Monographs in Theoretical Computer Science
An EATCS Series

Springer

Berlin
Heidelberg
New York
Barcelona
Hong Kong
London
Milan
Paris
Singapore
Tokyo

Aldo de Luca · Stefano Varricchio

Finiteness and Regularity in Semigroups and Formal Languages

Springer

Authors

Prof. Dr. Aldo de Luca
Dipartimento di Matematica
Università di Roma „La Sapienza“
Piazzale Aldo Moro, 2
I-00185 Roma, Italy
deluca@mat.uniroma1.it

Prof. Dr. Stefano Varricchio
Dipartimento di Matematica
Università di Roma „Torvergata“
Viale della Ricerca Scientifica
I-00133 Roma, Italy
varricchio@mat.uniroma1.it

Series Editors

Prof. Dr. Wilfried Brauer
Department of Computer Science
Technical University of Munich
Arcisstrasse 21
D-80333 München, Germany
brauer@informatik.tu-muenchen.de

Prof. Dr. Grzegorz Rozenberg
Department of Computer Science
University of Leiden
Niels Bohrweg 1, P.O. Box 9512
NL-2300 RA Leiden, The Netherlands
rozenber@wi.leidenuniv.nl

Prof. Dr. Arto Salomaa
Data City
Turku Centre for Computer Science
FIN-20520 Turku, Finland
asalomaa@utu.fi

Library of Congress Cataloging-in-Publication Data

de Luca, A. (Aldo)
Finiteness and regularity in semigroups and formal languages /
Aldo de Luca, Stefano Varricchio.
p. cm. -- (Monographs in theoretical computer science)
Includes bibliographical references and index.
ISBN 3-540-63771-0 (hc.: alk. paper)
1. Semigroups. 2. Formal languages. I. Varricchio, Stefano, 1960- . II. Title.
III. Series.
QA182.D4 1998
512'. 2--dc21 98-42554 CIP

ACM Computing Classification (1998): F.4.2-3, G.2.1

ISBN 3-540-63771-0 Springer-Verlag Berlin Heidelberg New York

Typesetting: Camera ready by authors
Production: PRO EDIT GmbH, D-69126 Heidelberg
Cover Design: *design & production*, D-69121 Heidelberg
SPIN: 10552017 45/3142 5 4 3 2 1 0 – Printed on acid-free paper

Preface

The aim of this monograph is to present some recent research work on the combinatorial aspects of the theory of semigroups which are of great interest for both algebra and theoretical computer science. This research mainly concerns that part of combinatorics of finite and infinite words over a finite alphabet which is usually called the theory of "unavoidable" regularities.

The unavoidable regularities of sufficiently large words over a finite alphabet are very important in the study of finiteness conditions for semigroups. This problem consists in considering conditions which are satisfied by a finite semigroup and are such as to assure that a semigroup satisfying them is finite. The most natural requirement is that the semigroup is *finitely generated.* If one supposes that the semigroup is also *periodic* the study of finiteness conditions for these semigroups (or groups) is called the *Burnside problem for semigroups* (or groups). There exists an important relationship with the theory of *finite automata* because, as is well known, *a language L over a finite alphabet is regular* (that is, recognizable by a finite automaton) *if and only if its syntactic monoid $S(L)$ is finite.* Hence, in principle, any finiteness condition for semigroups can be translated into a regularity condition for languages. The study of finiteness conditions for periodic languages (i.e., such that the syntactic semigroup is periodic) has been called the *Burnside problem for languages.*

Several finiteness conditions for finitely generated semigroups have been given in recent years based on different concepts such as: *permutation properties, iteration conditions, minimal conditions on ideals, and repetitivity.*

These conditions are analyzed in some detail in Chap. 3. They are based, as we said before, on the existence of some different unavoidable regularities on very large words over a finite alphabet. As we shall see the permutation conditions are related to Shirshov's theorem and the iteration conditions to bi-ideal factorizations. A very recent result shows that these two regularities "appear" simultaneously in a suitable way in very large words. This fact gives rise to a new finiteness condition in which any sequence of $n > 1$ elements of the semigroup can be either permutable or iterable (on the right). Repetitivity is a concept related to the unavoidable regularities expressed in the theorems of van der Waerden and of Brown.

We present also finiteness conditions for semigroups based on *chain conditions*. In particular, we consider some remarkable generalizations of a theorem of Hotzel and of a theorem of Coudrain and Schützenberger. From these one derives an extension of the theorem of Green and Rees relating the bounded Burnside problem for semigroups with the corresponding problem for groups, and a new simple proof of the theorem of McNaughton and Zalcstein which gives a positive answer to the Burnside problem in the case of $n \times n$ matrices with elements in a field. The proof of these results requires also some structure theorems on semigroups based on the Green relations, as the $\mathcal{J}$-depth decomposition theorem, which are given in Chap. 3.

Chapter 4 concerns the following general problem: given a semigroup S, under what conditions can we say that the finite parts of S are recognizable sets? This is also equivalent to the following problem: let θ be a congruence in a finitely generated free semigroup A^+; when are the congruence classes of θ in A^+ *regular languages*? A semigroup whose finite parts are recognizable is called *finitely recognizable*. Some general results relating this problem to the $\mathcal{J}$-depth decomposition of S are shown. In particular we refer to the case when the semigroup S is the quotient semigroup $M_n = A^*/\theta_n$, where θ_n is the congruence generated by the relation $x^n = x^{n+1}$. The problem of the regularity of the congruence classes (noncounting classes) was posed for any $n > 0$ by Brzozowsky about thirty years ago. The authors have proved that this problem has a positive answer for $n > 4$. In this proof the finiteness condition for semigroups due to Hotzel and introduced in Chap. 3 has been used. The proof of this result allows one to show also that the *word problem* for the semigroup M_n when $n > 4$ is recursively solvable. This result was subsequently improved by other authors for the case $n > 2$ and extended to more general cases.

Chapter 5 deals with the Burnside problem for languages. From the finiteness conditions for semigroups one can easily find some *uniform* conditions which assure the regularity of a periodic language. However, the Burnside problem for languages is more complicated since the regularity conditions can be presented in a *non-uniform* way, that is, they depend on the contexts which complete the words in the language. The use of Ramsey's theorem is often a good tool to transform non-uniform conditions into uniform ones. Some important regularity conditions such as the *block-pumping* property of Ehrenfeucht, Parikh, and Rozenberg and the *permutative property* of Restivo and Reutenauer are proved. Moreover, the existence of a non-uniform and *positive* block pumping property is shown.

In Chap. 6 we present some further combinatorial aspects of semigroups related to the notion of *well quasi-order* which gives a new insight into the combinatorics of the free monoids.

Classical theorems of automata theory, such as the Myhill theorem, are extended by some regularity conditions based on well quasi-orders. For instance, one has that *a language is regular if and only if it is a closed part of a* *monotone well quasi-order*. Some applications of these notions and techniques for the regularity conditions are given.

In conclusion, this monograph, which greatly extends and completes a chapter written on the subject for a Handbook on formal languages [54], presents very recent research work, including new unpublished results, on those combinatorial aspects of the theory of semigroups which are intimately related to the fundamental property of finite automata, namely finiteness. This relation with automata gives rise to regularity conditions for formal languages. Of course there exist other regularity conditions, based on different techniques and concepts, which are not covered by this volume (see, e.g., [12, 35, 61, 79]).

The methods used are of a combinatorial and algebraic nature. However, the book presupposes no prior and deep knowledge of any particular mathematical topic, but only the background of an undergraduate student in mathematics or computer science, and the knowledge of some rudiments of the theory of automata and semigroups.

We thank deeply Dr. Flavio D'Alessandro, who patiently read the entire manuscript and helped us in improving the presentation.

Rome, February 1999 *A. de Luca, S. Varricchio*

Springer

http://www.springer.com/978-3-540-63771-4

Finiteness and Regularity in Semigroups and Formal Languages
Luca, A. de; Varricchio, S.
1999, X, 240 p., Hardcover
ISBN: 978-3-540-63771-4