Preface

This book provides an introduction to complex analysis for students with some familiarity with complex numbers from high school. Students should be familiar with the Cartesian representation of complex numbers and with the algebra of complex numbers, that is, they should know that $i^2 = -1$. A familiarity with multivariable calculus is also required, but here the fundamental ideas are reviewed. In fact, complex analysis provides a good training ground for multivariable calculus. It allows students to consolidate their understanding of parametrized curves, tangent vectors, arc length, gradients, line integrals, independence of path, and Green's theorem. The ideas surrounding independence of path are particularly difficult for students in calculus, and they are not absorbed by most students until they are seen again in other courses.

The book consists of sixteen chapters, which are divided into three parts. The first part, Chapters I–VII, includes basic material covered in all undergraduate courses. With the exception of a few sections, this material is much the same as that covered in Cauchy's lectures, except that the emphasis on viewing functions as mappings reflects Riemann's influence. The second part, Chapters VIII–XI, bridges the nineteenth and the twentieth centuries. About half this material would be covered in a typical undergraduate course, depending upon the taste and pace of the instructor. The material on the Poisson integral is of interest to electrical engineers, while the material on hyperbolic geometry is of interest to pure mathematicians and also to high school mathematics teachers. The third part, Chapters XII–XVI, consists of a careful selection of special topics that illustrate the scope and power of complex analysis methods. These topics include Julia sets and the Mandelbrot set, Dirichlet series and the prime number theorem, and the uniformization theorem for Riemann surfaces. The final five chapters serve also to complete the coverage of all background necessary for passing PhD qualifying exams in complex analysis.

Note to the instructor

There is a glut of complex analysis textbooks on the market. It is a beautiful subject, so beautiful that a large number of experts have been moved to write their own accounts of the area. In spite of the plethora of textbooks, I have never found an introduction to complex analysis that is completely suitable for my own teaching style and audiences.

The students in each of my various audiences have begun the course with a wide range of backgrounds. Teaching to students with disparate backgrounds and preparations has posed a major teaching challenge. I respond by including early some topics that can be treated in an elementary way and yet are usually new and capture the imagination of students with already some background in complex analysis. For example, the stereographic projection appears early, the Riemann surface of the square root function is explained early at an intuitive level, and both conformality and fractional linear transformations are treated relatively early. Exercises range from the very simple to the quite challenging, in all chapters. Some of the exercises that appear early in the book can form the basis for an introduction to a more advanced topic, which can be tossed out to the more sophisticated students. Thus for instance the basis is laid for introducing students to the spherical metric already in the first chapter, though the topic is not taken up seriously until much later, in connection with Marty's theorem in Chapter XII.

The second problem addressed by the book has to do with flexibility of use. There are many routes through complex analysis, and many instructors hold strong opinions concerning the optimal route. I address this problem by laying out the material so as to allow for substantial flexibility in the ordering of topics. The instructor can defer many topics (for instance, the stereographic projection, or conformality, or fractional linear transformations) in order to reach Cauchy's theorem and power series relatively early, and then return to the omitted topics later, time permitting.

There is also flexibility with respect to adjusting the course to undergraduate students or to beginning graduate students. The bulk of the book was written with undergraduate students in mind, and I have used various preliminary course notes for Chapters I-XI at the undergraduate level. By adjusting the level of the lectures and the pace I have found the course notes for all sixteen chapters appropriate for a first-year graduate course sequence.

One of my colleagues wrote in commenting upon the syllabus of our undergraduate complex analysis course that "fractional powers should be postponed to the end of the course as they are very difficult for the students." My philosophy is just the reverse. If a concept is important but difficult, I prefer to introduce it early and then return to it several times, in order to give students time to absorb the idea. For example, the idea of a branch of a multivalued analytic function is very difficult for students, yet it is a central issue in complex analysis. I start early with a light introduction to the square root function. The logarithm function follows soon, followed by phase factors in connection with fractional powers. The basic idea is returned to several times throughout the course, as in the applications of residue theory to evaluate integrals. I find that by this time most students are reasonably comfortable with the idea.

A solid core for the one-semester undergraduate course is as follows:

Chapter I
Chapter II
Sections III.1-5
Sections IV.1-6
Sections V.1-7
Sections VI.1-4
Sections VII.1-4
Sections VIII.1-2
Sections IX.1-2
Sections X.1-2
Sections XI.1-2

To reach power series faster I would recommend postponing I.3, II.6-7, III.4-5, and going light on Riemann surfaces. Sections II.6-7 and III.4-5 should be picked up again before starting Chapter IX.

Which additional sections to cover depends on the pace of the instructor and the level of the students. My own preference is to add more contour integration (Sections VII.5 and VII.8) and hyperbolic geometry (Section IX.3) to the syllabus, and then to do something more with conformal mapping, as the Schwarz reflection principle (Section X.3), time permitting. To gain time, I mention some topics (as trigonometric and hyperbolic functions) only briefly in class. Students learn this material as well by reading and doing assigned exercises. Finishing with Sections XI.1-2 closes the circle and provides a good review at the end of the term, while at the same time it points to a fundamental and nontrivial theorem (the Riemann mapping theorem).

Note to the student

You are about to enter a fascinating and wonderful world. Complex analysis is a beautiful subject, filled with broad avenues and narrow backstreets leading to intellectual excitement. Before you traverse this terrain, let me provide you with some tips and some warnings, designed to make your journey more pleasant and profitable.

Above all, give some thought to strategies for study and learning. This is easier if you are aware of the difference between the "what," the "how," and the "why," (as Halmos calls them). The "what" consists of definitions, statements of theorems, and formulae. Determine which are most important and memorize them, at least in slogan form if not precisely. Just as one maintains in memory the landmark years 1066, 1453, and 1776 as markers in the continuum of history, so should you maintain in memory the definition of analytic function, the Cauchy-Riemann equations, and the residue formula. The simplest of the exercises are essentially restatements of "what."

The “how” consists in being able to apply the formulae and techniques to solve problems, as to show that a function is analytic by checking the Cauchy-Riemann equations, or to determine whether a polynomial has a zero in a certain region by applying the argument principle, or to evaluate a definite integral by contour integration. Before determining “how” you must know “what.” Many of the exercises are “how” problems. Working these exercises and discussing them with other students and the instructor are an important part of the learning process.

The “why” consists in understanding why a theorem is true or why a technique works. This understanding can be arrived at in many different ways and at various levels. There are several things you can do to understand why a result is true. Try it out on some special cases. Make a short synopsis of the proof. See where each hypothesis is used in the proof. Try proving it after altering or removing one of the hypotheses. Analyze the proof to determine which ingredients are absolutely essential and to determine its depth and level of difficulty. The slogan form of the Jordan curve theorem is that “every closed curve has an inside and an outside” (Section VIII.7). What is the level of difficulty of this theorem? Can you come up with a direct proof? Try it.

Finally, be aware that there is a language of formal mathematics that is related to but different from common English. We all know what “near” means in common English. In the language of formal mathematics the word carries with it a specific measure of distance or proximity, which is traditionally quantified by $\varepsilon > 0$ or a “for every neighborhood” statement. Look also for words like “eventually,” “smooth,” and “local.” Prepare to absorb not only new facts and ideas but also a different language. Developing some understanding of the language is not easy – it is part of growing up and becoming mathematically sophisticated.

Acknowledgments

This book stems primarily from courses I gave in complex analysis at the Interuniversity Summer School at Perugia (Italy). Each course was based on a series of exercises, for which I developed a computer bank. Gradually I deposited written versions of my lectures in the computer bank. When I finally decided to expand the material to book form, I also used notes based on lectures presented over the years at several places, including UCLA, Brown University, Valencia (Spain), and long ago at the university at La Plata (Argentina). I have enjoyed teaching this material. I learned a lot, both about the subject matter and about teaching, through my students. I would like to thank the many students who contributed, knowingly or unknowingly, to this book.

The origins of many of the mathematical ideas have been lost in the thickets of the history of mathematics. Let me mention the source for one item. The treatment of the parabolic case of the uniformization theorem follows a line of proof due to D. Marshall, and I am grateful for his sharing

his work. As far as I know, everything else is covered by the bibliography, and I apologize for any omissions.

Each time I reread a segment of the book manuscript I found various mathematical blunders, grammatical infringements, and stylistic travesties. Undoubtedly mistakes have persisted into the printed book. I would appreciate receiving your email about any egregious errors you come across, together with your comments about any passages you perceive to be particularly dense or unenlightening. My email address, while I am around, is twg@math.ucla.edu. I thank you, dear reader, in advance.

Julie Honig and Mary Edwards helped with the preparation of class notes that were used for parts of the book, and for this I thank them. Finally, I am happy to acknowledge the skilled assistance of the publishing staff, who turned my doodles into figures and otherwise facilitated publication of the book.

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