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## WHAT CAN WE LEARN FROM EDUCATIONAL RESEARCH AT THE UNIVERSITY LEVEL?<sup>1</sup>

### 1. INTRODUCTION

For more than 20 years, educational research has dealt with mathematics learning and teaching processes at the university level. It has tried to improve our understanding of the difficulties encountered by students and the dysfunction of the educational system; it has also tried to find ways to overcome these problems. What can such research offer to an international study? This is the issue I will address in this article, but first I would like to stress that it is not an easy question to answer, for several reasons including at least the following:

1. Educational research is far from being a unified field. This characteristic was clearly shown in the recent ICMI study entitled "What is research in mathematics education and what are its results?" (See Sierpinska and Kilpatrick, 1996.) The diversity of existing paradigms certainly contributes to the richness of the field but, at the same time, it makes the use and synthesis of research findings more difficult.
2. Learning and teaching processes depend partly on the cultural and social environments in which they develop. Up to a certain point, results obtained are thus time- and space- dependent, their field of validity is necessarily limited. However, these limits are not generally easy to identify.
3. Finally, research-based knowledge is not easily transformed into effective educational policies.

I will come back to this last point later on. Nevertheless, I am convinced that existing research can greatly help us today, if we make its results accessible to a large audience and make the necessary efforts to better link research and practice. I hope that this article will contribute to making this conviction not just a personal one. Before continuing, I would like to point out that the diversity mentioned above does not mean that general tendencies cannot be observed. At the theoretical level, these are indicated, for instance, by the dominating influence of constructivist approaches inspired by Piaget's genetic epistemology, or by the recent move

<sup>1</sup> A shorter version of this paper, Artigue (1999), was published in the *Notices of the American Mathematical Society*.

attempt to take more account of the social and cultural dimensions of learning and teaching processes (see Sierpinska and Lerman, 1996). But within these general perspectives, researchers have developed a multiplicity of local theoretical frames and methodologies, which differently shape the way research questions are selected and expressed, and the ways they are worked on – thus affecting the kind of results which can be obtained, and the ways they are described. At the cultural level, such general tendencies are also observed. Strong regularities in students' behaviour and difficulties as well as in the teaching problems met by educational institutions, have been observed. These, up to a point, apparently transcend the diversity of cultural environments.

In the following, after characterizing the beginnings of the research enterprise, I will try to overcome some of the above-mentioned difficulties presenting research findings along two main dimensions of learning processes: qualitative changes, reconstructions and breaches on the one hand, cognitive flexibility on the other hand. These dimensions can to some degree, be considered 'transversal' with respect to theoretical and cultural diversities as well as to mathematical domains. No doubt this is a personal choice, induced by my own experience as a university teacher, as a mathematician, and as a education researcher; it shapes the vision I give of research findings, a vision which does not pretend to be objective or exhaustive.

## 2. FIRST RESEARCH RESULTS: SOME NEGATIVE REPORTS

The first research results obtained at university level can be considered negative ones. Research began by investigating students' knowledge in specific mathematical areas, with particular emphasis on elementary analysis (or calculus in the Anglo-Saxon culture), an area perceived as the main source of failure at the undergraduate level. The results obtained gave statistical evidence of the limitations both of traditional teaching practices and of teaching practices which, reflecting the Bourbaki style, favoured formal and theoretical approaches. The structure and content of the book, *Advanced Mathematical Thinking* (Tall, 1991), gives clear evidence of these facts, noting that:

- by the early eighties, Orton (1980), in his doctoral thesis, showed the reasonable mastery English students had of what can be labelled as 'mere algebraic calculus': calculation of derivatives and primitives (anti-derivatives), but the significant difficulty they had in conceptualizing the limit processes underlying the notions of derivative and integral;
- at about the same time, Tall and Vinner (1981), highlighted the discrepancy between the formal definitions students were able to quote and the criteria they used in order to check properties such as functionality, continuity, derivability. This discrepancy led to the introduction of the notions of concept definition and concept image in order to analyze students' conceptions;

- very early, different authors documented students' difficulties with logical reasoning and proofs, with graphical representations, and especially with connecting analytic and graphical work in flexible ways.

Schoenfeld (1985), also documented the fact that, faced with non-routine tasks, students – even apparently bright students – were unable to efficiently use their mathematical resources.

Research also showed, quite early, that the spontaneous reactions of educational systems to the above-mentioned difficulties were likely to induce vicious circles such as the following. In order to guarantee an acceptable rate of success, an increasingly important issue for political reasons, teachers tended to increase the gap between what was taught and what was assessed. As the content of assessments is considered by students to be what should be learnt, this situation had dramatic effects on their beliefs about mathematics and mathematical activity. This, in turn, did not help them to cope with the complexity of advanced mathematical thinking.

Fortunately, research results are far from being limited to such negative reports. Thanks to an increasing use of qualitative methodologies allowing better explorations of students' thinking and the functioning of didactic institutions (Schoenfeld, 1994), research developed and tested global and local cognitive models. It also organized in coherent structures the many difficulties students encounter with specific mathematical areas, or in the secondary/tertiary transition. It led to research-based teaching designs (or engineering products) which, implemented in experimental environments and progressively refined, were proved to be effective. Without pretending to be exhaustive, let us give some examples, classified according to the two main dimensions given above. (For more details, the reader can refer to the different syntheses in Artigue, 1996, Dorier, 2000, Schoenfeld, 1994, Tall, 1991 and 1996; to the special issues dedicated to advanced mathematical thinking by the journal *Educational Studies in Mathematics* in 1995 edited by Dreyfus; by the journal *Recherches en Didactique des Mathématiques* in 1998 edited by Rogalski; to some of the diverse monographs published by the Mathematical Association of America about calculus reform, innovative teaching practices; and to research about specific undergraduate topics to be found in the MAA Notes on Collegiate Mathematics Education.)

### 3. QUALITATIVE CHANGES, RECONSTRUCTIONS AND BREACHES IN THE MATHEMATICAL DEVELOPMENT OF KNOWLEDGE AT UNIVERSITY LEVEL

One general and crosscutting finding in mathematics education research is the fact that mathematical learning is a cognitive process that necessarily includes 'discontinuities.' But, depending on the researcher this attention to discontinuities is expressed in different ways. In order to reflect this diversity and the different insights it allows, I will describe three different approaches: the first one, in terms of

process/object duality, the second one in terms of epistemological obstacles, the third one in terms of reconstructions of relationships to objects of knowledge.

### 3.1 *Qualitative changes in the transition from processes to objects: APOS theory*

As mentioned above, research at the university level is the source of theoretical models. The case of APOS theory, initiated by Dubinsky (see Tall 1991) and progressively refined (see Dubinsky and McDonald, this volume, pp. 275-282), is typical. This theory, which is an adaptation of the Piagetian theory of reflective abstraction, aims at modelling the mental constructions used in advanced mathematical learning. It considers that "understanding a mathematical concept begins with manipulating previously constructed mental or physical objects to form actions; actions are then interiorized to form processes which are then encapsulated to form objects. Objects can be de-encapsulated back to the processes from which they were formed. Finally, actions, processes and objects can be organized in schemas" Asiala et al, 1996. Of course, this does not occur all at once and objects, once constructed, can be engaged in new processes and so on. Researchers following this theory use it in order to construct genetic decomposition of concepts taught at university level (in calculus, abstract algebra, etc.) and design teaching processes reflecting the genetic structures they have constructed and tested.

As with any model, the APOS model only gives a partial vision of cognitive development in mathematics, but one cannot deny today that it put to the fore a crucial qualitative discontinuity in the relationships students develop with respect to mathematical concepts. This discontinuity is the transition from a process conception to an object one, the complexity of its acquisition and the dramatic effects of its underestimation by standard teaching practices.<sup>2</sup> Research related to APOS theory also gives experimental evidence of the positive role which can be played by programming activities in adequate languages (such as the language ISETL, cf. Tall, 1991) in order to help students encapsulate processes as objects.

*Breaches in the development of mathematical knowledge: Epistemological obstacles.* The theory of epistemological obstacles, firstly introduced by Bachelard (1938) and imported into educational research by Brousseau (1997), proposes an approach complementary to cognitive evolution, focussing on its necessary breaches. The fundamental principle of this theory is that scientific knowledge is not built in a continuous process but results from the rejection of previous forms of knowledge: the so-called epistemological obstacles. Researchers following this theory hypothesize that some learning difficulties, often the more resistant ones, result from forms of knowledge which are coherent and have been for a time effective in social and/or educational contexts. They also hypothesize that epistemological obstacles have some kind of universality and thus can be traced in the historical development of the corresponding concepts. At the university level,

<sup>2</sup> Note that a very similar approach was developed independently by Sfard, with more emphasis on the dialectic between the operational and structural dimensions of mathematical concepts in mathematical activity (Sfard, 1991).

such an approach has been fruitfully used in research concerning the concept of limit (cf. Artigue 1998 and Tall 1991 for synthetic views). Researchers such as Sierpinska, (1985), Cornu, (1991) and Schneider, (1991) provide us with historical and experimental evidence of the existence of epistemological obstacles, mainly the following:

- the everyday meaning of the word 'limit', which induces resistant conceptions of the limit as a barrier or as the last term of a process, or tends to restrict convergence to monotonic convergence;
- the overgeneralization of properties of finite processes to infinite processes, following the continuity principle stated by Leibniz;
- the strength of a geometry of forms which prevents students from clearly identifying the objects involved in the limit process and their underlying topology. This makes it difficult for students to appreciate the subtle interaction between the numerical and geometrical settings in the limit process.

Let us give one example (taken from Artigue, 1998) of this last resistance, which occurs even in advanced and bright students. In a research project about differential and integral processes, advanced students were asked the following non-standard question: "How can you explain the following: using the classical decomposition of a sphere into small cylinders in order to find its volume and area, one obtains the expected answer for the volume  $\frac{4}{3}\pi R^3$ , but  $\pi^2 R^2$  for the area instead  $4\pi R^2$ ?" It was observed that, faced with this question, the great majority of advanced students tested got stuck. And, even if they were able to make a correct calculation for the area (which they were not always able to do) they remained unable to resolve the conflict.

As the students eventually said, because the pile of cylinders, geometrically, tends towards the sphere, the magnitudes associated with the cylinders behave in the same way and thus have as a limit the corresponding magnitude for the sphere. Such a resistance may look strange but it appears more normal if we consider the effect produced on mathematicians by the famous Schwarz counterexample showing that, for a surface as simple as a cylinder, limits of areas of triangulations when the size of the triangles tends towards 0, can take any value greater than or equal to the area up to infinity, depending on the choices made in the triangulation process, an effect nicely described by in Lebesgue, (1956). The historical and universal commitments of the theory which leads to such results can be discussed and are presently discussed (see, for instance, Radford, 1997). However, what cannot be negated is the fact that the above-mentioned forms of knowledge constitute resistant difficulties for today's students; moreover, that mathematical learning necessarily implies partial rejection of previous forms of knowledge, which is not easy for students.



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