker(i_*^{r-1})),$$



and then define

$$E_r^{p,q} = \frac{Z_r^{p,q}}{B_r^{p,q}}.$$

for  $r \geq 1$ . Here, we adopt the convention that  $i_*^0 = 1$ , so that

$$E_1^{p,q} = H_{p+q}(F_p/F_{p-1})$$

This is cheating slightly (this only works for bi-complexes), but set

$$E_\infty^{p,q} = \frac{\ker(\partial)}{p_*(\ker(H_{p+q}(F_p) \rightarrow H_{p+q}(C)))}.$$

Finally, define

$$F_p H_{p+q}(C) = \text{im}(H_{p+q}(F_p) \rightarrow H_{p+q}(C)).$$

Given  $[x] \in E_r^{p,q}$  represented by  $x \in Z_r^{p,q}$  choose  $y \in H_{p+q}(F_{p-r})$  such that  $i_*^{r-1}(y) = \partial(x)$ . Then the assignment  $[x] \mapsto [p_*(y)]$  defines a homomorphism

$$d_r : E_r^{p,q} \rightarrow E_r^{p-r, q+r-1},$$

and this homomorphism is natural in filtered complexes.

Then we have the following:

**Lemma 25.1.** 1) We have the relation  $d_r^2 = 0$ , and there is an isomorphism

$$E_{r+1}^{p,q} \cong \frac{\ker(d_r : E_r^{p,q} \rightarrow E_r^{p-r,q+r-1})}{\operatorname{im}(d_r : E_r^{p+r,q-r+1} \rightarrow E_r^{p,q})}.$$

2) There are isomorphisms

$$E_r^{p,q} \cong E_\infty^{p,q}$$

for  $r > p, q + 2$ .

3) There are short exact sequences

$$0 \rightarrow F_{p-1}H_{p+q}(C) \rightarrow F_pH_{p+q}(C) \rightarrow E_\infty^{p,q} \rightarrow 0.$$

The proof is an exercise — chase some elements.

In general (ie. for general filtered complexes),

$$E_1^{p,q} = H_{p+q}(F_p/F_{p-1}),$$

and  $E_2^{p,q}$  is the homology of the complex with differentials  $E_1^{p,q} \rightarrow E_1^{p-1,q}$  given by the composites

$$H_{p+q}(F_p/F_{p-1}) \xrightarrow{\partial} H_{p+q-1}(F_{p-1}) \xrightarrow{p_*} H_{p+q-1}(F_{p-1}/F_{p-2})$$

In the case of the horizontal filtration  $F_p \operatorname{Tot}(C)$  for a bicomplex  $C$ , there is a natural isomorphism

$$F_p \operatorname{Tot}(C)/F_{p-1} \operatorname{Tot}(C) \cong C_{p,*}[p],$$

so there is an isomorphism

$$E_1^{p,q} \cong H_q(C_{p,*})$$

The differential  $d_1$  is the homomorphism

$$H_q(C_{p,*}) \xrightarrow{\partial_{h*}} H_q(C_{p-1,*})$$

which is induced by the horizontal differential.

It follows, that for the horizontal filtration on the total complex  $\text{Tot}(C)$  of a bicomplex  $C$ , there is a spectral sequence with

$$E_2^{p,q} = H_p^h(H_q^v C) \Rightarrow H_{p+q}(\text{Tot}(C)).$$

In particular, the spectral sequence converges to  $H_*(\text{Tot}(C))$  in the sense that the filtration quotients  $E_\infty^{p,q}$  determine  $H_*(\text{Tot}(C))$ .

Here's an example of how it all works:

**Lemma 25.2.** *Suppose  $f : C \rightarrow D$  is a morphism of bicomplexes such that for some  $r \geq 1$  the induced morphisms  $E_r^{p,q}(C) \rightarrow E_r^{p,q}(D)$  are isomorphisms for all  $p, q \geq 0$ .*

*Then the induced map  $\text{Tot}(C) \rightarrow \text{Tot}(D)$  is a homology isomorphism.*

Lemma 25.2 is sometimes called the *Zeeman comparison theorem*.

*Proof.* The map  $f$  induces isomorphisms

$$E_s^{p,q}(C) \xrightarrow{\cong} E_s^{p,q}(D)$$

for all  $s \geq r$  (because all such  $E_s$ -terms are computed by taking homology groups, inductively in  $s \geq r + 1$ ). It follows that all induced maps

$$E_\infty^{p,q}(C) \rightarrow E_\infty^{p,q}(D)$$

are isomorphisms. But then, starting with the morphism

$$\begin{array}{ccc} E_\infty^{0,p+q}(C) & \xrightarrow{\cong} & E_\infty^{0,p+q}(D) \\ \cong \downarrow & & \downarrow \cong \\ F_0 H_{p+q}(\text{Tot}(C)) & \longrightarrow & F_0 H_{p+q}(\text{Tot}(D)) \end{array}$$

and using the fact that the induced maps on successive filtration quotients are the isomorphisms

$$E_\infty^{r,p+q-r}(C) \xrightarrow{\cong} E_\infty^{r,p+q-r}(D),$$

one shows inductively that all maps

$$F_r H_{p+q}(\text{Tot}(C)) \rightarrow F_r H_{p+q}(\text{Tot}(D))$$

are isomorphisms, including the case  $r = p + q$  which is the map

$$H_{p+q}(\text{Tot}(C)) \rightarrow H_{p+q}(\text{Tot}(D)).$$

This is true for all total degrees  $p + q$ , so the map  $\text{Tot}(C) \rightarrow \text{Tot}(D)$  is a quasi-isomorphism.  $\square$

**Example:** Suppose  $A$  is a bisimplicial abelian group. Then the Generalized Eilenberg-Zilber Theorem (Theorem 26.1) asserts that there is a natural chain homotopy equivalence of chain complexes

$$d(A) \simeq \text{Tot}(A)$$

where  $\text{Tot}(A)$  is the total complex of the associated (Moore) bicomplex. Filtering  $A$  in the horizontal direction therefore gives a spectral sequence with

$$E_2^{p,q} = \pi_p^h(\pi_q^v(A)) \Rightarrow \pi_{p+q}d(A). \quad (2)$$

This spectral sequence is natural in bisimplicial abelian groups  $A$ .

This spectral sequence can be used to give an alternate proof of Lemma 24.4. If  $A \rightarrow B$  is a level equivalence of bisimplicial abelian groups, then there is an  $E_1$ -level isomorphism

$$\pi_q(A_{p,*}) \xrightarrow{\cong} \pi_q(B_{p,*})$$

for all  $p, q \geq 0$ .

Use Lemma 25.2 for the spectral sequence (2) to show that the map  $d(A) \rightarrow d(B)$  is a weak equivalence.

**Application:** The Lyndon-Hochschild-Serre spectral sequence

Suppose  $f : C \rightarrow D$  is a functor between small categories, and recall the bisimplicial set map

$$\bigsqcup_{d_0 \rightarrow \cdots \rightarrow d_n} B(f/d_0) \rightarrow BC$$

of Section 23. Lemma 23.3 says that this map is a diagonal weak equivalence.

The free abelian group functor preserves diagonal weak equivalences, so there is a spectral sequence

$$E_2^{p,q} = L(\varinjlim)_p H_q(B(f/?), \mathbb{Z}) \Rightarrow H_{p+q}(BC, \mathbb{Z}). \quad (3)$$

The derived colimit functors are computed over the base category  $D$ .

In the special case where  $f$  is a surjective group homomorphism  $G \rightarrow H$  with kernel  $K$ , this is a form of the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{p,q} = H_p(H, H_q(BK, \mathbb{Z})) \Rightarrow H_{p+q}(BG, \mathbb{Z}). \quad (4)$$

We can put in other coefficients if we want.

To see that the  $E_2$ -term of (4) has the indicated form, take a set-theoretic section  $\sigma : H \rightarrow G$  of the group homomorphism  $f$  such that  $\sigma(e) = e$ .

Conjugation  $x \mapsto \sigma(h)x\sigma(h)^{-1}$ , defines a group isomorphism  $c_{\sigma(h)} : K \rightarrow K$  and hence an isomorphism

$$h_* : H_*(BK, \mathbb{Z}) \rightarrow H_*(BK, \mathbb{Z})$$

in homology. The map  $h_*$  is independent of the choice of section  $\sigma$  because any two pre-images of  $h$  determine homotopic maps  $K \rightarrow K$  (ie. the two isomorphisms differ by conjugation by an element of  $K$ ).

This action of  $H$  on  $H_*(BK, \mathbb{Z})$  is the one appearing in the description of the  $E_2$ -term of (4).

The objects of  $f/*$  are the elements of  $H$ , and a morphism  $g : h \rightarrow h'$  in  $f/*$  is an element  $g \in G$  such that  $h'f(g) = h$ .

There is a functor  $K \rightarrow f/*$  defined by sending  $k \in K$  to the morphism  $k : e \rightarrow e$ .

There is a functor  $f/* \rightarrow K$  which is defined by sending the morphism  $g : h \rightarrow h'$  to the element  $\sigma(h')g\sigma(h)^{-1}$ .

$\sigma(e) = e$ , so the composite functor

$$K \rightarrow f/* \rightarrow K$$

is the identity, while the elements  $\sigma(h)$ ,  $h \in H$  de-

find a homotopy from the composite

$$f/* \rightarrow K \rightarrow f/*$$

to the identity on  $f/*$ .

Finally, composition with  $\alpha \in H$  defines the functor  $\alpha_* : f/* \rightarrow f/*$  in the description of the bisimplicial set for  $f : G \rightarrow H$ , and there is a homotopy commutative diagram

$$\begin{array}{ccc} K & \longrightarrow & f/* \\ c_{\sigma(\alpha)} \downarrow & & \downarrow \alpha_* \\ K & \longrightarrow & f/* \end{array}$$

Thus, the action of  $\alpha$  on  $H_*(B(f/*), \mathbb{Z})$  coincides with the morphism  $\alpha_* : H_*(BK, \mathbb{Z}) \rightarrow H_*(BK, \mathbb{Z})$  displayed above, up to isomorphism.

**Example:** Consider the short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0.$$

All three groups are abelian, so all conjugation actions are trivial and there is a spectral sequence with

$$E_2^{p,q} = H_p(B(\mathbb{Q}/\mathbb{Z}), H_q(B\mathbb{Z}, \mathbb{Q})) \Rightarrow H_{p+q}(B\mathbb{Q}, \mathbb{Q}).$$

$S^1 \simeq B\mathbb{Z}$ , so there are isomorphisms

$$H_q(B\mathbb{Z}, \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } q = 0, 1, \text{ and} \\ 0 & \text{if } q > 1. \end{cases}$$



The group  $\mathbb{Q}/\mathbb{Z}$  is all torsion, so that

$$H_q(B(\mathbb{Q}/\mathbb{Z}), \mathbb{Q}) = 0$$

for  $q \geq 1$ .

The  $E_2$ -term for the spectral sequence therefore collapses, so the “edge homomorphism”

$$H_*(B\mathbb{Z}, \mathbb{Q}) \rightarrow H_*(B\mathbb{Q}, \mathbb{Q})$$

is an isomorphism.

To see the claim about torsion groups, observe that torsion abelian groups are filtered colimits of finitely generated torsion abelian groups, and a finitely generated torsion abelian group is a finite direct sum of cyclic groups.

It therefore suffices, by a Künneth formula argument (see (11) below) to show that

$$H_p(\mathbb{Z}/n, \mathbb{Q}) = H_p(B(\mathbb{Z}/n), \mathbb{Q}) = 0$$

for  $p > 0$ .

The abelian group  $\mathbb{Z}$ , as a trivial  $\mathbb{Z}/n$ -module, has a free resolution by  $\mathbb{Z}/n$ -modules

$$\dots \xrightarrow{N} \mathbb{Z}(\mathbb{Z}/n) \xrightarrow{1-t} \mathbb{Z}(\mathbb{Z}/n) \rightarrow \mathbb{Z} \rightarrow 0 \quad (5)$$

where  $1 - t$  is multiplication by group-ring element  $1 - t$  and  $t$  is the generator of the group  $\mathbb{Z}/n$ .

The map  $N$  is multiplication by the “norm element”

$$N = 1 + t + t^2 + \cdots + t^{n-1}.$$

Tensoring the resolution with the trivial  $\mathbb{Z}/n$ -module  $\mathbb{Z}$  gives the chain complex

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z},$$

and it follows that

$$H_p(B\mathbb{Z}/n, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & p = 0, \\ 0 & \text{if } p = 2n, n > 0, \text{ and} \\ \mathbb{Z}/n & \text{if } p = 2n + 1, n \geq 0. \end{cases} \quad (6)$$

Tensoring with  $\mathbb{Q}$  (which is exact) therefore shows that  $H_p(B\mathbb{Z}/n, \mathbb{Q}) = 0$  for  $p > 0$ .

We could equally well tensor the resolution (5) with the trivial  $\mathbb{Z}/n$ -module  $\mathbb{Q}$  and get the same answer, because  $\mathbb{Q}$  is uniquely  $n$ -divisible.

## 26 The Eilenberg-Zilber Theorem

Every bisimplicial abelian group  $A$  has a naturally associated bicomplex  $M(A)$  with

$$M(A)_{m,n} = A_{m,n},$$

and with horizontal boundaries

$$\partial_h = \sum_{i=0}^m (-1)^i d_i : A_{m,n} \rightarrow A_{m-1,n}$$

and vertical boundaries

$$\partial_v = \sum_{i=0}^n (-1)^{m+i} d_i : A_{m,n} \rightarrow A_{m,n-1}.$$

One checks that

$$\partial_h \partial_v + \partial_v \partial_h = 0$$

in all bidegrees — the signs were put in to achieve this formula.

Here is the Generalized Eilenberg-Zilber Theorem of Dold-Puppe [2], [3, IV.2.2]:

**Theorem 26.1** (Dold-Puppe). *Suppose that  $A$  is a bisimplicial abelian group.*

*Then the chain complexes  $d(A)$  and  $\text{Tot}(A)$  are naturally chain homotopy equivalent.*

*Proof.* The standard Eilenberg-Zilber Theorem says that there are natural chain maps

$$f : \mathbb{Z}(K \times L) \rightarrow \text{Tot}(\mathbb{Z}(K) \otimes \mathbb{Z}(L))$$

(Moore complexes) and

$$g : \text{Tot}(\mathbb{Z}(K) \otimes \mathbb{Z}(L)) \rightarrow \mathbb{Z}(K \times L),$$

and there are natural chain homotopies  $fg \simeq 1$  and  $gf \simeq 1$  for simplicial sets  $K$  and  $L$ .

The Eilenberg-Zilber Theorem specializes to (is equivalent to — exercise) the existence of chain maps

$$d(\mathbb{Z}(\Delta^{p,q})) = \mathbb{Z}(\Delta^p \times \Delta^q) \xrightarrow{f} \text{Tot}(\mathbb{Z}(\Delta^p) \otimes \mathbb{Z}(\Delta^q)) = \text{Tot}(\mathbb{Z}(\Delta^{p,q}))$$

and

$$g : \text{Tot}(\mathbb{Z}(\Delta^{p,q})) \rightarrow d(\mathbb{Z}(\Delta^{p,q}))$$

and chain homotopies  $fg \simeq 1$  and  $gf \simeq 1$  which are natural in bisimplices  $\Delta^{p,q}$ .

Every bisimplicial abelian group  $A$  is a natural colimit of the diagrams

$$\begin{array}{ccc} A_{p,q} \otimes \mathbb{Z}(\Delta^{r,s}) & \xrightarrow{1 \otimes (\gamma, \theta)} & A_{p,q} \otimes \mathbb{Z}(\Delta^{p,q}) \\ (\gamma, \theta)^* \otimes 1 \downarrow & & \\ A_{r,s} \otimes \mathbb{Z}(\Delta^{r,s}) & & \end{array}$$

where  $(\gamma, \theta) : (\mathbf{r}, \mathbf{s}) \rightarrow (\mathbf{p}, \mathbf{q})$  varies over the morphisms of  $\Delta \times \Delta$ , and the maps

$$\gamma_{p,q} : A_{p,q} \otimes \mathbb{Z}(\Delta^{p,q}) \rightarrow A$$

given by  $(a, (\gamma, \theta)) \mapsto (\gamma, \theta)^*(a)$  define the colimit.

There are isomorphisms

$$\begin{aligned} d(B \otimes A) &\cong B \otimes d(A), \\ \text{Tot}(B \otimes A) &\cong B \otimes \text{Tot}(A) \end{aligned}$$

for bisimplicial abelian groups  $A$  and abelian groups  $B$ , and these isomorphisms are natural in both  $A$  and  $B$ . The functors  $\text{Tot}$  and  $d$  are also right exact.

It follows that  $f$  and  $g$  induce natural chain maps

$$f_* : d(A) \rightarrow \text{Tot}(A), \quad g_* : \text{Tot}(A) \rightarrow d(A)$$

for all simplicial abelian groups  $A$ .

The chain homotopies  $fg \simeq 1$  and  $gf \simeq 1$  induce natural chain homotopies

$$\begin{aligned} f_* g_* &\simeq 1 : \text{Tot}(A) \rightarrow \text{Tot}(A), \\ g_* f_* &\simeq 1 : d(A) \rightarrow d(A) \end{aligned}$$

for all bisimplicial abelian groups  $A$ . □

**Remarks:** 1) The proof of Theorem 26.1 which appears in [3, p.205] contains an error: the sequence

$$\bigoplus_{\tau \rightarrow \sigma} \mathbb{Z}(\Delta^{r,s}) \rightarrow \bigoplus_{\Delta^{p,q} \rightarrow A} \mathbb{Z}(\Delta^{p,q}) \rightarrow A \rightarrow 0$$

is not exact, which means that  $A$  is not a colimit of its bisimplices in general. The problem is fixed by using the co-end description of  $A$  that you see above.

2) The maps  $f$  and  $g$  have classical explicit models, namely the Alexander-Whitney map and shuffle map, respectively. See [6, VIII.8] for a full discussion.

Recall that  $M(A)$  denotes the Moore chain complex of a simplicial abelian group  $A$ .

The Alexander-Whitney map

$$f : M(A \otimes B) \rightarrow \text{Tot}(M(A) \otimes M(B)) \quad (7)$$

is defined, for simplicial abelian groups  $A$  and  $B$  by

$$f(a \otimes b) = \sum_{0 \leq p \leq n} a|_{[0, \dots, p]} \otimes b|_{[p, \dots, n]}.$$

Here,  $a \in A_n$  and  $b \in B_n$  are  $n$ -simplices. The

“front  $p$ -face”  $a|_{[0,\dots,p]}$  is defined by

$$\Delta^p \xrightarrow{[0,\dots,p]} \Delta^n \xrightarrow{a} A.$$

The “back  $(n - p)$ -face”  $b|_{[p,\dots,n]}$  is defined by

$$\Delta^{n-p} \xrightarrow{[p,\dots,n]} \Delta^n \xrightarrow{b} B.$$

The Eilenberg-Zilber Theorem follows from

**Lemma 26.2.** *1) The object*

$$(\mathbf{p}, \mathbf{q}) \mapsto \mathbb{Z}(\Delta^p \times \Delta^q)$$

*is a projective cofibrant  $(\Delta \times \Delta)$ -diagram of simplicial abelian groups.*

*2) The object*

$$(\mathbf{p}, \mathbf{q}) \mapsto \text{Tot}(N\mathbb{Z}(\Delta^p) \otimes N\mathbb{Z}(\Delta^q))$$

*is a projective cofibrant  $(\Delta \times \Delta)$ -diagram of chain complexes.*

To see that Lemma 26.2 implies the Eilenberg-Zilber Theorem, observe that there is a natural chain homotopy equivalence

$$\text{Tot}(N\mathbb{Z}(\Delta^p) \otimes N\mathbb{Z}(\Delta^q)) \simeq \text{Tot}(M\mathbb{Z}(\Delta^p) \otimes M\mathbb{Z}(\Delta^q))$$

of bicosimplicial chain complexes which is induced by the natural chain homotopy equivalence of Theorem 15.4 (Lecture 06) between normalized and Moore chain complexes.

There is a similar natural chain homotopy equivalence

$$M\mathbb{Z}(\Delta^p \times \Delta^q) \simeq N\mathbb{Z}(\Delta^p \times \Delta^q).$$

Finally, there is a natural chain homotopy equivalence

$$N\mathbb{Z}(\Delta^p \times \Delta^q) \simeq \text{Tot}(N\mathbb{Z}(\Delta^p) \otimes N\mathbb{Z}(\Delta^q)),$$

since both objects are projective cofibrant resolutions of the constant diagram of chain complexes  $\mathbb{Z}(0)$  on  $\Delta \times \Delta$  by Lemma 26.2.

We use the following result to prove Lemma 26.2:

**Lemma 26.3.** *Suppose  $p : A \rightarrow B$  is a trivial projective fibration of cosimplicial simplicial abelian groups. Then all induced maps*

$$(p, s) : A^{n+1} \rightarrow B^{n+1} \times_{M^n B} M^n A$$

*are trivial fibrations of simplicial abelian groups.*

*Proof.* The map  $s : B^{n+1} \rightarrow M^n B$  is surjective for all cosimplicial abelian groups  $B$ . In effect, if  $x =$



$(0, \dots, 0, x_i, \dots, x_n) \in M^n B$  then

$$s(d^{i+1}x_i) = (0, \dots, 0, x_i, y_{i+1}, \dots, y_n),$$

and  $x - s(d^{i+1}x_i)$  is of the form

$$x - s(d^{i+1}x_i) = (0, \dots, 0, z_{i+1}, \dots, z_n) =: z.$$

Thus, inductively, if  $z = s(v)$  for some  $v \in A^{n+1}$  then  $x = s(d^{i+1}x_i + v)$ .

Write

$$M_{(0,i)}^n A = \{(x_0, \dots, x_i) \mid x_i \in A^n, s^i x_j = s^{j-1} x_i \text{ for } i < j\}.$$

Then  $M^n A = M_{(0,n)}^n A$ , and there are pullback diagrams

$$\begin{array}{ccc} M_{(0,i+1)}^n & \longrightarrow & A^n \\ \downarrow & & \downarrow s \\ M_{(0,i)}^n & \xrightarrow{s^i} & M_{(0,i)}^{n-1} A \end{array}$$

in which the two unnamed arrows are projections.

Suppose  $K$  is a cosimplicial object in  $s\mathbf{Ab}$  such that all objects  $K^n$  are acyclic.

Then under the inductive assumption that  $s : K^n \rightarrow M_{(0,i)}^{n-1} K$  is a trivial fibration we see that the projection  $M_{(0,i+1)}^n K \rightarrow K^n$  is a trivial fibration, and so the map  $s : K^{n+1} \rightarrow M_{(0,i+1)}^n K$  is a weak equivalence.

This is true for all  $i < n$ , and it follows that the map  $s : K^{n+1} \rightarrow M^n K$  is a trivial fibration.

If  $K$  is the kernel of the projective trivial fibration  $p : A \rightarrow B$ , then there is an induced comparison of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & K^{n+1} & \longrightarrow & A^{n+1} & \longrightarrow & B^{n+1} \longrightarrow 0 \\ & & \downarrow s & & \downarrow (p,s) & & \downarrow 1 \\ 0 & \longrightarrow & M^n K & \longrightarrow & B^{n+1} \times_{M^n B} M^n A & \longrightarrow & B^{n+1} \longrightarrow 0 \end{array}$$

so the map  $(p, s)$  is a weak equivalence.  $\square$

**Corollary 26.4.** *The cosimplicial simplicial abelian group  $\mathbf{n} \mapsto \mathbb{Z}(\Delta^n)$  is projective cofibrant.*

*Proof.* Suppose  $p : A \rightarrow B$  is a projective trivial fibration. Solving a lifting problem

$$\begin{array}{ccc} & & A \\ & \nearrow & \downarrow p \\ \mathbb{Z}(\Delta) & \longrightarrow & B \end{array}$$

amounts to inductively solving lifting problems

$$\begin{array}{ccc} \partial \Delta^{n+1} & \longrightarrow & A^{n+1} \\ \downarrow & \nearrow & \downarrow (p,s) \\ \Delta^{n+1} & \longrightarrow & B^{n+1} \times_{M^n B} M^n A \end{array}$$

and this can be done by the previous Lemma.  $\square$

*Proof of Lemma 26.2.* Suppose  $q : C \rightarrow D$  is a projective trivial fibration of  $(\Delta \times \Delta)$ -diagrams of simplicial abelian groups. Then all maps

$$(q, s) : C^{n+1} \rightarrow D^{n+1} \times_{M^n D} M^n C$$

are projective trivial fibrations of cosimplicial simplicial abelian groups, by Lemma 26.3.

Write  $\Delta \times \Delta$  for the bicosimplicial diagram

$$(\mathbf{p}, \mathbf{q}) \mapsto \Delta^p \times \Delta^q$$

of simplicial sets.

Then lifting problems

$$\begin{array}{ccc} & & C \\ & \nearrow & \downarrow q \\ \Delta \times \Delta & \longrightarrow & D \end{array}$$

can be solved by inductively solving the lifting problems

$$\begin{array}{ccc} \mathbb{Z}(\Delta \times \partial \Delta^{n+1}) & \longrightarrow & C^{n+1} \\ \downarrow & \nearrow & \downarrow (q, s) \\ \mathbb{Z}(\Delta \times \Delta^{n+1}) & \longrightarrow & D^{n+1} \times_{M^n D} M^n C \end{array}$$

in cosimplicial simplicial abelian groups.

For that, it suffices to show that the map

$$\mathbb{Z}(\Delta \times \partial \Delta^{n+1}) \rightarrow \mathbb{Z}(\Delta \times \Delta^{n+1})$$

is a projective cofibration, but this follows from the observation that the maps

$$\mathbb{Z}(\partial\Delta^m \times \Delta^{n+1}) \cup \mathbb{Z}(\Delta^m \times \partial\Delta^{m+1}) \rightarrow \mathbb{Z}(\Delta^m \times \Delta^{n+1})$$

are cofibrations of simplicial abelian groups xfor  $m \geq 0$ , with Lemma 26.3.

We have proved statement 1) of Lemma 26.2.

The second statement of Lemma 26.2 has a very similar proof. If  $q : C \rightarrow D$  is a projective trivial fibration of bicosimplicial chain complexes, then all maps

$$(p, s) : C^{n+1} \rightarrow D^{n+1} \times_{M^n D} M^n C$$

are projective trivial fibrations of cosimplicial chain complexes, by Lemma 26.3. Write

$$\text{Tot}(N\mathbb{Z}(\Delta) \otimes N\mathbb{Z}(\Delta))$$

for the bicosimplicial chain complex

$$(\mathbf{p}, \mathbf{q}) \mapsto \text{Tot}(N\mathbb{Z}(\Delta^p) \otimes N\mathbb{Z}(\Delta^q)).$$

Then solving lifting problems

$$\begin{array}{ccc} & & C \\ & \nearrow & \downarrow q \\ \text{Tot}(N\mathbb{Z}(\Delta) \otimes N\mathbb{Z}(\Delta)) & \longrightarrow & D \end{array}$$

amounts to inductively solving lifting problems

$$\begin{array}{ccc}
\mathrm{Tot}(N\mathbb{Z}(\Delta) \otimes N\mathbb{Z}(\partial\Delta^{n+1})) & \longrightarrow & C^{n+1} \\
\downarrow i & \nearrow & \downarrow (q,s) \\
\mathrm{Tot}(N\mathbb{Z}(\Delta) \otimes N\mathbb{Z}(\Delta^{n+1})) & \longrightarrow & D^{n+1} \times_{M^n D} M^n C
\end{array}$$

For this, we show that all maps  $i$  are projective cofibrations of cosimplicial chain complexes, but this reduces to showing that each of the maps

$$\begin{aligned}
& \mathrm{Tot}(N\mathbb{Z}(\Delta^m) \otimes N\mathbb{Z}(\partial\Delta^{n+1})) \cup \mathrm{Tot}(N\mathbb{Z}(\partial\Delta^m) \otimes N\mathbb{Z}(\Delta^{n+1})) \\
& \rightarrow \mathrm{Tot}(N\mathbb{Z}(\Delta^m) \otimes N\mathbb{Z}(\Delta^{n+1}))
\end{aligned}$$

are cofibrations of chain complexes.

This last morphism is defined by freely adjoining the chain  $\iota_m \otimes \iota_n$ , so it is a cofibration.  $\square$

**Remark:** The proof of the Eilenberg-Zilber Theorem that one finds in old textbooks uses the method of acyclic models.

## 27 Universal coefficients, Künneth formula

Suppose  $X$  is a simplicial set, and that  $A$  is an abelian group.

Recall that the  $n^{th}$  homology group  $H_n(X, A)$  of  $X$  with coefficients in  $A$  is defined by

$$H_n(X, A) = H_n(\mathbb{Z}(X) \otimes_{\mathbb{Z}} A),$$

where  $\mathbb{Z}(X)$  denotes both a free simplicial abelian group and its associated Moore complex.

The ring  $\mathbb{Z}$  is a principal ideal domain, so  $A$  (a  $\mathbb{Z}$ -module) has a free resolution

$$0 \rightarrow F_2 \xrightarrow{i} F_1 \xrightarrow{p} A \rightarrow 0.$$

All abelian groups  $\mathbb{Z}(X_n)$  are free, and tensoring with a free abelian group is exact, so there is a short exact sequence of chain complexes

$$0 \rightarrow \mathbb{Z}(X) \otimes F_2 \xrightarrow{1 \otimes i} \mathbb{Z}(X) \otimes F_1 \xrightarrow{1 \otimes p} \mathbb{Z}(X) \otimes A \rightarrow 0.$$

The long exact sequence in  $H_*$  has the form

$$\dots \xrightarrow{\partial} H_n(X, F_2) \xrightarrow{(1 \otimes i)_*} H_n(X, F_1) \xrightarrow{(1 \otimes p)_*} H_n(X, A) \xrightarrow{\partial} \dots$$

There are commutative diagrams

$$\begin{array}{ccc} H_n(X, F_2) & \xrightarrow{(1 \otimes i)_*} & H_n(X, F_1) \\ \cong \downarrow & & \downarrow \cong \\ H_n(X, \mathbb{Z}) \otimes F_2 & \xrightarrow{1 \otimes i} & H_n(X, \mathbb{Z}) \otimes F_1 \end{array}$$

It follows that there are short exact sequences

$$0 \rightarrow H_n(X, \mathbb{Z}) \otimes A \rightarrow H_n(X, A) \rightarrow \text{Tor}(H_{n-1}(X, \mathbb{Z}), A) \rightarrow 0. \quad (8)$$

These are the **universal coefficients** exact sequences.

Both  $Z_n$  ( $n$ -cycles) and  $B_n$  ( $n$ -boundaries) are free abelian groups, and so there is a map  $\phi_n : B_n \rightarrow \mathbb{Z}(X)_{n+1}$  such that the diagram of abelian group homomorphisms

$$\begin{array}{ccc} B_n & \xrightarrow{\phi_n} & \mathbb{Z}(X)_{n+1} \\ j \downarrow & & \downarrow \\ Z_n & \xrightarrow{i} & \mathbb{Z}(X)_n \end{array} \quad (9)$$

commutes, where  $i$  and  $j$  are canonical inclusions.

Write  $\tilde{Z}_n$  for the chain complex which is concentrated in degrees  $n$  and  $n + 1$  and with boundary morphism given by the inclusion  $j$ .

The diagram (9) defines a chain map

$$\phi_n : \tilde{Z}_n \rightarrow \mathbb{Z}(X),$$

which induces an isomorphism

$$H_n(\tilde{Z}_n) \cong H_n(\mathbb{Z}(X)),$$

while  $H_k(\tilde{Z}_n) = 0$  for  $k \neq n$ .

Adding up the maps  $\phi_n$  therefore determines a (non-natural) weak equivalence

$$\phi : \bigoplus_{n \geq 0} \tilde{Z}_n \rightarrow \mathbb{Z}(X).$$

The two complexes are cofibrant, so  $\phi$  is a chain homotopy equivalence and in particular there is a chain homotopy inverse

$$\psi : \mathbb{Z}(X) \rightarrow \bigoplus_{n \geq 0} \tilde{Z}_n.$$

The map  $\psi$  and projection onto the complex  $\tilde{Z}_n$  therefore determine a chain map

$$\mathbb{Z}(X) \otimes A \rightarrow \tilde{Z}_n \otimes A$$

Comparing universal coefficients sequences gives a commutative diagram

$$\begin{array}{ccc} H_n(X) \otimes A & \longrightarrow & H_n(X, A) \\ \cong \downarrow & & \downarrow \\ H_n(X) \otimes A & \xrightarrow{\cong} & H_n(\tilde{Z}_n \otimes A) \end{array}$$

It follows that the natural map

$$H_n(X) \otimes A \rightarrow H_n(X, A)$$

from the universal coefficients sequence (8) is non-naturally split.



We have proved

**Theorem 27.1** (Universal Coefficients Theorem).  
*Suppose  $X$  is a simplicial set and  $A$  is an abelian group.*

*There is a short exact sequence*

$$\begin{aligned} 0 \rightarrow H_n(X, \mathbb{Z}) \otimes A \rightarrow H_n(X, A) \\ \rightarrow \operatorname{Tor}(H_{n-1}(X, \mathbb{Z}), A) \rightarrow 0. \end{aligned}$$

*for each  $n \geq 1$ . This sequence is natural in  $X$ , and has a non-natural splitting.*

Here's a different take on universal coefficients:

The chain complex  $\tilde{Z}_n \otimes A$  has homology

$$H_k(\tilde{Z}_n \otimes A) \cong \begin{cases} \operatorname{Tor}(H_n(X), A) & \text{if } k = n + 1, \\ H_n(X) \otimes A & \text{if } k = n, \\ 0 & \text{if } k \neq n, n + 1, \end{cases}$$

and the chain homotopy equivalence  $\phi$  induces isomorphisms

$$\begin{aligned} H_n(X, A) &\cong H_n\left(\bigoplus_{n \geq 0} \tilde{Z}_n \otimes A\right) \\ &\cong (H_n(X) \otimes A) \oplus \operatorname{Tor}(H_{n-1}(X), A). \end{aligned}$$

**Remark:** The simplicial set underlying a simplicial abelian group has the homotopy type (non-naturally) of a product of Eilenberg-Mac Lane spaces — see [3, III.2.20].

Suppose  $C$  is a chain complex.

Then the chain homotopy equivalence  $\phi$  induces a homology isomorphism

$$\mathrm{Tot}((\bigoplus_{n \geq 0} \tilde{Z}_n) \otimes C) \xrightarrow{\cong} \mathrm{Tot}(\mathbb{Z}(X) \otimes C).$$

We can assume that  $C$  is cofibrant, even free in each degree.

Form cofibrant chain complexes  $F_k C$  and maps  $F_k C \rightarrow C$  such that the maps  $H_k F_k C \rightarrow H_k C$  are isomorphisms, and such that  $H_p(F_k C) = 0$  for  $p \neq k$ .

It follows that there is a chain homotopy equivalence

$$\bigoplus_{k \geq 0} F_k C \xrightarrow{\cong} C.$$

There are isomorphisms

$$H_p(\tilde{Z}_n \otimes F_k C) \cong \begin{cases} H_n(X) \otimes H_k(C) & \text{if } p = n + k, \\ \mathrm{Tor}(H_n(X), H_k(C)) & \text{if } p = n + k + 1, \\ 0 & \text{otherwise} \end{cases}$$

**Exercise:** Do you need a spectral sequence?

Hint: Filter  $F_k C$ .

Adding up these isomorphisms gives split short exact sequences

$$\begin{aligned} 0 \rightarrow H_n(X) \otimes H_k(C) &\rightarrow H_{n+k} \text{Tot}(\tilde{Z}_n \otimes C) \\ &\rightarrow \text{Tor}(H_n(X), H_{k-1}(C)) \rightarrow 0 \end{aligned} \quad (10)$$

for  $k \geq 0$ , and  $H_k \text{Tot}(\tilde{Z}_n \otimes C) = 0$  for  $k < 0$ .

Taking a direct sum of the sequences (10) (and reindexing) gives short exact sequences

$$\begin{aligned} 0 \rightarrow \bigoplus_{0 \leq p \leq n} H_{n-p}(X) \otimes H_p(C) &\rightarrow H_n \text{Tot}(\mathbb{Z}(X) \otimes C) \\ &\rightarrow \bigoplus_{0 \leq q \leq n-1} \text{Tor}(H_{n-1-q}(X), H_q(C)) \rightarrow 0 \end{aligned} \quad (11)$$

The sequence (11) and the Eilenberg-Zilber Theorem (Theorem 26.1) together imply the following:

**Theorem 27.2** (Künneth Theorem). *Suppose  $X$  and  $Y$  are simplicial sets. Then there is a natural short exact sequence*

$$\begin{aligned} 0 \rightarrow \bigoplus_{0 \leq p \leq n} H_{n-p}(X) \otimes H_p(Y) &\rightarrow H_n(X \times Y) \\ &\rightarrow \bigoplus_{0 \leq q \leq n-1} \text{Tor}(H_{n-1-q}(X), H_q(Y)) \rightarrow 0. \end{aligned}$$

*This sequence splits, but not naturally.*

The coefficient ring  $\mathbb{Z}$  in the statement of Theorem 27.2 can be replaced by a principal ideal domain  $R$ . The same theorem holds for  $H_*(X \times Y, R)$ , with the same proof.

If  $R = F$  is a field, all  $F$ -modules are free and the Tor terms in the Theorem vanish, so

$$H_n(X \times Y, F) \cong \bigoplus_{0 \leq p \leq n} H_{n-p}(X, F) \otimes_F H_p(Y, F). \quad (12)$$

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## Lecture 10: Serre spectral sequence

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### 28 The fundamental groupoid, revisited

The **path category**  $PX$  for a simplicial set  $X$  is the category generated by the graph  $X_1 \rightrightarrows X_0$  of 1-simplices  $x : d_1(x) \rightarrow d_0(x)$ , subject to the relations

$$d_1(\sigma) = d_0(\sigma) \cdot d_2(\sigma)$$

given by the 2-simplices  $\sigma$  of  $X$ .

There is a natural bijection

$$\mathrm{hom}(PX, C) \cong \mathrm{hom}(X, BC),$$

so the functor  $P : s\mathbf{Set} \rightarrow \mathbf{cat}$  is left adjoint to the nerve functor.

Write  $GPX$  for the groupoid freely associated to the path category. The functor  $X \mapsto GP(X)$  is left adjoint to the nerve functor

$$B : \mathbf{Gpd} \rightarrow s\mathbf{Set}.$$

Say that a functor  $f : G \rightarrow H$  between groupoids is a **weak equivalence** if the induced map  $f : BG \rightarrow BH$  is a weak equivalence of simplicial sets.

Observe that  $\text{sk}_2(X) \subset X$  induces an isomorphism  $P(\text{sk}_2(X)) \cong P(X)$ , and hence an isomorphism

$$GP(\text{sk}_2(X)) \cong GP(X).$$

Nerves of groupoids are Kan complexes, so  $f : G \rightarrow H$  is a weak equivalence if and only if

- 1)  $f$  induces bijections

$$f : \text{hom}(a, b) \rightarrow \text{hom}(f(a), f(b))$$

for all objects  $a, b$  of  $G$ , (ie.  $f$  is full and faithful) and

- 2) for every object  $c$  of  $H$  there is a morphism  $c \rightarrow f(a)$  in  $H$  for some object  $a$  of  $G$  ( $f$  is surjective on  $\pi_0$ ).

Thus,  $f$  is a weak equivalence of groupoids if and only if it is a categorical equivalence (exercise).

**Lemma 28.1.** *The functor  $X \mapsto GP(X)$  takes weak equivalences of simplicial sets to weak equivalences of groupoids.*

*Proof.* 1) Claim: The inclusion  $\Lambda_k^n \subset \Delta^n$  induces an isomorphism  $GP(\Lambda_k^n) \cong GP(\Delta^n)$  if  $n \geq 2$ .

This is obvious if  $n \geq 3$ , for then  $\text{sk}_2(\Lambda_k^n) = \text{sk}_2(\Delta^n)$ .

If  $n = 2$ , then  $GP(\Lambda_k^2)$  has a contracting homotopy onto the vertex  $k$  (exercise). It follows that  $GP(\Lambda_k^2) \rightarrow GP(\Delta^2)$  is an isomorphism.

If  $n = 1$ , then  $\Lambda_k^1$  is a point, and  $GP\Lambda_k^1$  is a strong deformation retraction of  $GP(\Delta^1)$ .

2) In all cases,  $GP(\Lambda_k^n)$  is a strong deformation retraction of  $GP(\Delta^n)$ .

Strong deformation retractions are closed under pushout in the groupoid category (exercise).

Thus, every trivial cofibration  $i : A \rightarrow B$  induces a weak equivalence  $GP(A) \rightarrow GP(B)$ , so every weak equivalence  $X \rightarrow Y$  induces a weak equivalence  $GP(X) \rightarrow GP(Y)$ .  $\square$

Suppose  $Y$  is a Kan complex, and recall that the fundamental groupoid  $\pi(Y)$  for  $Y$  has objects given by the vertices of  $Y$ , morphisms given by homotopy classes of paths (1-simplices)  $x \rightarrow y$  rel end points, and composition law defined by extending maps

$$(\beta, \ , \alpha) : \Lambda_1^2 \rightarrow Y$$

to maps  $\sigma : \Delta^2 \rightarrow Y$ :  $[d_1(\sigma)] = [\beta] \cdot [\alpha]$ .

There is a natural functor

$$GP(Y) \rightarrow \pi(Y)$$

which is the identity on vertices and takes a simplex  $\Delta^1 \rightarrow Y$  to the corresponding homotopy class. This functor is an isomorphism of groupoids (exercise).

If  $X$  is a topological space then the combinatorial fundamental groupoid  $\pi(S(X))$  coincides up to isomorphism with the usual fundamental groupoid  $\pi(X)$  of  $X$ .

**Corollary 28.2.** *Suppose  $i : X \rightarrow Z$  is a weak equivalence, such that  $Z$  is a Kan complex.*

*Then  $i$  induces a weak equivalence of groupoids*

$$GP(X) \xrightarrow{i_*} GP(Z) \xrightarrow{\cong} \pi(Z).$$

There is a functor

$$u_X : GP(X) \rightarrow G(\Delta/X)$$

that takes a 1-simplex  $\omega : d_1(\omega) \rightarrow d_0(\omega)$  to the morphism  $(d^0)^{-1}(d^1)$  in  $G(\Delta/X)$  defined by the diagram

$$\begin{array}{ccccc} \Delta^0 & \xrightarrow{d^1} & \Delta^1 & \xleftarrow{d^0} & \Delta^0 \\ & \searrow & \downarrow \omega & \swarrow & \\ & d_1(\omega) & X & d_0(\omega) & \end{array}$$



This assignment takes 2-simplices to composition laws of  $G(\Delta/X)$  [1, p.141].

There is a functor

$$v_X : G(\Delta/X) \rightarrow GP(X)$$

which associates to each object  $\sigma : \Delta^n \rightarrow X$  its last vertex

$$\Delta^0 \xrightarrow{n} \Delta^n \xrightarrow{\sigma} X.$$

Then any map between simplices of  $\Delta/X$  is mapped to a canonically defined path between last vertices, and compositions of  $\Delta/X$  determine 2-simplices relating last vertices.

Then  $v_X u_X$  is the identity on  $GP(X)$  and the maps

$$\begin{array}{ccc} \Delta^0 & & \\ \downarrow n & \searrow \sigma \cdot n & \\ & & X \\ \uparrow \sigma & \nearrow & \\ \Delta^n & & \end{array}$$

determine a natural isomorphism (aka. homotopy)

$$u_X v_X \cong 1_{G(\Delta/X)}.$$

We have proved

**Lemma 28.3.** *There is an equivalence of groupoids*

$$u_X : GP(X) \rightleftarrows G(\Delta/X) : v_X,$$

*which is natural in simplicial sets  $X$ .*

Here's a summary. Suppose  $X$  is a simplicial set with fibrant model  $i : X \rightarrow Z$ . Then there is a picture of natural equivalences

$$\begin{array}{ccc} GP(X) & \xrightarrow[\simeq]{i_*} & GP(Z) \xrightarrow{\cong} \pi(Z) \\ u_X \downarrow \simeq & & \simeq \uparrow \varepsilon_* \\ G(\Delta/X) & & \pi(S|Z|) \xrightarrow[\simeq]{} \pi(|Z|) \end{array}$$

You need the Milnor theorem (Theorem 13.2) to show that  $\varepsilon_*$  is an equivalence.

I refer to any of the three equivalent models  $\pi(Z)$ ,  $GP(X)$  or  $G(\Delta/X)$  as the **fundamental groupoid** of  $X$ , and write  $\pi(X)$  to denote any of these objects.

The adjunction map  $X \rightarrow BGP(X)$  is often written

$$\eta : X \rightarrow B\pi(X).$$

**Lemma 28.4.** *Suppose  $C$  is a small category.*

*Then there is an isomorphism*

$$GP(BC) \cong G(C),$$

*which is natural in  $C$ .*

*Proof.* The adjunction functor  $\varepsilon : P(BC) \rightarrow C$  is an isomorphism (exercise).  $\square$

**Remark:** This result leads to a fast existence proof for the isomorphism

$$\pi_1(BQ\mathbf{M}, 0) \cong K_0(\mathbf{M})$$

(due to Quillen [3]) for an exact category  $\mathbf{M}$ , in algebraic  $K$ -theory.

It also follows that the adjunction functor

$$\varepsilon : GP(BG) \rightarrow G$$

is an isomorphism for all groupoids  $G$ .

**Lemma 28.5.** *Suppose  $X$  is a Kan complex.*

*Then the adjunction map  $\eta : X \rightarrow BGP(X)$  induces a bijection  $\pi_0(X) \cong \pi_0(BGP(X))$  and isomorphisms*

$$\pi_1(X, x) \xrightarrow{\cong} \pi_1(BGP(X), x)$$

*for each vertex  $x$  of  $X$ .*

*Proof.* This result is another corollary of Lemma 28.4.

There is a commutative diagram

$$\begin{array}{ccc} \pi(X) & \xrightarrow{\pi(\eta)} & \pi(BGP(X)) \\ \cong \uparrow & & \uparrow \cong \\ GP(X) & \xrightarrow{GP(\eta)} & GPBGP(X) \\ & \searrow 1 & \downarrow \cong \varepsilon \\ & & GP(X) \end{array}$$

It follows that  $\eta$  induces an isomorphism

$$\pi(\eta) : \pi(X) \xrightarrow{\cong} \pi(BGPX).$$

Finish by comparing path components and automorphism groups, respectively.  $\square$

Say that a morphism  $p : G \rightarrow H$  of groupoids is a **fibration** if the induced map  $BG \rightarrow BH$  is a fibration of simplicial sets.

**Exercise:** Show that a functor  $p$  is a fibration if and only if it has the **path lifting property** in the sense that all lifting problems

$$\begin{array}{ccc} \mathbf{0} & \longrightarrow & G \\ \downarrow & \nearrow & \downarrow p \\ \mathbf{1} & \longrightarrow & H \end{array}$$

(involving functors) can be solved.

**Cofibrations** of groupoids are defined by a left lifting property in the usual way.

There is a **function complex** construction  $\mathbf{hom}(G, H)$  for groupoids, with

$$\mathbf{hom}(G, H) := \mathbf{hom}(BG, BH).$$

**Lemma 28.6.** *1) With these definitions, the category **Gpd** satisfies the axioms for a closed simplicial model category. This model structure is cofibrantly generated and right proper.*

## 2) The functors

$$GP : s\mathbf{Set} \rightleftarrows \mathbf{Gpd} : B$$

form a Quillen adjunction.

*Proof.* Use Lemma 28.1 and its proof.  $\square$

## 29 The Serre spectral sequence

Suppose  $f : X \rightarrow Y$  is a map of simplicial sets, and consider all pullback diagrams

$$\begin{array}{ccc} f^{-1}(\sigma) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \Delta^n & \xrightarrow{\sigma} & Y \end{array}$$

defined by the simplices of  $Y$ .

We know (Lemma 23.1) that the bisimplicial set map

$$\bigsqcup_{\sigma_0 \rightarrow \dots \rightarrow \sigma_n} f^{-1}(\sigma_0) \rightarrow X$$

defines a (diagonal) weak equivalence

$$\underline{\mathrm{holim}}_{\sigma : \Delta^n \rightarrow Y} f^{-1}(\sigma) \rightarrow X$$

where the homotopy colimit defined on the simplex category  $\Delta/Y$ .

The induced bisimplicial abelian group map

$$\bigoplus_{\sigma_0 \rightarrow \dots \rightarrow \sigma_n} \mathbb{Z}(f^{-1}(\sigma_0)) \rightarrow \mathbb{Z}(X)$$

is also a diagonal weak equivalence.

It follows (see Lemma 24.4) that there is a spectral sequence with

$$E_2^{p,q} = L(\varinjlim_{\sigma: \Delta^n \rightarrow Y})_p H_q(f^{-1}(\sigma)) \Rightarrow H_{p+q}(X, \mathbb{Z}), \quad (1)$$

often called the *Grothendieck spectral sequence*.

Making sense of the spectral sequence (1) usually requires more assumptions on the map  $f$ .

A) Suppose  $f : X \rightarrow Y$  is a fibration and that  $Y$  is connected.

By properness, the maps

$$\theta_* : f^{-1}(\sigma) \rightarrow f^{-1}(\tau)$$

induced by simplex morphisms  $\theta : \sigma \rightarrow \tau$  are weak equivalences, and the maps

$$\theta_* : H_k(f^{-1}(\sigma), \mathbb{Z}) \rightarrow H_k(f^{-1}(\tau), \mathbb{Z})$$

are isomorphisms.

It follows that the functors  $H_k : \Delta/Y \rightarrow \mathbf{Ab}$  which are defined by

$$\sigma \mapsto H_k(f^{-1}(\sigma), \mathbb{Z})$$

factor through an action of the fundamental groupoid of  $Y$ , in the sense that these functors extend uniquely to functors

$$H_k : G(\Delta/Y) \rightarrow \mathbf{Ab}.$$

Suppose  $x$  is a vertex of  $Y$ , and write  $F = p^{-1}(x)$  for the fibre of  $f$  over  $x$ .

Since  $Y$  is connected there is a morphism  $\omega_\sigma : x \rightarrow \sigma$  in  $G(\Delta/Y)$  for each object  $\sigma$  of the simplex category. The maps  $\omega_\sigma$ , induce isomorphisms

$$\omega_{\sigma*} : H_k(F, \mathbb{Z}) \rightarrow H_k(f^{-1}(\sigma), \mathbb{Z}),$$

and hence define a functor

$$H_k(F, \mathbb{Z}) : G(\Delta/Y) \rightarrow \mathbf{Ab}$$

which is naturally isomorphic to the functor  $H_k$ .

It follows that the spectral sequence (1) is isomorphic to

$$E_2^{p,q} = L(\varinjlim_{\Delta/Y} H_q(F, \mathbb{Z})) \Rightarrow H_{p+q}(X, \mathbb{Z}) \quad (2)$$

under the assumption that  $f : X \rightarrow Y$  is a fibration and  $Y$  is connected.

This is the general form of the **Serre spectral sequence**.

This form of the Serre spectral sequence is used, but calculations often involve more assumptions.

B) The fundamental groupoid  $G(\Delta/Y)$  **acts trivially** on the homology fibres  $H_k(f^{-1}(\sigma), \mathbb{Z})$  of  $f$  if any two morphisms  $\alpha, \beta : \sigma \rightarrow \tau$  in  $G(\Delta/Y)$  induce the same map

$$\alpha_* = \beta_* : H_k(f^{-1}(\sigma), \mathbb{Z}) \rightarrow H_k(f^{-1}(\tau), \mathbb{Z})$$

for all  $k \geq 0$ .

This happens, for example, if the fundamental group (or groupoid) of  $Y$  is trivial.

In that case, all maps  $x \rightarrow x$  in  $G(\Delta/Y)$  induce the identity

$$H_k(F, \mathbb{Z}) \rightarrow H_k(F, \mathbb{Z})$$

for all  $k \geq 0$ , and there are isomorphisms (exercise)

$$\begin{aligned} L(\varinjlim)_p H_q(F, \mathbb{Z}) &\cong H_p(B(\Delta/Y), H_q(F, \mathbb{Z})) \\ &\cong H_p(Y, H_q(F, \mathbb{Z})). \end{aligned}$$



Thus, we have the following:

**Theorem 29.1.** *Suppose  $f : X \rightarrow Y$  is a fibration with  $Y$  connected, and let  $F$  be the fibre of  $f$  over a vertex  $x$  of  $Y$ . Suppose the fundamental groupoid  $G(\Delta/Y)$  of  $Y$  acts trivially on the homology fibres of  $f$ .*

*Then there is a spectral sequence with*

$$E_2^{p,q} = H_p(Y, H_q(F, \mathbb{Z})) \Rightarrow H_{p+q}(X, \mathbb{Z}). \quad (3)$$

*This spectral sequence is natural in all such fibre sequences.*

The spectral sequence given by Theorem 29.1 is the standard form of the homology Serre spectral sequence for a fibration.

Integral coefficients were used in the statement of Theorem 29.1 for display purposes —  $\mathbb{Z}$  can be replaced by an arbitrary abelian group of coefficients.

**Examples:** Eilenberg-Mac Lanes spaces

Say that  $X$  is  **$n$ -connected** ( $n \geq 0$ ) if  $\pi_0 X = *$ , and  $\pi_k(X, x) = 0$  for all  $k \leq n$  and all vertices  $x$ .

One often says that  $X$  is **simply connected** if it is 1-connected.

$X$  is simply connected if and only if it has a trivial fundamental groupoid  $\pi(X)$  (exercise).

Here's a general fact:

**Lemma 29.2.** *Suppose  $X$  is a Kan complex,  $n \geq 0$ , and that  $X$  is  $n$ -connected. Pick a vertex  $x \in X$ .*

*Then  $X$  has a subcomplex  $Y$  such that  $Y_k = \{x\}$  for  $k \leq n$ , and  $Y$  is a strong deformation retract of  $X$ .*

The proof is an exercise.

**Corollary 29.3.** *Suppose  $X$  is  $n$ -connected. Then there are isomorphisms*

$$H_k(X, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ 0 & \text{if } 0 < k \leq n. \end{cases}$$

**Example:** There is a fibre sequence

$$K(\mathbb{Z}, 1) \rightarrow WK(\mathbb{Z}, 1) \rightarrow K(\mathbb{Z}, 2) \quad (4)$$

such that  $WK(\mathbb{Z}, 1) \simeq *$ .

$K(\mathbb{Z}, 2)$  is simply connected, so the Serre spectral sequence for (4) has the form

$$H_p(K(\mathbb{Z}, 2), H_q(K(\mathbb{Z}, 1), \mathbb{Z})) \Rightarrow H_{p+q}(*, \mathbb{Z}).$$

1)  $H_1(K(\mathbb{Z}, 2), A) = 0$  by Corollary 29.3, so  $E_2^{1,q} = 0$  for all  $q$ .

2)  $K(\mathbb{Z}, 1) \simeq S^1$ , so  $E_2^{p,q} = 0$  for  $q > 1$ .

The quotient of the differential

$$d_2 : E_2^{2,0} \rightarrow E_2^{0,1} \cong \mathbb{Z}$$

survives to  $E_\infty^{0,1} \subset H_1(*) = 0$ , so  $d_2$  is surjective.

The kernel of  $d_2$  survives to  $E_\infty^{2,0} = 0$ , so  $d_2$  is an isomorphism and

$$H_2(K(\mathbb{Z}, 2), \mathbb{Z}) \cong \mathbb{Z}.$$

Inductively, we find isomorphisms

$$H_n(K(\mathbb{Z}, 2), \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 2k, k \geq 0, \text{ and} \\ 0 & \text{if } n = 2k + 1, k \geq 0. \end{cases}$$

**Example:** There is a fibre sequence

$$K(\mathbb{Z}/n, 1) \rightarrow WK(\mathbb{Z}/n, 1) \rightarrow K(\mathbb{Z}/n, 2) \quad (5)$$

such that  $WK(\mathbb{Z}/n, 1) \simeq *$ .

$K(\mathbb{Z}/n, 2)$  is simply connected, so the Serre spectral sequence for (5) has the form

$$H_p(K(\mathbb{Z}/n, 2), H_q(K(\mathbb{Z}/n, 1), \mathbb{Z})) \Rightarrow H_{p+q}(*, \mathbb{Z}).$$

We showed (see (6) of Section 25) that there are isomorphisms

$$H_p(B\mathbb{Z}/n, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & p = 0, \\ 0 & \text{if } p = 2n, n > 0, \text{ and} \\ \mathbb{Z}/n & \text{if } p = 2n + 1, n \geq 0. \end{cases}$$

There are isomorphisms

$$E_2^{1,q} \cong 0$$

for  $q \geq 0$  and

$$H_2(K(\mathbb{Z}/n, 2), \mathbb{Z}) \xrightarrow[\cong]{d_2} H_1(K(\mathbb{Z}/n, 1), \mathbb{Z}) \cong \mathbb{Z}/n.$$

$E_2^{0,2} = H_2(K(\mathbb{Z}/n, 1), \mathbb{Z}) = 0$ , so all differentials on  $E_2^{3,0}$  are trivial. Thus,  $E_2^{3,0} = E_\infty^{3,0} = 0$  because  $H_3(*) = 0$ , and

$$H_3(K(\mathbb{Z}/n, 2), \mathbb{Z}) = E_2^{3,0} = 0.$$

We shall need the following later:

**Lemma 29.4.** *Suppose  $A$  is an abelian group. Then there is an isomorphism*

$$H_3(K(A, 2), \mathbb{Z}) \cong 0.$$

*Proof.* Suppose  $X$  and  $Y$  are connected spaces such that

$$H_i(X, \mathbb{Z}) \cong 0 \cong H_i(Y, \mathbb{Z})$$

for  $i = 1, 3$ . Then a Künneth formula argument (exercise — use Theorem 27.2) shows that  $X \times Y$  has the same property.

The spaces  $K(\mathbb{Z}, 2)$  and  $K(\mathbb{Z}/n, 2)$  are connected and have vanishing integral  $H_1$  and  $H_3$ , so the same holds for all  $K(A, 2)$  if  $A$  is finitely generated.

Every abelian group is a filtered colimit of its finitely generated subgroups, and the functors  $H_*(\ , \mathbb{Z})$  preserve filtered colimits.  $\square$

**Lemma 29.5.** *Suppose  $A$  is an abelian group and that  $n \geq 2$ . Then there is an isomorphism*

$$H_{n+1}(K(A, n), \mathbb{Z}) \cong 0.$$

*Proof.* The proof is by induction on  $n$ . The case  $n = 2$  follows from Lemma 29.4.

Consider the fibre sequence

$$K(A, n) \rightarrow WK(A, n) \rightarrow K(A, n+1),$$

with contractible total space  $WK(A, n)$ .

$E_2^{p, n+1-p} = 0$  for  $p < n+1$  (the case  $p = 0$  is the inductive assumption). All differentials defined on  $E_2^{n+2, 0}$  are therefore 0 maps, so

$$H_{n+2}(K(A, n+1), \mathbb{Z}) \cong E_2^{n+2, 0} \cong E_\infty^{n+2, 0} = 0,$$

since  $E_\infty^{n+2, 0}$  is a quotient of  $H_{n+2}(\ast) = 0$ .  $\square$

### 30 The transgression

Suppose  $p : X \rightarrow Y$  is a fibration with connected base space  $Y$ , and let  $F = p^{-1}(*)$  be the fibre of  $p$  over some vertex  $*$  of  $Y$ . Suppose that  $F$  is connected.

Consider the bicomplex

$$\bigoplus_{\sigma_0 \rightarrow \dots \rightarrow \sigma_n} \mathbb{Z}(p^{-1}(\sigma_0))$$

defining the Serre spectral sequence for  $H_*(X, \mathbb{Z})$ , and write  $F_p$  for its horizontal filtration stages.

$\mathbb{Z}(F)$  is a subobject of  $F_0$ .

The differential  $d_n : E_n^{n,0} \rightarrow E_n^{0,n-1}$  is called the **transgression**, and is represented by the picture

$$\begin{array}{ccc} H_{n-1}F_0 & \xrightarrow{\cong} & H_{n-1}(F_0/F_{-1}) \longrightarrow E_n^{0,n-1} \\ \downarrow i_* & & \\ H_n(F_n/F_{n-1}) & \xrightarrow{\partial} & H_{n-1}F_{n-1} \end{array}$$

Here,

$$\begin{aligned} E_n^{n,0} &= \partial^{-1}(\text{im}(i_*)) / \text{im}(\ker(i_*)), \\ E_n^{0,n-1} &= H_{n-1}(F_0) / \ker(i_*), \end{aligned}$$

and  $d_n([x]) = [y]$  where  $i_*(y) = \partial(x)$ .

One says (in old language) that  $[x]$  **transgresses** to  $[y]$  if  $d_n([x]) = [y]$ .

Note that

$$E_n^{0,n-1} \cong H_{n-1}(F_0)/\ker(i_*).$$

Given  $[x] \in E_n^{n,0}$  and  $z \in E_n^{0,n-1}$ , then  $d_n([x]) = z$  if and only if there is an element  $y \in H_{n-1}(F_0)$  such that  $i_*(y) = \partial(x)$  and  $y \mapsto z$  under the composite

$$H_{n-1}(F_0) \xrightarrow{\cong} H_{n-1}(F_0/F_{-1}) \rightarrow E_n^{0,n-1}.$$

The inclusion  $j : \mathbb{Z}(F) \subset F_0$  induces a composite map

$$j' : H_{n-1}(F) \rightarrow \varinjlim_{\sigma} H_{n-1}(F_{\sigma}) = E_2^{0,n-1} \twoheadrightarrow E_n^{0,n-1},$$

and  $j'$  is surjective since  $Y$  is connected (exercise).

Suppose  $x \in H_n(F_n/F_{n-1})$  represents an element of  $E_n^{n,0}$ . Then  $\partial(x) = i_*(y)$  for some  $y \in H_{n-1}(F_0)$ . Write  $z$  for the image of  $y$  in  $E_n^{0,n-1}$ .

Choose  $v \in H_{n-1}(F)$  such that  $j'(v) = z$ . Then  $j_*(v)$  and  $y$  have the same image in  $E_n^{0,n-1}$  so  $i_*j_*(z) = i_*(y)$  in  $H_{n-1}(F_{n-1})$ . This means that  $\partial(x)$  is in the image of the map  $H_{n-1}(F) \rightarrow H_{n-1}(F_{n-1})$ .

It follows from the comparison of exact sequences

$$\begin{array}{ccccccc} H_n(F_n) & \longrightarrow & H_n(F_n/F) & \xrightarrow{\partial} & H_{n-1}(F) & \longrightarrow & H_{n-1}(F_n) \\ \downarrow = & & \downarrow & & \downarrow & & \downarrow = \\ H_n(F_n) & \longrightarrow & H_n(F_n/F_{n-1}) & \xrightarrow{\partial} & H_{n-1}(F_{n-1}) & \longrightarrow & H_{n-1}(F_n) \end{array}$$

that  $x$  is in the image of the map

$$H_n(F_n/F) \rightarrow H_n(F_n/F_{n-1}).$$

In particular, the induced map

$$H_n(F_n/F) \rightarrow E_n^{n,0}$$

is surjective.

Thus,  $d_n(x) = y$  if and only if there is an element  $w$  of  $H_n(F_n/F)$  such that  $w$  maps to  $x$  and  $y$ , respectively, under the maps

$$E_n^{n,0} \leftarrow H_n(F_n/F) \xrightarrow{\partial} H_{n-1}(F) \xrightarrow{j'} E_n^{0,n-1}.$$

$H_n(F_n) \rightarrow H_n(X)$  is surjective, and  $H_{n-1}(F_n) \rightarrow H_{n-1}(X)$  is an isomorphism, so a comparison of long exact sequences also shows that the map

$$H_n(F_n/F) \rightarrow H_n(X/F)$$

is surjective.

In summary, there is a commutative diagram

$$\begin{array}{ccccccc}
E_2^{n,0} & \longleftarrow & E_n^{n,0} & \longleftarrow & H_n(F_n/F) & \xrightarrow{\partial} & H_{n-1}(F) \xrightarrow{j'} E_n^{0,n-1} \\
& & \uparrow & & \nearrow & & \nearrow \\
& & H_n(F_n) & & \downarrow & & \downarrow \\
& & \swarrow & \downarrow & \downarrow & \nearrow & \nearrow \\
& & E_\infty^{n,0} & \longleftarrow & H_n(X) & \longrightarrow & H_n(X/F)
\end{array}
\tag{6}$$



This diagram is natural in fibrations  $p$ .

There is a comparison of Serre spectral sequences arising from the diagram

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ p \downarrow & & \downarrow 1 \\ Y & \xrightarrow{1} & Y \end{array} \quad (7)$$

All fibres of  $p$  are connected, so it follows that the map

$$p_* : E_2^{n,0} \rightarrow H_n(Y)$$

is an isomorphism.

Write  $F_n(Y)$  and  $E_r^{p,q}(Y)$  for the filtration and spectral sequences, respectively, for the total complex associated to the map  $1 : Y \rightarrow Y$ .

There is a commutative diagram

$$\begin{array}{ccc} E_2^{n,0} & \longleftarrow & E_n^{n,0} \\ \cong \downarrow & & \downarrow p_* \\ E_2^{n,0}(Y) & \xleftarrow{\cong} & E_n^{n,0}(Y) \end{array}$$

that is induced by the comparison (7).

It follows that  $p_* : E_n^{n,0} \rightarrow E_n^{n,0}(Y)$  is injective, and that  $E_n^{n,0}$  is identified with a subobject of  $H_n(Y/*)$  via the composite

$$E_n^{n,0} \xrightarrow{p_*} E_n^{n,0}(Y) \xleftarrow{\cong} E_\infty^{n,0}(Y) \xleftarrow{\cong} H_n(Y) \xrightarrow{\cong} H_n(Y/*) .$$

**Lemma 30.1.** *Suppose  $p : X \rightarrow Y$  is a fibration with connected base  $Y$  and connected fibre  $F$  over  $*$   $\in Y_0$ . Suppose  $x \in E_n^{n,0} \subset H_n(Y/*)$ ,  $n \geq 1$ , and that  $y \in E_n^{0,n-1}$ .*

*Then  $d_n(x) = y$  if and only if there is an element  $z \in H_n(X/F)$  such that  $p_*(z) = x \in H_n(Y/*)$  and  $z \mapsto y$  under the composite*

$$H_n(X/F) \xrightarrow{\partial} H_{n-1}(F) \xrightarrow{j'} E_n^{0,n-1}.$$

*Proof.* Use the fact that the map

$$H_n(F_n/F) \rightarrow H_n(X/F)$$

is surjective, and chase elements through the comparison induced by (7) of the diagram (6) with the diagram

$$\begin{array}{ccccc} E_n^{n,0}(Y) & \xleftarrow{\quad} & H_n(F_n(Y)/*) & & \\ & \nearrow \cong & \downarrow & & \\ & H_n(F_n(Y)) & & & \\ & \swarrow & \downarrow & & \\ E_\infty^{n,0}(Y) & \xleftarrow{\cong} & H_n(Y) & \xrightarrow{\cong} & H_n(Y/*) \end{array}$$

to prove the result. □

### 31 The path-loop fibre sequence

We will use the model structure for the category  $s_*\mathbf{Set}$  of pointed simplicial sets (aka. pointed spaces).

This model structure is easily constructed, since  $s_*\mathbf{Set} = */s\mathbf{Set}$  is a slice category: a pointed simplicial set is a simplicial set map  $* \rightarrow X$ , and a pointed map is a diagram

$$\begin{array}{ccc} & & X \\ & \nearrow x & \downarrow g \\ * & & \\ & \searrow y & \\ & & Y \end{array} \quad (8)$$

In general, if  $\mathcal{M}$  is a closed model category, with object  $A$ , then the slice category  $A/\mathcal{M}$  has a closed model structure, for which a morphism

$$\begin{array}{ccc} & & X \\ & \nearrow & \downarrow f \\ A & & \\ & \searrow & \\ & & Y \end{array}$$

is a weak equivalence (resp. fibration, cofibration) if the map  $f : X \rightarrow Y$  is a weak equivalence (resp. fibration, cofibration).

**Exercise:** 1) Verify the existence of the model structure for the slice category  $A/\mathcal{M}$ .

2) The dual assertion is the existence of a model structure for the category  $\mathcal{M}/B$  for all objects  $B \in \mathcal{M}$ . Formulate the result.

**Warning:** A map  $g : X \rightarrow Y$  of pointed simplicial sets is a weak equivalence if and only if it induces a bijection  $\pi_0(X) \cong \pi_0(Y)$  and isomorphisms

$$\pi_n(X, z) \cong \pi_n(Y, g(z))$$

for **all** base points  $z \in X_0$ .

The model structure for  $s_*\mathbf{Set}$  is a closed simplicial model structure, with function complex  $\mathbf{hom}_*(X, Y)$  defined by

$$\mathbf{hom}_*(X, Y)_n = \mathbf{hom}(X \wedge \Delta_+^n, Y),$$

where

$$\Delta_+^n = \Delta^n \sqcup \{*\}$$

is the simplex  $\Delta^n$  with a disjoint base point.

The **smash product** of pointed spaces  $X, Y$  is defined by

$$X \wedge Y = \frac{X \times Y}{X \vee Y},$$

where the **wedge**  $X \vee Y$  or **one-point union** of  $X$  and  $Y$  is the coproduct of  $X$  and  $Y$  in the pointed category.

The **loop space**  $\Omega X$  of a pointed Kan complex  $X$  is the pointed function complex

$$\Omega X = \mathbf{hom}_*(S^1, X),$$

where  $S^1 = \Delta^1 / \partial \Delta^1$  is the simplicial circle with the obvious choice of base point.

Write  $\Delta_*^1$  for the simplex  $\Delta^1$ , pointed by the vertex 1, and let

$$S^0 = \partial \Delta^1 = \{0, 1\},$$

pointed by 1. Then the cofibre sequence

$$S^0 \subset \Delta_*^1 \xrightarrow{\pi} S^1 \tag{9}$$

of pointed spaces induces a fibre sequence

$$\Omega X = \mathbf{hom}_*(S^1, X) \rightarrow \mathbf{hom}_*(\Delta_*^1, X) \xrightarrow{p} \mathbf{hom}_*(S^0, X) \cong X \tag{10}$$

provided  $X$  is fibrant.

The pointed inclusion  $\{1\} \subset \Delta_*^1$  is a weak equivalence, so the space

$$PX = \mathbf{hom}_*(\Delta_*^1, X)$$

is contractible if  $X$  is fibrant.

The simplicial set  $PX$  is the **pointed path space** for  $X$ , and the fibre sequence (10) is the **path-loop fibre sequence** for  $X$ .

It follows that, if  $X$  is fibrant and  $*$  denotes the base point for all spaces in the fibre sequence (10), then there are isomorphisms

$$\pi_n(X, *) \cong \pi_{n-1}(\Omega X, *)$$

for  $n \geq 2$  and a bijection

$$\pi_1(X, *) \cong \pi_0(\Omega X).$$

Dually, one can take a pointed space  $Y$  and smash with the cofibre sequence (9) to form a natural cofibre sequence

$$Y \cong S^0 \wedge Y \rightarrow \Delta_*^1 \wedge Y \rightarrow S^1 \wedge Y.$$

The space  $\Delta_*^1 \wedge Y$  is contractible (exercise) — it is the **pointed cone** for  $Y$ , and one writes

$$CX = X \wedge \Delta_*^1.$$

One often writes

$$\Sigma X = X \wedge S^1.$$

This object is called the **suspension** of  $X$ , although saying this is a bit dangerous because there's more than one suspension construction for simplicial sets — see [1, III.5], [2, 4.4].

The suspension functor is left adjoint to the loop functor. More generally, there is a natural isomor-

phism

$$\mathbf{hom}_*(X \wedge K, Y) \cong \mathbf{hom}_*(K, \mathbf{hom}_*(X, Y))$$

of pointed simplicial sets (exercise).

**Lemma 31.1.** *Suppose  $f : X \rightarrow \Omega Y$  is a pointed map, and let  $f' : \Sigma X \rightarrow Y$  denote its adjoint. Then there is a commutative diagram*

$$\begin{array}{ccccc} X & \longrightarrow & CX & \longrightarrow & \Sigma X \\ f \downarrow & & \downarrow h(f) & & \downarrow f' \\ \Omega Y & \longrightarrow & PY & \xrightarrow{p} & Y \end{array}$$

*Proof.* We'll say how  $h(f)$  is defined. Checking that the diagram commutes is an exercise.

The pointed map (contracting homotopy)

$$h : \Delta_*^1 \wedge \Delta_*^1 \rightarrow \Delta_*^1$$

is defined by the relations

$$\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & 1 \end{array}$$

Then the map

$$h(f) : X \wedge \Delta_*^1 \rightarrow \mathbf{hom}_*(\Delta_*^1, Y)$$

is adjoint to the composite

$$\begin{aligned} X \wedge \Delta_*^1 \wedge \Delta_*^1 &\xrightarrow{1 \wedge h} X \wedge \Delta_*^1 \xrightarrow{f \wedge 1} \mathbf{hom}_*(S^1, Y) \wedge \Delta_*^1 \\ &\xrightarrow{1 \wedge \pi} \mathbf{hom}_*(S^1, Y) \wedge S^1 \xrightarrow{ev} Y. \end{aligned}$$

□

**Lemma 31.2.** *Suppose  $Y$  is a pointed Kan complex which is  $n$ -connected for  $n \geq 1$ .*

*Then the transgression  $d_i$  induces isomorphisms*

$$H_i(Y) \cong H_{i-1}(\Omega Y)$$

*for  $2 \leq i \leq 2n$ .*

*Proof.*  $Y$  is at least simply connected, and the homotopy groups  $\pi_i(Y, *)$  vanish for  $i \leq n$ .

The Serre spectral sequence for the path-loop fibration for  $Y$  has the form

$$E_2^{p,q} = H_p(Y, H_q(\Omega Y)) \Rightarrow H_{p+q}(PY).$$

The space  $\Omega Y$  is  $(n-1)$ -connected, so  $E_2^{p,q} = 0$  for  $0 < q \leq n-1$  or  $0 < p \leq n$ .

Thus, the first possible non-trivial group off the edges in the  $E_2$ -term is in bidegree  $(n+1, n)$ .

All differentials reduce total degree by 1 so

- the differentials  $d_r : E_r^{i,0} \rightarrow E_r^{i-r,r-1}$  vanish for  $i \leq 2n$  and  $r < i$ ,
- the differentials  $d_r : E_r^{r,i-r} \rightarrow E_r^{0,i-1}$  vanish for  $r < i$  and  $i \leq 2n$ .



It follows that there is an exact sequence

$$0 \rightarrow E_\infty^{i,0} \rightarrow E_i^{i,0} \xrightarrow{d_i} E_i^{0,i-1} \rightarrow E_\infty^{0,i-1} \rightarrow 0$$

for  $0 < i \leq 2n$ , and

$$E_i^{i,0} \cong E_2^{i,0} \cong H_i(Y), \text{ and}$$

$$E_i^{0,i-1} \cong E_2^{0,i-1} \cong H_{i-1}(\Omega Y)$$

for  $0 < i \leq 2n$ .

All groups  $E_\infty^{p,q}$  vanish for  $(p,q) \neq (0,0)$ . □

**Lemma 31.3.** *Suppose  $f : X \rightarrow \Omega Y$  is a map of pointed simplicial sets, where  $Y$  is fibrant. Suppose  $Y$  is  $n$ -connected, where  $n \geq 1$ .*

*Then for  $2 \leq i \leq 2n$  there is a commutative diagram*

$$\begin{array}{ccc} H_i(\Sigma X) & \xrightarrow[\cong]{\partial} & H_{i-1}(X) \\ f'_* \downarrow & & \downarrow f_* \\ H_i(Y) & \xrightarrow[\cong]{d_i} & H_{i-1}(\Omega Y) \end{array} \quad (11)$$

where  $f' : \Sigma X \rightarrow Y$  is the adjoint of  $f$ .

*Proof.* From the diagram of Lemma 31.1, there is a commutative diagram

$$\begin{array}{ccccc} H_i(\Sigma X / *) & \xleftarrow[\cong]{} & H_i(CX / X) & \xrightarrow{\partial} & H_{i-1}(X) \\ f'_* \downarrow & & \downarrow h(f')_* & & \downarrow f_* \\ H_i(Y / *) & \xleftarrow[p_*]{} & H_i(PY / \Omega Y) & \xrightarrow[\partial]{} & H_{i-1}(\Omega Y) \end{array} \quad (12)$$

After the standard identifications

$$\begin{aligned} E_i^{i,0} &\cong H_i(Y / *), \text{ and} \\ E_i^{0,i-1} &\cong H_{i-1}(\Omega Y). \end{aligned}$$

and given  $x \in H_i(Y / *)$  and  $y \in H_{i-1}(\Omega Y)$ , Lemma 30.1 implies that  $d_i(x) = y$  if there is a  $z \in H_i(PY / \Omega Y)$  such that  $p_*(z) = x$  and  $\partial(z) = y$ .

This is true for  $f'_*(v)$  and  $f_*(\partial(v))$  for  $v \in H_i(\Sigma X)$ , given the isomorphism in the diagram (12).

The map  $d_i$  is an isomorphism for  $2 \leq i \leq 2n$  by Lemma 31.2.  $\partial$  is always an isomorphism.  $\square$

**Corollary 31.4.** *Suppose  $Y$  is an  $n$ -connected pointed Kan complex with  $n \geq 1$ .*

*Then there is a commutative diagram*

$$\begin{array}{ccc} H_i(\Sigma\Omega Y) & \xrightarrow[\cong]{\partial} & H_{i-1}(\Omega Y) \\ \varepsilon_* \downarrow & \nearrow d_i & \\ H_i(Y) & & \end{array}$$

for  $2 \leq i \leq 2n$ .

The adjunction map  $\varepsilon : \Sigma\Omega Y \rightarrow Y$  induces an isomorphism  $H_i(\Sigma\Omega Y) \cong H_i(Y)$  for  $2 \leq i \leq 2n$ .

*Proof.* This is the case  $f = 1_{\Omega Y}$  of Lemma 31.3.  $\square$

If  $Y$  is a 1-connected pointed Kan complex, then  $\Omega Y$  is connected.

We can say more about the map  $\varepsilon_*$ . The following result implies that  $\Sigma\Omega Y$  is simply connected, so the adjunction map  $\varepsilon$  in the statement of Corollary 31.4 induces isomorphisms

$$\varepsilon_* : H_i(\Sigma\Omega Y) \xrightarrow{\cong} H_i(Y)$$

for  $0 \leq i \leq 2n$ .

**Lemma 31.5.** *Suppose  $X$  is a connected pointed simplicial set.*

*Then the fundamental groupoid  $\pi(\Sigma X)$  is a trivial groupoid.*

*Proof.* The proof is an exercise.

Use the assumption that  $X$  is connected to show that the functor  $\pi(CX) \rightarrow \pi(\Sigma X)$  is full.  $\square$

## References

- [1] P. G. Goerss and J. F. Jardine. *Simplicial Homotopy Theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.
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# Lecture 11: Postnikov towers, some applications

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## 32 Postnikov towers

Suppose  $X$  is a simplicial set, and that  $x, y : \Delta^n \rightarrow X$  are  $n$ -simplices of  $X$ .

Say that  $x$  is  **$k$ -equivalent** to  $y$  and write  $x \sim_k y$  if there is a commutative diagram

$$\begin{array}{ccc} \mathrm{sk}_k \Delta^n & \xrightarrow{i} & \Delta^n \\ i \downarrow & & \downarrow y \\ \Delta^n & \xrightarrow{x} & X \end{array}$$

or if

$$x|_{\mathrm{sk}_k \Delta^n} = y|_{\mathrm{sk}_k \Delta^n}.$$

Write  $X(k)_n$  for the set of equivalence classes of  $n$ -simplices of  $X_n$  mod  $k$ -equivalence.

Every morphism  $\Delta^m \rightarrow \Delta^n$  induces a morphism  $\mathrm{sk}_k \Delta^m \rightarrow \mathrm{sk}_k \Delta^n$ . Thus, if  $x \sim_k y$  then  $\theta^*(x) \sim_k \theta^*(y)$ .

The sets  $X(k)_m$ ,  $m \geq 0$ , therefore assemble into a simplicial set  $X(k)$ .

The map

$$\pi_k : X \rightarrow X(k)$$

is the canonical surjection. It is natural in simplicial sets  $X$ , and is defined for  $k \geq 0$ .

$X(k)$  is the  $k^{th}$  **Postnikov section** of  $X$ .

If  $x \sim_{k+1} y$  then  $x \sim_k y$ . It follows that there are natural commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{\pi_{k+1}} & X(k+1) \\ & \searrow \pi_k & \downarrow p \\ & & X(k) \end{array}$$

The system of simplicial set maps

$$X(0) \xleftarrow{p} X(1) \xleftarrow{p} X(2) \xleftarrow{p} \dots$$

is called the **Postnikov tower** of  $X$ .

The map  $\pi_k : X_n \rightarrow X(k)_n$  of  $n$ -simplices is a bijection for  $n \leq k$ , since  $\text{sk}_k \Delta^n = \Delta^n$  in that case.

It follows that the induced map

$$X \rightarrow \varprojlim_k X(k)$$

is an isomorphism of simplicial sets.

**Lemma 32.1.** *Suppose  $X$  is a Kan complex. Then*

- 1)  $\pi_k : X \rightarrow X(k)$  is a fibration and  $X(k)$  is a Kan complex for  $k \geq 0$ .
- 2)  $\pi_k : X \rightarrow X(k)$  induces a bijection  $\pi_0(X) \cong \pi_0 X(k)$  and isomorphisms

$$\pi_i(X, x) \xrightarrow{\cong} \pi_i(X(k), x)$$

for  $1 \leq i \leq k$ .

- 3)  $\pi_i(X(k), x) = 0$  for  $i > k$ .

*Proof.* Suppose given a commutative diagram

$$\begin{array}{ccc} \Lambda_r^n & \xrightarrow{(x_0, \dots, \hat{x}_r, \dots, x_n)} & X \\ \downarrow & & \downarrow \pi_k \\ \Delta^n & \xrightarrow{[y]} & X(k) \end{array}$$

If  $n \leq k$  the lift  $y : \Delta^n \rightarrow X$  exists because  $\pi_k$  is an isomorphism in degrees  $\leq k$ .

If  $n = k + 1$  then  $d_i(y) = d_i([y]) = x_i$  for  $i \neq r$ , so that the representative  $y$  is again a suitable lift.

If  $n > k + 1$  there is a simplex  $x \in X_n$  such that  $d_i x = x_i$  for  $i \neq r$ , since  $X$  is a Kan complex.

There is an identity  $\text{sk}_k(\Lambda_r^n) = \text{sk}_k(\Delta^n)$  for since  $n \geq k + 2$ , and it follows that  $[x] = [y]$ .

We have proved that  $\pi_k$  is a Kan fibration.

Generally, if  $p : X \rightarrow Y$  is a surjective fibration and  $X$  is a Kan complex, then  $Y$  is a Kan complex (exercise).

It follows that all Postnikov sections  $X(k)$  are Kan complexes.

If  $n > k$ ,  $x \in X_0 = X(k)_0$  and the picture

$$\begin{array}{ccc} \partial\Delta^n & & \\ \downarrow & \searrow x & \\ \Delta^n & \xrightarrow{[\alpha]} & X(k) \end{array}$$

defines an element of  $\pi_n(X(k), x)$ , then all faces of the representative  $\alpha : \Delta^n \rightarrow X$  and all faces of the element  $x : \Delta^n \rightarrow X$  have the same  $k$ -skeleton,  $\alpha$  and  $x$  have the same  $k$ -skeleton, and so  $[\alpha] = [x]$ .

We have proved statements 1) and 3). Statement 2) is an exercise.  $\square$

The fibration trick used in the proof of Lemma 32.1 is a special case of the following:

**Lemma 32.2.** *Suppose given a commutative diagram of simplicial set maps*

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ & \searrow q & \downarrow \pi \\ & & Z \end{array}$$



such that  $p$  and  $q$  are fibrations and  $p$  is surjective in all degrees.

Then  $\pi$  is a fibration.

*Proof.* The proof is an exercise.  $\square$

**Remarks:**

1) If  $X$  is a Kan complex, it follows from Lemma 32.1 and Lemma 32.2 that all maps

$$p : X(k+1) \rightarrow X(k)$$

in the Postnikov tower for  $X$  are fibrations.

2) There is a natural commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta} & B\pi(X) \\ \pi_1 \downarrow & & \simeq \downarrow \pi_{1*} \\ X(1) & \xrightarrow[\eta]{\simeq} & B\pi(X(1)) \end{array}$$

for Kan complexes  $X$ , in which the indicated maps  $\eta$  and  $\pi_{1*}$  are weak equivalences by Lemma 28.5

3) Suppose that  $X$  is a connected Kan complex. The fibre  $F_n(X)$  of the fibration  $\pi_n : X \rightarrow X(n)$  is the  **$n$ -connected cover** of  $X$ . The space  $F_n(X)$  is  $n$ -connected, and the maps

$$\pi_k(F_n(X), z) \rightarrow \pi_k(X, z)$$

are isomorphisms for  $k \geq n+1$ , by Lemma 32.1.

The homotopy fibres of the map  $\pi_1 : X \rightarrow X(1)$ , equivalently of the map  $X \rightarrow B(\pi(X))$  are the **universal covers** of  $X$ .

All universal covers of  $X$  are simply connected, and are weakly equivalent because  $X$  is connected.

More is true. Replace  $\eta : X \rightarrow B\pi(X)$  by a fibration  $p : Z \rightarrow B(\pi(X))$ , and form the pullbacks

$$\begin{array}{ccc} p^{-1}(x) & \longrightarrow & Z \\ \downarrow & & \downarrow p \\ B(\pi(X)/x) & \longrightarrow & B\pi(X) \end{array}$$

All spaces  $p^{-1}(x)$  are universal covers, and there are weak equivalences

$$\operatorname{holim}_{x \in \pi(X)} p^{-1}(x) \xrightarrow{\sim} Z \xleftarrow{\sim} X.$$

Thus, every space  $X$  is a homotopy colimit of universal covers, indexed over its fundamental groupoid  $\pi(X)$ .

### 33 The Hurewicz Theorem

Suppose  $X$  is a pointed space.

The **Hurewicz map** for  $X$  is the composite

$$X \xrightarrow{\eta} \mathbb{Z}(X) \rightarrow \mathbb{Z}(X)/\mathbb{Z}(*)$$

where  $*$  denotes the base point of  $X$ .

The homology groups of the quotient

$$\tilde{\mathbb{Z}}(X) := \mathbb{Z}(X)/\mathbb{Z}(*)$$

are the **reduced homology groups** of  $X$ , and one writes

$$\tilde{H}_n(X) = H_n(\tilde{\mathbb{Z}}(X)).$$

The reduced homology groups  $\tilde{H}_n(X, A)$  are defined by

$$\tilde{H}_n(X, A) = H_n(\tilde{\mathbb{Z}}(X) \otimes A)$$

for any abelian group  $A$ .

The Hurewicz map is denoted by  $h$ . We have

$$h : X \rightarrow \tilde{\mathbb{Z}}(X).$$

**Lemma 33.1.** *Suppose that  $\pi$  is a group.*

*The homomorphism*

$$h_* : \pi_1(B\pi) \rightarrow \tilde{H}_1(B\pi)$$

*is isomorphic to the homomorphism*

$$\pi \rightarrow \pi/[\pi, \pi].$$

*Proof.* From the Moore chain complex  $\mathbb{Z}(B\pi)$ , the group  $H_1(B\pi) = \tilde{H}_1(B\pi)$  is the free abelian group  $\mathbb{Z}(\pi)$  on the elements of  $\pi$  modulo the relations  $g_1 g_2 - g_1 - g_2$  and  $e = 0$ .

The composite

$$\pi \xrightarrow{\cong} \pi_1(B\pi) \xrightarrow{h_*} H_1(B\pi)$$

is the canonical map. □

Consequence: If  $A$  is an abelian group, the map

$$h_* : \pi_1(BA) \rightarrow \tilde{H}_1(BA)$$

is an isomorphism.

**Lemma 33.2.** *Suppose  $X$  is a connected pointed space.*

*Then  $\eta : X \rightarrow B\pi(X)$  induces an isomorphism*

$$H_1(X) \xrightarrow{\cong} H_1(B\pi(X)).$$

*Proof.* The homotopy fibre  $F$  of  $\eta$  is simply connected, so  $H_1(F) = 0$  by Lemma 33.1 (or otherwise — exercise).

It follows that  $E_2^{0,1} = 0$  in the (general) Serre spectral sequence for the fibre sequence

$$F \rightarrow X \rightarrow B\pi(X)$$

Thus,  $E_\infty^{0,1} = 0$ , while  $E_2^{1,0} = E_\infty^{1,0} = H_1(B\pi(X))$ .

The edge homomorphism

$$H_1(X) \rightarrow H_1(B\pi(X)) = E_\infty^{1,0}$$

is therefore an isomorphism. □

The proof of the following result is an exercise:

**Corollary 33.3.** *Suppose  $X$  is a connected pointed Kan complex.*

*The Hurewicz homomorphism*

$$h_* : \pi_1(X) \rightarrow \tilde{H}_1(X)$$

*is an isomorphism if  $\pi_1(X)$  is abelian.*

The following result gives the relation between the path-loop fibre sequence and the Hurewicz map.

**Lemma 33.4.** *Suppose  $Y$  is an  $n$ -connected pointed Kan complex, with  $n \geq 1$ .*

*For  $2 \leq i \leq 2n$  there is a commutative diagram*

$$\begin{array}{ccc} \pi_i(Y) & \xrightarrow[\cong]{\partial} & \pi_{i-1}(\Omega Y) \\ h_* \downarrow & & \downarrow h_* \\ \tilde{H}_i(Y) & \xrightarrow[d_i]{\cong} & \tilde{H}_{i-1}(\Omega Y) \end{array}$$

*Proof.* Form the diagram

$$\begin{array}{ccccc} \tilde{\mathbb{Z}}(\Omega Y) & = & \tilde{\mathbb{Z}}(\Omega Y) & \xleftarrow{h} & \Omega Y \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{\mathbb{Z}}(C\Omega Y) & \longrightarrow & \tilde{\mathbb{Z}}(PY) & \xleftarrow{h} & PY \\ \downarrow & & \downarrow p_* & & \downarrow p \\ \tilde{\mathbb{Z}}(\Sigma\Omega Y) & \xrightarrow{\varepsilon_*} & \tilde{\mathbb{Z}}(Y) & \xleftarrow{h} & Y \end{array}$$

By replacing  $p_*$  by a fibration one finds a comparison diagram of fibre sequences and there is an induced diagram

$$\begin{array}{ccc} \pi_i(Y) & \xrightarrow{\partial} & \pi_{i-1}(\Omega Y) \\ h_* \downarrow & & \downarrow h_* \\ \tilde{H}_i(Y) & \xleftarrow[\varepsilon_*]{\cong} \tilde{H}_i(\Sigma\Omega Y) & \xrightarrow[\partial]{\cong} \tilde{H}_{i-1}(\Omega Y) \end{array}$$

The bottom composite is the transgression  $d_i$  by Corollary 31.4. □

**Theorem 33.5** (Hurewicz Theorem). *Suppose  $X$  is an  $n$ -connected pointed Kan complex, and that  $n \geq 1$ .*

*Then the Hurewicz homomorphism*

$$h_* : \pi_i(X) \rightarrow \tilde{H}_i(X)$$

*is an isomorphism if  $i = n + 1$  and is an epimorphism if  $i = n + 2$ .*

The proof of the Hurewicz Theorem requires some preliminary observations about Eilenberg-Mac Lane spaces:

The **good truncation**  $T_m C$  for a chain complex  $C$  is the chain complex

$$C_0 \xleftarrow{\partial} \dots \xleftarrow{\partial} C_{m-1} \xleftarrow{\partial^*} C_m / \partial(C_{m+1}) \leftarrow 0 \dots$$

The canonical map

$$C \rightarrow T_m(C)$$

induces isomorphisms  $H_i(C) \cong H_i(T_m(C))$  for  $i \leq m$ , while  $H_i(T_m(C)) = 0$  for  $i > m$ .

The isomorphism  $H_m(C) \cong H_m(T_m(C))$  is the “goodness”. It means that the functor  $C \mapsto T_m(C)$  preserves homology isomorphisms.

It follows that the composite

$$Y \xrightarrow{h_*} \tilde{\mathbb{Z}}(Y) \cong \Gamma N \tilde{\mathbb{Z}}(Y) \rightarrow \Gamma T_m N \tilde{\mathbb{Z}}(Y)$$

is a weak equivalence for a space  $Y$  of type  $K(A, m)$ , if  $A$  is abelian.

For this, we need to show that  $h_*$  induces an isomorphism  $\pi_m(Y) \rightarrow \tilde{H}_m(Y)$ .

This seems like a special case of the Hurewicz theorem, but it is true for  $m = 1$  by Corollary 33.3, and then true for all  $m \geq 1$  by an inductive argument that uses Lemma 33.4.

We have shown that there is a weak equivalence  $Y \rightarrow B$  where  $B$  is a simplicial abelian group of type  $K(A, m)$ .

It is an exercise to show that  $B$  is weakly equivalent as a simplicial abelian group to the simplicial abelian group

$$K(A, m) = \Gamma(A(m)).$$

*Proof of Theorem 33.5.* The space  $X(n+1)$  is an Eilenberg-Mac Lane space of type  $K(A, n+1)$ , where  $A = \pi_{n+1}(X)$ .

The Hurewicz map

$$h_* : \pi_m(Y) \rightarrow \tilde{H}_m(Y)$$

is an isomorphism for all spaces  $Y$  of type  $K(A, m)$ , for all  $m \geq 1$ .



We know from Lemma 29.5 and the remarks above that there is an isomorphism

$$H_{m+1}(Y) = 0$$

for all spaces  $Y$  of type  $K(A, m)$ , for all  $m \geq 2$ .

It follows that

$$H_{n+2}(X(n+1)) = 0.$$

Now suppose that  $F$  is the homotopy fibre of the map  $\pi_{n+1} : X \rightarrow X(n+1)$ .

There are diagrams

$$\begin{array}{ccc} \pi_{n+1}(X) & \xrightarrow{\cong} & \pi_{n+1}(X(n+1)) & \pi_{n+2}(F) & \xrightarrow{\cong} & \pi_{n+2}(X) \\ h_* \downarrow & & \cong \downarrow h_* & h_* \downarrow & & \downarrow h_* \\ \tilde{H}_{n+1}(X) & \longrightarrow & \tilde{H}_{n+1}(X(n+1)) & \tilde{H}_{n+2}(F) & \longrightarrow & \tilde{H}_{n+2}(X) \end{array}$$

The Serre spectral sequence for the fibre sequence

$$F \rightarrow X \rightarrow X(n+1)$$

is used to show that

- 1) the map  $H_{n+1}(X) \rightarrow H_{n+1}(X(n+1))$  is an isomorphism since  $F$  is  $(n+1)$ -connected, and
- 2) the map  $H_{n+2}(F) \rightarrow H_{n+2}(X)$  is surjective, since  $H_{n+2}(X(n+1)) = 0$ .

The isomorphism statement in the Theorem is a consequence of statement 1).

It follows that the map  $h_* : \pi_{n+2}(F) \rightarrow \tilde{H}_{n+2}(F)$  is an isomorphism since  $F$  is  $(n+1)$ -connected.

The surjectivity statement of the Theorem is then a consequence of statement 2).  $\square$

### 34 Freudenthal Suspension Theorem

Here's a first consequence of the Hurewicz Theorem (Theorem 33.5):

**Corollary 34.1.** *Suppose  $X$  is an  $n$ -connected space where  $n \geq 0$ .*

*Then the suspension  $\Sigma(X)$  is  $(n+1)$ -connected.*

*Proof.* The case  $n = 0$  has already been done, as an exercise. Suppose that  $n \geq 1$ .

Then  $\Sigma X$  is at least simply connected since  $X$  is connected, and  $\tilde{H}_k(\Sigma X) = 0$  for  $k \leq n+1$ .

Thus, the first non-vanishing homotopy group  $\pi_r(\Sigma X)$  is in degree at least  $n+2$ .  $\square$

**Theorem 34.2.** [*Freudenthal Suspension Theorem*]  
 Suppose  $X$  is an  $n$ -connected pointed Kan complex  
 where  $n \geq 0$ .

The homotopy fibre  $F$  of the canonical map

$$\eta : X \rightarrow \Omega\Sigma X$$

is  $2n$ -connected.

**Remark:** “The canonical map” in the statement of  
 the Theorem is actually the “derived” map, mean-  
 ing the composite

$$X \rightarrow \Omega(\Sigma X) \xrightarrow{j_*} \Omega(\Sigma X_f),$$

where  $j : \Sigma X \rightarrow \Sigma X_f$  is a fibrant model, ie. a weak  
 equivalence such that  $\Sigma X_f$  is fibrant.

*Proof.* In the triangle identity

$$\begin{array}{ccc} \Sigma X & \xrightarrow{\Sigma\eta} & \Sigma\Omega\Sigma(X) \\ & \searrow 1 & \downarrow \varepsilon \\ & & \Sigma X \end{array}$$

the space  $\Sigma X$  is  $(n+1)$ -connected (Corollary 34.1)  
 so that the map  $\varepsilon$  induces isomorphisms

$$\tilde{H}_i(\Sigma\Omega\Sigma X) \xrightarrow{\cong} \tilde{H}_i(\Sigma X)$$

for  $i \leq 2n+2$ , by Corollary 31.4.

It follows that  $\eta$  induces isomorphisms

$$\tilde{H}_i(X) \xrightarrow{\cong} \tilde{H}_i(\Omega\Sigma X) \quad (1)$$

for  $i \leq 2n + 1$ .

In the diagram

$$\begin{array}{ccc} \pi_{n+1}(X) & \xrightarrow{\eta_*} & \pi_{n+1}(\Omega\Sigma X) \\ h \downarrow & & \cong \downarrow h \\ H_{n+1}(X) & \xrightarrow[\eta_*]{\cong} & H_{n+1}(\Omega\Sigma X) \end{array}$$

the indicated Hurewicz map is an isomorphism for  $n > 0$  since  $\pi_1(\Omega\Sigma X)$  is abelian (Corollary 33.3), while the map  $h : \pi_1(X) \rightarrow H_1(X)$  is surjective by Lemma 33.1 and Lemma 33.2. It follows that  $\eta_* : \pi_{n+1}(X) \rightarrow \pi_{n+1}(\Omega\Sigma X)$  is surjective, so  $F$  is  $n$ -connected.

A Serre spectral sequence argument for the fibre sequence

$$F \rightarrow X \xrightarrow{\eta} \Omega\Sigma X$$

shows that that  $\tilde{H}_i(F) = 0$  for  $i \leq 2n$ , so the Hurewicz Theorem implies that  $F$  is  $2n$ -connected.

In effect,  $E_2^{i,0} \cong E_\infty^{i,0}$  for  $i \leq 2n + 1$  and  $E_\infty^{p,q} = 0$  for  $q > 0$  and  $p + q \leq 2n + 1$ , all by the isomorphisms in (1).

It follows that the first non-vanishing  $H_k(F)$  is in degree greater than  $2n$ .  $\square$

**Example:** The *suspension homomorphism*

$$\Sigma : \pi_i(S^n) \rightarrow \pi_i(\Omega(S^{n+1})) \cong \pi_{i+1}(S^{n+1})$$

is an isomorphism if  $i \leq 2(n-1)$  and is an epimorphism if  $i = 2n-1$ .

In effect, the homotopy fibre of  $S^n \rightarrow \Omega S^{n+1}$  is  $(2n-1)$ -connected.

In particular, the maps  $\Sigma : \pi_{n+k}(S^n) \rightarrow \pi_{n+1+k}(S^{n+1})$  are isomorphisms (ie. the groups stabilize) for  $n \geq k+2$ , ie.  $n+k \leq 2n-2$ .

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## Lecture 12: Cohomology: an introduction

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### 35 Cohomology

Suppose that  $C \in Ch_+$  is an ordinary chain complex, and that  $A$  is an abelian group.

There is a *cochain complex*  $\text{hom}(C, A)$  with

$$\text{hom}(C, A)^n = \text{hom}(C_n, A)$$

and *coboundary*

$$\delta : \text{hom}(C_n, A) \rightarrow \text{hom}(C_{n+1}, A)$$

defined by precomposition with  $\partial : C_{n+1} \rightarrow C_n$ .

Generally, a **cochain complex** is an unbounded complex which is concentrated in negative degrees. See Section 1.

We use classical notation for  $\text{hom}(C, A)$ : the corresponding complex in negative degrees is specified by

$$\text{hom}(C, A)_{-n} = \text{hom}(C_n, A).$$

The **cohomology group**  $H^n \operatorname{hom}(C, A)$  is specified by

$$H^n \operatorname{hom}(C, A) := \frac{\ker(\delta : \operatorname{hom}(C_n, A) \rightarrow \operatorname{hom}(C_{n+1}, A))}{\operatorname{im}(\delta : \operatorname{hom}(C_{n-1}, A) \rightarrow \operatorname{hom}(C_n, A))}.$$

This group coincides with the group  $H_{-n} \operatorname{hom}(C, A)$  for the complex in negative degrees.

**Exercise:** Show that there is a natural isomorphism

$$H^n \operatorname{hom}(C, A) \cong \pi(C, A(n))$$

where  $A(n)$  is the chain complex consisting of the group  $A$  concentrated in degree  $n$ , and  $\pi(C, A(n))$  is chain homotopy classes of maps.

**Example:** If  $X$  is a space, then the cohomology group  $H^n(X, A)$  is defined by

$$H^n(X, A) = H^n \operatorname{hom}(\mathbb{Z}(X), A) \cong \pi(\mathbb{Z}(X), A(n)),$$

where  $\mathbb{Z}(X)$  is the Moore complex for the free simplicial abelian group  $\mathbb{Z}(X)$  on  $X$ .

Here is why the classical definition of  $H^n(X, A)$  is not silly: all ordinary chain complexes are fibrant, and the Moore complex  $\mathbb{Z}(X)$  is free in each degree, hence cofibrant, and so there is an isomorphism

$$\pi(\mathbb{Z}(X), A(n)) \cong [\mathbb{Z}(X), A(n)],$$

where the square brackets determine morphisms in the homotopy category for the standard model structure on  $Ch_+$  (Theorem 3.1).

The normalized chain complex  $N\mathbb{Z}(X)$  is naturally weakly equivalent to the Moore complex  $\mathbb{Z}(X)$ , and there are natural isomorphisms

$$\begin{aligned} [\mathbb{Z}(X), A(n)] &\cong [N\mathbb{Z}(X), A(n)] \\ &\cong [\mathbb{Z}(X), K(A, n)] \quad (\text{Dold-Kan correspondence}) \\ &\cong [X, K(A, n)] \quad (\text{Quillen adjunction}) \end{aligned}$$

Here,  $[X, K(A, n)]$  is morphisms in the homotopy category for simplicial sets. We have proved the following:

**Theorem 35.1.** *There is a natural isomorphism*

$$H^n(X, A) \cong [X, K(A, n)]$$

*for all simplicial sets  $X$  and abelian groups  $A$ .*

In other words,  $H^n(X, A)$  is representable by the Eilenberg-Mac Lane space  $K(A, n)$  in the homotopy category.

Suppose that  $C$  is a chain complex and  $A$  is an abelian group. Define the **cohomology groups** (or hypercohomology groups)  $H^n(C, A)$  of  $C$  with coefficients in  $A$  by

$$H^n(C, A) = [C, A(n)].$$



This is the derived functor definition of cohomology.

**Example:** Suppose that  $A$  and  $B$  are abelian groups. We compute the groups  $H^n(A(0), B) = [A(0), B(n)]$ . This is done by replacing  $A(0)$  by a cofibrant model. There is a short exact sequence

$$0 \rightarrow F_1 \rightarrow F_0 \rightarrow A \rightarrow 0$$

with  $F_i$  free abelian. The chain complex  $F_*$  given by

$$\cdots \rightarrow 0 \rightarrow 0 \rightarrow F_1 \rightarrow F_0$$

is cofibrant, and the chain map  $F_* \rightarrow A(0)$  is a weak equivalence, hence a cofibrant replacement for the complex  $A(0)$ .

It follows that there are isomorphisms

$$[A(0), B(n)] \cong [F_*, B(n)] \cong \pi(F_*, B(n)) = H^n \operatorname{hom}(F_*, A),$$

and there is an exact sequence

$$\begin{aligned} 0 \rightarrow H^0 \operatorname{hom}(F_*, B) \rightarrow \operatorname{hom}(F_0, B) \rightarrow \operatorname{hom}(F_1, B) \\ \rightarrow H^1 \operatorname{hom}(F_*, B) \rightarrow 0. \end{aligned}$$

It follows that

$$[A(0), B(n)] = H^n \operatorname{hom}(F_*, B) = \begin{cases} \operatorname{hom}(A, B) & \text{if } n = 0, \\ \operatorname{Ext}^1(A, B) & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Similarly, there are isomorphisms

$$[A(p), B(n)] = \begin{cases} \text{hom}(A, B) & \text{if } n = p, \\ \text{Ext}^1(A, B) & \text{if } n = p + 1, \\ 0 & \text{if } n > p + 1 \text{ or } n < p. \end{cases}$$

Most generally, for ordinary chain complexes, we have the following:

**Theorem 35.2.** *Suppose that  $C$  is a chain complex, and  $B$  is an abelian group.*

*There is a short exact sequence*

$$0 \rightarrow \text{Ext}^1(H_{n-1}(C), B) \rightarrow H^n(C, B) \xrightarrow{p} \text{hom}(H_n(C), B) \rightarrow 0. \quad (1)$$

*The map  $p$  is natural in  $C$  and  $B$ . This sequence is split, with a non-natural splitting.*

Theorem 35.2 is the **universal coefficients theorem** for cohomology.

*Proof.* Let  $Z_p = \ker(\partial : C_p \rightarrow C_{p-1})$ . Pick a surjective homomorphism,  $F_0^p \rightarrow Z_p$  with  $F_0^p$  free, and  $F_1^p$  be the kernel of the (surjective) composite

$$F_0^p \rightarrow Z_p \rightarrow H_p(C).$$

Then  $F_1^p$  is free, and there is a map  $F_1^p \rightarrow C_{p+1}$

such that the diagram

$$\begin{array}{ccccc} F_1^p & \longrightarrow & C_{p+1} & & \\ \downarrow & & \downarrow \partial & & \\ F_0^p & \longrightarrow & Z_p & \longrightarrow & C_p \end{array}$$

commutes. Write  $\phi_p$  for the resulting chain map  $F_*^p[-p] \rightarrow C$ . Then the sum

$$\phi : \bigoplus_{p \geq 0} F_*^p[-p] \rightarrow C$$

( $\phi_n$  on the  $n^{th}$  summand) is a cofibrant replacement for the complex  $C$ .

At the same time, we have cofibrant resolutions  $F_*^p[-p] \rightarrow H_p(C)(p)$ , for  $p \geq 0$ .

It follows that there are isomorphisms

$$\begin{aligned} [C, B(n)] &\cong [\bigoplus_{p \geq 0} H_p(C)(p), B(n)] \\ &\cong \prod_{p \geq 0} [H_p(C)(p), B(n)] \\ &\cong \text{hom}(H_n(C), B) \oplus \text{Ext}^1(H_{n-1}(C), B). \end{aligned}$$

The induced map  $p : [C, B(n)] \rightarrow \text{hom}(H_p(C), B)$  is defined by restricting a chain map  $F \rightarrow B(n)$  to the group homomorphism  $Z_n(F) \subset F_n \rightarrow B$ , where  $F \rightarrow C$  is a cofibrant model of  $C$ .  $\square$

Recall that there are various models for the space  $K(A, n)$  in simplicial abelian groups. These include the object  $\Gamma A(n)$  arising from the Dold-Kan correspondence, and the space

$$A \otimes S^n \cong A \otimes (S^1)^{\otimes n}$$

where

$$S^n = (S^1)^{\wedge n} = S^1 \wedge \cdots \wedge S^1 \quad (n \text{ smash factors}).$$

In general, if  $K$  is a pointed simplicial set and  $A$  is a simplicial abelian group, we write

$$A \otimes K = A \otimes \tilde{\mathbb{Z}}(K),$$

where  $\tilde{\mathbb{Z}}(K)$  is the reduced Moore complex for  $K$ .

Suppose given a short exact sequence

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 0 \quad (2)$$

of simplicial abelian groups.

The diagram

$$\begin{array}{ccc} A & \longrightarrow & A \otimes \Delta_*^1 \\ \downarrow & & \downarrow 0 \\ B & \xrightarrow{p} & C \end{array}$$

is homotopy cocartesian, so there is a natural map  $\delta : C \rightarrow A \otimes S^1$  in the homotopy category. Pro-

ceeding inductively gives the **Puppe sequence**

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \xrightarrow{\delta} A \otimes S^1 \xrightarrow{i \otimes 1} B \otimes S^1 \xrightarrow{p \otimes 1} \dots \quad (3)$$

and a long exact sequence

$$[E, A] \rightarrow [E, B] \rightarrow [E, C] \xrightarrow{\delta} [E, A \otimes S^1] \rightarrow [E, B \otimes S^1] \rightarrow \dots$$

or equivalently

$$H^0(E, A) \rightarrow H^0(E, B) \rightarrow H^0(E, C) \xrightarrow{\delta} H^1(E, A) \rightarrow H^1(E, B) \rightarrow \dots \quad (4)$$

in cohomology, for arbitrary simplicial abelian groups (or chain complexes)  $E$ .

The morphisms  $\delta$  in the long exact sequence (4) are called **boundary maps**.

Specializing to  $E = \mathbb{Z}(X)$  for a space  $X$  and a short exact sequence of groups (2) gives the standard long exact sequence

$$H^0(X, A) \rightarrow H^0(X, B) \rightarrow H^0(X, C) \xrightarrow{\delta} H^1(X, A) \rightarrow H^1(X, B) \rightarrow \dots \quad (5)$$

in cohomology for the space  $X$ .

There are other ways of constructing the long exact sequence (5) — exercise.

### 36 Cup products

**Lemma 36.1.** *The twist automorphism*

$$\tau : S^1 \wedge S^1 \xrightarrow{\cong} S^1 \wedge S^1, \quad x \wedge y \mapsto y \wedge x.$$

*induces*

$$\tau_* = \times(-1) : H_2(S^1 \wedge S^1, \mathbb{Z}) \rightarrow H_2(S^1 \wedge S^1, \mathbb{Z}).$$

*Proof.* There are two non-degenerate 2-simplices  $\sigma_1, \sigma_2$  in  $S^1 \wedge S^1$  and a single non-degenerate 1-simplex  $\gamma = d_1 \sigma_1 = d_1 \sigma_2$ .

It follows that the normalized chain complex  $N\mathbb{Z}(S^1 \wedge S^1)$  has the form

$$\cdots \rightarrow 0 \rightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\nabla} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

where  $\nabla(m, n) = m + n$ . Thus,  $H_2(S^1 \wedge S^1, \mathbb{Z}) \cong \mathbb{Z}$ , generated by  $\sigma_1 - \sigma_2$ .

The twist  $\tau$  satisfies  $\tau(\sigma_1) = \sigma_2$  and fixes their common face  $\gamma$ .

Thus,  $\tau_*(\sigma_1 - \sigma_2) = \sigma_2 - \sigma_1$ . □

**Corollary 36.2.** *Suppose that  $\sigma \in \Sigma_n$  acts on  $(S^1)^{\wedge n}$  by shuffling smash factors.*

*Then the induced automorphism*

$$\sigma_* : H_n((S^1)^{\wedge n}, \mathbb{Z}) \rightarrow H_n((S^1)^{\wedge n}, \mathbb{Z}) \cong \mathbb{Z}$$

*is multiplication by the sign of  $\sigma$ .*

Explicitly, the action of  $\sigma$  on  $(S^1)^{\wedge n}$  is specified by

$$\sigma(x_1 \wedge \cdots \wedge x_n) = x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)}.$$

Suppose that  $A$  and  $B$  are abelian groups. There are natural isomorphisms of simplicial abelian groups

$$K(A, n) \otimes K(B, m) \xrightarrow{\cong} A \otimes B \otimes (S^1)^{\otimes(n+m)} = K(A \otimes B, n+m)$$

where the displayed isomorphism

$$(S^1)^{\otimes n} \otimes A \otimes (S^1)^{\otimes m} \otimes B \xrightarrow{\cong} (S^1)^{\otimes n} \otimes (S^1)^{\otimes m} \otimes A \otimes B$$

is defined by permuting the middle tensor factors.

Suppose that  $X$  and  $Y$  are simplicial sets, and suppose that  $f : X \rightarrow K(A, n)$  and  $g : Y \rightarrow K(B, m)$  are simplicial set maps.

There is a natural map

$$X \times Y \xrightarrow{\eta} \mathbb{Z}(X) \otimes \mathbb{Z}(Y),$$

which is defined by  $(x, y) \mapsto x \otimes y$ .

The composite

$$X \times Y \xrightarrow{\eta} \mathbb{Z}(X) \otimes \mathbb{Z}(Y) \xrightarrow{f_* \otimes g_*} K(A, n) \otimes K(B, m) \cong K(A \otimes B, n+m)$$

represents an element of  $H^{n+m}(X \times Y, A \otimes B)$ .

**Warning:** The isomorphism above has the form

$$\begin{aligned} a \otimes (x_1 \wedge \cdots \wedge x_n) \otimes b \otimes (y_1 \wedge \cdots \wedge y_m) \\ \mapsto a \otimes b \otimes (x_1 \wedge \cdots \wedge x_n \wedge y_1 \wedge \cdots \wedge y_m). \end{aligned}$$

Do **not** shuffle smash factors.

We have defined a pairing

$$\cup : H^n(X, A) \otimes H^m(Y, B) \rightarrow H^{n+m}(X \times Y, A \otimes B),$$

called the **external cup product**.

If  $R$  is a unitary ring, then the ring multiplication  $m : R \otimes R \rightarrow R$  and the diagonal  $\Delta : X \rightarrow X \times X$  together induce a composite

$$H^n(X, R) \otimes H^m(X, R) \xrightarrow{\cup} H^{n+m}(X \times X, R \otimes R) \xrightarrow{\Delta^* \cdot m_*} H^{n+m}(X, R)$$

which is the cup product

$$\cup : H^n(X, R) \otimes H^m(X, R) \rightarrow H^{n+m}(X, R)$$

for  $H^*(X, R)$ .

**Exercise:** Show that the cup product gives the cohomology  $H^*(X, R)$  the structure of a graded commutative ring with identity. This ring structure is natural in spaces  $X$  and rings  $R$ .

The graded commutativity follows from Corollary 36.2.



Suppose that we have a short exact sequence of simplicial abelian groups

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

and that  $D$  is a flat simplicial abelian group in the sense that the functor  $?\otimes D$  is exact. The sequence

$$\begin{aligned} 0 \rightarrow A \otimes D \xrightarrow{i \otimes 1} B \otimes D \xrightarrow{p \otimes 1} C \otimes D \xrightarrow{\delta \otimes 1} A \otimes S^1 \otimes D \\ \xrightarrow{i \otimes 1} B \otimes S^1 \otimes D \xrightarrow{p \otimes 1} \dots \end{aligned}$$

is equivalent to the Puppe sequence for the short exact sequence

$$0 \rightarrow A \otimes D \rightarrow B \otimes D \rightarrow C \otimes D \rightarrow 0$$

It follows that there is a commutative diagram

$$\begin{array}{ccc} [E, C] \otimes [F, D] & \xrightarrow{\cup} & [E \otimes F, C \otimes D] \\ \delta \otimes 1 \downarrow & & \downarrow \delta \\ [E, A \otimes S^1] \otimes [F, D] & \xrightarrow{\cup} & [E \otimes F, A \otimes D \otimes S^1] \end{array}$$

In particular, if  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is a short exact sequence of  $R$ -modules and  $X$  is a space, then there is a commutative diagram

$$\begin{array}{ccc} H^p(X, C) \otimes H^q(X, R) & \xrightarrow{\cup} & H^{p+q}(X, C) \\ \delta \otimes 1 \downarrow & & \downarrow \delta \\ H^{p+1}(X, A) \otimes H^q(X, R) & \xrightarrow{\cup} & H^{p+q+1}(X, A) \end{array} \quad (6)$$

It an exercise to show that the diagram

$$\begin{array}{ccc}
H^q(X, R) \otimes H^p(X, C) & \xrightarrow{\cup} & H^{q+p}(X, C) \\
1 \otimes \delta \downarrow & & \downarrow (-1)^q \delta \\
H^q(X, R) \otimes H^{p+1}(X, A) & \xrightarrow{\cup} & H^{q+p+1}(X, A)
\end{array} \quad (7)$$

commutes.

The diagrams (6) and (7) are cup product formulas for the boundary homomorphism.

### 37 Cohomology of cyclic groups

Suppose that  $\ell$  is a prime  $\neq 2$ . What follows is directly applicable to cyclic groups of  $\ell$ -primary roots of unity in fields.

We shall sketch the proof of the following:

**Theorem 37.1.** *There is a ring isomorphism*

$$H^*(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[x] \otimes \Lambda(y)$$

where  $|x| = 2$  and  $|y| = 1$ .

We write  $|z| = n$  for  $z \in H^n(X, A)$ .  $|z|$  is the **degree** of  $z$ .

In the statement of Theorem 37.1,  $x \in H^2(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell)$  and  $y \in H^1(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell)$ .

$\mathbb{Z}/\ell[x]$  is a graded polynomial ring with generator  $x$  in degree 2, and  $\Lambda(y)$  is an exterior algebra with generator  $y$  in degree 1.

**Fact:** If  $z \in H^{2k+1}(X, \mathbb{Z}/\ell)$  and  $\ell \neq 2$ , then

$$z \cdot z = (-1)^{(2k+1)(2k+1)} z \cdot z = (-1) z \cdot z,$$

so that  $2(z \cdot z) = 0$ , and  $z \cdot z = 0$ .

We know, from the Example at the end of Section 25, that there are isomorphisms

$$H_p(B\mathbb{Z}/\ell^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p = 0, \\ \mathbb{Z}/\ell^n & \text{if } p = 2k + 1, k \geq 0, \\ 0 & \text{if } p = 2k, k > 0. \end{cases}$$

It follows (exercise) that there are isomorphisms

$$H_p(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell, \text{ for } p \geq 0.$$

There is an isomorphism

$$H^p(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell) \cong \text{hom}(H_p(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell), \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell$$

for  $p \geq 0$  (Theorem 35.2).

1)  $x \in H^2(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell)$  is dual to the generator of the  $\ell$ -torsion subgroup of

$$\mathbb{Z}/\ell^n = H_1(B\mathbb{Z}/\ell^n, \mathbb{Z}).$$

2)  $y \in H^1(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell)$  is dual to the generator of

$$\mathbb{Z}/\ell \cong \mathbb{Z}/\ell^n \otimes \mathbb{Z}/\ell = H_1(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell).$$

Here's an integral coefficients calculation:

**Theorem 37.2.** *There is a ring isomorphism*

$$H^*(B\mathbb{Z}/\ell^n, \mathbb{Z}) \cong \mathbb{Z}[x]/(\ell^n \cdot x)$$

where  $|x| = 2$ .

This result appears in a book of Snaith, [1]. The argument uses explicit cocycles, with the Alexander-Whitney map ((7) of Section 26).

We can verify the underlying additive statement, namely that

$$H^p(B\mathbb{Z}/\ell^n, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } p = 0, \\ \mathbb{Z}/\ell^n & \text{if } p = 2k, k > 0, \\ 0 & \text{if } p \text{ odd} \end{cases}$$

Apply  $\text{hom}(\_, \mathbb{Z})$  to the exact sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{\ell^n} \mathbb{Z} \rightarrow \mathbb{Z}/\ell^n \rightarrow 0$$

to get the exact sequence

$$0 \rightarrow \text{hom}(\mathbb{Z}/\ell^n, \mathbb{Z}) \rightarrow \mathbb{Z} \xrightarrow{\ell^n} \mathbb{Z} \rightarrow \text{Ext}^1(\mathbb{Z}/\ell^n, \mathbb{Z}) \rightarrow 0$$

to show that  $\text{hom}(\mathbb{Z}/\ell^n, \mathbb{Z}) = 0$  (we knew this) and  $\text{Ext}^1(\mathbb{Z}/\ell^n, \mathbb{Z}) \cong \mathbb{Z}/\ell^n$ .

Then

$$H^{2k}(B\mathbb{Z}/\ell^n, \mathbb{Z}) \cong \text{Ext}^1(H_{2k-1}(B\mathbb{Z}/\ell^n, \mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}/\ell^n$$

for  $k > 0$  and

$$H^{2k+1}(B\mathbb{Z}/\ell^n, \mathbb{Z}) \cong \text{hom}(H_{2k+1}(B\mathbb{Z}/\ell^n, \mathbb{Z}), \mathbb{Z}) = 0$$

for  $k \geq 0$ .

*Proof of Theorem 37.1.* The exact sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{\times \ell} \mathbb{Z} \rightarrow \mathbb{Z}/\ell \rightarrow 0$$

is an exact sequence of  $\mathbb{Z}$ -modules, so that the Puppe sequence

$$0 \rightarrow K(\mathbb{Z}, 0) \xrightarrow{\times \ell} K(\mathbb{Z}, 0) \rightarrow K(\mathbb{Z}/\ell, 0) \xrightarrow{\delta} K(\mathbb{Z}, 1) \xrightarrow{\times \ell} K(\mathbb{Z}, 1) \rightarrow \dots$$

has an action by  $K(\mathbb{Z}, 2)$ .

It follows that there are commutative diagrams

$$\begin{array}{ccccc} H^p(B\mathbb{Z}/\ell^n, \mathbb{Z}) & \xrightarrow{\times \ell} & H^p(B\mathbb{Z}/\ell^n, \mathbb{Z}) & \longrightarrow & H^p(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell) \\ \cdot x \downarrow \cong & & \cong \downarrow \cdot x & & \downarrow \cdot x \\ H^{p+2}(B\mathbb{Z}/\ell^n, \mathbb{Z}) & \xrightarrow{\times \ell} & H^{p+2}(B\mathbb{Z}/\ell^n, \mathbb{Z}) & \longrightarrow & H^{p+2}(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell) \end{array}$$

and

$$\begin{array}{ccccc} H^p(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell) & \xrightarrow{\delta} & H^{p+1}(B\mathbb{Z}/\ell^n, \mathbb{Z}) & \xrightarrow{\times \ell} & H^{p+1}(B\mathbb{Z}/\ell^n, \mathbb{Z}) \\ \cdot x \downarrow & & \cdot x \downarrow \cong & & \cong \downarrow \cdot x \\ H^{p+2}(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell) & \xrightarrow{\delta} & H^{p+3}(B\mathbb{Z}/\ell^n, \mathbb{Z}) & \xrightarrow{\times \ell} & H^{p+3}(B\mathbb{Z}/\ell^n, \mathbb{Z}) \end{array}$$

for  $p > 0$ .

Thus, the cup product map

$$\cdot x : H^p(B\mathbb{Z}/\ell^n, \mathbb{Z}) \rightarrow H^{p+2}(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell)$$

is an isomorphism for all  $p$ .

Finally, the map

$$H^2(B\mathbb{Z}/\ell^n, \mathbb{Z}) \rightarrow H^2(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell)$$

is surjective, so the generator  $x \in H^2(B\mathbb{Z}/\ell^n, \mathbb{Z})$  maps to a generator  $x$  of  $H^2(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell)$ .

The ring homomorphism

$$\mathbb{Z}/\ell[x] \otimes \Lambda(y) \rightarrow H^*(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell)$$

defined by  $x \in H^2(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell)$  and a generator  $y \in H^1(B\mathbb{Z}/\ell^n, \mathbb{Z}/\ell)$  is then an isomorphism of  $\mathbb{Z}/\ell$ -vector spaces in all degrees.  $\square$

## References

- [1] Victor P. Snaith. *Topological methods in Galois representation theory*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons Inc., New York, 1989. A Wiley-Interscience Publication.

# Lecture 13: Spectra and stable equivalence

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## 38 Spectra

The approach to stable homotopy that follows was introduced in a seminal paper of Bousfield and Friedlander [2], which appeared in 1978.

A **spectrum**  $X$  consists of pointed (level) simplicial sets  $X^n$ ,  $n \geq 0$ , together with **bonding maps**

$$\sigma : S^1 \wedge X^n \rightarrow X^{n+1}.$$

A **map of spectra**  $f : X \rightarrow Y$  consists of pointed maps  $f : X^n \rightarrow Y^n$  which respect structure, in that the diagrams

$$\begin{array}{ccc} S^1 \wedge X^n & \xrightarrow{\sigma} & X^{n+1} \\ S^1 \wedge f \downarrow & & \downarrow f \\ S^1 \wedge Y^n & \xrightarrow{\sigma} & Y^{n+1} \end{array}$$

commute.

The category of spectra is denoted by **Spt**. This category is complete and cocomplete.

**Examples:**

1) Suppose  $Y$  is a pointed simplicial set. The **suspension spectrum**  $\Sigma^\infty Y$  consists of the pointed simplicial sets

$$Y, S^1 \wedge Y, S^1 \wedge S^1 \wedge Y, \dots, S^n \wedge Y, \dots$$

where

$$S^n = S^1 \wedge \dots \wedge S^1$$

( $n$ -fold smash power).

The bonding maps of  $\Sigma^\infty Y$  are the canonical isomorphisms

$$S^1 \wedge S^n \wedge Y \cong S^{n+1} \wedge Y.$$

There is a natural bijection

$$\text{hom}(\Sigma^\infty Y, X) \cong \text{hom}(X, Y^0).$$

The suspension spectrum functor is left adjoint to the “level 0” functor  $X \mapsto X^0$ .

2)  $S = \Sigma^\infty S^0$  is the **sphere spectrum**.

3) Suppose  $X$  is a spectrum and  $K$  is a pointed simplicial set.

The spectrum  $X \wedge K$  has level spaces

$$(X \wedge K)^n = X^n \wedge K,$$



and bonding maps

$$\sigma \wedge K : S^1 \wedge X^n \wedge K \rightarrow X^{n+1} \wedge K.$$

There is a natural isomorphism

$$\Sigma^\infty K \cong S \wedge K.$$

3)  $X \wedge S^1$  is the **suspension** of a spectrum  $X$ .

The **fake suspension**  $\Sigma X$  of  $X$  has level spaces  $S^1 \wedge X^n$  and bonding maps

$$S^1 \wedge \sigma : S^1 \wedge S^1 \wedge X^n \rightarrow S^1 \wedge X^{n+1}.$$

**Remark:** There is a commutative diagram

$$\begin{array}{ccc} S^1 \wedge S^1 \wedge X^n & \xrightarrow{S^1 \wedge \sigma} & S^1 \wedge X^{n+1} \\ \tau \wedge X^n \downarrow & \nearrow & \downarrow \cong \tau \\ S^1 \wedge S^1 \wedge X^n & & \\ S^1 \wedge \tau \downarrow \cong & & \\ S^1 \wedge X^n \wedge S^1 & \xrightarrow{\sigma \wedge S^1} & X^{n+1} \wedge S^1 \end{array}$$

where  $\tau$  flips adjacent smash factors:

$$\tau(x \wedge y) = y \wedge x.$$

The dotted arrow (bonding map induced by  $\sigma \wedge S^1$ ) differs from  $S^1 \wedge \sigma$  by precomposition by  $\tau \wedge X^n$ .

The flip  $\tau : S^1 \wedge S^1 \rightarrow S^1 \wedge S^1$  is non-trivial: it is multiplication by  $-1$  in  $H_2(S^2)$  (Lemma 36.1).

We recall some definitions and results from Section 15:

Suppose that  $X$  is a simplicial set, and write  $\tilde{\mathbb{Z}}(X)$  for the kernel of the map  $\mathbb{Z}(X) \rightarrow \mathbb{Z}(*)$ .

Then  $H_n(X, \mathbb{Z}) = \pi_n(\mathbb{Z}(X), 0)$  (see Theorem 15.4), and  $\tilde{H}_n(X, \mathbb{Z}) = \pi_n(\tilde{\mathbb{Z}}(X), 0)$  (reduced homology).

If  $X$  is pointed there is a natural isomorphism

$$\tilde{\mathbb{Z}}(X) \cong \mathbb{Z}(X)/\mathbb{Z}(*),$$

and there is a natural pointed map

$$h : X \xrightarrow{\eta} \mathbb{Z}(X) \rightarrow \tilde{\mathbb{Z}}(X)$$

(the Hurewicz map).

If  $A$  is a simplicial abelian group, there is a natural simplicial map

$$\gamma : S^1 \wedge A \rightarrow \tilde{\mathbb{Z}}(S^1) \otimes A =: S^1 \otimes A,$$

defined by  $x \wedge a \mapsto x \otimes a$ .

4) The **Eilenberg-Mac Lane spectrum**  $H(A)$  associated to a simplicial abelian group  $A$  consists of the spaces

$$A, S^1 \otimes A, S^2 \otimes A, \dots$$

with bonding maps

$$S^1 \wedge (S^n \otimes A) \xrightarrow{\gamma} S^1 \otimes (S^n \otimes A) \cong S^{n+1} \otimes A.$$

5) Suppose  $X$  is a spectrum and  $K$  is a pointed simplicial set.

The spectrum  $\mathbf{hom}_*(K, X)$  has

$$\mathbf{hom}_*(K, X)^n = \mathbf{hom}_*(K, X^n),$$

with bonding map

$$S^1 \wedge \mathbf{hom}_*(K, X^n) \rightarrow \mathbf{hom}_*(K, X^{n+1})$$

adjoint to the composite

$$S^1 \wedge \mathbf{hom}_*(K, X^n) \wedge K \xrightarrow{S^1 \wedge ev} S^1 \wedge X^n \xrightarrow{\sigma} X^{n+1}.$$

There is a natural bijection

$$\mathbf{hom}(X \wedge K, Y) \cong \mathbf{hom}(X, \mathbf{hom}_*(K, Y)).$$

Suppose  $X$  is a spectrum and  $n \in \mathbb{Z}$ .

The **shifted spectrum**  $X[n]$  has

$$X[n]^m = \begin{cases} * & m+n < 0 \\ X^{m+n} & m+n \geq 0 \end{cases}$$

**Examples:**  $X[-1]^0 = *$  and  $X[-1]^n = X^{n-1}$  for  $n \geq 1$ .

$X[1]^n = X^{n+1}$  for all  $n \geq 0$ .

**Remarks:** 1) The bonding maps define a natural map

$$\Sigma X \rightarrow X[1].$$

We'll see later that this map is a stable equivalence, and that there is a stable equivalence

$$\Sigma X \simeq X \wedge S^1.$$

2) There is a natural bijection

$$\text{hom}(X[n], Y) \cong \text{hom}(X, Y[-n])$$

and a stable equivalence  $X[n][-n] \rightarrow X$ , so that all shift operators are invertible in the stable category.

3) There is a natural bijection

$$\text{hom}(\Sigma^\infty K[-n], Y) \cong \text{hom}(K, Y^n)$$

for  $n \geq 0$ , so that the  $n^{\text{th}}$  level functor  $Y \mapsto Y^n$  has a left adjoint.

4) The  $n^{\text{th}}$  **layer**  $L_n X$  of a spectrum  $X$  consists of the spaces

$$X^0, \dots, X^n, S^1 \wedge X^n, S^2 \wedge X^n, \dots$$

There are obvious maps

$$L_n X \rightarrow L_{n+1} X \rightarrow X$$

and a natural isomorphism

$$\varinjlim_n L_n X \cong X.$$

The functor  $X \mapsto L_n X$  is left adjoint to truncation up to level  $n$ .

The system of maps

$$\Sigma^\infty X^0 = L_0 X \rightarrow L_1 X \rightarrow \dots$$

is called the **layer filtration** of  $X$ .

Here's an exercise: show that there are natural pushout diagrams

$$\begin{array}{ccc} \Sigma^\infty(S^1 \wedge X^n)[-n-1] & \longrightarrow & L_n X \\ \sigma_* \downarrow & & \downarrow \\ \Sigma^\infty X^{n+1}[-n-1] & \longrightarrow & L_{n+1} X \end{array}$$

### 39 Strict model structure

A map  $f : X \rightarrow Y$  is a **strict (levelwise) weak equivalence** (resp. **strict (levelwise) fibration**) if all maps  $f : X^n \rightarrow Y^n$  are weak equivalences (resp. fibrations) of pointed simplicial sets.

A **cofibration** is a map  $i : A \rightarrow B$  such that

- 1)  $i : A^0 \rightarrow B^0$  is a cofibration of (pointed) simplicial sets, and
- 2) all maps

$$(S^1 \wedge B^n) \cup_{(S^1 \wedge A^n)} A^{n+1} \rightarrow B^{n+1}$$

are cofibrations.

**Exercise:** Show that all cofibrations are levelwise cofibrations.

Given spectra  $X, Y$ , the function complex  $\mathbf{hom}(X, Y)$  is a simplicial set with

$$\mathbf{hom}(X, Y)_n = \mathbf{hom}(X \wedge \Delta_+^n, Y).$$

Recall that  $\Delta_+^n = \Delta^n \sqcup \{*\}$  is the simplex with a disjoint base point attached.

**Proposition 39.1.** *With these definitions, the category **Spt** of spectra satisfies the axioms for a proper closed simplicial model category.*

*This model structure is also cofibrantly generated.*

*Proof.* Suppose given a lifting problem

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ i \downarrow & \nearrow \theta^0 & \downarrow p \\ B & \xrightarrow{\beta} & Y \end{array}$$

where  $i$  is a cofibration and  $p$  is a strict fibration and strict weak equivalence.

The lifting  $\theta^0$  exists in the diagram

$$\begin{array}{ccc} A^0 & \xrightarrow{\alpha} & X^0 \\ i \downarrow & \nearrow \theta^0 & \downarrow p \\ B^0 & \xrightarrow{\beta} & Y^0 \end{array}$$

and then  $\theta^1$  exists in the diagram

$$\begin{array}{ccc} (S^1 \wedge B^0) \cup_{(S^1 \wedge A^0)} A^1 & \xrightarrow{(\theta_*^0, \alpha)} & X^1 \\ cof \downarrow & \nearrow \theta^1 & \downarrow p \\ B^1 & \xrightarrow{\beta} & Y^1 \end{array}$$

Proceed inductively to show that the lifting problem can be solved.

The lifting problem is solved in a similar way if  $i$  is a trivial cofibration and  $p$  is a strict fibration.

We have proved the lifting axiom **CM4**.

Suppose that  $f : X \rightarrow Y$  is a map of spectra, and find a factorization

$$\begin{array}{ccc} X^0 & \xrightarrow{i^0} & Z^0 \\ & \searrow f & \downarrow p^0 \\ & & Y^0 \end{array}$$

in level 0, where  $i^0$  is a cofibration and  $p^0$  is a fibration.

Form the diagram

$$\begin{array}{ccc} S^1 \wedge X^0 & \xrightarrow{\quad} & X^1 \\ S^1 \wedge i^0 \downarrow & \nearrow i_* & \downarrow f \\ S^1 \wedge Z^0 & \xrightarrow{\quad} & (S^1 \wedge Z^0) \cup X^1 \\ S^1 \wedge p^0 \downarrow & \searrow f_* & \downarrow \\ S^1 \wedge Y^0 & \xrightarrow{\quad} & Y^1 \end{array}$$

and find a factorization

$$\begin{array}{ccc} (S^1 \wedge Z^0) \cup X^1 & \xrightarrow{j} & Z^1 \\ & \searrow f_* & \downarrow p^1 \\ & & Y^1 \end{array}$$

where  $j$  is a cofibration and  $p^1$  is a trivial fibration. Write  $i^1 = j \cdot i_*$ .

We have factorized  $f$  as a cofibration followed by a trivial fibration up to level 1. Proceed inductively to show that  $f = p \cdot i$  where  $p$  is a trivial strict fibration and  $i$  is a cofibration.



The other factorization statement has the same proof, giving **CM5**.

The simplicial model structure is inherited from pointed simplicial sets, as is properness (exercise).

The generating sets for the cofibrations and trivial cofibrations, respectively are the maps

$$\Sigma^\infty(\Lambda_k^n)_+[-m] \rightarrow \Sigma^\infty \Delta_+^n[-m]$$

and

$$\Sigma^\infty(\partial \Delta^n)_+[-m] \rightarrow \Sigma^\infty \Delta_+^n[-m]$$

respectively. □

## 40 Stable equivalences

Suppose  $X$  is a pointed simplicial set, and recall that the loop space  $\Omega X$  is defined by

$$\Omega X = \mathbf{hom}_*(S^1, X).$$

The construction only makes homotopy theoretic sense (ie. preserves weak equivalences) if  $X$  is fibrant — in that case there are isomorphisms

$$\pi_{i+1}(X, *) \cong \pi_i(\Omega X, *), \quad i \geq 0,$$

of simplicial homotopy groups ( $*$  is the base point for  $X$ ), by a standard long exact sequence argument (see Section 31).

If  $X$  is not fibrant, then  $\Omega X$  is most properly a derived functor:

$$\Omega X := \Omega X_f$$

where  $j : X \rightarrow X_f$  is a fibrant model for  $X$  in the sense that  $j$  is a weak equivalence and  $X_f$  is fibrant.

This construction can be made functorial, since the category  $\mathbf{sSet}_*$  of pointed simplicial sets has functorial fibrant replacements.

There is a natural bijection

$$\mathbf{hom}(Z \wedge S^1, X) \cong \mathbf{hom}(Z, \Omega X).$$

so that every morphism  $f : Z \wedge S^1 \rightarrow X$  has a uniquely determined adjoint  $f_* : Z \rightarrow \Omega X$ .

We can say that a spectrum  $X$  consists of pointed simplicial sets  $X^n, n \geq 0$ , and **adjoint bonding maps**  $\sigma_* : X^n \rightarrow \Omega X^{n+1}$

Here are two constructions::

- 1) There is a natural (levelwise) fibrant model  $j : Y \rightarrow FY$  in the strict model structure for **Spt**.
- 2) Suppose  $X$  is a spectrum. Set

$$\Omega^\infty X^n = \varinjlim (X^n \xrightarrow{\sigma_*} \Omega X^{n+1} \xrightarrow{\Omega \sigma_*} \Omega^2 X^{n+2} \rightarrow \dots).$$

The comparison diagram

$$\begin{array}{ccccccc} X^n & \xrightarrow{\sigma_*} & \Omega X^{n+1} & \xrightarrow{\Omega \sigma_*} & \Omega^2 X^{n+2} & \longrightarrow & \dots \\ \sigma_* \downarrow & & \downarrow \Omega \sigma_* & & \downarrow \Omega^2 \sigma_* & & \\ \Omega X^{n+1} & \xrightarrow{\Omega \sigma_*} & \Omega^2 X^{n+2} & \xrightarrow{\Omega^2 \sigma_*} & \Omega^3 X^{n+3} & \longrightarrow & \dots \end{array}$$

determines a spectrum structure  $\Omega^\infty X$  and a natural map  $\omega : X \rightarrow \Omega^\infty X$ .

The adjoint bonding map

$$\Omega^\infty X^n \xrightarrow{\sigma_*} \Omega(\Omega^\infty X^{n+1})$$

is an isomorphism (exercise).

Write  $QY = \Omega^\infty FY$  and let  $\eta : Y \rightarrow QY$  be the composite

$$Y \xrightarrow{j} FY \xrightarrow{\omega} \Omega^\infty FY = QY.$$

The spectrum  $QY$  is the **stabilization** of  $Y$ .

Say that a map  $f : X \rightarrow Y$  is a **stable equivalence** if the map  $f_* : QX \rightarrow QY$  is a strict equivalence.

**Remarks:**

1) All spaces  $QY^n$  are fibrant (NB: this is a special property of “ordinary” spectra), and the map  $\sigma_* : QY^n \rightarrow \Omega QY^{n+1}$  is an isomorphism.

2) All  $QY^n$  are  $H$ -spaces with groups  $\pi_0 QY^n$  of path components. All induced maps  $f_* : QX^n \rightarrow QY^n$  are  $H$ -maps.

It follows that the maps  $f_* : QX^n \rightarrow QY^n$  are weak equivalences (or that  $f$  is a stable equivalence) if and only if all maps

$$\pi_i(QX^n, *) \rightarrow \pi_i(QY^n, *)$$

*based at the distinguished base point* are isomorphisms.

Define the **stable homotopy groups**  $\pi_k^s Y$ ,  $k \in \mathbb{Z}$  by

$$\pi_k^s Y = \varinjlim_{n+k \geq 0} (\cdots \rightarrow \pi_{n+k} F Y^n \rightarrow \pi_{n+k+1} F Y^{n+1} \rightarrow \cdots),$$

where the maps of homotopy groups are induced by the maps  $\sigma_* : F Y^n \rightarrow \Omega F Y^{n+1}$ .

There are isomorphisms

$$\pi_k(QY^n, *) \cong \pi_{k-n}^s Y,$$

so  $f : X \rightarrow Y$  is a stable equivalence if and only if  $f$  induces an isomorphism in all stable homotopy groups.

The strict model structure on the category of spectra **Spt** and the stabilization functor  $Q$  fits into a general framework.

Suppose **M** is a right proper closed model category with a functor  $Q : \mathbf{M} \rightarrow \mathbf{M}$ , and suppose there is a natural map  $\eta_X : X \rightarrow QX$ .

Say that a map  $f : X \rightarrow Y$  of **M** is a  **$Q$ -equivalence** if the induced map  $Qf : QX \rightarrow QY$  is a weak equivalence of **M**.

**$Q$ -cofibrations** are cofibrations of **M**.

A  **$Q$ -fibration** is a map which has the RLP wrt all maps which are cofibrations and  $Q$ -equivalences.

Here are some conditions:

**A4** The functor  $Q$  preserves weak equivalences of  $\mathbf{M}$ .

**A5** The maps  $\eta_{QX}, Q(\eta_X) : QX \rightarrow QQX$  are weak equivalences of  $\mathbf{M}$ .

**A6'**  $Q$ -equivalences are stable under pullback along  $Q$ -fibrations.

**Theorem 40.1** (Bousfield-Friedlander). *Suppose that  $\mathbf{M}$  is a right proper closed model category. Suppose that we have a functor  $Q : \mathbf{M} \rightarrow \mathbf{M}$ , and natural map  $\eta : X \rightarrow QX$ . Suppose that the  $Q$ -equivalences, cofibrations and  $Q$ -fibrations satisfy the axioms **A4**, **A5** and **A6'**.*

*Then  $\mathbf{M}$ , together with these three classes of maps, has the structure of a right proper closed model category.*

**Proposition 40.2.** *The category **Spt** of spectra and the stabilization functor  $Q$  satisfy the axioms **A4**, **A5** and **A6'**.*

For the proof of Proposition 40.2, the condition **A4** is a consequence of the following:

**Lemma 40.3.** *Suppose  $I$  is a filtered category, and suppose given a natural transformation  $f : X \rightarrow Y$  of functors  $X, Y : I \rightarrow \mathbf{sSet}$  such that each component map  $f_i : X_i \rightarrow Y_i$  is a weak equivalence.*

*Then the map  $f_* : \varinjlim_i X_i \rightarrow \varinjlim_i Y_i$  is a weak equivalence.*

*Proof.* Exercise. □

To verify condition **A5** for the category of spectra, consider the diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{j} & FX & \xrightarrow{\omega} & \Omega^\infty FX \\
 j \downarrow & & \simeq \downarrow j & & \simeq \downarrow j \\
 FX & \xrightarrow[Fj]{\simeq} & FFX & \xrightarrow{F\omega} & F\Omega^\infty FX \\
 \omega \downarrow & & \omega \downarrow & & \omega \downarrow \\
 \Omega^\infty FX & \xrightarrow[\Omega^\infty Fj]{\simeq} & \Omega^\infty FFX & \xrightarrow[\Omega^\infty F\omega]{\simeq} & \Omega^\infty F\Omega^\infty FX
 \end{array}$$

The indicated maps are strict weak equivalences, so it suffices to show that  $\Omega^\infty F\omega$  and

$$\omega : F\Omega^\infty FX \rightarrow \Omega^\infty F\Omega^\infty FX$$

are strict weak equivalences.

Here's another picture:

$$\begin{array}{ccccc}
FX & \xrightarrow{\omega} & \Omega^\infty FX & & \\
\downarrow j & \searrow \omega & \downarrow j \simeq & \searrow \omega & \\
& \Omega^\infty FX & \xrightarrow{\Omega^\infty \omega} & \Omega^\infty \Omega^\infty FX & \\
& \downarrow \Omega^\infty j \simeq & & \downarrow \Omega^\infty j & \\
FFX & \xrightarrow{F\omega} & F\Omega^\infty FX & & \\
& \searrow \omega & \downarrow \omega & \searrow \omega & \\
& \Omega^\infty FFX & \xrightarrow{\Omega^\infty F\omega} & \Omega^\infty F\Omega^\infty FX &
\end{array}$$

It's an exercise to show that  $\Omega^\infty \omega$  is an isomorphism: actually

$$\omega = \Omega^\infty \omega : \Omega^\infty FX \rightarrow \Omega^\infty \Omega^\infty FX.$$

Then the required maps are strict equivalences.

To verify **A6'**, use the fact that every strict fibre sequence  $F \rightarrow X \rightarrow Y$  induces a long exact sequence

$$\cdots \rightarrow \pi_k^s F \rightarrow \pi_k^s X \rightarrow \pi_k^s Y \xrightarrow{\partial} \pi_{k-1}^s F \rightarrow \cdots$$

(exercise). “Right properness” follows from an exact sequence comparison.

This completes the proof of Proposition 40.2

The model structure on **Spt** arising from the Bousfield-Friedlander Theorem via Proposition 40.2 and Theorem 40.1 is called the **stable model structure** for the category of spectra.



The homotopy category  $\mathrm{Ho}(\mathbf{Spt})$  is the **stable category**.

This is traditional usage, but also a misnomer, because there are many stable categories.

The proof of Theorem 40.1 is accomplished with a series of lemmas.

Recall that  $\mathbf{M}$  is a right proper closed model category with functor  $Q : \mathbf{M} \rightarrow \mathbf{M}$  and natural transformation  $\eta : X \rightarrow QX$  such that the following conditions hold:

**A4** The functor  $Q$  preserves weak equivalences of  $\mathbf{M}$ .

**A5** The maps  $\eta_{QX}, Q(\eta_X) : QX \rightarrow QQX$  are weak equivalences of  $\mathbf{M}$ .

**A6'**  $Q$ -equivalences are stable under pullback along  $Q$ -fibrations.

**Lemma 40.4.** *A map  $p : X \rightarrow Y$  is a  $Q$ -fibration and a  $Q$ -equivalence if and only if it is a trivial fibration of  $\mathbf{M}$ .*

*Proof.* Every trivial fibration  $p$  has the RLP wrt all cofibrations, and is therefore a  $Q$ -fibration.  $p$  is also a  $Q$ -equivalence, by **A4**.

Suppose that  $p : X \rightarrow Y$  is a  $Q$ -fibration and a  $Q$ -equivalence.

There is a factorization

$$\begin{array}{ccc} X & \xrightarrow{j} & Z \\ & \searrow p & \downarrow \pi \\ & & Y \end{array}$$

where  $j$  is a cofibration and  $\pi$  is a trivial fibration of  $\mathbf{M}$ .

$\pi$  is a  $Q$ -equivalence by **A4**, so  $j$  is a  $Q$ -equivalence.

There is a diagram

$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ j \downarrow & \nearrow & \downarrow p \\ Z & \xrightarrow{\pi} & Y \end{array}$$

since  $j$  is a cofibration and a  $Q$ -equivalence and  $p$  is a  $Q$ -fibration.

Then  $p$  is a retract of  $\pi$  and is therefore a trivial fibration of  $\mathbf{M}$ . □

**Lemma 40.5.** *Suppose  $p : X \rightarrow Y$  is a fibration of  $\mathbf{M}$  and the maps  $\eta : X \rightarrow QX$ ,  $\eta : Y \rightarrow QY$  are weak equivalences of  $\mathbf{M}$ .*

*Then  $p$  is a  $Q$ -fibration.*

*Proof.* Consider the lifting problem

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ i \downarrow & \nearrow & \downarrow p \\ B & \xrightarrow{\beta} & Y \end{array}$$

There is a diagram

$$\begin{array}{ccccc} QA & \xrightarrow{Q\alpha} & QX & & \\ Qi \downarrow & j_\alpha \searrow & Z & \nearrow p_\alpha & \downarrow Qp \\ QB & \xrightarrow{\pi} & QY & & \\ j_\beta \searrow & W & \nearrow p_\beta & & \end{array}$$

where  $j_\alpha, j_\beta$  are trivial cofibrations of  $\mathbf{M}$  and  $p_\alpha, p_\beta$  are fibrations.

There is an induced diagram

$$\begin{array}{ccccc} A & \longrightarrow & Z \times_{QX} X & \longrightarrow & X \\ i \downarrow & & \pi_* \downarrow & & \downarrow p \\ B & \longrightarrow & W \times_{QY} Y & \longrightarrow & Y \end{array}$$

and the lifting problem is solved if we can show that  $\pi_*$  is a weak equivalence.

But there is finally a diagram

$$\begin{array}{ccccc} QA & \xrightarrow{j_\alpha} & Z & \xleftarrow{pr} & Z \times_{QX} X \\ Qi \downarrow & & \downarrow \pi & & \downarrow \pi_* \\ QB & \xrightarrow{j_\beta} & W & \xleftarrow{pr} & W \times_{QY} Y \end{array}$$

The maps  $Qi$ ,  $j_\alpha$  and  $j_\beta$  are weak equivalences of  $\mathbf{M}$  so that  $\pi$  is a weak equivalence.

The maps  $pr$  are weak equivalences by right properness of  $\mathbf{M}$  and the assumptions on  $p$ .

It follows that  $\pi_*$  is a weak equivalence of  $\mathbf{M}$ .  $\square$

**Lemma 40.6.** *Every map  $f : QX \rightarrow QY$  has a factorization  $f = q \cdot j$ , where  $j$  is a cofibration and  $Q$ -equivalence and  $q$  is a  $Q$ -fibration.*

*Proof.*  $f$  has a factorization  $f = q \cdot j$  where  $j$  is a trivial cofibration and  $q$  is a fibration of  $\mathbf{M}$ .

$j$  is a  $Q$ -equivalence by **A4**, and  $q$  is a  $Q$ -fibration by Lemma 40.5.

In effect, there is a diagram

$$\begin{array}{ccccc} QX & \xrightarrow[\simeq]{j} & Z & \xrightarrow{p} & QY \\ \eta \downarrow \simeq & & \eta \downarrow & & \simeq \downarrow \eta \\ QQX & \xrightarrow[\simeq]{Qj} & QZ & \xrightarrow[\simeq]{Qp} & QQY \end{array}$$

so  $\eta : Z \rightarrow QZ$  is a weak equivalence of  $\mathbf{M}$ .  $\square$

**Lemma 40.7.** *Every map  $f : X \rightarrow Y$  has a factorization  $f = q \cdot j$ , where  $j$  is a cofibration and  $Q$ -equivalence and  $q$  is a  $Q$ -fibration.*

*Proof.* The map  $f_* : QX \rightarrow QY$  has a factorization

$$\begin{array}{ccc} QX & \xrightarrow{f_*} & QY \\ & \searrow i & \nearrow p \\ & X & \end{array}$$

where  $p$  is a  $Q$ -fibration and  $i$  is a cofibration and a  $Q$ -equivalence, by Lemma 40.6.

Form the diagram

$$\begin{array}{ccccc} X & \xrightarrow{i_*} & Z \times_{QY} Y & \xrightarrow{p_*} & Y \\ \eta \downarrow & & \downarrow \eta_* & & \downarrow \eta \\ QX & \xrightarrow{i} & Z & \xrightarrow{p} & QY \end{array}$$

The maps  $\eta$  are  $Q$ -equivalence by **A5**, so  $\eta_*$  is a  $Q$ -equivalence by **A6'**. It follows that  $i_*$  is a  $Q$ -equivalence.

The map  $i_*$  has a factorization

$$\begin{array}{ccc} X & \xrightarrow{i_*} & Z \times_{QY} Y \\ & \searrow j & \nearrow \pi \\ & W & \end{array}$$

where  $j$  is a cofibration and  $\pi$  is a trivial strict fibration.

Then  $\pi$  is a  $Q$ -equivalence and a  $Q$ -fibration by Lemma 40.4, so  $j$  is a  $Q$ -equivalence, and the composite  $p_* \cdot \pi$  is a  $Q$ -fibration.  $\square$

*Proof of Theorem 40.1.* The non-trivial closed model statements are the lifting axiom **CM4** and the factorization axiom **CM5**.

**CM5** is a consequence of Lemma 40.4 and Lemma 40.7. **CM4** follows from Lemma 40.4.

The right properness of the model structure is the statement **A6'**.  $\square$

Say that the model structure on  $\mathbf{M}$  given by Theorem 40.1 is the *Q-structure*.

**Lemma 40.8.** *Suppose that, in addition to the assumptions of Theorem 40.1, that the model structure  $\mathbf{M}$  is left proper.*

*Then the Q-structure on  $\mathbf{M}$  is left proper.*

*Proof.* Suppose given a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ i \downarrow & & \downarrow \\ B & \xrightarrow{f_*} & B \cup_A C \end{array}$$

where  $f$  is a  $Q$ -equivalence and  $i$  is a cofibration. We must show that  $f_*$  is a  $Q$ -equivalence (see Definition 17.4).

Find a factorization

$$\begin{array}{ccc} A & \xrightarrow{j} & D \\ & \searrow f & \downarrow \pi \\ & & C \end{array}$$

where  $j$  is a cofibration and  $\pi$  is a trivial fibration of  $\mathbf{M}$ .

The map  $\pi_* : B \cup_A D \rightarrow B \cup_A C$  is a weak equivalence of  $\mathbf{M}$  by left properness for  $\mathbf{M}$ , so  $\pi_*$  is a  $Q$ -equivalence by **A4**.

$j$  is a  $Q$ -equivalence as well as a cofibration, so that  $j_* : B \rightarrow B \cup_A D$  is a cofibration and a  $Q$ -equivalence.

Then the composite  $f_* = \pi_* \cdot j_*$  is a  $Q$ -equivalence.

□

Here's the other major abstract result in this game, again from [2]:

**Theorem 40.9.** *Suppose the model category  $\mathbf{M}$  and the functor  $Q$  satisfy the conditions for Theorem 40.1*

*Then a map  $p : X \rightarrow Y$  of  $\mathbf{M}$  is a stable fibration if and only if the following conditions hold:*

- 1)  $p$  is a fibration of  $\mathbf{M}$ , and

2) the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta} & QX \\ p \downarrow & & \downarrow Qp \\ Y & \xrightarrow{\eta} & QY \end{array}$$

is homotopy cartesian in  $\mathbf{M}$ .

**Corollary 40.10.** 1) An object  $X$  of  $\mathbf{M}$  is  $Q$ -fibrant if and only if it is fibrant and the map  $\eta : X \rightarrow QX$  is a weak equivalence of  $\mathbf{M}$ .

2) A spectrum  $X$  is stably fibrant if and only if it is strictly fibrant and all adjoint bonding maps  $\sigma_* : X^n \rightarrow \Omega X^{n+1}$  are weak equivalences of pointed simplicial sets.

Fibrant spectra are often called  **$\Omega$ -spectra**.

**Corollary 40.11.** Suppose given a diagram

$$\begin{array}{ccc} X & \xrightarrow{\simeq} & X' \\ p \downarrow & & \downarrow p' \\ Y & \xrightarrow{\simeq} & Y' \end{array}$$

in which  $p, p'$  are fibrations and the horizontal maps are weak equivalences of  $\mathbf{M}$ .

Then  $p$  is a  $Q$ -fibration if and only if  $p'$  is a  $Q$ -fibration.



*Proof of Theorem 40.9.* Suppose  $p : X \rightarrow Y$  is a fibration of  $\mathbf{M}$ , and that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta} & QX \\ p \downarrow & & \downarrow Qp \\ Y & \xrightarrow{\eta} & QY \end{array}$$

is homotopy cartesian in  $\mathbf{M}$ .

Then  $Qp$  has a factorization

$$\begin{array}{ccc} QX & \xrightarrow{i} & Z \\ & \searrow Qp & \downarrow q \\ & & QY \end{array}$$

where  $i$  is a trivial cofibration and  $q$  is a fibration. Then  $q$  is a  $Q$ -fibration by Lemma 40.5.

Factorize the weak equivalence  $\theta : X \rightarrow Y \times_{QY} Z$  (the square is homotopy cartesian) as

$$\begin{array}{ccc} X & \xrightarrow{i} & W \\ & \searrow \theta & \downarrow \pi \\ & & Y \times_{QY} Z \end{array}$$

where  $\pi$  is a trivial fibration of  $\mathbf{M}$  and  $i$  is a trivial cofibration.

Then  $q_* \cdot \pi$  is a  $Q$ -fibration (Lemma 40.4), and the

lifting exists in the diagram

$$\begin{array}{ccc} X & \xrightarrow{1} & X \\ i \downarrow & \nearrow & \downarrow p \\ W & \xrightarrow{q_* \pi} & Y \end{array}$$

Thus,  $p$  is a retract of a  $Q$ -fibration, and is therefore a  $Q$ -fibration.

Conversely, suppose that  $p : X \rightarrow Y$  is a  $Q$ -fibration, and factorize  $Qp = q \cdot i$  as above.

The map  $\eta_* : Y \times_{QY} Z \rightarrow Z$  is a  $Q$ -equivalence by **A6'**, so  $\theta$  is a  $Q$ -equivalence.

The picture

$$\begin{array}{ccc} X & \xrightarrow{\theta} & Y \times_{QY} Z \\ p \searrow & & \nearrow q_* \\ & Y & \end{array}$$

is a weak equivalence of fibrant objects in the category  $\mathbf{M}/Y$  of objects fibred over  $Y$ , for the  $Q$ -structure on  $\mathbf{M}$ .

The usual category of fibrant objects trick (see Section 13) implies that  $\theta$  has a factorization

$$\begin{array}{ccc} X & \xrightarrow{i} & V \\ & \searrow \theta & \downarrow \pi \\ & & Y \times_{QY} Z \end{array}$$

in  $\mathbf{Spt}/Y$ , where  $\pi$  is a  $Q$ -fibration and a  $Q$ -equivalence, and  $i$  is a section of a map  $V \rightarrow X$  which is a  $Q$ -fibration and a  $Q$ -equivalence.

Thus,  $\pi$  and  $i$  are weak equivalences of  $\mathbf{M}$  by Lemma 40.4, so that  $\theta$  is a weak equivalence of  $\mathbf{M}$ .  $\square$

Write

$$A \otimes K = A \wedge K_+,$$

for a spectrum  $A$  and a simplicial set  $K$ .

**Lemma 40.12.** *Suppose  $i : A \rightarrow B$  is a stably trivial cofibration of spectra.*

*Then all induced maps*

$$(B \otimes \partial \Delta^n) \cup (A \otimes \Delta^n) \rightarrow B \otimes \Delta^n$$

*are stably trivial cofibrations.*

Quillen's axiom **SM7** for the stable model structure on  $\mathbf{Spt}$  follows easily: if  $j : K \rightarrow L$  is a cofibration of simplicial sets and  $i : A \rightarrow B$  is a cofibration of spectra, then the induced map

$$(B \otimes K) \cup (A \otimes L) \subset B \otimes L$$

is a cofibration which is a stable equivalence if either  $i$  is a stable equivalence (Lemma 40.12) or  $j$  is a weak equivalence of simplicial sets (use the simplicial model axiom for the strict structure).

*Proof of Lemma 40.12.* It suffices to show that

$$i \otimes \partial \Delta^n : A \otimes \partial \Delta^n \rightarrow B \otimes \partial \Delta^n$$

is a stable equivalence.

There is a pushout diagram

$$\begin{array}{ccc} A \otimes \partial \Delta^{n-1} & \longrightarrow & A \otimes \Lambda_k^n \\ \downarrow & & \downarrow \\ A \otimes \Delta^{n-1} & \longrightarrow & A \otimes \partial \Delta^n \end{array}$$

There is also a corresponding diagram for  $B$  and an obvious comparison.

The simplicial sets  $\Lambda_k^n$  and  $\Delta^{n-1}$  are both weakly equivalent to a point, so it suffices to show that the comparison

$$i \otimes \partial \Delta^{n-1} : A \otimes \partial \Delta^{n-1} \rightarrow B \otimes \partial \Delta^{n-1}$$

is a stable equivalence.

This is the inductive step in an argument that starts with the case

$$i \otimes \partial \Delta^1 : A \otimes \partial \Delta^1 \rightarrow B \otimes \partial \Delta^1$$

and this map is isomorphic to the map

$$i \wedge i : A \wedge A \rightarrow B \wedge B.$$

Finally, a wedge (coproduct) of stably trivial cofibrations is stably trivial.  $\square$

**Note:** Bousfield gives a different proof of the Lemma 40.12 in [1]. The result is also mentioned in Remark X.4.7 (on p.496) of [3], without proof.

**Remark:** The Bousfield-Friedlander Theorem (Theorem 40.1) admits a localization style of proof for the category of spectra. The result itself is sometimes called the “Bousfield-Friedlander Localization Theorem”.

For this, one uses Lemma 40.4 (which is a formality), together with a “bounded monomorphism” statement given by Lemma 44.1 below. It follows (Lemma 44.2) that the class of stably trivial cofibrations has a set of generators given by countable stably trivial cofibrations.

The factorization axiom for the stable model structure on **Spt** follows from the cofibrant generation statement and Lemma 40.4, and the lifting axiom is a formal consequence.

See Chapter 10 of [4] for more detail.

## References

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## Lecture 14: Basic properties

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### 41 Suspensions and shift

The suspension  $X \wedge S^1$  and the fake suspension  $\Sigma X$  of a spectrum  $X$  were defined in Section 38 — the constructions differ by a non-trivial twist of bonding maps.

The **loop** spectrum for  $X$  is the function complex object

$$\mathbf{hom}_*(S^1, X).$$

There is a natural bijection

$$\mathrm{hom}(X \wedge S^1, Y) \cong \mathrm{hom}(X, \mathbf{hom}_*(S^1, Y))$$

so that the suspension and loop functors are adjoint.

The **fake loop** spectrum  $\Omega Y$  for a spectrum  $Y$  consists of the pointed spaces  $\Omega Y^n$ ,  $n \geq 0$ , with adjoint

bonding maps

$$\Omega\sigma_* : \Omega Y^n \rightarrow \Omega^2 Y^{n+1}.$$

There is a natural bijection

$$\text{hom}(\Sigma X, Y) \cong \text{hom}(X, \Omega Y),$$

so the fake suspension functor is left adjoint to fake loops.

The adjoint bonding maps  $\sigma_* : Y^n \rightarrow \Omega Y^{n+1}$  define a natural map

$$\gamma : Y \rightarrow \Omega Y[1].$$

for spectra  $Y$ .

The map  $\omega : Y \rightarrow \Omega^\infty Y$  of Section 40 is the filtered colimit of the maps

$$Y \xrightarrow{\gamma} \Omega Y[1] \xrightarrow{\Omega\gamma[1]} \Omega^2 Y[2] \xrightarrow{\Omega^2\gamma[2]} \dots$$

Recall the statement of the Freudenthal suspension theorem (Theorem 34.2):

**Theorem 41.1.** *Suppose that a pointed space  $X$  is  $n$ -connected, where  $n \geq 0$ .*

*Then the homotopy fibre  $F$  of the canonical map  $\eta : X \rightarrow \Omega(X \wedge S^1)$  is  $2n$ -connected.*



In particular, the suspension homomorphism

$$\pi_i X \rightarrow \pi_i(\Omega(X \wedge S^1)) \cong \pi_{i+1}(X \wedge S^1)$$

is an isomorphism for  $i \leq 2n$  and is an epimorphism for  $i = 2n + 1$ , provided that  $X$  is  $n$ -connected.

In general (ie. with no connectivity assumptions on  $Y$ ), the space  $S^n \wedge Y$  is  $(n - 1)$ -connected, by Lemma 31.5 and Corollary 34.1.

Thus, the suspension homomorphism

$$\pi_{i+k}(S^{n+k} \wedge Y) \rightarrow \pi_{i+k+1}(S^{n+k+1} \wedge Y)$$

is an isomorphism if  $i \leq 2n - 2 + k$ , and it follows that the map

$$\pi_i(S^n \wedge Y) \rightarrow \pi_{i-n}^s(\Sigma^\infty Y)$$

is an isomorphism for  $i \leq 2(n - 1)$ .

Here's an easy observation:

**Lemma 41.2.** *The natural map  $\gamma : X \rightarrow \Omega X[1]$  is a stable equivalence if  $X$  is strictly fibrant.*

*Proof.* This is a cofinality argument, which uses the fact that  $\Omega^\infty X$  is the filtered colimit of the system

$$X \rightarrow \Omega X[1] \rightarrow \Omega^2 X[2] \rightarrow \dots$$

□

**Lemma 41.3.** *Suppose that  $Y$  is a pointed space.*

*Then the canonical map*

$$\eta : \Sigma^\infty Y \rightarrow \Omega\Sigma(\Sigma^\infty Y)$$

*is a stable homotopy equivalence.*

*Proof.* The map

$$\pi_k(S^n \wedge Y) \rightarrow \pi_{k-n}^s(\Sigma^\infty Y)$$

is an isomorphism for  $k \leq 2(n-1)$ .

Similarly (exercise), the map

$$\pi_k(\Omega(S^{n+1} \wedge X)) \rightarrow \pi_{k-n}^s(\Omega\Sigma(\Sigma^\infty X))$$

is an isomorphism for  $k+1 \leq 2n$  or  $k \leq 2n-1$ .

There is a commutative diagram

$$\begin{array}{ccc} \pi_k(S^n \wedge Y) & \xrightarrow{\cong} & \pi_{k-n}^s(\Sigma^\infty Y) \\ \cong \downarrow & & \downarrow \\ \pi_k(\Omega(S^{n+1} \wedge Y)) & \xrightarrow{\cong} & \pi_{k-n}^s(\Omega\Sigma(\Sigma^\infty Y)) \end{array}$$

in which the indicated maps are isomorphisms for  $k \leq 2(n-1)$ .

It follows that the map

$$\pi_p^s(\Sigma^\infty Y) \rightarrow \pi_p^s(\Omega\Sigma(\Sigma^\infty Y))$$

is an isomorphism for  $p \leq n-2$ .

Finish by letting  $n$  vary. □

**Remark:** What we've really shown in Lemma 41.3 is that the composite

$$\Sigma^\infty X \xrightarrow{\eta} \Omega \Sigma(\Sigma^\infty X) \xrightarrow{\Omega j} \Omega F(\Sigma(\Sigma^\infty X))$$

is a natural stable equivalence.

**Lemma 41.4.** *Suppose that  $Y$  is a spectrum. Then the composite*

$$Y \xrightarrow{\eta} \Omega \Sigma Y \xrightarrow{\Omega j} \Omega F(\Sigma Y)$$

*is a stable equivalence.*

*Proof.* We show that the maps

$$L_n Y \xrightarrow{\eta} \Omega \Sigma L_n Y \xrightarrow{\Omega j} \Omega F(\Sigma L_n Y)$$

arising from the layer filtration for  $Y$  are stable equivalences.

In the layer filtration

$$L_n Y : Y^0, \dots, Y^n, S^1 \wedge Y^n, S^2 \wedge Y^n, \dots$$

the maps

$$(\Sigma^\infty Y^n[-n])^r \rightarrow L_n Y^r$$

are isomorphisms for  $r \geq n$ .

Thus, the maps

$$(\Omega F(\Sigma(\Sigma^\infty Y^n[-n])))^r \rightarrow \Omega F(\Sigma(L_n Y))^r$$

are weak equivalences for  $r \geq n$ , so that

$$\Omega F(\Sigma(\Sigma^\infty Y^n[-n])) \rightarrow \Omega F(\Sigma(L_n Y))$$

is a stable equivalence.

The map  $\eta : X \rightarrow \Omega \Sigma X$  respects shifts, so Lemma 41.3 implies that the composite

$$\Sigma^\infty Y^n[-n] \rightarrow \Omega \Sigma(\Sigma^\infty Y^n[-n]) \rightarrow \Omega F(\Sigma(\Sigma^\infty Y^n[-n]))$$

is a stable equivalence.  $\square$

**Theorem 41.5.** *Suppose that  $X$  is a spectrum.*

*Then the canonical map*

$$\sigma : \Sigma X \rightarrow X[1]$$

*is a stable equivalence.*

*Proof.* The map  $\sigma$  is adjoint to the map  $\sigma_* : X \rightarrow \Omega X[1]$ , so that there is a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\eta} & \Omega \Sigma X & \xrightarrow{\Omega j} & \Omega F(\Sigma X) \\ & \searrow \sigma_* & \downarrow \Omega \sigma & & \downarrow \Omega F \sigma \\ & & \Omega X[1] & \xrightarrow{\Omega j} & \Omega F(X[1]) \end{array}$$

where  $j : \Sigma X \rightarrow F(\Sigma X)$  is a strictly fibrant model.

The composite

$$X \xrightarrow{\sigma_*} \Omega X[1] \xrightarrow{\Omega j[1]} \Omega(FX)[1]$$

is a stable equivalence by Lemma 41.2, and the shifted map  $j[1] : X[1] \rightarrow (FX)[1]$  is a strictly fibrant model of  $X[1]$ .

It follows that the composite

$$X \xrightarrow{\sigma_*} \Omega X[1] \xrightarrow{\Omega j} \Omega F(X[1])$$

is a stable equivalence.

The composite

$$X \xrightarrow{\eta} \Omega \Sigma X \xrightarrow{\Omega j} \Omega F(\Sigma X)$$

is a stable equivalence by Lemma 41.4.

The map  $\Omega F \sigma$  is therefore a stable equivalence, so Lemma 41.2 implies that

$$F \sigma : F(\Sigma X) \rightarrow F(X[1])$$

is a stable equivalence. □

Here's another, still elementary but fussier result:

**Theorem 41.6.** *The functors  $X \mapsto X \wedge S^1$  and  $X \mapsto \Sigma X$  are naturally stably equivalent.*

**Sketch Proof:** ([1], Lemma 1.9, p.7) The isomorphisms  $\tau : S^1 \wedge X^n \rightarrow X^n \wedge S^1$  and the bonding maps  $\sigma \wedge S^1$  together define a spectrum with the space

$S^1 \wedge X^n$  in level  $n$ , and with bonding maps  $\tilde{\sigma}$  defined by the diagrams

$$\begin{array}{ccc} S^1 \wedge S^1 \wedge X^n & \xrightarrow{\tilde{\sigma}} & S^1 \wedge X^{n+1} \\ S^1 \wedge \tau \downarrow \cong & & \cong \downarrow \tau \\ S^1 \wedge X^n \wedge S^1 & \xrightarrow{\sigma \wedge S^1} & X^{n+1} \wedge S^1 \end{array}$$

There are commutative diagrams

$$\begin{array}{ccc} S^1 \wedge S^1 \wedge X^n & \xrightarrow{S^1 \wedge \sigma} & S^1 \wedge X^{n+1} \\ \tau \wedge X^n \downarrow & & \uparrow \\ S^1 \wedge S^1 \wedge X^n & \xrightarrow{\tilde{\sigma}} & \end{array}$$

Composing then gives a diagram

$$\begin{array}{ccc} S^1 \wedge S^1 \wedge S^1 \wedge X^n & & \\ \downarrow (3,2,1) \wedge X^n & \searrow (S^1 \wedge \sigma)(S^1 \wedge S^1 \wedge \sigma) & \\ S^1 \wedge S^1 \wedge S^1 \wedge X^n & \xrightarrow{\tilde{\sigma} \cdot (S^1 \wedge \tilde{\sigma})} & S^1 \wedge X^{n+2} \end{array}$$

where  $(3, 2, 1)$  is induced on the smash factors making up  $S^3$  by the corresponding cyclic permutation of order 3.

The spaces  $S^1 \wedge X^0, S^1 \wedge X^2, \dots$  and the respective composite bonding maps  $(S^1 \wedge \sigma)(S^1 \wedge S^1 \wedge \sigma)$  and  $\tilde{\sigma} \cdot (S^1 \wedge \tilde{\sigma})$  define “partial” spectrum structures from which the stable homotopy types of the original spectra can be recovered.

The self map  $(3, 2, 1)$  of the 3-sphere  $S^3$  has degree 1 and is therefore homotopic to the identity.

This homotopy can be used to describe a telescope construction (see [1], p.11-15, and the next section) which is stably equivalent to both of these partial spectra.  $\square$

**Remark:** The proof of Theorem 41.6 that is sketched here is essentially classical. See Prop. 10.53 of [2] for an alternative.

**Corollary 41.7.** *1) The functors  $X \mapsto X[1]$ ,  $X \mapsto \Sigma X$  and  $X \mapsto X \wedge S^1$  are naturally stably equivalent.*

*2) The functors  $X \mapsto X[-1]$ ,  $X \mapsto \Omega X$  and  $X \mapsto \mathbf{hom}_*(S^1, X)$  are naturally stably equivalent.*

*Proof.* Lemma 41.2 implies that the composite

$$X \xrightarrow{\sigma_*} \Omega X[1] \xrightarrow{\Omega j[1]} \Omega F X[1]$$

is a stable equivalence for all spectra  $X$ , where  $j : X \rightarrow F X$  is a strictly fibrant model.

Shift preserves stable equivalences, so the induced composite

$$X[-1] \xrightarrow{\sigma_*[-1]} \Omega X \xrightarrow{\Omega j} \Omega F X$$

is a stable equivalence.

The natural stable equivalence  $\Sigma Y \simeq Y \wedge S^1$  induces a natural stable equivalence

$$\Omega X \simeq \mathbf{hom}_*(S^1, X)$$

for all strictly fibrant spectra  $X$ . □

In other words, the suspension and loop functors (real or fake) are equivalent to shift functors, and define equivalences  $\mathrm{Ho}(\mathbf{Spt}) \rightarrow \mathrm{Ho}(\mathbf{Spt})$  of the stable category.

## 42 The telescope construction

Observe that a spectrum  $Y$  is cofibrant if and only if all bonding maps  $\sigma : S^1 \wedge Y^n \rightarrow Y^{n+1}$  are cofibrations.

The **telescope**  $TX$  for a spectrum  $X$  is a natural cofibrant replacement, equipped with a natural strict equivalence  $s : TX \rightarrow X$ .

The construction is an iterated mapping cylinder. We find natural trivial cofibrations

$$X^k \xrightarrow{j_k} CX^k \xrightarrow{\alpha_k} TX^k, \quad k \leq n,$$



and  $t_k : TX^k \rightarrow X^k$  such that  $t_k \cdot (\alpha_k \cdot j_k) = 1$  and the maps  $t_k$  define a strict weak equivalence of spectra  $t : TX \rightarrow X$ .

- $X^0 = CX^0 = TX^0$  and  $j_0$  and  $\alpha_0$  are identities,
- $CX^n$  is the mapping cylinder for  $\sigma : S^1 \wedge X^n \rightarrow X^{n+1}$ , meaning that there is a pushout diagram

$$\begin{array}{ccc} S^1 \wedge X^n & \xrightarrow{\sigma} & X^{n+1} \\ d^0 \downarrow & & \downarrow j_{n+1} \\ (S^1 \wedge X^n) \wedge \Delta_+^1 & \xrightarrow[\zeta_{n+1}]{} & CX^{n+1} \end{array}$$

for each  $n$ .

Write  $\sigma_*$  for the composite

$$S^1 \wedge X^n \xrightarrow{d^1} (S^1 \wedge X^n) \wedge \Delta_+^1 \xrightarrow{\zeta_{n+1}} CX^{n+1}$$

and observe that  $\sigma_*$  is a cofibration.

The projection map

$$s : (S^1 \wedge X^n) \wedge \Delta_+^1 \rightarrow S^1 \wedge X^n$$

satisfies  $s \cdot d^0 = 1$  and induces a map  $s_{n+1} : CX^{n+1} \rightarrow X^{n+1}$  such that  $s_{n+1} \cdot j_{n+1} = 1$ . Further  $s_{n+1} \cdot \sigma_* = \sigma$ .

- Form the pushout diagram

$$\begin{array}{ccc}
S^1 \wedge X^n & \xrightarrow{\sigma_*} & CX^{n+1} \\
S^1 \wedge j_n \downarrow & & \downarrow \alpha_{n+1} \\
S^1 \wedge CX^n & & \\
S^1 \wedge \alpha_n \downarrow & & \\
S^1 \wedge TX^n & \xrightarrow{\tilde{\sigma}} & TX^{n+1}
\end{array}$$

Then  $\tilde{\sigma}$  is a cofibration, and the maps  $j_{n+1}$ ,  $\alpha_{n+1}$  are trivial cofibrations.

The maps  $S^1 \wedge t_n$  and  $s_{n+1}$  together induce  $t_{n+1} : TX^{n+1} \rightarrow X^{n+1}$  such that  $t_{n+1} \cdot (\alpha_{n+1} \cdot j_{n+1}) = 1$ , and the  $t_k : TX^k \rightarrow X^k$  define a map of spectra up to level  $n + 1$ .

The projection maps  $s$  can be replaced with homotopies  $h : (S^1 \wedge X^n) \wedge \Delta_+^1 \rightarrow Z^n$  in the construction above, giving the following:

**Lemma 42.1.** *Suppose  $X$  is a spectrum with bonding maps  $\sigma : S^1 \wedge X^n \rightarrow X^{n+1}$ . Suppose  $X'$  is a spectrum with the same objects as  $X$ , with bonding maps  $\sigma' : S^1 \wedge X^n \rightarrow X^{n+1}$ . Suppose  $j : X' \rightarrow X$*

is a map of spectra such that there are homotopies

$$\begin{array}{ccc}
 S^1 \wedge X^n & & \\
 d^1 \downarrow & \searrow j\sigma' & \\
 (S^1 \wedge X^n) \wedge \Delta_+^1 & \xrightarrow{h} & Z^{n+1} \\
 d^0 \uparrow & \nearrow j\sigma & \\
 S^1 \wedge X^n & & 
 \end{array}$$

Then the homotopies  $h$  define a map  $h_* : TX \rightarrow Z$ , giving a morphism

$$X \xleftarrow[\simeq]{t} TX \xrightarrow{h_*} Z$$

from  $X$  to  $Z$  in the stable category.

If  $j : X' \rightarrow Z$  is a strict weak equivalence then the map  $h_*$  is a strict weak equivalence.

**Remarks:**

1) The construction of Lemma 42.1 is natural, and hence applies to diagrams of spectra.

Suppose that  $i \mapsto X_i$  and  $i \mapsto X'_i$  are spectrum valued functors defined on an index category  $I$  such that  $X_i^n = X_i'^n$  for all  $i \in I$ . Let  $j : X' \rightarrow Z$  be a natural choice of strict fibrant model for the diagram  $X'$  and suppose finally that there are natural

homotopies

$$\begin{array}{ccc}
 S^1 \wedge X_i^n & & \\
 d^1 \downarrow & \searrow j\sigma' & \\
 (S^1 \wedge X_i^n) \wedge \Delta_+^1 & \xrightarrow{h} & Z_i^n \\
 d^0 \uparrow & \nearrow j\sigma & \\
 S^1 \wedge X_i^n & & 
 \end{array}$$

where  $\sigma$  and  $\sigma'$  are the bonding maps for  $X$  and  $X'$  respectively.

Then the homotopies  $h$  canonically determine a natural strict equivalence  $h_* : TX \rightarrow Z$ , and there are natural strict equivalences

$$X \leftarrow TX \xrightarrow{h_*} Z \xleftarrow{j} X'.$$

2) Suppose given  $S^2$ -spectra  $X(1)$  and  $X(2)$  having objects  $S^1 \wedge X^{2n}$  and bonding maps

$$\sigma_1, \sigma_2 : S^2 \wedge S^1 \wedge X^{2n} = S^3 \wedge X^{2n} \rightarrow S^1 \wedge X^{2n+2}$$

respectively, such that the diagram

$$\begin{array}{ccc}
 S^3 \wedge X^{2n} & & \\
 c \wedge 1 \downarrow & \searrow \sigma_1 & \\
 S^3 \wedge X^{2n} & \xrightarrow{\sigma_2} & S^1 \wedge X^{2n+2}
 \end{array}$$

commutes, where  $c$  is induced by the cyclic permutation  $(3, 2, 1)$ .

The map  $c$  has degree 1 and is therefore the identity in the homotopy category.

Choose a strict fibrant model  $j : X(2) \rightarrow FX(2)$  in  $S^2$ -spectra for  $X(2)$ . Then

$$j \cdot \sigma_1 \simeq j \cdot \sigma_2 : S^3 \wedge X^{2n} \rightarrow F(S^1 \wedge X^{2n+2}),$$

and it follows that there are strict equivalences

$$X(1) \xleftarrow{t} TX(1) \xrightarrow{h_*} FX(2) \xleftarrow{j} X(2).$$

If  $X(1)$  and  $X(2)$  are the outputs of functors defined on spectra (eg. the comparison of fake and real suspension in Theorem 41.6), then these equivalences are natural.

### 43 Fibrations and cofibrations

Suppose  $i : A \rightarrow X$  is a levelwise cofibration of spectra with cofibre  $\pi : X \rightarrow X/A$ .

Suppose  $\alpha : S^r \rightarrow X^n$  represents a homotopy element such that the composite

$$S^r \xrightarrow{\alpha} X^n \xrightarrow{\pi} X^n/A^n$$

represents  $0 \in \pi_r(X/A)^n$ .

Comparing cofibre sequences gives a diagram

$$\begin{array}{ccccccc}
S^r & \longrightarrow & CS^r & \longrightarrow & S^1 \wedge S^r & \xrightarrow{\cong} & S^1 \wedge S^r \\
\alpha \downarrow & & \downarrow & & \downarrow & & \downarrow S^1 \wedge \alpha \\
X^n & \xrightarrow{\pi} & (X/A)^n & \longrightarrow & S^1 \wedge A^n & \xrightarrow{S^1 \wedge i} & S^1 \wedge X^n \\
& & & & \sigma \downarrow & & \downarrow \sigma \\
& & & & A^{n+1} & \xrightarrow{i} & X^{n+1}
\end{array}$$

where  $CS^r \simeq *$  is the cone on  $S^r$ .

It follows that the image of  $[\alpha]$  under the suspension map

$$\pi_r X^n \rightarrow \pi_{r+1} X^{n+1}$$

is in the image of the map  $\pi_{r+1} A^{n+1} \rightarrow \pi_{r+1} X^{n+1}$ .

We have proved the following:

**Lemma 43.1.** *Suppose  $A \rightarrow X \rightarrow X/A$  is a level cofibre sequence of spectra.*

*Then the sequence*

$$\pi_k^s A \rightarrow \pi_k^s X \rightarrow \pi_k^s (X/A)$$

*is exact.*

**Corollary 43.2.** *Any levelwise cofibre sequence*

$$A \rightarrow X \rightarrow X/A$$

*induces a long exact sequence*

$$\dots \xrightarrow{\partial} \pi_k^s A \rightarrow \pi_k^s X \rightarrow \pi_k^s (X/A) \xrightarrow{\partial} \pi_{k-1}^s A \rightarrow \dots \quad (1)$$

The sequence (1) is the **long exact sequence** in stable homotopy groups for a level cofibre sequence of spectra.

*Proof.* The map  $X/A \rightarrow A \wedge S^1$  in the Puppe sequence induces the boundary map

$$\pi_k^s(X/A) \rightarrow \pi_k^s(A \wedge S^1) \cong \pi_k^s(A[1]) \cong \pi_{k-1}^s A,$$

since  $A \wedge S^1$  is naturally stably equivalent to the shifted spectrum  $A[1]$  by Corollary 41.7.  $\square$

**Corollary 43.3.** *Suppose that  $X$  and  $Y$  are spectra. Then the inclusion*

$$X \vee Y \rightarrow X \times Y$$

*is a natural stable equivalence.*

*Proof.* The sequence

$$0 \rightarrow \pi_k^s X \rightarrow \pi_k^s(X \vee Y) \rightarrow \pi_k^s Y \rightarrow 0$$

arising from the level cofibration  $X \subset X \vee Y$  is split exact, as is the sequence

$$0 \rightarrow \pi_k^s X \rightarrow \pi_k^s(X \times Y) \rightarrow \pi_k^s Y \rightarrow 0$$

arising from the fibre sequence  $X \rightarrow X \times Y \rightarrow Y$ .

It follows that the map  $X \vee Y \rightarrow X \times Y$  induces an isomorphism in all stable homotopy groups.  $\square$

**Corollary 43.4.** *The stable homotopy category  $\mathrm{Ho}(\mathbf{Spt})$  is additive.*

*Proof.* The sum of two maps  $f, g : X \rightarrow Y$  is represented by the composite

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y \xleftarrow{\cong} Y \vee Y \xrightarrow{\nabla} Y.$$

□

**Corollary 43.5.** *Suppose that*

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \alpha \downarrow & & \downarrow \beta \\ C & \xrightarrow{j} & D \end{array}$$

*is a pushout in  $\mathbf{Spt}$  where  $i$  is a levelwise cofibration. Then there is a long exact sequence in stable homotopy groups*

$$\dots \xrightarrow{\partial} \pi_k^s A \xrightarrow{(i, \alpha)} \pi_k^s C \oplus \pi_k^s B \xrightarrow{j - \beta} \pi_k^s D \xrightarrow{\partial} \pi_{k-1}^s A \rightarrow \dots \quad (2)$$

The sequence (2) is the **Mayer-Vietoris sequence** for the cofibre square.

The boundary map  $\partial : \pi_k^s D \rightarrow \pi_{k-1}^s A$  is the composite

$$\pi_k^s D \rightarrow \pi_k^s (D/C) = \pi_k^s (B/A) \xrightarrow{\partial} \pi_{k-1}^s A.$$



**Lemma 43.6.** *Suppose*

$$A \xrightarrow{i} X \xrightarrow{\pi} X/A$$

*is a level cofibre sequence in **Spt**, and let  $F$  be the strict homotopy fibre of the map  $\pi$ .*

*Then the map  $i_* : A \rightarrow F$  is a stable equivalence.*

*Proof.* Choose a strict fibration  $p : Z \rightarrow X/A$  such that  $Z \rightarrow *$  is a strict weak equivalence.

Form the pullback

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\pi_*} & Z \\ p_* \downarrow & & \downarrow p \\ X & \xrightarrow{\pi} & X/A \end{array}$$

Then  $\tilde{X}$  is the homotopy fibre of  $\pi$  and the maps  $i : A \rightarrow X$  and  $*$  :  $A \rightarrow Z$  together determine a map  $i_* : A \rightarrow \tilde{X}$ . We show that  $i_*$  is a stable equivalence.

Pull back the cofibre square

$$\begin{array}{ccc} A & \longrightarrow & * \\ i \downarrow & & \downarrow \\ X & \xrightarrow{\pi} & X/A \end{array}$$

along the fibration  $p$  to find a (levelwise) cofibre

square

$$\begin{array}{ccc} \tilde{A} & \longrightarrow & U \\ \tilde{i} \downarrow & & \downarrow \\ \tilde{X} & \longrightarrow & Z \end{array}$$

A Mayer-Vietoris sequence argument (Corollary 43.5) implies that the map  $\tilde{A} \rightarrow \tilde{X} \times U$  is a stable equivalence.

From the fibre square

$$\begin{array}{ccc} \tilde{A} & \longrightarrow & U \\ \downarrow & & \downarrow \\ A & \longrightarrow & * \end{array}$$

we see that the map  $\tilde{A} \rightarrow A \times U$  is a stable equivalence.

The map  $i_* : A \rightarrow \tilde{X}$  induces a section  $\theta : A \rightarrow \tilde{A}$  of the map  $\tilde{A} \rightarrow A$  which composes with the projection  $\tilde{A} \rightarrow U$  to give the trivial map  $* : A \rightarrow U$ .

Thus, there is a commutative diagram

$$\begin{array}{ccccc} & A & \xrightarrow{i_*} & \tilde{X} & \\ & \downarrow \theta & & \downarrow (1_{\tilde{X}}, *) & \\ (1_A, *) \swarrow & & & & \\ A \times U & \xleftarrow{\cong} & \tilde{A} & \xrightarrow{\cong} & \tilde{X} \times U \\ & \searrow pr & \downarrow & \swarrow & \\ & & U & & \end{array}$$

and it follows that  $A$  is the stable fibre of the map  $\tilde{A} \rightarrow U$ , so  $i_*$  is a stable equivalence.  $\square$

**Lemma 43.7.** *Suppose that*

$$F \xrightarrow{i} E \xrightarrow{p} B$$

*is a strict fibre sequence, where  $i$  is a level cofibration.*

*Then the map  $E/F \rightarrow B$  is a stable equivalence.*

*Proof.* There is a diagram

$$\begin{array}{ccccccc}
 F & \xrightarrow{i} & E & \xrightarrow{\pi} & E/F & & \\
 \searrow j'_* & & \downarrow j' & & \downarrow \gamma & \searrow = & \\
 = \downarrow & & F' & \xrightarrow{i'} & U & \xrightarrow{p'} & E/F \\
 \swarrow \theta_* & & \downarrow \theta & & \downarrow \gamma & \swarrow \gamma & \\
 F & \xrightarrow{i} & E & \xrightarrow{p} & B & & 
 \end{array}$$

where  $p'$  is a strict fibration,  $j'$  is a cofibration and a strict equivalence, and  $\theta$  exists by a lifting property:

$$\begin{array}{ccc}
 E & \xrightarrow{=} & E \\
 j' \downarrow & \nearrow \theta & \downarrow p \\
 U & \xrightarrow{\gamma p'} & B
 \end{array}$$

The map  $j'_*$  is a stable equivalence by Lemma 43.6, so  $\theta_*$  is a stable equivalence.

The map  $\theta$  is a strict equivalence, and a comparison of long exact sequences shows that  $\gamma$  is a stable equivalence.  $\square$

**Remark:** Lemma 43.6 and Lemma 43.7 together say that fibre and cofibre sequences coincide in the stable category.

#### 44 Cofibrant generation

We will show that the stable model structure on the category **Spt** of spectra is cofibrantly generated.

This means that there are sets  $I$  and  $J$  of stably trivial cofibrations and cofibrations, such that  $p : X \rightarrow Y$  is a stable fibration (resp. stably trivial fibration) if and only if it has the RLP wrt all members of the set  $I$  (resp. all members of  $J$ ).

Recall that a map  $p : X \rightarrow Y$  is a stably trivial fibration if and only if it is a strict fibration and a strict weak equivalence.

Thus  $p$  is a stably trivial fibration if and only if it has the RLP wrt all maps

$$\Sigma^\infty \partial \Delta_+^n[m] \rightarrow \Sigma^\infty \Delta_+^n[m].$$

We have found our set of maps  $J$ .

It remains to find a set of stably trivial cofibrations  $I$  which generates the full class of stably trivial cofibrations. We do this in a sequence of lemmas.

Say that a spectrum  $A$  is **countable** if all constituent simplicial sets  $A^n$  are countable in the sense that they have countably many simplices in each degree — see Section 11.

It follows from Lemma 11.2 that a countable spectrum  $A$  has countable stable homotopy groups.

The following “bounded cofibration lemma” is the analogue of Lemma 11.3 for the category of spectra.

**Lemma 44.1.** *Suppose given level cofibrations of spectra*

$$\begin{array}{ccc} & X & \\ & \downarrow j & \\ A & \xrightarrow{i} & Y \end{array}$$

*such that  $A$  is countable and  $j$  is a stable equivalence.*

*Then there is a countable subobject  $B \subset Y$  such that  $A \subset B \subset Y$  and the map  $B \cap X \rightarrow B$  is a stable equivalence.*

*Proof.* The map  $B \cap X \rightarrow B$  is a stable equivalence if and only if all stable homotopy groups

$$\pi_n^s(B/(B \cap X))$$

vanish, by Corollary 43.2.

Write  $A_0 = A$ .  $Y$  is a filtered colimit of its countable subobjects, and the countable set of elements of the homotopy groups  $\pi_n^s(A_0/(A_0 \cap X))$  vanish in  $\pi_n^s(A_1/(A_1 \cap X))$  for some countable subobject  $A_1 \subset X$  with  $A_0 \subset A_1$ .

Repeat the construction inductively to find countable subcomplexes

$$A = A_0 \subset A_1 \subset A_2 \subset \dots$$

of  $Y$  such that all induced maps

$$\pi_n^s(A_i/(A_i \cap X)) \rightarrow \pi_n^s(A_{i+1}/(A_{i+1} \cap X))$$

are 0. Set  $B = \cup_i A_i$ . Then  $B$  is countable and all groups  $\pi_n^s(B/(B \cap X))$  vanish.  $\square$

Consider the set of all stably trivial level cofibrations  $j : C \rightarrow D$  with  $D$  countable, and find a factorization

$$\begin{array}{ccc} C & \xrightarrow{in_j} & E_j \\ & \searrow j & \downarrow p_j \\ & & D \end{array}$$

for each such  $j$  such that  $in_j$  is a stably trivial cofibration and  $p_j$  is a stably trivial fibration.

Make fixed choices of the factorizations  $j = p_j \cdot in_j$ , and let  $I$  be the set of all stably trivial cofibrations  $in_j$ .

**Lemma 44.2.** *The set  $I$  generates the class of stably trivial cofibrations.*

*Proof.* Suppose given a diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ j \downarrow & & \downarrow f \\ B & \longrightarrow & Y \end{array}$$

where  $j$  is a cofibration,  $f$  is a stable equivalence and  $B$  is countable.

Then  $f$  has a factorization  $f = q \cdot i$  where  $i$  is a stably trivial cofibration and  $q$  is a stably trivial fibration.

There is a diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ j \downarrow & & \downarrow i \\ & \nearrow \theta & Z \\ B & \longrightarrow & Y \end{array}$$

$\downarrow q$

where the lift  $\theta$  exists since  $j$  is a cofibration and  $q$  is a stably trivial fibration.

The image  $\theta(B)$  of  $B$  is a countable subobject of  $Z$ , so Lemma 44.1 says that there is a subobject  $D \subset Z$  such that  $D$  is countable and the level cofibration  $j : D \cap X \rightarrow D$  is a stable equivalence.

What we have, then, is a factorization

$$\begin{array}{ccccc} A & \longrightarrow & D \cap X & \longrightarrow & X \\ j \downarrow & & \downarrow j & & \downarrow f \\ B & \longrightarrow & D & \longrightarrow & Y \end{array}$$

of the original diagram, such that  $j$  is a countable, stably trivial level countable.

We can further assume (by lifting to  $E_j$ ) that the original diagram has a factorization

$$\begin{array}{ccccc} A & \longrightarrow & D \cap X & \longrightarrow & X \\ j \downarrow & & \downarrow in_j & & \downarrow f \\ B & \longrightarrow & E_j & \longrightarrow & Y \end{array}$$

where the map  $in_j$  is a member of the set  $I$ .

Now suppose that  $i : U \rightarrow V$  is a stably trivial cofibration. Then  $i$  has a factorization

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & W \\ & \searrow i & \downarrow q \\ & & V \end{array}$$

where  $\alpha$  is a member of the saturation of  $I$  and  $q$  has the RLP wrt all members of  $I$ .



But then  $q$  has the RLP wrt all countable cofibrations by the construction above, so that  $q$  has the RLP wrt all cofibrations.

In particular, there is a diagram

$$\begin{array}{ccc} U & \xrightarrow{j} & W \\ i \downarrow & \nearrow & \downarrow q \\ V & \xrightarrow{1} & V \end{array}$$

so that  $i$  is a retract of  $j$ . □

**Remark:** Compare the proof of Lemma 44.2 with the proof of Lemma 11.5 — they are the same.

## References

- [1] J. F. Jardine. *Generalized Étale Cohomology Theories*, volume 146 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1997.
- [2] J.F. Jardine. *Local Homotopy Theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2015.

## Lecture 15: Spectrum objects

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### 45 Spectra in simplicial modules

Suppose  $A$  is a simplicial  $R$ -module and  $K$  is a pointed simplicial set.

The simplicial  $R$ -module  $A \otimes K$  is defined by

$$A \otimes K = A \otimes_R \tilde{R}(K),$$

where  $\tilde{R}(K) = R(K)/R(*)$  defines the reduced free  $R$ -module functor

$$\tilde{R} : s_*\mathbf{Set} \rightarrow s(R\text{-}\mathbf{Mod}).$$

(Compare with Section 15.)

There are natural isomorphisms

$$\tilde{R}(K \wedge L) \cong \tilde{R}(K) \otimes \tilde{R}(L) = K \otimes \tilde{R}(L),$$

and there is a natural map

$$\gamma : u(A) \wedge K \rightarrow u(A \otimes K).$$

Here,

$$u : s(R - \mathbf{Mod}) \rightarrow s_* \mathbf{Set}$$

is the forgetful functor, where  $u(A)$  is the simplicial set underlying  $A$ , pointed by 0.

The functor  $u$  is right adjoint to  $\tilde{R}$ .

We frequently write  $A$  for both a simplicial  $R$ -module  $A$  and its underlying pointed simplicial set.

**Lemma 45.1.** *Suppose  $A$  is a simplicial abelian group.*

*Then the canonical map*

$$\eta : A \rightarrow \mathbf{hom}_*(S^1, A \otimes S^1)$$

*is a weak equivalence.*

*Proof.*  $\Delta_*^1$  is the simplicial set  $\Delta^1$ , pointed by the vertex 0.

There is a contracting homotopy  $h : \Delta_*^1 \wedge \Delta_*^1 \rightarrow \Delta_*^1$  given by the picture

$$\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 1 \end{array}$$

and this map  $h$  determines a contracting homotopy

$$h_* : \mathbf{hom}_*(\Delta_*^1, B) \otimes \Delta_*^1 \rightarrow \mathbf{hom}_*(\Delta_*^1, B).$$

for all simplicial abelian groups  $B$ .

$B \otimes \Delta_*^1$  is a model for the cone on  $B$ , and there is a natural short exact sequence

$$0 \rightarrow B \rightarrow B \otimes \Delta_*^1 \rightarrow B \otimes S^1 \rightarrow 0.$$

The homotopy  $h_*$  induces a composite morphism

$$\begin{array}{ccc} A \otimes \Delta_*^1 & \xrightarrow{\eta \otimes 1} & \mathbf{hom}_*(S^1, A \otimes S^1) \otimes \Delta_*^1 \\ & \searrow \gamma & \downarrow \\ & & \mathbf{hom}_*(\Delta_*^1, A \otimes S^1) \otimes \Delta_*^1 \\ & & \downarrow h_* \\ & & \mathbf{hom}_*(\Delta_*^1, A \otimes S^1) \end{array}$$

and there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta} & \mathbf{hom}_*(S^1, A \otimes S^1) \\ \downarrow & & \downarrow \\ A \otimes \Delta_*^1 & \xrightarrow[\gamma]{\cong} & \mathbf{hom}_*(\Delta_*^1, A \otimes S^1) \\ \downarrow & & \downarrow \\ A \otimes S^1 & \xrightarrow{1} & A \otimes S^1 \end{array}$$

This is a comparison of fibre sequences, so the map  $\eta$  is a weak equivalence.  $\square$

Compare with the proof of Lemma 31.1.

**Corollary 45.2.** *The natural map*

$$\varepsilon_A : \mathbf{hom}_*(S^1, A) \otimes S^1 \rightarrow A$$

*induces isomorphisms in  $\pi_k$  for  $k \geq 1$ .*

*Write  $\Omega A = \mathbf{hom}_*(S^1, A)$ .*

*Proof.* There is a diagram

$$\begin{array}{ccc} \Omega A & \xrightarrow[\simeq]{\eta} & \Omega(\Omega A \otimes S^1) \\ & \searrow 1 & \downarrow \Omega \varepsilon \\ & & \Omega A \end{array}$$

Thus,  $\Omega \varepsilon_A$  is a weak equivalence, so that  $\varepsilon_A$  has the claimed effect in homotopy groups.  $\square$

A **spectrum** (or spectrum object)  $A$  in simplicial  $R$ -modules consists of simplicial  $R$ -modules  $A^n$ ,  $n \geq 0$ , together with bonding maps

$$\sigma : S^1 \otimes A^n \rightarrow A^{n+1}, \quad n \geq 0.$$

A morphism  $f : A \rightarrow B$  of spectrum objects consists of simplicial  $R$ -module maps  $A^n \rightarrow B^n$ ,  $n \geq 0$ , which respect the bonding homomorphisms.

$\mathbf{Spt}(R)$  is the corresponding category. This category is complete and cocomplete.

The maps

$$\gamma : S^1 \wedge u(A^n) \rightarrow u(S^1 \otimes A^n)$$

give the pointed simplicial sets  $u(A^n)$  the structure of a spectrum, and define a **forgetful** functor

$$u : \mathbf{Spt}(R) \rightarrow \mathbf{Spt}.$$

The reduced free  $R$ -module functor  $\tilde{R}$  determines a left adjoint to  $u$ . Explicitly,

$$(\tilde{R}X)^n = \tilde{R}(X^n),$$

and the bonding morphisms are the composites

$$S^1 \otimes \tilde{R}(X^n) \cong \tilde{R}(S^1 \wedge X^n) \xrightarrow{\sigma_*} \tilde{R}(X^{n+1}).$$

A map  $f : A \rightarrow B$  of spectrum objects is a **stable equivalence** (respectively **stable fibration**) if the underlying map  $u(f) : uA \rightarrow uB$  of spectra is a stable equivalence (respectively stable fibration).

A **cofibration** in  $\mathbf{Spt}(R)$  is a map which has the LLP wrt all morphisms which are stable fibrations and stable equivalences.

By adjointness, if  $A \rightarrow B$  is a cofibration of spectra, then the induced map  $\tilde{R}(A) \rightarrow \tilde{R}(B)$  is a cofibration of spectrum objects.

**Lemma 45.3.** *The functor  $\tilde{R} : \mathbf{Spt} \rightarrow \mathbf{Spt}(R)$  preserves stable equivalences.*

*Proof.* The functor  $\tilde{R}$  preserves level equivalences, so it suffices to show that if  $A \rightarrow B$  is a stably trivial

cofibration of spectra, then  $\tilde{R}(A/B) \rightarrow 0$  is a stable equivalence.

We show that  $\tilde{R}(X) \rightarrow 0$  is a stable equivalence if  $X \rightarrow *$  is a stable equivalence. We can assume that  $X$  is level fibrant.

Since  $X$  is level fibrant, the assumption that  $X \rightarrow *$  is a stable equivalence implies that all spaces  $\Omega^\infty X^n$  are contractible. Thus, if  $K \subset X^n$  is a finite subcomplex of  $X^n$ , there is a  $k \geq 0$  such that the composite

$$S^k \wedge K \rightarrow S^k \wedge X^n \xrightarrow{\sigma^k} X^{n+k}$$

is homotopically trivial. This means that the induced map

$$S^k \otimes \tilde{R}(K) \rightarrow S^k \otimes \tilde{R}(X^n) \rightarrow \tilde{R}(X^{n+k})$$

is also homotopically trivial, and so the morphism

$$\Sigma^\infty \tilde{R}(K)[-n] \rightarrow \tilde{R}(X)$$

induces 0 in all stable homotopy groups. Every element in  $\pi_k^s(\tilde{R}(X))$  is in the image of such a map, so all stable homotopy groups of  $\tilde{R}(X)$  are 0.  $\square$

Suppose that  $i : A \rightarrow B$  is a level monomorphism in  $\mathbf{Spt}(R)$  (as are all level cofibrations). Then there

is a short exact sequence

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{\pi} B/A \rightarrow 0$$

and the map  $\pi$  is a level surjection, hence a level fibration. In particular, the sequence is a level fibre sequence, and so there is a long exact sequence

$$\dots \pi_{k+1}^s(B/A) \xrightarrow{\partial} \pi_k^s A \xrightarrow{i_*} \pi_k^s(B) \xrightarrow{\pi_*} \pi_k^s(B/A) \rightarrow \dots$$

**Theorem 45.4.** *With these definitions, the category  $\mathbf{Spt}(R)$  of spectrum objects in simplicial  $R$ -modules has the structure of a proper closed simplicial model category.*

*Proof.* The category  $\mathbf{Spt}$  is cofibrantly generated (Lemma 41.2). Thus, a map  $p : A \rightarrow B$  is a stable fibration if and only if it has the right lifting property with respect to the maps

$$\tilde{R}(U) \rightarrow \tilde{R}(V)$$

induced by a set  $J$  of stably trivial cofibrations  $U \rightarrow V$ .

All induced maps  $\tilde{R}(U) \rightarrow \tilde{R}(V)$  are stable equivalences by Lemma 45.3.

The class of level inclusions which are stable equivalences is closed under pushout, by a long exact sequence argument.



It follows from a (transfinite) small object argument that every map  $f : A \rightarrow B$  in  $\mathbf{Spt}(R)$  has a factorization

$$\begin{array}{ccc} A & \xrightarrow{j} & C \\ & \searrow f & \downarrow p \\ & & D \end{array}$$

where  $j$  is a stably trivial cofibration which has the LLP wrt all fibrations and  $p$  is a fibration.

The proof of the other statement of the factorization axiom **CM5** uses the fact (Lemma 40.4) that a map  $p : A \rightarrow B$  is a stable fibration and a stable equivalence if and only if it has the right lifting property with respect to all morphisms

$$\tilde{R}(\Sigma^\infty \partial \Delta_+^n[k]) \rightarrow \tilde{R}(\Sigma^\infty \Delta_+^n[k]).$$

If  $i : A \rightarrow B$  is a stably trivial cofibration, then  $i$  is a retract of a map which has the LLP wrt all fibrations, on account of a factorization for  $i$  in the style displayed above. Thus, every stably trivial cofibration has the LLP wrt all fibrations, proving **CM4**.

The function complex  $\mathbf{hom}(A, B)$  is the simplicial  $R$ -module with  $n$ -simplices

$$\mathbf{hom}(A, B)_n = \{A \otimes \Delta^n \rightarrow B\}.$$

There is a natural isomorphism

$$\mathbf{hom}(\tilde{R}(K), A) \cong \mathbf{hom}(K, u(A)),$$

so that Quillen's axiom **SM7** follows from the corresponding statement for spectra. Thus, **Spt**( $R$ ) has a simplicial model structure.

Right properness follows from right properness for **Spt**, and left properness is proved by comparing long exact sequences.  $\square$

Here are some things to notice:

0) Every spectrum object in simplicial  $R$ -modules is level fibrant.

1) The forgetful functor  $u$  and its left adjoint  $\tilde{R}$  determine a Quillen adjunction

$$\tilde{R} : \mathbf{Spt} \rightleftarrows \mathbf{Spt}(R) : u$$

If  $R = \mathbb{Z}$  the canonical map  $X \rightarrow u(\tilde{\mathbb{Z}}(X))$  is the **Hurewicz homomorphism** for spectra.

2) There is a Quillen adjunction

$$\Sigma^\infty : s(R - \mathbf{Mod}) \rightleftarrows \mathbf{Spt}(R) : 0\text{-level}$$

where

$$(\Sigma^\infty A)^n = S^n \otimes A$$

(suspension spectrum) and the “0-level” functor is defined by  $B \mapsto B^0$ .

We also write  $H(A) = \Sigma^\infty A$ , and call it an Eilenberg-Mac Lane spectrum.

3) Suppose that  $A$  is a simplicial  $R$ -module, and consider the suspension spectrum object  $\Sigma^\infty A$ .

The bonding maps  $S^1 \otimes S^n \otimes A \rightarrow S^{n+1} \otimes A$  are canonical isomorphisms, with adjoints

$$S^n \otimes A \rightarrow \mathbf{hom}(S^1, S^1 \otimes S^n \otimes A)$$

given by adjunction maps  $\eta$ .

All of these maps  $\eta$  are weak equivalences by Lemma 45.1, and so  $\Sigma^\infty A$  is stably fibrant, ie.  $u(\Sigma^\infty A)$  is an  $\Omega$ -spectrum. It also follows that there are isomorphisms

$$\pi_n^s(A) = \begin{cases} \pi_n(A) & \text{if } n \geq 0, \text{ and} \\ 0 & \text{if } n < 0. \end{cases}$$

In particular,

$$\pi_n^s(\tilde{R}(\Sigma^\infty(X))) \cong \pi_n^s(\Sigma^\infty \tilde{R}(X))$$

coincides with the reduced homology group  $\tilde{H}_n(X, R)$  for  $n \geq 0$  and is 0 otherwise.

Recall that there is a natural map

$$\gamma: u(A) \wedge K \rightarrow u(A \otimes K)$$

for pointed simplicial sets  $K$  and simplicial  $R$ -modules  $R$ . In simplicial degree  $n$  it is the obvious function

$$\bigvee_{K_n-*} A_n \rightarrow \bigoplus_{K_n-*} A_n.$$

The construction can be iterated, meaning that there are commutative diagrams

$$\begin{array}{ccc} L \wedge u(A) \wedge K & \xrightarrow{1 \wedge \gamma} & L \wedge u(A \otimes K) \\ \gamma \wedge 1 \downarrow & & \downarrow \gamma \\ u(L \otimes A) \wedge K & \xrightarrow{\gamma} & u(L \otimes A \otimes K) \end{array}$$

The map  $\gamma$  may therefore be promoted to the spectrum level, so there is a natural map

$$\gamma : u(B) \wedge K \rightarrow u(B \otimes K)$$

for spectrum objects  $B$  and pointed simplicial sets  $K$ .

**Theorem 45.5.** *The map*

$$\gamma : u(B) \wedge K \rightarrow u(B \otimes K)$$

*is a stable equivalence for all spectrum objects  $B$  and pointed simplicial sets  $K$ .*

*Proof.* The simplicial set  $K$  has a (pointed) skeletal decomposition  $\text{sk}_n K \subset K$ , and there are pushout

diagrams

$$\begin{array}{ccc} \bigvee_{x \in NK_n} \partial \Delta_+^n & \longrightarrow & \mathrm{sk}_{n-1} K \\ \downarrow & & \downarrow \\ \bigvee_{x \in NK_n} \Delta_+^n & \longrightarrow & \mathrm{sk}_n K \end{array}$$

of pointed simplicial sets.

Smashing with  $u(B)$  gives a homotopy cocartesian diagram, which can be compared to the diagram of spectra underlying the pushout diagram

$$\begin{array}{ccc} \bigoplus_{x \in NK_n} (B \otimes \partial \Delta_+^n) & \longrightarrow & B \otimes \mathrm{sk}_{n-1} B \\ \downarrow & & \downarrow \\ \bigoplus_{x \in NK_n} (B \otimes \Delta_+^n) & \longrightarrow & B \otimes \mathrm{sk}_n K \end{array}$$

in  $\mathbf{Spt}(R)$  via the map  $\gamma$ . The underlying diagram of spectra is homotopy cocartesian since both vertical maps have the same cofibres.

Inductively, one assumes that

$$u(B) \wedge \mathrm{sk}_{n-1} K \rightarrow u(B \otimes \mathrm{sk}_{n-1} K)$$

is a stable equivalence for all  $K$ . The statement for 0-skeleta is a consequence of additivity (Corollary 43.3), with a filtered colimit argument.

It therefore suffices to show that the map

$$\gamma : u(B) \wedge \left( \bigvee_{NK_n} \Delta_+^n \right) \rightarrow u(B \otimes \left( \bigvee_{NK_n} \Delta_+^n \right)).$$

is a stable equivalence. By additivity, this reduces to the statement that

$$\gamma: u(B) \wedge \Delta_+^n \rightarrow u(B \otimes \Delta_+^n)$$

is a stable equivalence.

Both displayed functors preserve homotopy equivalences, so this particular instance of  $\gamma$  is equivalent to

$$\gamma: u(B) \wedge S^0 \rightarrow u(B \otimes S^0),$$

which is an isomorphism. □

**Example:** There is a natural isomorphism

$$H_n(X, R) \cong \pi_n^s(H(R) \wedge X).$$

Here  $H(R)$  is the Eilenberg-Mac Lane spectrum  $\tilde{R}(\mathbf{S}) = \Sigma^\infty R(S^0)$ ; it's also the sphere spectrum for  $\mathbf{Spt}(R)$ .

More generally, the groups

$$E_*(X) = \pi_*(E \wedge X)$$

are the  **$E$ -homology groups** of the space  $X$ , for a spectrum  $E$ .

## 46 Chain complexes

Given a chain complex  $D$  in  $Ch_+$ , define the shifted complex  $D[k]$  by

$$D[k]_p = \begin{cases} D_{k+p} & \text{if } p > 0, \\ \ker(\partial : D_k \rightarrow D_{k-1}) & \text{if } p = 0. \end{cases}$$

For  $n \geq 0$ ,  $D[-n]$  shifts up (“suspends”)  $n$  times while  $D[n]$  is the good truncation of a shift down.

There are two suspension constructions for simplicial  $R$ -modules:

- the standard suspension  $S^1 \otimes A = \tilde{R}(S^1) \otimes A$ ,
- the Eilenberg-Mac Lane (or Kan) suspension  $\overline{W}A = \Gamma(NA[-1])$ .

There is an alternative construction for  $\overline{W}A$ , as follows.

Every simplicial abelian group can be written as a coequalizer

$$\bigoplus_{\theta: \mathbf{m} \rightarrow \mathbf{n}} A_n \otimes \Delta_+^m \rightrightarrows \bigoplus_{n \geq 0} A_n \otimes \Delta_+^n \rightarrow A$$

There is a pointed cosimplicial set  $\mathbf{n} \mapsto \Delta_*^{n+1}$ , where  $\Delta_*^{n+1}$  is  $\Delta^{n+1}$  pointed by 0, and  $\theta : \mathbf{m} \rightarrow \mathbf{n}$  induces

$\theta_* : \mathbf{m} + \mathbf{1} \rightarrow \mathbf{n} + \mathbf{1}$  which is defined by

$$\theta_*(j) = \begin{cases} 0 & j = 0, \\ \theta(j-1) + 1 & j > 0. \end{cases}$$

The simplicial set maps  $d^0 : \Delta^n \rightarrow \Delta^{n+1}$  determine a map of cosimplicial spaces, and a pointwise monomorphism of cosimplicial simplicial modules

$$\tilde{R}(\Delta_+^n) \rightarrow \tilde{R}(\Delta_*^{n+1})$$

One checks that there is an isomorphism of cosimplicial chain complexes

$$N(\tilde{R}\Delta_*^{n+1}/N\tilde{R}\Delta_+^n) \cong N\tilde{R}\Delta_+^n[-1]$$

that is natural in ordinal numbers  $\mathbf{n}$  (exercise).

Thus,  $\Gamma NA[-1]$  is defined by the coequalizer

$$\bigoplus_{\theta: \mathbf{m} \rightarrow \mathbf{n}} A_n \otimes N\tilde{R}\Delta_+^m[-1] \rightrightarrows \bigoplus_{n \geq 0} A_n \otimes N\tilde{R}\Delta_+^n[-1] \rightarrow \Gamma NA[-1]$$

There is a natural short exact sequence

$$0 \rightarrow A \xrightarrow{d^0} CA \rightarrow \overline{W}A \rightarrow 0$$

where the “cone”  $CA$  is defined by the coequalizer

$$\bigoplus_{\theta: \mathbf{m} \rightarrow \mathbf{n}} A_n \otimes \Delta_*^{m+1} \rightrightarrows \bigoplus_{n \geq 0} A_n \otimes \Delta_*^{n+1} \rightarrow CA$$



The inclusion  $d^0 : \Delta^n \rightarrow \Delta^{n+1}$  contracts to the vertex  $0 \in \mathbf{n} + \mathbf{1}$ , via the homotopy

$$h : \Delta_+^n \wedge \Delta_*^1 \rightarrow \Delta_*^{n+1}$$

( $\Delta^1$  is pointed by 0) which is given by the picture

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & \downarrow \\ 1 & \longrightarrow & 2 & \longrightarrow & \dots & \longrightarrow & n+1 \end{array}$$

The homotopies  $h$  form a map of cosimplicial spaces, and hence determine a natural map

$$A \otimes \Delta_*^1 \rightarrow CA,$$

which in turn induces a natural map

$$h : S^1 \otimes A \rightarrow \overline{W}A.$$

This map  $h$  is a natural equivalence, since  $A \otimes \Delta_*^1$  and  $CA$  are both contractible.

The map  $h$  is even a natural homotopy equivalence, since the cosimplicial objects  $S^1 \otimes \tilde{R}(\Delta_+)$  and  $\overline{W}\tilde{R}(\Delta_+)$  are projective cofibrant.

For this last claim, we use Corollary 26.4, and its proof to show that the cosimplicial map

$$\tilde{R}(\Delta_+^n) \rightarrow \tilde{R}(\Delta_*^{n+1})$$

is a projective cofibration.

Write  $g$  for the natural homotopy inverse for  $f$ .

Every spectrum object  $\sigma : S^1 \otimes A^n \rightarrow A^{n+1}$  in simplicial  $R$ -modules determines a “Kan” spectrum object

$$\overline{W}A^n \xrightarrow{g} S^1 \otimes A^n \xrightarrow{\sigma} A^{n+1}$$

and hence a spectrum object

$$\tilde{\sigma} : NA^n[-1] \cong N(\overline{W}A^n) \rightarrow NA^{n+1}$$

in chain complexes.

Let  $\sigma_* : A^n \rightarrow \Omega A^{n+1}$  be the adjoint of  $\sigma$ .

Corollary 45.2 says that the evaluation map

$$ev : \Omega A^{n+1} \otimes S^1 \rightarrow A^{n+1}$$

is a homology isomorphism above degree 0, and further that there is an induced equivalence

$$ev_*[1] : N\Omega A^{n+1} \rightarrow NA^{n+1}[1]$$

(as  $\mathbb{Z}$ -graded chain complexes), on account of the diagram

$$\begin{array}{ccc} N(S^1 \otimes \Omega A^{n+1}) & \xrightarrow{Nev} & NA^{n+1} \\ Ng \uparrow & & \uparrow ev_* \\ N(\overline{W}\Omega A^{n+1}) & \xrightarrow{\cong} & N(\Omega A^{n+1})[-1] \end{array}$$

There is, finally, a natural commutative diagram of

chain complex maps

$$\begin{array}{ccc}
 NA^n & \xrightarrow{N\sigma_*} & N\Omega A^{n+1} \\
 & \searrow \sigma & \downarrow \simeq ev_*[1] \\
 & & NA^{n+1}[1]
 \end{array}$$

which defines the map  $\sigma$ .

Identify all chain complexes  $NA^n$  with  $\mathbb{Z}$ -graded chain complexes, and let  $QNA$  be the colimit of the diagram

$$NA^0 \xrightarrow{\sigma} NA^1[1] \xrightarrow{\sigma[1]} NA^2[2] \xrightarrow{\sigma[2]} \dots$$

Then one can show the following:

**Proposition 46.1.** *A map  $f : A \rightarrow B$  is a stable equivalence of spectrum objects in simplicial  $R$ -modules if and only if the induced map  $f_* : QNA \rightarrow QNB$  is a quasi-isomorphism of  $\mathbb{Z}$ -graded chain complexes.*

One can go further [1], to show that the Dold-Kan equivalence induces a Quillen equivalence

$$N : \mathbf{Spt}(R) \rightleftarrows Ch(R) : \Gamma$$

of the stable model structure on  $\mathbf{Spt}(R)$ , with the model structure on the category  $Ch(R)$  of  $\mathbb{Z}$ -graded chain complexes of  $R$ -modules of Section 3, from the beginning of the course.

The weak equivalences in  $Ch(R)$  are the quasi-isomorphisms, and the fibrations are the surjective homomorphisms of chain complexes.

This equivalence further induces an equivalence of the stable homotopy category for  $\mathbf{Spt}(R)$  with the full derived category  $\mathrm{Ho}(Ch(R))$  for chain complexes of  $R$ -modules.

This is the start of a long story — see also [1], [2].

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