Chapter 2

Differential Galois Theory

The differential Galois theory for linear differential equations is the Picard-Vessiot Theory. In this theory there is a very nice concept of "integrability" i.e., solutions in closed form: an equation is integrable if the general solution is obtained by a combination of algebraic functions (over the coefficient field), exponentiation of quadratures and quadratures. Furthermore, all information about the integrability of the equation is coded in the identity component of the Galois group: the equation is integrable if, and only if, the identity component of its Galois group is solvable. It is a powerful theory in the sense that, in some favorable cases (for instance, for equations of order 2), it is possible to construct algorithms to determine whether a given linear differential equation is integrable or not.

We shall present only the essential definitions. Results shall be stated without proofs, unless the author has some contribution to them or if they are not easily found in the references. Three different approachs shall be used: ([12, 21, 50, 51, 54, 71, 69, 94, 102]): the classical approach, the Tannakian approach and the monodromy and Stokes's multipliers approach. As will become clear, all of them will be useful in this monograph.

In Sections 2.3, 2.4, 2.5 and 2.6 we will follow [77].

2.1 Algebraic groups

In this section the necessary results of linear algebraic groups are presented. An introduction to linear algebraic groups is given in [19]. For more information see the monographs [45, 14].

A linear algebraic group G (over $\mathbf{C}$) is a subgroup of $GL(m,\mathbf{C})$ whose matrix coefficients satisfy polynomial equations over $\mathbf{C}$. It has structures of an algebraic variety (non-singular) as well as of a group, and these two structures

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are compatible: the group operation and taking of inverses are morphisms of algebraic varieties. We note that in a linear algebraic group there are two different topologies: the Zariski topology, where the closed sets are the algebraic sets, and the usual real topology. In particular, an algebraic group is a complex analytical Lie group and we can consider the Lie algebra of this group. Therefore the dimension of G is the dimension of the Lie algebra of G. Given a linear algebraic group G, the identity component (or the neutre component) G^0 is the (unique) irreducible component which contains the identity element of G.

We remark that an algebraic linear (or affine) group G is usually defined as an affine algebraic variety with a group structure, with the compatibility condition above: the group multiplication and taking of inverses are morphisms of algebraic varieties. Then, given a such G, there is a rational faithful representation of G as a closed subgroup of $GL(m, \mathbf{C})$, for some m, and we obtain the equivalence with our definition.

It is clear that the classical linear complex groups are linear algebraic groups. For instance $SL(n, \mathbf{C})$, $SO(n, \mathbf{C})$ (rotation group) and $Sp(n, \mathbf{C}) \subset Gl(2n, \mathbf{C})$ (symplectic group) are linear algebraic groups since they are defined by polynomial identities.

Proposition 2.1 *The identity component G^0 of a linear algebraic group G is a closed (with respect to the two above topologies) normal subgroup of G of finite index and it is connected with respect to the two above topologies. Furthermore the classes of G/G^0 are the irreducible connected components of G.*

We note that by the above proposition G^0 is also a linear algebraic group and the Lie algebra of G, $\mathrm{Lie}(G) = \mathcal{G}$ coincides with the Lie algebra of G^0, $\mathrm{Lie}(G^0) = \mathcal{G}$. As for every Lie group, G^0 is solvable (abelian) if, and only if, $\mathcal{G}$ is solvable (respectively abelian). Furthermore, G is connected if, and only if $G = G^0$.

The characterization of the connected solvable linear algebraic groups is given by the Lie-Kolchin theorem.

Theorem 2.1 (Lie-Kolchin Theorem) *A connected linear algebraic group is solvable if, and only if, it is conjugated to a triangular group.*

In the context of linear algebraic groups a torus is a group isomorphic to the multiplicative group $(\mathbf{C}^*)^k$. The dimension of the above torus is k. Equivalently, it is a linear algebraic group conjugated to a diagonal group. It is clear that a torus is connected and abelian.

Let G be a linear algebraic group. A maximal torus in G is a torus of maximal dimension contained in G. As a maximal torus is connected, it is contained in the identity component G^0.

Example. Let $Sp(n,\mathbf{C}) \subset Gl(2n,\mathbf{C})$ be the symplectic group. It is easy to see that the maximal tori in $Sp(n,\mathbf{C})$ are all the groups conjugated to

$$T = \{\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n,\lambda_1^{-1},\lambda_2^{-1},\ldots,\lambda_n^{-1}), \lambda_i \in \mathbf{C}^*, i=1,2,\ldots,n\}.$$

Indeed as is well known, the eigenvalues of the symplectic matrices $\sigma \in Sp(n,\mathbf{C})$ appear in pairs (λ,λ^{-1}) (see for instance [3]) and we get the above.

Given a subset $S \subset GL(n,\mathbf{C})$, let M be the group generated by S and G be the Zariski closure of the group M. By definition the group G is a linear algebraic group and we will say that this group is topologically generated by the set S. Sometimes we will emphasize in the difference between M and G and we will say that M is algebraically generated by S.

Since the examples of irreducible equations that we shall meet will be of second order and symplectic, we end this section with a classification of the algebraic subgroups of $SL(2,\mathbf{C})$. We shall need two lemmas.

Lemma 2.1 ([50]) *Let G be an algebraic group contained in $SL(2,\mathbf{C})$. Assume that the identity component G^0 of G is solvable. Then G is conjugate to one of the following types:*

(1) *G is finite,*
(2) $G = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & 0 \end{pmatrix} \lambda,\beta \in \mathbf{C}^* \right\},$
(3) *G is triangular.*

Lemma 2.2 *Let G be an algebraic subgroup of $SL(2,\mathbf{C})$ such that the identity component G^0 is not solvable. Then $G = SL(2,\mathbf{C})$.*

The last lemma is well known and it follows easily from consideration of the Lie algebra of $G \subset SL(2,\mathbf{C})$. Indeed, if G^0 is not solvable then the dimension of G must be equal to 3, because all 2-dimensional Lie algebras are solvable.

Proposition 2.2 ([81]) *Any algebraic subgroup G of $SL(2,\mathbf{C})$ is conjugated to one of the following types:*

1. *Finite, $G^0 = \{\mathbf{1}\}$, where* $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$
2. $G = G^0 = \left\{ \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}, \mu \in \mathbf{C} \right\}.$
3. $G_k = \left\{ \begin{pmatrix} \lambda & 0 \\ \mu & \lambda^{-1} \end{pmatrix}, \lambda \text{ is a } k\text{-root of unity}, \mu \in \mathbf{C} \right\},$

$$G^0 = \left\{ \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}, \mu \in \mathbf{C} \right\}.$$

4. $G = G^0 = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \in \mathbf{C}^* \right\}$.
5. $G = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & 0 \end{pmatrix} \lambda, \beta \in \mathbf{C}^* \right\}$,
$$G^0 = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \in \mathbf{C}^* \right\}.$$
6. $G = G^0 = \left\{ \begin{pmatrix} \lambda & 0 \\ \mu & \lambda^{-1} \end{pmatrix}, \lambda \in \mathbf{C}^*, \mu \in \mathbf{C} \right\}$.
7. $G = G^0 = SL(2, \mathbf{C})$.

Proof. Assume G to be infinite and conjugated to a triangular group, i.e., it is contained in the total triangular group (isomorphic to the semidirect product of the additive group $\mathbf{C}$ and of the multiplicative group $\mathbf{C}^*$)

$$\left\{ \begin{pmatrix} \lambda & 0 \\ \mu & \lambda^{-1} \end{pmatrix}, \lambda \in \mathbf{C}^*, \mu \in \mathbf{C} \right\}.$$

Let ψ be the morphism of algebraic groups

$$\psi : G \longrightarrow \mathbf{C}^*,$$

defined by

$$\psi \begin{pmatrix} \lambda & 0 \\ \mu & \lambda^{-1} \end{pmatrix} = \lambda.$$

If $\ker \psi$ is trivial then G must be the diagonal group

$$G = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \in \mathbf{C}^* \right\},$$

because then $G \approx \psi(G)$, $\psi(G)$ being an algebraic subgroup of the multiplicative group $\mathbf{C}^*$. But then $\psi(G)$ must be equal to $\mathbf{C}^*$ (the only possible non-trivial subgroups of $\mathbf{C}^*$ are the cyclic finite groups).

If $\ker \psi$ is non-trivial then, as it is (isomorphic to) an algebraic subgroup of the additive group $\mathbf{C}$, it is the total unipotent group

$$\left\{ \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}, \mu \in \mathbf{C} \right\}.$$

Now as above we have two possibilities: either $\psi(G)$ is equal to the multiplicative group $\mathbf{C}^*$ or it is a finite cyclic group. The proposition follows from the two lemmas above. $\square$

The above proposition is analogous (but more precise: we need to know when the identity component of the Galois group is not only solvable, but abelian) to the proposition in [56], p. 7. We remark that the identity component G^0 is abelian in cases (1)–(5) and is solvable in cases (1)–(6).

2.2 Classical approach

A differential field K is a field with a derivative (or derivation) $\delta = {}'$, i.e., an additive mapping that satisfies the Leibniz rule. Examples are $\mathcal{M}(\overline{\Gamma})$ (meromorphic functions over a connected Riemann surface $\overline{\Gamma}$, the reason for this notation will be clear below: $\overline{\Gamma} - \Gamma$ will be the set of singular points of the linear differential equation) with a non-trivial meromorphic tangent vector field X as derivation, in particular $\mathbf{C}(z) = \mathcal{M}(\mathbf{P^1})$ with $\frac{d}{dz}$ or $z\frac{d}{dz}$ as derivation, $\mathbf{C}\{x\}[x^{-1}]$ (convergent Laurent series), or $\mathbf{C}[[x]][x^{-1}]$ (formal Laurent series) with $x\frac{d}{dx}$ as derivation. We observe that there are some inclusions between the above differential fields.

We can define (differential) subfields, (differential) extensions in a direct way by requiring that inclusions must commute with the derivations. Analogously, a (differential) automorphism in K is an automorphism that commutes with the derivative. The field of constants of K is the kernel of the derivative. In the above examples $\mathbf{C}$ is such a kernel. From now on we will suppose that this is the case.

Let

$$\xi' = A\xi, \quad A \in Mat(m, K). \tag{2.1}$$

We shall proceed to associate to (2.1) the so-called Picard-Vessiot extension of K. The Picard-Vessiot extension L of (2.1) is an extension of K, such that if $u_1, \ldots, u_m$ is a "fundamental" system of solutions of the equation (2.1) (i.e., linearly independent over $\mathbf{C}$), then $L = K(u_{ij})$ (rational functions in K in the coefficients of the "fundamental" matrix $(u_1 \cdots\ u_m)$). This is the extension of K generated by K together with u_{ij}. We observe that L is a differential field (by (2.1)). The existence and unicity (except by isomorphism) of the Picard-Vessiot extensions is proved by Kolchin (in the analytical case, $K = \mathcal{M}(\overline{\Gamma})$, and this result is essentially the existence and uniqueness theorem for linear differential equations).

As in classical Galois theory, we define the Galois group of (2.1) $G := \mathrm{Gal}_K(L) = \mathrm{Gal}(L/K)$ as the group of all the (differential) automorphisms of L which leave fixed the elements of K. This group is isomorphic to an algebraic linear group over $\mathbf{C}$. We say that the extension L/K is normal if any element of L, invariant by the Galois group $\mathrm{Gal}_K(L)$, necessarily belongs to K. The Picard-Vessiot extensions are normal and by this property of the Picard-Vessiot extensions it is proved that the Galois correspondence (between groups and extensions) works well in this theory.

Theorem 2.2 *Let L/K be the Picard-Vessiot extension associated to a linear differential equation. Then there is a $1-1$ correspondence between the intermediary differential fields $K \subset M \subset L$ and the algebraic subgroups $H \subset G :=$*

$\mathrm{Gal}_K(L)$, *such that* $H = \mathrm{Gal}_M(L)$ *(the extension* L/M *is a Picard-Vessiot extension). Furthermore, we have*

(i) *The normal extensions* M/K *correspond to the normal subgroups* $H \subset G$. *Then the group* G/H *is a linear algebraic group, the extension* M/K *is a Picard-Vessiot extension and* $G/H = \mathrm{Gal}_K(M)$.

(ii) *Let* F *be a subgroup of* G *and* K_F *the subfield of* L *given by the elements of* L *fixed by* F. *Then* $H := \mathrm{Gal}_{K_F}(L)$ *is the Zariski closure (over the field of constants* $\mathbf{C}$*) of* F *(i.e.,*H *is topologically generated by* F*).*

As a corollary, when we consider the algebraic closure $\overline{K}$ (of K in L), we obtain $\mathrm{Gal}_K(\overline{K}) = G/G^0$, where $G^0 = \mathrm{Gal}_{\overline{K}}(L)$ is the identity component of the Galois group G which corresponds to the transcendental part of the Picard-Vessiot extension, i.e., by definition, the extension $L/\overline{K}$ is the maximal transcendental extension between the extensions L/L_1, with L_1 an extension of K. If $\overline{K} = K$ (i.e., if $G = G^0$), we say that L/K is a purely transcendental extension.

Another consequence of Theorem 2.2 is that if $\Lambda \subset \overline{\Gamma}$ is a Riemann surface contained in $\overline{\Gamma}$ and L is a Picard-Vessiot extension of $\mathcal{M}(\overline{\Gamma})$, then $\mathrm{Gal}_{\mathcal{M}(\Lambda)}(L) \subset \mathrm{Gal}_{\mathcal{M}(\overline{\Gamma})}(L)$. We will apply this in Chapter 7. In a similar way, the local Galois group at a singular point $s \in \overline{\Gamma} - \Gamma$, $\mathrm{Gal}_{\mathbf{C}\{x\}[x^{-1}]}(L) := \mathrm{Gal}_{k_s}(L)$, is a subgroup of the global Galois group $\mathrm{Gal}_{\mathcal{M}(\Gamma)}(L)$ (as usual, we identify the germs of meromorphic functions at a singular point s with Laurent series centered at this point).

We will say that a linear differential equation is (Picard-Vessiot) integrable (or solvable) if we can obtain its Picard-Vessiot extension $K \subset L$ and, hence, the general solution, by adjunction to K of integrals, exponentiation of integrals or algebraic functions of elements of K. In other words, there exists a chain of differential extensions $K_1 := K \subset K_2 \subset \cdots \subset K_r := L$, where each extension is given by the adjunction of one element a, $K_i \subset K_{i+1} = K_i(a, a', a'', \ldots)$, such that a satisfies one of the following conditions:

(i) $a' \in K_i$,

(ii) $a' = ba$, $b \in K_i$,

(iii) a is algebraic over K_i.

The usual terminology is that the Picard-Vessiot extension is Liouvillian. Then, it can be proved that a linear differential equation is integrable if, and only if, the identity component of the Galois group, G^0, is a solvable group. In particular, if the identity component is abelian, the equation is integrable.

Furthermore, the relation between the monodromy and the Galois group is as follows.

Let $\overline{\Gamma} - \Gamma$ be the set of singular points of the equation i.e., the poles of the coefficients on $\overline{\Gamma}$. We recall that the monodromy group of the equation is a subgroup of the linear group, given by the image of a representation of the fundamental group $\pi_1(\Gamma)$ into the linear group $GL(m, \mathbf{C})$. This representation is obtained by analytical continuation of the solutions along the elements of $\pi_1(\Gamma)$ (see for instance [36]). The monodromy group is contained in the Galois group and if the equation is of Fuchsian class (i.e., it has regular singular singularities only), then the Galois group is dense in the monodromy group (Zariski topology) i.e., the Galois group is topologically generated by the monodromy group (see [69]). In the general case, Ramis found a generalization of the above and, for example, he proved that the Stokes matrices associated to an irregular singularity belong to the (local) Galois group (see Section 2.5 below). We will formulate a generalization of this result in Theorem 2.4 (see Appendix B for the proof).

A useful criterion for unimodularity is the following. The second order equation with coefficients p and q in a differential field K

$$\xi'' + p\xi' + q\xi = 0, \tag{2.2}$$

has a Galois group contained in $SL(2, \mathbf{C})$ if, and only if, $p = nd/d'$, for some $n \in \mathbf{Z}$, $d \in K$. To show this we note that for all σ in the Galois group, the Wronskian w belongs to K if, and only if, $w = \sigma(w) = \det(\sigma)w$, which is equivalent to $\det(\sigma) = 1$. We get this result by Abel's formula $w' + pw = 0$ (we take $w = Cd^n$, with $C \in \mathbf{C}$).

By the above criterion, the equation

$$\xi'' + g\xi = 0, \tag{2.3}$$

(where $g \in K$) has a Galois group contained in $SL(2, \mathbf{C})$. Now the classical change $v = -\xi'/\xi$ leads to the associated Riccati equation

$$v' = g + v^2. \tag{2.4}$$

Then

Proposition 2.3 ([81]) *If the equation $\xi'' + g\xi = 0$ is integrable then we are in one of the situations* 1 *to* 6 *of Proposition* 2.2*, and if we assume that the Galois group is not finite, then one has for the Riccati equation* (2.2) *the following:*

1. *Cases* 2, 3 *and* 6*: it has exactly one solution in K.*
2. *Case* 4*: it has two solutions in K.*
3. *Case* 5*: it has two solutions in a quadratic extension of K but they do not belong to K.*

Proof. (A less detailed statement can be found in [50].) In cases 2, 3 and 6 there exists a solution, ξ_1, such that

$$\sigma(\frac{\xi'}{\xi}) = \frac{\xi'}{\xi},$$

for any σ in the Galois group. By the normality of the Picard-Vessiot extensions, one has that $v_1 = -\xi_1'/\xi_1$ belongs to K. Therefore v_1 is a solution of the Riccati equation in K. Let us assume that there is another solution, v_2 of the Riccati equation in K. Let ξ_2 be defined by $v_2 = -\xi_2'/\xi_2$. Then $\{\xi_1, \xi_2\}$ is a fundamental system of solutions of $\xi'' + g\xi = 0$, because

$$v_2 - v_1 = \frac{w}{\xi_1\xi_2},$$

w being the Wronskian of $\{\xi_1, \xi_2\}$. But for each element

$$\sigma = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

in the Galois group G, $\sigma(v_i) = v_i$, $i = 1, 2$, implies that G is diagonal. Indeed, if

$$\frac{\xi}{\xi'} = \sigma(\frac{\xi}{\xi'}) = \frac{\alpha\xi_1' + \beta\xi_2'}{\alpha\xi_1 + \beta\xi_2},$$

then βw=0 and, therefore, $\beta = 0$. In an analogous way we obtain $\gamma = 0$. Then G would be diagonal and this contradicts the hypothesis.

In case 4, let $\{\xi_1, \xi_2\}$ be a fundamental system of solutions such that $\sigma(\xi_1) = \lambda\xi_1$, $\sigma(\xi_2) = \lambda^{-1}\xi_2$, with σ an element of the Galois group. Hence $\sigma(\xi'/\xi) = \xi'/\xi$, and by normality we get $v_i = -\xi'/\xi \in K$, for $i = 1, 2$. Of course one has $v_1 \neq v_2$ because

$$v_2 - v_1 = \frac{w}{\xi_1\xi_2}.$$

In case 5, G/G^0 is the Galois group of the quadratic extension $\overline{K}/K$ and G^0 is the Galois group of the extension $L/\overline{K}$ (see the remark after Theorem 2.2). As G^0 is diagonal, the proof that the Riccati equation has two different solutions v_1, v_2 in $\overline{K}$ proceeds as in case 4. They do not belong to K because then G would be diagonal. We note that, in case 5, if the Riccati equation has one solution $\overline{K}$ then it has two solutions in $\overline{K}$. Indeed, let $v_1 = \kappa + \sqrt{\omega}$ be a solution in $\overline{K}$, with $\kappa, ,\omega \in \overline{K}$, $\sqrt{\omega} \notin K$, then $v_2 = \kappa - \sqrt{\omega}$ is another solution in $\overline{K}$. $\square$

We remark that although the above proposition is closely related to Kovacic's algorithm (see Section 2.6), we establish it in an independent way, because for some particular equations (as for Lamé's equation, see Section 2.8.4) it gives us good results without using all the machinery of Kovacic's algorithm.

We finish this section with a remark about differential extensions by quadratures. Let L/K be a differential extension by integrals, i.e., $L = K(a_1, a_2, \ldots, a_s)$, where $a_i' \in K$, $i = 1, 2, \ldots, s$. Then L/K is a Picard-Vessiot extension and the Galois group $\mathrm{Gal}(L/K)$ is isomorphic to an additive group

$G_a^r := (\mathbf{C}^r, +)$, for some $r \leq s$. For $s = 1$ (see [50, 71]), the corresponding linear differential equation is a second order equation. For arbitrary s, we write the corresponding linear differential equation as a direct sum of s second order equations and we obtain the linear representation of the Galois group as an additive subgroup of the unipotent linear group contained in $GL(2s, \mathbf{C})$. In particular, $\mathrm{Gal}_K(L)$ is connected, and L/K is a purely transcendental extension.

2.3 Meromorphic connections

Linear connections are the intrinsic version of systems of linear differential equations. Moreover, with connections it is possible to work with necessarily non-trivial fibre bundles. A good reference for this section is [104] (see also [29, 30, 51, 69]).

Let Γ be a (connected) Riemann surface. We denote by $\mathcal{O}_\Gamma$ its sheaf of holomorphic functions, by Ω_Γ its sheaf of holomorphic 1-forms (corresponding to the canonical bundle) and by $\mathcal{X}_\Gamma$ its sheaf of holomorphic vector fields. We will identify vector fields with derivations on $\mathcal{O}_\Gamma$. We have a sheaf structure of Lie-algebras on $\mathcal{X}_\Gamma$. There exist, clearly, natural structures of $\mathcal{O}_\Gamma$-modules on Ω_Γ and $\mathcal{X}_\Gamma$, respectively. There exists a natural map (contraction)

$$\Omega_\Gamma \otimes_{\mathcal{O}_\Gamma} \mathcal{X}_\Gamma \to \mathcal{O}_\Gamma,$$

$$\omega \otimes v \to \langle \omega, v \rangle.$$

Let V be a holomorphic vector bundle of rank m on Γ. We denote by $\mathcal{O}_V$ its sheaf of holomorphic sections. Then a *holomorphic* connection is by definition a map

$$\nabla : \mathcal{O}_V \to \Omega_\Gamma \otimes_{\mathcal{O}_\Gamma} \mathcal{O}_V,$$

satisfying the Leibniz rule

$$\nabla(v + w) = \nabla v + \nabla w$$

$$\nabla f v = df \otimes v + f \nabla v,$$

where v, w are holomorphic sections of the fibre bundle V and f is a holomorphic function.

By definition a section v of the fibre bundle V is horizontal for the connection ∇ if $\nabla v = 0$.

If the connection ∇ is fixed, then to each holomorphic vector field X over Γ, we can associate the covariant derivative along X

$$\nabla_X : \mathcal{O}_V \to \mathcal{O}_V,$$
$$\nabla_X : v \to \langle \nabla v, X \rangle.$$

It is clearly a $\mathbf{C}$-linear map. If we denote by $\mathrm{End}_{\mathbf{C}}(\mathcal{O}_V)$ the sheaf of spaces of $\mathbf{C}$-linear endomorphisms of the sheaf of complex vector spaces $\mathcal{O}_V$, then we get a map

$$\nabla : \mathcal{X}_\Gamma \to \mathrm{End}_{\mathbf{C}}(\mathcal{O}_V),$$

$$X \mapsto \nabla_X,$$

such that

$$\nabla_X(v+w) = \nabla_X v + \nabla_X w,$$

$$\nabla_X(fv) = X(f)v + f\nabla_X v, \quad f \in \mathcal{O}_\Gamma.$$

We are going to compute ∇ in local coordinates. Let X be a holomorphic vector field over an open subset U of the Riemann surface Γ. Restricting U, if necessary, we can suppose that there exists a holomorphic local coordinate t over U such that

$$X = \frac{d}{dt}.$$

Let $e = \{e_1, \ldots, e_m\}$ be a holomorphic frame of U, i.e., the data of m holomorphic sections of V over U, such that $e_1(p), \ldots, e_m(p) \in V_p$ are linearly independent at every point $p \in U$. Then we can set

$$\nabla e_j = -\sum_{i=1}^{m} a_{ij} e_i,$$

(a_{ij}) being a square matrix of order m whose entries are holomorphic functions over U. We write $\nabla e = -Ae$.

The matrix $A = (a_{ij})$ is by definition the connection matrix and it determines completely the connection: if v is a holomorphic section over U, then we can write it in coordinates

$$v = \sum_{i=1}^{m} \xi_i e_i,$$

where the ξ_i's are holomorphic functions over U, and we have

$$\nabla v = \sum_{i=1}^{m} \Big(\frac{d\xi_i}{dt} - \sum_{j=1}^{m} a_{ij}\xi_j\Big) e_i,$$

i.e., the connection ∇ is represented in the local coordinate t and the frame e by the linear differential operator

$$\nabla := \nabla_{\frac{d}{dt}} = \frac{d}{dt} - A.$$

Hence, we can associate to the solutions $\xi \in \mathcal{O}_U^m$ of the linear differential system

$$\frac{d\xi_i}{dt} = \sum_{j=1}^{m} a_{ij}\xi_j, \quad i = 1, \dots, m,$$

the horizontal sections v of the connection

$$\nabla v = 0.$$

More precisely the map

$$\xi \mapsto \sum_{i=1}^{m} \xi_i e_i$$

induces an isomorphism of m-dimensional complex vector spaces between the space of solutions and the space of horizontal sections.

In fact we are interested not only in differential equations (or systems) with *holomorphic* coefficients, but also in differential equations (or systems) with *meromorphic* coefficients. Therefore we need to extend the above concept of holomorphic connection in order to deal with poles and consequently to introduce *meromorphic* connections. We shall follow Section 4 of [104] (a more elaborated analysis in the context of free coherent sheaves can be found in [67]).

Let $\overline{\Gamma}$ be a Riemann surface and V a *holomorphic* vector bundle on $\overline{\Gamma}$. In our applications, the following specific conditions will hold. Let $\Gamma \subset \overline{\Gamma}$ be an open subset such that $S = \overline{\Gamma} - \Gamma$ is a discrete subset (the singular set). We will consider meromorphic sections of the bundle V, and in general we will limit ourselves to sections whose restriction to Γ is *holomorphic*. Then at any point $s \in S$ their components, in coordinates with respect to a holomorphic local frame, are meromorphic functions in a neighborhood U_s, which are holomorphic on $U_s - \{s\}$, with a pole at s. Using a local holomorphic coordinate t, vanishing at s, we can identify these functions with elements of the field $\mathbf{C}\{t\}[t^{-1}]$. That is the field $\mathbf{C}\{t\}[t^{-1}]$ with the field k_s of germs at s of meromorphic functions.

We denote by $\mathcal{M}_{\overline{\Gamma}}$ the sheaf of meromorphic functions over $\overline{\Gamma}$, by $\mathcal{M}^1_{\overline{\Gamma}} = \mathcal{M}_{\overline{\Gamma}} \otimes_{\mathcal{O}_{\overline{\Gamma}}} \Omega_{\overline{\Gamma}}$ the sheaf of meromorphic 1-forms, and by $\mathcal{L}_{\overline{\Gamma}} = \mathcal{M}_{\overline{\Gamma}} \otimes_{\mathcal{O}_{\overline{\Gamma}}} \mathcal{X}_{\overline{\Gamma}}$ its sheaf of meromorphic vector fields. We have a sheaf structure of Lie algebras on $\mathcal{L}_{\overline{\Gamma}}$. Clearly there exist natural structures of sheaves of $\mathcal{M}_{\overline{\Gamma}}$-vector spaces on $\mathcal{M}^1_{\overline{\Gamma}}$ and $\mathcal{L}_{\overline{\Gamma}}$, respectively. There exists a natural map (contraction)

$$\mathcal{M}^1_{\overline{\Gamma}} \otimes_{\mathcal{M}_{\overline{\Gamma}}} \mathcal{L}_{\overline{\Gamma}} \to \mathcal{M}_{\overline{\Gamma}},$$

$$\mu \otimes v \to \langle \mu, v \rangle.$$

Let V be a holomorphic vector bundle of rank m on $\overline{\Gamma}$. Then a *meromorphic* connection on V is by definition a map

$$\nabla : \mathcal{M}_V \to \mathcal{M}^1_{\overline{\Gamma}} \otimes_{M_{\overline{\Gamma}}} \mathcal{M}_V,$$

satisfying the Leibniz rule

$$\nabla(v+w) = \nabla v + \nabla w$$

$$\nabla f v = df \otimes v + f \nabla v,$$

where v, w are holomorphic sections of the fibre bundle V and f is a meromorphic function.

If the meromorphic connection ∇ is fixed, then to each meromorphic vector field X over Γ we can associate the covariant derivative along X

$$\nabla_X : \mathcal{M}_V \to \mathcal{M}_V,$$

$$\nabla_X : v \mapsto \langle \nabla v, X \rangle.$$

It is clearly a $\mathbf{C}$-linear map. Then if we denote by $\mathrm{End}_{\mathbf{C}}(\mathcal{M}_V)$ the sheaf of $\mathbf{C}$-linear endomorphisms of the sheaf of complex vector spaces $\mathcal{M}_V$, we get a map

$$\nabla : \mathcal{L}_{\overline{\Gamma}} \to \mathrm{End}_{\mathbf{C}}(\mathcal{M}_V),$$

$$X \mapsto \nabla_X,$$

such that

$$\nabla_X(v+w) = \nabla_X v + \nabla_X w,$$

$$\nabla_X(fv) = X(f)v + f\nabla_X v, \quad f \in \mathcal{M}_\Gamma.$$

Let ∇ be a meromorphic connection over $\overline{\Gamma}$. We will say that it is *holomorphic* at a point $p \in \overline{\Gamma}$ if, for every germ at p of the *holomorphic* vector field X, the space of germs at p of *holomorphic* sections of the fibre bundle V is invariant by the covariant derivative ∇_X. Later we will consider connections that are meromorphic on $\overline{\Gamma}$ and holomorphic on Γ. They can have poles on the singular set S.

If we want to compute in local coordinates in a neighborhood of a singular point $s \in S$, then we choose a holomorphic coordinate t at s (vanishing at s) and we write our given vector field $X = f(t)\frac{d}{dt}$, where $f \in k_s$ (in general we cannot write X as $\frac{d}{dt}$, because the field X may vanish or admit a pole at the point s, as we shall see later in the applications). Then using a holomorphic frame e of V as above, we get a differential system

$$\nabla = f(t)\frac{d}{dt} - A(t).$$

We can introduce the meromorphically equivalent differential system

$$\frac{d}{dt} - B(t),$$

where $B = f^{-1}A$ is a *meromorphic* matrix over U.

We denote the field of global meromorphic functions over $\overline{\Gamma}$ by $k_{\overline{\Gamma}}$. It is important to notice that every holomorphic fibre bundle over a Riemann surface $\overline{\Gamma}$ is *meromorphically* trivial over $\overline{\Gamma}$ (i.e., globally, see Appendix A). Therefore its space of global meromorphic sections is isomorphic to some $k_{\overline{\Gamma}}^m$. In particular, we can choose a non-trivial meromorphic vector field X over $\overline{\Gamma}$. It will define a *derivation* δ over the field $k_{\overline{\Gamma}}$ and we will get a differential field $(k_{\overline{\Gamma}}, \delta)$. If V is a holomorphic vector bundle over $\overline{\Gamma}$ and if $\mathcal{M}(\overline{\Gamma}) \approx k_{\overline{\Gamma}}^m$ is its $k_{\overline{\Gamma}}$-vector space of meromorphic sections, then the covariant derivative ∇_X induces a $\mathbf{C}$-linear endomorphism of the space $\mathcal{M}(\overline{\Gamma})$ and therefore it can be interpreted as a $\mathbf{C}$-linear endomorphism of the space $k_{\overline{\Gamma}}^m$. We can choose as a local coordinate t over $\overline{\Gamma}$ a non-trivial global meromorphic function over $\overline{\Gamma}$ (it will be a true local coordinate, i.e., a local biholomorphism, but perhaps over a discrete subset). We can write $X = f(t)\frac{d}{dt}$, where $f \in k_{\overline{\Gamma}}$. Then we can choose a global meromorphic frame of V over $\overline{\Gamma}$, that is a set $e = \{e_1, \ldots, e_m\}$ of meromorphic sections of V inducing a true holomorphic frame over a non-trivial open subset (necessarily dense). Finally, proceeding as above, we can interpret our connection as a global meromorphic differential system

$$\nabla = f(t)\frac{d}{dt} - A(t), \quad \text{or equivalently} \quad \frac{d}{dt} - B(t),$$

where $B = f^{-1}A$ is a global meromorphic matrix whose entries belong to $k_{\overline{\Gamma}}$.

In the preceding process it is in general necessary to introduce new poles. We will keep our notations, always denoting by S the new singular set and by Γ the new regular set (i.e., the set S can be bigger than the set of poles of our connection).

We will also need meromorphic connections on *meromorphic* bundles over a Riemann surface $\overline{\Gamma}'$. It is easy, using Appendix A, to adapt the preceding definitions. We leave the details to the reader. In our applications the more general situation will be the following. The symbol ∇ will be a meromorphic connection on a meromorphic bundle over $\overline{\Gamma}'$. By restriction, we will get a meromorphic connection on a holomorphic bundle over an open dense subset $\overline{\Gamma} \subset \overline{\Gamma}'$, and by a new restriction a holomorphic connection on a holomorphic bundle over an open dense subset $\Gamma \subset \overline{\Gamma}$. The sets $\overline{\Gamma} - \Gamma$ and $\overline{\Gamma}' - \overline{\Gamma}$ will be discrete (frequently finite in the applications) subsets and they will correspond to the introduction of *equilibrium points* and *points at infinity*, respectively.

In the rest of this section we fix the (connected) Riemann surface $\overline{\Gamma}$, and the non-trivial meromorphic vector field X over $\overline{\Gamma}$. We interpret this field as a derivation on the field of global meromorphic functions $k_{\overline{\Gamma}} = \mathcal{M}(\overline{\Gamma})$ over $\overline{\Gamma}$. As we explained above, we can consider a meromorphic vector bundle as a vector space over $k_{\overline{\Gamma}}$.

From a given meromorphic connection ∇ defined on the vector bundle V, we can obtain an infinite number of induced meromorphic connections ([29, 30, 51, 69, 104]) by natural geometric processes. The idea is to extend naturally the connection to the tensor levels by requiring that the Leibniz rule be satisfied by the tensor products ($\nabla(u \otimes v) = \nabla u \otimes v + v \otimes \nabla v$) and that the action on a direct sum is the evident one (i.e., $\nabla(U \oplus V) = \nabla U \oplus \nabla V$). So, we can construct connections: $\nabla^*, \quad \otimes^k \nabla, \quad \wedge^k \nabla, \quad S^k \nabla$, acting on the bundles $V^*, \quad \otimes^k V, \quad \wedge^k V, \quad S^k V$, respectively. By definition, $\otimes^0 V$ is the field of meromorphic functions and we endow it with the connection X (interpreted as a derivation on this field). With all these constructions we can build various direct sums and we can iterate the process. So, for example, $\wedge^3(\nabla^* \oplus S^2\nabla)$ is an induced connection. If a subbundle is invariant by a connection, this connection is by definition a subconnection. We can also introduce subconnections and quotients in our machinery.

We observe the similarity of the above definitions to derivations in differential geometry (Lie derivative, etc...). This is not merely a coincidence as we will see in Section 4.1, where we will consider a connection as a Lie derivative.

In a natural way we can generalize the above in order to consider constructions using a family of given connections. For instance, let ∇_1 and ∇_2 be two meromorphic connections over the vector bundles V_1 and V_2, respectively. The tensor product $\nabla_1 \otimes \nabla_2$ is defined by the Leibniz rule as above, $\nabla_1 \otimes \nabla_2(u \otimes v) = \nabla_1 u \otimes v + v \otimes \nabla_2 v$, where $v \in V_1$ and $u \in V_2$. In an analogous way we define the direct sum of connections, etc.... Finally, we get the tensor category of the meromorphic connections over $\overline{\Gamma}$. The homomorphisms of this category are defined in the following way. A homomorphism ϕ between ∇_1 and ∇_2 is a homomorphism of the underlying vector spaces (over the field $k_{\overline{\Gamma}}$) $\phi : V_1 \to V_2$, such that $\phi\nabla_1 = \nabla_2\phi$ (for more details and formal definitions, which are not needed here, the interested reader can look at [29]). Now it is clear how to extend the usual definitions on homomorphisms of vector spaces to homomorphisms of connections. For instance, an exact sequence of connections is given by an exact sequence of vector spaces, where the homomorphisms that define the sequence are homomorphisms of connections.

Now, we will obtain the connection matrices for some examples.

Example 1. The dual connection ∇^* is defined from the Leibniz rule by

$$X\langle \alpha, v\rangle = \langle \nabla^*\alpha, v\rangle + \langle \alpha, \nabla v\rangle,$$

where $v \in V$, $\alpha \in V^*$, and $\langle , \rangle$ denotes the duality. If e and e^* are dual frames in V and V^*, respectively, then we have

$$\langle \nabla^* e^*, e \rangle = \frac{d}{dt}\langle e^*, e \rangle + \langle e^*, eA \rangle = \langle e^* A^t, e \rangle,$$

A is the connection matrix of ∇ in the frame e, i.e., $\nabla e = -Ae$. Hence, we have obtained just the adjoint differential equation: the adjoint differential equation of

$$\frac{d\xi}{dt} = A\xi \quad \text{is by definition} \quad \frac{d\eta}{dt} = -A^t \eta.$$

We observe that, in order for $\alpha = \sum_{i=1}^m \eta_i e_i^*$ to be a linear first integral of

$$\nabla v = 0,$$

it is necessary and sufficient that

$$\nabla^* \alpha = 0.$$

This is a well-known property of the adjoint. In a similar way, it is possible to prove that the horizontal sections of $S^k \nabla^*$ are the homogeneous polynomial first integrals of the linear equation defined by the initial connection on V.

It is usual to write ∇ instead of ∇^*, $\otimes^k \nabla$, etc..., if the vector spaces on which they act are clear enough. We will follow this convention.

Example 2. The connection $\wedge^m \nabla$ $(\dim V = m)$ is defined by

$$\nabla(v_1 \wedge \cdots \wedge v_m) = \sum_{i=1}^m v_1 \wedge \cdots \wedge \nabla v_i \wedge \cdots \wedge v_m.$$

Then $\nabla(e_1 \wedge \cdots \wedge e_m) = \mathrm{tr} A\ e_1 \wedge \cdots \wedge e_m$. We have obtained the differential equation for the determinant of a fundamental matrix, i.e., the so-called Jacobi-Abel formula ($v_1 \wedge \cdots \wedge v_m = \det(v_1, \ldots, v_m)\, e_1 \wedge \cdots \wedge e_m$).

Example 3. Let $0 \longrightarrow (V_1, \nabla_1) \longrightarrow (V, \nabla) \longrightarrow (V_2, \nabla_2) \longrightarrow 0$ be an exact sequence of connections. In other words the connection (V_1, ∇_1) is a subconnection of (V, ∇) (i.e., isomorphic to the restriction of ∇ over an invariant subspace of V by ∇), and (V_2, ∇_2) is isomorphic to the "normal" connection $(V/V_1, \tilde{\nabla})$ to (V_1, ∇_1), defined in the natural way. It is easy to verify that this normal connection is well defined. Then if we take a basis $e_1, \ldots, e_k, e_{k+1}, \ldots, e_n$ of V such that $e_1, \ldots, e_k$ is a basis of V_1, the matrix of the connection ∇ (we write in a more informal way ∇ instead of (V, ∇), etc...) is given by

$$\begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix},$$

A_1 and A_2 being the matrices of the connections ∇_1 and ∇_2 respectively.

Now the method for solving the linear equation of (V, ∇) is the following. Let U_1, U_2 be fundamental matrices of the connections ∇_1, ∇_2 respectively, then a fundamental matrix of ∇ is given by

$$\begin{pmatrix} U_1 & V \\ 0 & U_2 \end{pmatrix}.$$

By writing explicitly the differential equation of U,

$$\frac{dU}{dt} = AU,$$

it is clear that the matrix V is obtained from U_1 and U_2 by the method of variation of constants. Then we have the chain of Picard-Vessiot extensions

$$K \subset K(U_1) \subset K(U_1, U_2) \subset K(U_1, U_2, V),$$

the last one being obtained by variation of constants.

As we will see later in Chapter 4, a similar method (but more involved due to the additional structure given by the symplectic form) will be used in order to reduce the variational equation (along a particular solution of a Hamiltonian system) to the normal variational equation. This will be also useful in Chapter 8.

In this book we are mainly interested in the following particular vector bundles and connections. A (meromorphic) symplectic vector bundle is a (meromorphic) vector bundle V such that there is a holomorphic section $\Omega \in \wedge^2 V^*$ whose restrictions to the fibres of V are not degenerated (the rank m of V is $2n$).

Then we have also the following result on the trivialization of a symplectic vector bundle.

Proposition 2.4 ([77]) *A symplectic vector bundle V over a Riemann surface is (symplectic) meromorphically trivial (i.e., there exists a global symplectic frame given by meromorphic sections).*

We will give a proof of the above proposition in Appendix A.

As above we denote by $k_{\overline{\Gamma}}$ the field of meromorphic functions over $\overline{\Gamma}$. We denote by $\mathcal{E}$ the $k_{\overline{\Gamma}}$-vector space of global meromorphic sections of V. The form Ω induces a $k_{\overline{\Gamma}}$-bilinear antisymmetric map

$$\Omega : \mathcal{E} \otimes \mathcal{E} \to k_{\overline{\Gamma}}, \qquad (v, w) \mapsto \Omega(v, w).$$

If v, w are holomorphic sections of V in a neighborhood of a point $p \in \overline{\Gamma}$, then $\Omega(v, w)(p) = \Omega(v(p), w(p)) \in \mathbf{C}$. Consequently the $k_{\overline{\Gamma}}$-bilinear map

$$\Omega; \mathcal{E} \otimes \mathcal{E} \to k_{\overline{\Gamma}}$$

is non-degenerate.

For many applications, we can identify the symplectic bundle V with the symplectic vector space $\mathcal{E}$ over the field $k_{\overline{\Gamma}}$. In this situation all the purely algebraic results on symplectic vector spaces over the numerical fields $\mathbf{R}$ or $\mathbf{C}$ remain also true [6]. In particular, there are symplectic bases i.e., canonical frames given by global meromorphic sections, and, with respect to a symplectic base, Ω is represented by the canonical form

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Furthermore, changes of symplectic bases are given by elements of the symplectic group $Sp(n, k_{\overline{\Gamma}}) \subset GL(2n, k_{\overline{\Gamma}})$.

By definition we will say that a (holomorphic or more generally meromorphic) connection ∇ over the *symplectic* bundle V (or (∇, V, Ω) in a more formal way) is *symplectic* if Ω is a horizontal section of $\wedge^2\nabla^*$, i.e., it satisfies $\nabla\Omega = 0$ (for a related definition see [8]). Then, it is easy to see that, after a choice of coordinates, if we compute the connection matrix A of ∇ in a symplectic frame e, it satisfies

$$A^t J + JA = 0$$

(to show this, it is sufficient to note that $0 = \nabla\Omega = \nabla(e^* \otimes Je^{*t})$). This condition is equivalent to the existence of a meromorphic symmetric matrix S such that $A = JS$, and the matrix A belongs to the Lie algebra of the symplectic Lie group with coefficients in the field $k_{\overline{\Gamma}}$. Then the equation

$$\nabla v = 0$$

is the intrinsic expression of the linear Hamiltonian system

$$\dot{\xi} = JS\xi,$$

where $\xi = (\xi_1, \dots, \xi_{2n})^t$ are the coordinates of v in the symplectic base and, as usual in dynamical systems, we denote the temporal derivative by a dot.

Conversely, if the matrix of the connection ∇ computed in a symplectic frame is symplectic, then $\nabla\Omega = 0$ and this connection is symplectic. Therefore our definition of a symplectic connection is equivalent to the definition of a connection with structure group $G = Sp(2n; \mathbf{C})$ given in Appendix A.

All the above constructions remain valid if we start with a local meromorphic connection on the vector space V over the field $\mathbf{C}\{t\}[t^{-1}]$ with the suitable dictionary: $\frac{d}{dt}$ instead X, etc. . . .

2.4 The Tannakian approach

We present now the Galois theory from the intrinsic connection perspective [21, 29, 51, 69]. Let (V, ∇) be, as in the above section, a meromorphic connection over a fibre bundle of rank m. Then, we consider the horizontal sections, $\mathrm{Sol}\,\nabla := \mathrm{Sol}_{p_0}\nabla$ of this connection at a fixed non-singular point $p_0 \in \Gamma$ (they correspond to solutions of the corresponding linear equation). By the general existence theory of linear differential equations, $\mathrm{Sol}\,\nabla$ is a vector space over $\mathbf{C}$ of dimension m (if we consider the solutions in a simply connected domain that contain p_0). Then the mapping

$$(V, \nabla) \longrightarrow \mathrm{Sol}\,\nabla$$

is called a functor fibre (it is a functor between the tensor category of the meromorphic connections and the tensor category of complex vector spaces).

Now, as in the previous section, we obtain the family of tensor constructions: $(V, \nabla), (V^*, \nabla^*)$, etc. . . , from a given connection. In this family we include the subconnections. A subconnection of a construction $(C(V), C(\nabla))$ is an object $(W, C(\nabla)_{|W})$, W being a subbundle of $C(V)$ invariant by $C(\nabla)$. The next step is to consider the corresponding spaces of solutions by the functor Sol, for all the elements of this extended family. Then $C(\mathrm{Sol}\,\nabla) = \mathrm{Sol}(C\nabla)$, and the Galois group of the initial connection (V, ∇), $\mathrm{Gal}\,\nabla$, is defined as the subgroup of $GL(\mathrm{Sol}\,\nabla) \approx GL(m,\ C)$, which leaves invariant the spaces corresponding to all constructions $C(V)$. We remark that $GL(\mathrm{Sol}\,\nabla)$ acts on any construction by the usual pull-back. The key point is that the above group is isomorphic (as an algebraic group) to the Galois group G of the corresponding linear equation. This approach to the Picard-Vessiot theory is called the Tannakian point of view.

Example. Let (V, ∇, Ω) be a symplectic connection with rank $V = 2n$ and X the holomorphic vector field over $\overline{\Gamma}$. We make the construction $(\mathcal{M}_\Gamma(\overline{\Gamma}) \oplus \wedge^2 V^*, X \oplus \wedge^2\nabla^*)$, $\mathcal{M}_\Gamma(\overline{\Gamma})$ being the (global) meromorphic functions over $\overline{\Gamma}$, holomorphic on Γ. The line subbundle generated by $1 + \Omega$, $\mathcal{M}_\Gamma(\overline{\Gamma})(1 + \Omega)$, is invariant, because $\nabla\Omega = 0$ and $\nabla(f(1 + \Omega)) = X(f)(1 + \Omega)$, $f \in \mathcal{M}_\Gamma(\overline{\Gamma})$. Hence, the corresponding construction by Sol, $\mathbf{C}(1 + \Omega_0)$ (Ω is a horizontal section of $\wedge^2\nabla^*$) is invariant by the Galois group. Therefore, the Galois group is contained in the symplectic group $Sp(\mathrm{Sol}(V)) \approx Sp(n, \mathbf{C})$. A different proof of this in a more general context will be given in Appendix C.

2.5 Stokes multipliers

The objective now is to state a theorem of Ramis which relates the Picard-Vessiot theory with the Stokes multipliers at an irregular singular point [90, 69, 74, 16]. For simplicity, we will explain only the main concepts necessary to understand the theorem, for the case of a second order differential equation (equivalently, for a system of dimension two). The reader can find a good introduction in [74] and the complete proof is in [16].

We start with the local case and we will consider that the singular point is at infinity, $x_0 = \infty$. Furthermore, we denote by $\hat{K} := \mathbf{C}[[x^{-1}]][x]$, $K := \mathbf{C}\{x^{-1}\}[x]$, the field of formal and convergent Laurent series respectively. Then, the objective is to calculate the Galois group of the equation

$$\frac{d}{dx}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = A\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad A \in Mat(2, K). \tag{2.5}$$

We also assume that the Newton polygon of the above equation has only one integer slope $k \in \mathbf{N}^*$. This is called the non-ramified case and the general case can be reduced to this one. By the Newton polygon of (2.5) we mean the Newton polygon of the equivalent second order single differential equation in $z = \frac{1}{x}$, i.e., the Newton polygon of the differential polynomial $P[D] = pD^2 + qD + r$ (the equation is $P[D]\xi = 0$), with $D = \frac{d}{dz}$, $p \in C[z]$, $q, r \in C\{z\}$, $p(0) = q(0) = 0$. By the Fuchs theory, it is not difficult to see that the point $z_0 = \infty$ is an irregular singular point if, and only if, the Newton polygon has a side with slope in $(0, \infty)$.

By the classical theory (Huhukara-Turritin, [105]), there is a fundamental matrix U of (2.5), such that,

$$U = x^L e^Q H, \quad L \in M(2, C),$$

where $Q = \operatorname{diag}(q_1, q_2)$, $q_1, q_2 \in C[x]$, $LQ = QL$, and H is holomorphic in any open angular sector at ∞ of opening angle $< \pi/k$,

$$S_d(\pi/k) := \{t : |x| > a, \arg x \in (d - \pi/2k, d + \pi/2k)\},$$

with a a suitable constant and $k := \deg(q_1 - q_2)$. Then H has an asymptotic expansion (whose entries are formal series):

$$H \sim \hat{H}, \ \hat{H} \in GL(2, \hat{K}).$$

Here d is the argument of the bisecting line of the sector. A sector is characterized by d, α, where α is the opening. Then we will denote this sector by $S_d(\alpha)$.

This means that equation (2.5) has the formal solution

$$\hat{U} = x^L e^Q \hat{H}, \quad U \sim \hat{U}.$$

We observe that for $q_1 = q_2 = 0$, we are in the regular situation (the singular point is a singular regular one, and the formal series $\hat{U}$ is convergent).

In order to state the Ramis theorem we need some terminology: the exponential torus, the formal monodromy and the Stokes multipliers.

The exponential torus of (2.5) is defined (up to an isomorphism) as the Galois differential group of the Picard-Vessiot extension

$$\hat{K} \subset \hat{K}(e^{q_1}, e^{q_2}).$$

We see that this group is the Galois group of the trivial equation, considered over $\hat{K}$,

$$\frac{d\xi_i}{dx} = \frac{dq_i}{dx}\xi_i, \;\; i = 1, 2.$$

This exponential torus is (isomorphic to) C^* or $(C^*)^2$ if the rank of the **Z**-module M_Q generated by $\{q_1, q_2\}$ is one or two, respectively. In the first case, the action of C^* is defined by

$$\lambda : e^{q_i} = e^{n_i s} \mapsto \lambda^{n_i} e^{n_i s}, \quad ,\lambda \in C^*, \quad \langle s \rangle = M_Q,$$

and, in the second case,

$$\lambda_i : e^{q_i} \mapsto \lambda_i e^{q_i}, \quad i = 1, 2.$$

By definition this action is constant on the coefficient field $\hat{K}$.

The formal monodromy is the transformation $\hat{M} \in GL(2, \mathbf{C})$, such that

$$\hat{U} \mapsto \hat{U}\hat{M},$$

when we formally make the circuit

$$x \mapsto e^{2\pi i}x.$$

It is clear that, by analytic continuation, it is possible to continue the analytic solution U to sectors $S_d(\alpha)$, with $\alpha > \pi/k$. The problem is that, in this new sector, this solution is not necessarily asymptotic to $\hat{U}$. The lines that bound the sectors where the asymptotic relation (2.5) remains valid are called Stokes rays. These lines are characterized by

$$\lim |x|^{-\deg(q_1 - q_2)} Re(q_1 - q_2) = 0,$$

when $|x|$ tends to ∞ along this line. We can think that the analytic continuation from a sector $S_d(\pi/2)$, where the asymptotic expansion (2.5) is satisfied, is obtained by rotating the bisecting line d (in both directions), but preserving the opening π/k. Then we stop when a bounding side of the sector reaches a Stokes ray. The bisecting line d_s of this bounding sector $S_{d_s} := S_{d_s}(\pi/k)$ is called a singular line (sometimes it is called an anti-Stokes ray). They are characterized by the maximal exponential decay for $e^{q_1-q_2}$ or $e^{q_2-q_1}$. By the general theory, there are two sectors $S_{d_s+\epsilon}$, $S_{d_s-\epsilon}$, ($d_s+\epsilon$ means a small change in the argument of d_s by ϵ, $0<\epsilon<\pi/2k$, and keeping the opening less than π/k). Hence, we get two analytical solutions U^+, U^- defined over S_{d_s} (by analytical continuation to this sector, ϵ going to 0). Then we have

$$U^- = U^+ Sto_{d_s},$$

where, by definition, the matrix $Sto_{d_s} \in GL(2,\mathbf{C})$ is the Stokes matrix in the singular direction d_s. It is possible to see that these Stokes matrices are unipotents, i.e., of the form (in the suitable fundamental system)

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \qquad \text{or,} \qquad \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}.$$

In particular, they belong to $SL(2,\ C)$. The complex numbers $\mu,\ \lambda$ are called the Stokes multipliers. In particular they belong to $SL(2,\mathbf{C})$.

In an analogous, but more delicate, way (in this general case the phenomenon of multi-summability appears) we may describe the exponential torus, the formal monodromy and the Stokes matrices for a local system of differential equations of arbitrary dimension m

$$\frac{d\xi}{dx} = A\xi, \quad A \in Mat(m,K) \tag{2.6}$$

(see [74, 16]). Then

Theorem 2.3 ([90, 69, 16]) *The Galois (local) group of* (2.6) *is topologically generated by the exponential torus, the formal monodromy and the Stokes matrices (at* $x=0$*).*

We note that among these generators the main source of non-integrability comes from the Stokes multipliers. For example, it is not difficult to prove that the Zariski closure of the group (algebraically) generated by the two matrices

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix},$$

where $\lambda,\ \mu$ are both different from zero, is $SL(2,\ C)$ [18].

It is possible to generalize the above theorem to a global linear differential equation in the following way. Let K be the field of meromorphic functions on a Riemann surface X and $S \subset X$ a discrete set. Then (see Appendix B).

Theorem 2.4 ([77]) *Let*

$$\frac{d\xi}{dx} = A\xi, \quad A \in Mat(m, K) \tag{2.7}$$

be a linear differential equation, S being the set of singular points (i.e., poles of the entries of A). Let P_i be the set of Stokes matrices and exponential torus at each of the singular points $a_i \in S$, and let M be the usual monodromy group of (2.7). *Then the Galois group of* (2.7) *is topologically generated by P_i $(a_i \in S)$ and M.*

2.6 Coverings and differential Galois groups

In concrete differential equations it is useful, if possible, to replace the original differential equation over a compact Riemann surface, by a new differential equation over the Riemann sphere $\mathbf{P^1}$ (i.e., with rational coefficients) by a change of the independent variable. This equation on $\mathbf{P^1}$ is called the algebraic form of the equation. In a more general way we will consider the effect of a finite ramified covering on the Galois group of the original differential equation. In Appendix B the following theorem is proved.

Theorem 2.5 ([77]) *Let X be a (connected) Riemann surface. Let $f : X' \longrightarrow X)$ be a* finite *ramified covering of X by a Riemann surface X'. Let ∇ be a meromorphic connection on X. We set $\nabla' = f^*\nabla$. Then we have a natural injective homomorphism*

$$\mathrm{Gal}\ (\nabla') \to \mathrm{Gal}\ (\nabla)$$

of differential Galois groups which induces an isomorphism between their Lie algebras.

We observe that, in terms of the differential Galois groups, this theorem means that the identity component of the differential Galois group is invariant by the covering.

An algebraic version of the above theorem is given by Katz [51]. This result is also proved in [8] (Proposition 4.7) for the particular case of a Fuchsian differential equation (see also [24, 25, 27, 9]). It is the mapping version for the so-called (in the cited references) method of reduction by discrete symmetries. Then this method is also valid in our more general setting. It is important to notice that, if one of the connections in the proposition is symplectic, then the identity components of the Galois groups of both connections are symplectic too.

2.7 Kovacic's algorithm

The Kovacic algorithm gives us a procedure in order to compute the Picard-Vessiot extension (i.e., a fundamental system of solutions) of a second order differential equation, provided the differential equation is integrable. Reciprocally, if the differential equation is non-integrable, the algorithm does not work (see[56]). In this (necessarily brief) description of the algorithm we essentially follow the version of the algorithm given in [33, 32]. The author is indebted to Anne Duval for some clarifications about his papers.

Given a second order linear differential equation with coefficients in $\mathbf{C}(x)$, it is a classical fact that it can be transformed to the so-called reduced invariant form

$$\xi'' - g\xi = 0, \tag{2.8}$$

with $g = g(x) \in \mathbf{C}(x)$.

We remark that in this change we introduce the exponentiation of a quadrature and the integrability of the original equation is equivalent to the integrability of the above equation although, in general, the Galois groups are not the same.

The algorithm is based on the following two general facts:

(A) The classification of the algebraic subgroups of $SL(2, \mathbf{C})$ given in Proposition 2.2 (the Galois group of the equation (2.8) is contained in $SL(2, \mathbf{C})$: see Section 2.1).

(B) The well-known transformation to a Riccati equation, by the change $v = \xi'/\xi$,

$$v' = g + v^2. \tag{2.9}$$

Then (see Section 2.2) the differential equation (2.8) is integrable, if and only if, the equation (2.9) has an algebraic solution. The *key* point now is that the degree n of the associated minimal polynomial $Q(v)$ (with coefficients in $\mathbf{C}(x)$) belong to the set

$$L_{\max} = \{1, 2, 4, 6, 12\}.$$

The determination of the set L of possible values for n, is the **First Step** of the algorithm. We remark that for $n = 4$, $n = 6$ and $n = 12$, the Galois group of (2.8) is finite (hence these values are related to the crystalographic groups). The two other steps of the algorithm (**Second Step** and **Third Step**) are devoted to computation of the polynomial $Q(v)$ (if it exists). If the algorithm does not work (i.e., if the equation (2.9) has no algebraic solution) then equation (2.8) is non-integrable and its Galois group is $SL(2, \mathbf{C})$.

Now we will describe the algorithm.

Let

$$g = g(x) = \frac{s(x)}{t(x)},$$

with $s(x)$, $t(x)$ relatively prime polynomials, and $t(x)$ monic. We define the following function h on the set $L_{\max} = \{1, 2, 4, 6, 12\}$, $h(1) = 1$, $h(2) = 4$, $h(4) = h(6) = h(12) = 12$.

First Step

If $t(x) = 1$ we put $m = 0$, else we factorize $t(x)$ in monic relatively prime polynomials. Then

1.1. Let Γ' be the set of roots of $t(x)$ (i.e., the singular points at the finite complex plane) and let $\Gamma = \Gamma' \cup \infty$ be the set of singular points. Then the order at a singular point $c \in \Gamma'$ is, as usual, $o(c) = i$ if c is a root of multiplicity i of $t(x)$. The order at infinity is defined by $o(\infty) = \max(0, 4 + \deg(s) - \deg(t))$. We call m^+ the maximum value of the order that appears at the singular points in Γ, and Γ_i is the set of singular points of order $i \leq m^+$.

1.2. If $m^+ \geq 2$ then we write $\gamma_2 = \mathrm{card}(\Gamma_2)$, else $\gamma_2 = 0$. Then we compute

$$\gamma = \gamma_2 + \mathrm{card}(\cup_{\substack{k\ odd \\ 3 \leq k \leq m^+}} \Gamma_k).$$

1.3. For the singular points of order one or two, $c \in \Gamma_2 \cup \Gamma_1$, we compute the principal parts of g:

$$g = \alpha_c (x-c)^{-2} + \beta_c (x-c) + O(1),$$

if $c \in \Gamma'$, and

$$g = \alpha_\infty x^{-2} + \beta_\infty x^{-3} + O(x^{-4}),$$

for the point at infinity.

1.4. We define the subset L' (of possible values for the degree of the minimal polynomial $Q(v)$) as $\{1\} \subset L'$ if $\gamma = \gamma_2$, $\{2\} \subset L'$ if $\gamma \geq 2$ and $\{4, 6, 12\} \subset L'$ if $m^+ \leq 2$.

1.5. We have the three following mutually exclusive cases:

1.5.1. If $m^+ > 2$, then $L = L'$.

1.5.2. If $m^+ \leq 2$ and the two following conditions are satisfied:

1.5.2.1. For any $c \in \Gamma$, $\sqrt{1 + 4\alpha_c} \in \mathbf{Q}$, and $\sum_{c \in \Gamma'} \beta_c = 0$,

1.5.2.2. For any $c \in \Gamma$ such that $\sqrt{1 + 4\alpha_c} \in \mathbf{Z}$, logarithmic term does not appear in the local solutions in a neighborhood of c,
then $L = L'$.

1.5.3. If cases 1.5.1 and 1.5.2 do not hold then $L = L' - \{4, 6, 12\}$.

1.6. If $L = \emptyset$, then equation (2.8) is non-integrable with Galois group $SL(2, \mathbf{C})$, else one writes n for the minimum value in L.

(We remark that Condition 1.5.2.2 is new. As the reader can check, it follows trivially from the fact that the existence of a logarithm in a local solution is an obstruction to have a finite monodromy and Galois group. I decided to include this condition here because it has been applied with success in some important applications [49].)

For the **Second Step** and the **Third Step** of the algorithm we consider the value of n fixed.

Second Step

2.1. If ∞ has order 0 we write the set

$$E_\infty = \{0, \frac{h(n)}{n}, 2\frac{h(n)}{n}, 3\frac{h(n)}{n}, \dots, n\frac{h(n)}{n}\}.$$

2.2. If c has order 1, then $E_c = \{h(n)\}$.

2.3. If $n = 1$, for each c of order 2 we define

$$E_c = \{\frac{1}{2}(1 + \sqrt{1 + 4\alpha_c}, \frac{1}{2}(1 - \sqrt{1 + 4\alpha_c}\}.$$

2.4. If $n \geq 2$, for each c of order 2, we define

$$E_c = \mathbf{Z} \cap \{\frac{h(n)}{2}(1 - \sqrt{1 + 4\alpha_c}) + \frac{h(n)}{n}k\sqrt{1 + 4\alpha_c} : k = 0, 1, \dots, n\}.$$

2.5. If $n = 1$, for each singular point of even order 2ν, with $\nu > 1$, we compute the numbers α_c and β_c defined (up to a sign) by the following conditions:

2.5.1. If $c \in \Gamma'$,

$$g = \{\alpha_c(x - c)^{-\nu} + \sum_{i=2}^{\nu-1} \mu_{i,c}(x - c)^{-i}\}^2 + \beta_c(x - c)^{-\nu-1} + O((x - c)^{-\nu}),$$

and we write

$$\sqrt{g}_c := \alpha_c(x - c)^{-\nu} + \sum_{i=2}^{\nu-1} \mu_{i,c}(x - c)^{-i}.$$

2.5.2. If $c = \infty$,

$$g = \{\alpha_\infty x^{\nu-2} + \sum_{i=0}^{\nu-3} \mu_{i,\infty} x^i\}^2 - \beta_\infty x^{\nu-3} + O(x^{\nu-4}),$$

and we write

$$\sqrt{g}_\infty := \alpha_\infty x^{\nu-2} + \sum_{i=0}^{\nu-3} \mu_{i,\infty} x^i.$$

Then for each c as above, we compute

$$E_c = \{\frac{1}{2}(\nu + \epsilon\frac{\beta_c}{\alpha_c}) : \epsilon = \pm 1\},$$

and the sign function on E_c is defined by

$$\text{sign}(\frac{1}{2}(\nu + \epsilon\frac{\beta_c}{\alpha_c})) = \epsilon,$$

being $+1$ if $\beta_c = 0$.

2.6. If $n = 2$, for each c of order ν, with $\nu \geq 3$, we write $E_c = \{\nu\}$.

Third Step

3.1. For n fixed, we try to obtain elements $\mathbf{e} = (e_c)_{c\in\Gamma}$ in the cartesian product $\prod_{c\in\Gamma} E_c$, such that:

(i) $d(\mathbf{e}) := n - \frac{n}{h(n)}\sum_{c\in\Gamma} e_c$ is a non-negative integer,

(ii) If $n = 2$ then there is at least one odd number in $\mathbf{e}$.

If no element $\mathbf{e}$ is obtained, we select the next value in L and go to the **Second Step**, else n is the maximum value in L and the Galois group is $SL(2, \mathbf{C})$ (i.e., the equation (2.8) is non-integrable).

3.2. For each family $\mathbf{e}$ as above, we try to obtain a rational function Q and a polynomial P, such that

(i)

$$Q = \frac{n}{h(n)}\sum_{c\in\Gamma'}\frac{e_c}{x-c} + \delta_{n1}\sum_{c\in\cup_{\nu>1}\Gamma_{2\nu}} \text{sign}(e_c)\sqrt{g}_c,$$

where δ_{n1} is the Kronecker delta.

(ii) P is a polynomial of degree $d(\mathbf{e})$ and its coefficients are found as a solution of the (in general, overdetermined) system of equations

$$P_{-1} = 0,$$

$$P_{i-1} = -(P_i)' - QP_i - (n-i)(i+1)gP_{i+1},\ n \geq i \geq 0,$$

$$P_n = -P.$$

If a pair (P, Q) as above is found, then equation (2.8) is integrable and the Riccati equation (2.9) has an algebraic solution v given by any root v of the equation

$$\sum_{i=0}^{n}\frac{P_i}{(n-1)!}v^i = 0.$$

If no pair as above is found we take the next value in L and we go to the **Second Step**. If n is the greatest value in L then equation (2.8) is non-integrable and the Galois group is $SL(2, \mathbf{C})$.

We notice that a remarkable simplification of the above algorithm was obtained in [100] for irreducible differential equations, and an algorithm for third order differential equations is given in [97, 98].

2.8 Examples

We now illustrate the Picard-Vessiot Theory with some examples. As we are interested to know when the identity component of the Galois group is abelian (see Chapter 4), we make it explicit in the known cases.

2.8.1 The hypergeometric equation

The hypergeometric (or Riemann) equation is the more general second order linear differential equation over the Riemann sphere with three regular singular singularities. If we place the singularities at $x = 0, 1, \infty$ it is given by

$$\begin{aligned} \frac{d^2\xi}{dx^2} &+ (\frac{1-\alpha-\alpha'}{x} + \frac{1-\gamma-\gamma'}{x-1})\frac{d\xi}{dx} \\ &+ (\frac{\alpha\alpha'}{x^2} + \frac{\gamma\gamma'}{(x-1)^2} + \frac{\beta\beta'-\alpha\alpha'-\gamma\gamma'}{x(x-1)})\xi = 0, \end{aligned} \tag{2.10}$$

where (α, α'), (γ, γ'),(β, β') are the exponents at the singular points and must satisfy the Fuchs relation $\alpha + \alpha' + \gamma + \gamma' + \beta + \beta' = 1$. We denote the exponent differences by $\hat{\lambda} = \alpha - \alpha'$, $\hat{\nu} = \gamma - \gamma'$ and $\hat{\mu} = \beta - \beta'$.

We also use one of its reduced forms

$$\frac{d^2\xi}{dx^2} + \frac{c-(a+b+1)x}{x(x-1)}\frac{d\xi}{dx} - \frac{ab}{x(x-1)}\xi = 0, \tag{2.11}$$

where a, b, c are parameters, with the exponent differences $\hat{\lambda} = 1-c$, $\hat{\nu} = c-a-b$ and $\hat{\mu} = b - a$, respectively.

Now, we recall the theorem of Kimura that gives necessary and sufficient conditions for the hypergeometric equation to have integrability.

Theorem 2.6 ([52]) *The identity component of the Galois group of the hypergeometric equation* (2.10) *is solvable if, and only if, either*

(i) *at least one of the four numbers* $\hat{\lambda}+\hat{\mu}+\hat{\nu}$, $-\hat{\lambda}+\hat{\mu}+\hat{\nu}$, $\hat{\lambda}-\hat{\mu}+\hat{\nu}$, $\hat{\lambda}+\hat{\mu}-\hat{\nu}$ *is an odd integer, or*

(ii) *the numbers* $\hat{\lambda}$ *or* $-\hat{\lambda}$, $\hat{\mu}$ *or* $-\hat{\mu}$ *and* $\hat{\nu}$ *or* $-\hat{\nu}$ *belong (in an arbitrary order) to some of the following fifteen families*

1	$1/2+l$	$1/2+m$	*arbitrary complex number*	
2	$1/2+l$	$1/3+m$	$1/3+q$	
3	$2/3+l$	$1/3+m$	$1/3+q$	$l+m+q$ *even*
4	$1/2+l$	$1/3+m$	$1/4+q$	
5	$2/3+l$	$1/4+m$	$1/4+q$	$l+m+q$ *even*
6	$1/2+l$	$1/3+m$	$1/5+q$	
7	$2/5+l$	$1/3+m$	$1/3+q$	$l+m+q$ *even*
8	$2/3+l$	$1/5+m$	$1/5+q$	$l+m+q$ *even*
9	$1/2+l$	$2/5+m$	$1/5+q$	$l+m+q$ *even*
10	$3/5+l$	$1/3+m$	$1/5+q$	$l+m+q$ *even*
11	$2/5+l$	$2/5+m$	$2/5+q$	$l+m+q$ *even*
12	$2/3+l$	$1/3+m$	$1/5+q$	$l+m+q$ *even*
13	$4/5+l$	$1/5+m$	$1/5+q$	$l+m+q$ *even*
14	$1/2+l$	$2/5+m$	$1/3+q$	$l+m+q$ *even*
15	$3/5+l$	$2/5+m$	$1/3+q$	$l+m+q$ *even*

Here l, m *and* q *are integers.*

We recall that Schwarz's table gives us the cases for which the Galois (and monodromy) groups are finite (i.e., the identity component of the Galois group is reduced to the identity element) and is given by fifteen families. These families are given by families 2–15 of the table above and by the family $(1/2+\mathbf{Z}) \times (1/2+\mathbf{Z}) \times \mathbf{Q}$ (see, for instance, [88]). As this last family is already contained in family 1 of the above table, all of the Schwarz's families are, of course, contained in the above table.

2.8.2 The Bessel equation

The Bessel equation is

$$x^2 \frac{d^2\xi}{dx^2} + x\frac{d\xi}{dx} + (x^2 - n^2)\xi = 0, \tag{2.12}$$

with n a complex parameter. This equation is a particular confluent hypergeometric equation (by a limit process two of the singular points in a variant of the hypergeometric equation coincide).

As (2.12) is one of the most simple but non-trivial (i.e., in general, non-integrable) equations with Stokes phenomenon, we are going to make explicit for it the concepts introduced in Section 2.5.

First, we observe that the Galois group is contained in $SL(2,\mathbf{C})$, since $1/x$ is a logarithmic derivative (see Section 2.2). It is an equation with two singular points, 0, ∞, the first one being regular singular and the second one irregular. We are interested in the point at infinity.

There are several ways to compute the matrices Q and L of Section 2.5. For example, we can follow the general constructive method of the Huhukara-Turritin theory [105, 10]. First, we make a formal change

$$\begin{pmatrix} \xi \\ \xi' \end{pmatrix} = \hat{P} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

where $P \in Mat(2, \hat{K})$ ($\hat{K} := \mathbf{C}[[x^{-1}]][x]$) which formally diagonalizes the equation. The solution is precisely the formal solution in equation (2.12), and is found step by step in a recursive way ([105, 10]). In this way we obtain $q_1 = ix = -q_2$ and $L = -1/2I$. The exponential torus is $\mathbf{C}^*$ and the formal monodromy $\hat{M} = -I$.

The Stokes rays are $\mathbf{R}_+$ and $\mathbf{R}_-$, and the singular lines $i\mathbf{R}_+$, $i\mathbf{R}_-$. Hence, we have two Stokes multipliers (one for each singular line),

$$St_1 = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix},$$

$$St_2 = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}.$$

But, for this equation the global theory (coefficients in $\mathbf{C}(x)$) and the local one (coefficients in $K = \mathbf{C}\{\{x^{-1}\}\}[x]$) are essentially the same. We note that the actual monodromy M_0 around 0 and around ∞ are the same, therefore the differential Galois group at the origin can be interpreted as a subgroup of the differential Galois group at infinity. It is possible to compute the actual monodromy M_0 in the classical basis at the origin, which is of course different from the basis at infinity introduced in the previous computation. We get $M_0 = \text{diag}(e^{2\pi i n}, e^{-2\pi i n})$.

It is easy to relate the actual monodromy and the formal monodromy at infinity using the Stokes multipliers:

$$M_0 = St_1 \hat{M} St_2.$$

Now, as the trace is an invariant, we get

$$\operatorname{tr} M_0 = 2\cos(2\pi n) = -\lambda\mu - 2,\ \lambda\mu = -4\cos^2 \pi n.$$

Hence, if n does not belong to $\mathbf{Z} + 1/2$, the Bessel equation is non-integrable. In fact, this necessary condition for integrability is also sufficient. So by the classical theory (see, for example, [62]) it is well known that the Bessel functions for $n \in \mathbf{Z} + 1/2$ can be expressed by elementary functions: the Picard-Vessiot extension is obtained by exponentiation of integrals of elements of $\mathbf{C}(x)$.

2.8.3 The confluent hypergeometric equation

One of the forms of the general confluent hypergeometric equation is given by the Whittaker equation [109]

$$\frac{d^2\xi}{dz^2} - (\frac{1}{4} - \frac{\kappa}{z} + \frac{4\mu^2 - 1}{4z^2})\eta = 0, \tag{2.13}$$

with parameters κ and μ. The singular points are $z = 0$ (regular) and $z = \infty$ (irregular).

As in the case of the Bessel equation we have two singular lines associated to the irregular point for (2.13). For computation of the Galois group, the following proposition is useful ([69], Subsection 3.3))

Proposition 2.5 *There is a fundamental system of solutions such that if α, β are the two Stokes multipliers corresponding to the two singular lines, with corresponding Stokes matrices*

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix},$$

then

(i) $\alpha = 0$ *if, and only if, either* $\kappa - \mu \in \frac{1}{2} + \mathbf{N}$ *or* $\kappa + \mu \in \frac{1}{2} + \mathbf{N}$.

(ii) $\beta = 0$ *if, and only if, either* $-\kappa - \mu \in \frac{1}{2} + \mathbf{N}$ *or* $-\kappa + \mu \in \frac{1}{2} + \mathbf{N}$. *Furthermore (with respect to the same fundamental system of solutions), the group generated by the formal monodromy and the exponential torus is given by the multiplicative group*

$$\left\{ \begin{pmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{pmatrix} : \delta \in \mathbf{C}^* \right\}.$$

As a consequence, we get the abelian criterion expressed in terms of the parameters $p := \kappa + \mu - \frac{1}{2}$ and $q := \kappa - \mu - \frac{1}{2}$.

Corollary 2.1 *The identity component G^0 of* (2.13) *is abelian if, and only if, (p, q) belong to* $(\mathbf{N} \times (-\mathbf{N}^*)) \cup ((-\mathbf{N}^*) \times \mathbf{N})$ *(i.e., p, q are integers, one of them being positive and the other negative).*

We observe that the abelian case (G^0 abelian) for the Whittaker equation is only possible when the two Stokes multipliers are zero and this corresponds to the diagonal case 4 of the classification given by Proposition 2.2 (the Whittaker equation is a symplectic one). If only one of the Stokes multipliers is different from zero we are in case 6 of this classification, and we have integrability but the identity component of the Galois group is not abelian. If two of the Stokes multipliers are different from zero, we fall in case 7 with Galois group $SL(2, \mathbf{C})$, as we remarked in Section 2.5.

If in the Bessel equation (2.12) we make the change of the dependent variable $\xi = x^{-1/2}\eta$ and of the independent variable $x = z/2i$, we get a Whittaker equation

$$\frac{d^2\eta}{dz^2} - (\frac{1}{4} + \frac{4n^2-1}{4z^2})\eta = 0, \tag{2.14}$$

with parameters $\kappa = 0$ and $\mu = n$. As in the above change we only introduce algebraic functions, the identity component of the Galois group of the Bessel equation is preserved.

2.8.4 The Lamé equation

The algebraic form of the Lamé Equation is [88, 109]

$$\frac{d^2\eta}{dx^2} + \frac{f'(x)}{2f(x)}\frac{d\eta}{dx} - \frac{Ax+B}{f(x)}\eta = 0, \tag{2.15}$$

where $f(x) = 4x^3 - g_2 x - g_3$, with A, B, g_2 and g_3 parameters such that the discriminant of f, $27g_3^2 - g_2^3$ is non-zero. This equation is a Fuchsian differential equation with four singular points over the Riemann sphere.

With the well-known change $x = \mathcal{P}(z)$, we get the Weierstrass form of the Lamé equation

$$\frac{d^2\eta}{dz^2} - (A\mathcal{P}(z) + B)\eta = 0, \tag{2.16}$$

where $\mathcal{P}$ is the elliptic Weierstrass function with invariants g_2, g_3 (we recall that $\mathcal{P}(z)$ is a solution of the differential equation $(\frac{dx}{dz})^2 = f(x)$). Classically the equation is written with the parameter n instead of A, with $A = n(n+1)$. This equation is defined on a torus Π (a genus one Riemann surface) with only one singular point at the origin. Let $2w_1$, $2w_3$ be the two periods of the Weierstrass function $\mathcal{P}$ and $\mathbf{g}_1$, $\mathbf{g}_2$ their corresponding monodromies in the above equation. If $\mathbf{g}_*$ represents the monodromy around the singular point, then $\mathbf{g}_* = [\mathbf{g}_1, \mathbf{g}_2]$ ([109, 88]).

By the above theorem we know that the identity component of the Galois group is preserved by the covering $\Pi \to \mathbf{P}^1$, $t \mapsto x$.

The relation between the monodromy groups of equations (2.15) and (2.16) is discussed in [88], Chapter IX. From a modern point of view it is studied in [27].

Now we study the integrability of the Lamé equation (2.16) which is equivalent to the integrability of (2.15). In fact, we have a stronger result: the identity components of the Galois groups of both equations are isomorphic (by Theorem 2.5).

First, it is easy to see that a necessary and sufficient condition for the *total* Galois group of (2.16) to be abelian is that $n \in \mathbf{Z}$. We sketch the steps of the proof. Indeed, this is a classical well-known necessary and sufficient condition for the monodromy group M of the equation (2.16) to be abelian (it is clear that, as M is generated by $\mathbf{g}_1$ and $\mathbf{g}_2$, an equivalent condition for the abelianess of M is $\mathbf{g}_* = \mathbf{1}(identity)$, and the indicial equation at the singularity is $\rho^2 - \rho - n(n+1) = 0$, and there is no logarithmic term for n integer. Therefore, as G is topologically generated by M, it must also be abelian.

Now the known (mutually exclusive) cases of closed form solutions of the Lamé equation (2.16) are as follows:

(i) The Lamé and Hermite solutions [34, 42, 88, 109]. In this case n is an arbitrary integer and the three other parameters are arbitrary. In the case of the Lamé solutions there is one solution that is an elliptic function with the same periods as the function $\mathcal{P}$ (i.e., it belongs to the coefficient field K), hence, by the normality of the Picard-Vessiot extensions, in this case the Galois group of the equation (2.16) is of type 3 of Proposition 2.2. We will use this property in the last chapter.

(ii) The Brioschi-Halphen-Crawford solutions [7, 34, 42, 88]. Now $m := n + \frac{1}{2} \in \mathbf{N}$ and the parameters B, g_2 and g_3 must satisfy an algebraic equation

$$0 = Q_m(g_2/4, g_3/4, B) \in \mathbf{Z}[g_2/4, g_3/4, B],$$

where Q_m has degree m in B. This polynomial is known as the Brioschi determinant and we will construct it later in this section.

(iii) The Baldassarri solutions [7]. The condition on n is $n + \frac{1}{2} \in \frac{1}{3}\mathbf{Z} \cup \frac{1}{4}\mathbf{Z} \cup \frac{1}{5}\mathbf{Z} - \mathbf{Z}$, with additional (involved) algebraic restrictions on the other parameters.

We notice that, by the above arguments, case (i) exhaust all the possible abelian cases for the Galois group G of equation (2.16) (i.e., types 1 abelian, 2, 3 with $k = 1, 2$ and 4 in Proposition 2.2). Furthermore cases (ii) and (iii) exhaust all the other possibilities of purely algebraic solutions (i.e., G finite). In other words, the known solutions cover types 1, 2, 3 with $k = 1, 2$ and 4 of Proposition 2.2. We are left now with type 3 (with $k > 2$), type 5 and type 6, to complete the study of the integrability of equation (2.16).

Proposition 2.6 ([81]) *The equation* (2.16) *is integrable only in the cases (i), (ii) and (iii) above.*

Proof. For type 5 of Proposition 2.2, by Proposition 2.3 the associated Riccati equation, $v' = g + v^2$, $g(z) = -(n(n+1)\mathcal{P}(z) + B)$, must have two solutions, $v_{1,2} = \kappa \pm \sqrt{\omega}$, in a quadratic extension of $K = \mathcal{M}(\Pi)$ (field of meromorphic functions on the Riemann surface Π of genus one). Therefore $\kappa, \omega \in K$ satisfy the system

$$\kappa' = \kappa^2 + \omega + g,$$

$$\omega' = 4\kappa\omega.$$

These equations are found in [7] and in what follows we use some of the methods of this paper.

The above system is equivalent to

$$\frac{1}{4}\left(\frac{\omega'}{\omega}\right)' - \frac{1}{16}\left(\frac{\omega'}{\omega}\right)^2 - \omega = g \tag{2.17}$$

(this equation was well known in the classical literature, see [43], p. 35).

If $v_i = -\xi_i'/\xi_i$, $i = 1, 2$, proceeding as in the proof of Proposition 2.2 we get $w^2/(4\omega) = \xi_1^2\xi_2^2$. On the other hand let α_1, α_2 be a fundamental system of solutions corresponding to the indicial equation around the singular point $z = 0$ (modulo periods). That is, $\alpha_1 = z^{\rho_1}\phi_1(z)$, $\alpha_2 = z^{\rho_2}\phi_2(z)$ ($\phi_1(0) \neq 0$, $\phi_2(0) \neq 0$), where $\rho_1 = n+1$, $\rho_2 = -n$ are the roots of the indicial equations at the origin and logarithmic terms can not appear, because $w^2/(4\omega) = \xi_1^2\xi_2^2$ and $\omega \in K$.

Expressing ξ_1, ξ_2 as linear combinations of α_1, α_2 we obtain

$$\frac{w^2}{4\omega} = az^{4m+4}\phi_1^4 + bz^{2m+3}\phi_1^3\phi_2 + cz^2\phi_1^2\phi_2^2$$

$$+dz^{-2m+1}\phi_1\phi_2^3 + ez^{-4m}\phi_2^4.$$

Furthermore, $\xi_1^2\xi_2^2$ is an elliptic function whose only pole is $z = 0$ (modulo periods), because this is the only singular point of the solutions of the Lamé equation. Therefore $w^2/(4\omega) = z^{-k}\phi$, $k \in \mathbf{N}$, $\phi(0) \neq 0$, where ϕ is a holomorphic function in a neighborhood of $z = 0$. We have the following mutually exclusive possibilities:

(a) $2m \in \mathbf{Z}$ (and $m \neq \mathbf{Z}$, otherwise we fall into the Lamé or Hermite solutions), else

(b) $4m \in \mathbf{Z}$, $2m \in \mathbf{Z}$.

Case (a) corresponds to the Brioschi-Halphen-Crawford solutions since those are the only ones such that $2m \in \mathbf{Z}$, $m \neq \mathbf{Z}$ and they have no logarithmic term.

In case (b), $b = d = 0$ and we can take $m > 0$ because if $4m \leq -5$ then $4m + 4 \leq -1$, and the values $4m = -1$, $4m = -3$ are excluded because $w^2/4\omega$

must have a pole at $z = 0$. Therefore $4m = k \in \mathbf{N}$, k odd. Then $\omega = w^2/(4\xi_1^2\xi_2^2)$ is an elliptic function of odd order k, having a zero of order k in $z = 0$ and hence a pole of odd order at some point $z = z_0 \neq 0$ (module periods). On the other hand $(\omega'/\omega)^2$ and $(\omega'/\omega)'$ are elliptic functions with double poles at $z = 0$ and at the poles of ω. From (2.17) it follows that g has a pole at $z_0 \neq 0$ contradicting the fact that it has only one pole at $z = 0$ (module periods). Type 5 does not occur in the equation (2.16).

The impossibility of type 3 (with $k > 2$) and of type 6 of Proposition 2.2 is simpler. By a direct computation, the derived group G' is given by unipotent triangular matrices. But as the local monodromy around the singular point $\mathbf{g}_* \in G'$, the exponents (i.e., solutions of the indicial equation) must be integers and we are in the Lamé or Hermite solutions with an abelian Galois group, contradicting the assumption that the Galois group is of type 3 (with $k > 2$) or type 6. We have finished the proof. □

We will need two more results about equation (2.16). The first one is very elementary, we state it as a proposition for future references.

Proposition 2.7 *Assume that for equation* (2.16) *we have* $\mathbf{g}_1^2 = \mathbf{1}$ *(or* $\mathbf{g}_2^2 = \mathbf{1}$*),* $\mathbf{g}_i$*,* $i = 1, 2$*, being the monodromies along the periods. Then the Galois group of this equation is abelian.*

Proof. From $\mathbf{g}_1^2 = \mathbf{1}$ it follows that $\mathbf{g}_1 = \mathbf{1}$ or $\mathbf{g}_1 = -\mathbf{1}$ (because $\mathbf{g}_1$ is in $SL(2, \mathbf{C})$). If $\mathbf{g}_1 = \mathbf{1}$, it is clear that $\mathbf{g}_* = [\mathbf{g}_1, \mathbf{g}_2] = \mathbf{1}$ (the case $\mathbf{g}_1 = -\mathbf{1}$ is analogous). □

The second result is not so elementary and we need some terminology.

We recall that the moduli of the elliptic curve $v^2 = 4u^2 - g_2 - g_3$ (we write the elliptic curve in the canonical form, where as above g_2 and g_3 are the invariants) is characterized by the value of the modular function j,

$$j = j(g_2, g_3) = \frac{g_2^3}{g_2^3 - 27g_3^2}. \tag{2.18}$$

We recall that two elliptic curves are birationally equivalent if, and only if, they have the same value of the modular function (see, for instance [93]).

Although the conditions on g_2, g_3 and B for a finite Galois group (case (iii)) are difficult to systematize, there is, in this case, a general result by Dwork answering a question posed by Baldassarri in [7].

Proposition 2.8 ([35]) *Assume that the Galois group of equation* (2.16) *is finite. Then for a fixed value of* n*, the number of possible couples* (j, B) *is finite.*

We note that the proof of Dwork is given for the algebraic form of the Lamé equation (equation (2.15)). But as by a finite covering the identity component of the Galois group is preserved (Theorem 2.5), then the finiteness of the Galois group of equation (2.15) is equivalent to the finiteness of the Galois group of equation (2.16) (a linear algebraic group is finite if, and only if, its identity component is trivial) and the result is valid also for equation (2.16).

The author is indebted to B. Dwork for sending him the above result.

Finally, for the families of type (ii) we recall the computation of the Brioschi determinant following Baldassarri [7] (it will be important in the applications of Chapter 6). If in the Lamé equation we make the Halphen substitution [42] $z = 2\hat{z}$ and use the addition theorem for $\mathcal{P}$ (see [109]) we obtain

$$\frac{d^2\,\xi}{d\,\hat{z}^2} - 4\left[n(n+1)\Big(\frac{1}{4}\,\Big(\frac{\mathcal{P}''(\hat{z})}{\mathcal{P}'(\hat{z})}\Big)^2 - 2\mathcal{P}(\hat{z})\Big) + B\right]\xi = 0\,. \tag{2.19}$$

If $(2\omega_1,\, 2\omega_3)$ are the periods of $\mathcal{P}$, the singularity of (2.16) at $z = 0$ (modulo the periods) is transformed to the singularities of (2.19)

$$\hat{z} = 0, \omega_1, \omega_2, \omega_3$$

(modulo the periods), where $\omega_1 + \omega_2 + \omega_3 = 0$. Now, to complete the Halphen transformation, we perform the change $\xi = \big(\mathcal{P}'(\hat{z})\big)^{-n} w$, obtaining

$$\frac{d^2\,w}{d\,\hat{z}^2} - 2n\,\frac{\mathcal{P}''(\hat{z})}{\mathcal{P}'(\hat{z})}\,\frac{dw}{d\hat{z}} + 4\big(n(2n-1)\,\mathcal{P}(\hat{z}) - B\big)\;w = 0\,,$$

with singularities as above. Now let $x = \mathcal{P}(\hat{z})$ be a *new* independent variable (i.e., we have a finite covering $z \mapsto x$). We get the following algebraic form for the above equation

$$\Big(-x^3 + \frac{g_2}{4}x + \frac{g_3}{4}\Big)\frac{d^2\,w}{d\,x^2} + \Big(3x^2 - \frac{g_2}{4}\Big)\,(m-1)\frac{d\,w}{d\,x} + \big[B - (2m-1)(m-1)\,x\big]\,w = 0\,, \tag{2.20}$$

having singularities at ∞, e_1, e_2, e_3 (corresponding to the previous ones

$$0, \omega_1, \omega_2, \omega_2$$

in (2.19)). We recall that $m = n + \frac{1}{2}$.

The exponents associated to the singularities are $(0, m)$ at $e_i, i = 1, 2, 3$, and $(-2m+1, -m+1)$ at ∞. As the difference is $m \in \mathbf{N}$ there will appear, in general, logarithmic terms. But if in one of the singularities there are no logarithmic terms they do not appear in any of the other singularities, because all the singularities come from the unique singularity of (2.16) by means of the Halphen transformation. Furthermore, if in an equation over $\mathbf{P}^1$ all the exponents are integers and there are no logarithmic terms, then the general solution is rational. In particular, if this happens in (2.20), then we have integrability for the Lamé equation.

To avoid logarithmic terms at $x = \infty$, a necessary and sufficient condition is the existence of a Laurent series solution of the form

$$w = \sum_{j=0}^{\infty} c_j \, x^{2m-j-1} \ , \ c_0 \neq 0, \tag{2.21}$$

corresponding to the lower exponent $-2m+1$.

This leads to a recurrent system for the coefficients $c_0, c_1, \ldots$, which, in particular, gives the uncoupled system:

$$\begin{array}{lllll}
B\ c_0 & + & (m-1)c_1 & & =0, \\
(2m-1)(m-1)\frac{g_2}{4}c_0 & + & B\quad c_1 & +2(m-2)c_2 & =0, \\
(2m-1)(2m-2)\frac{g_3}{4}c_0 & +(2m-2)(m-2)\frac{g_2}{4}c_1 & +B\ c_2 & +3(m-3)c_3 & =0, \\
(2m-2)(2m-3)\frac{g_3}{4}c_1 & +(2m-3)(m-3)\frac{g_2}{4}c_2 & +B\ c_3 & +4(m-4)c_4 & =0, \\
\vdots & & & & \\
\vdots & & & & \\
(m+3)(m+2)\frac{g_3}{4}c_{m-4} & +(m+2)2\frac{g_2}{4}c_{m-3} & +B\ c_{m-2} & +(m-1)1c_{m-1} & =0, \\
(m+2)(m+1)\frac{g_3}{4}c_{m-3} & +(m+1)1\frac{g_2}{4}c_{m-2} & +B\ c_{m-1} & & =0.
\end{array}$$

Therefore, the necessary and sufficient condition to have a solution of the form (2.21) is

$$Q_m\left(\frac{g_2}{4}, \frac{g_3}{4}, B\right) = 0, \tag{2.22}$$

where $Q_m\left(\frac{g_2}{4}, \frac{g_3}{4}, B\right)$ is the determinant of dimension m of the coefficients of the above linear system in the variables $c_0, c_1, \ldots, c_{m-1}$. This is the Brioschi determinant.

We observe that all the examples in this section are second order differential equations over the Riemann sphere (in the case of the Lamé equation we consider its algebraic form), then it is theoretically possible to apply the Kovacic's algorithm [56, 33].

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