2
Classical Probability Spaces

2.1 Theory and Problems

Here, we consider one of the simplest models of a random experiment with m possible outcomes. We assume (believe) that these outcomes are equally likely, so the probability of an event consisting of n outcomes is simply $\frac{n}{m}$. In the framework of general probability spaces, this means that we assume that Ω is a finite set (with m elements), $\Sigma = 2^{\Omega}$, and $P(\{\omega\}) = \frac{1}{m}$ for each $\omega \in \Omega$; hence,

$$P(A) = \frac{\#A}{\#\Omega}.$$

We shall call such probability spaces *classical* and refer to P as the *uniform probability measure.*

It is useful to know some formulas for computing the numbers of elements of various finite sets.

2.1 What is the number of subsets of an n-element set?

2.2 In how many ways can you order an n-element set?

2.3 What is the number of all k-element subsets of an n-element set?

2.4 What is the number of all one-to-one mappings from an n-element set to an m-element set?

2.5 What is the number of all mappings from an n-element set to an m-element set?

We introduce some notation:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \times (n-1) \times \cdots \times (n-k+1)}{1 \times 2 \times 3 \times \cdots \times k}.$$

This gives, as we can see in Solution 2.3, the number of k-element subsets of an n-element set and is called the *binomial coefficient*.

2.6 Use the above problems to justify the following particular case of Newton's binomial formula

$$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

2.7 Prove the Van der Monde formula

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i}.$$

Now some typical exercises on finding probabilities of events. The scheme is as follows. First, we choose the set Ω and find the appropriate subset $A \subset \Omega$. Here lies the main difficulty because there are many possibilities and we must be careful to be consistent. (In the solution to the next problem, we provide three different approaches, all leading to the same result.) At the end, we count the elements of Ω and A and compute the probability of A.

2.8 Ten people are randomly seated at a round table. What is the probability that a particular couple will sit next to each other?

2.9 If boys and girls are born equally likely, what is the probability that in a family with three children, exactly one is a girl.

2.10 Two dice are thrown. What is the probability that the total number of dots is

a) equal to 7,
b) equal to 3,
c) greater than 5,
d) an even number.

2.11 In a lottery, 6 numbers are drawn out of 49. Find the probability that

a) 1, 2, 3, 4, 5, 6 are drawn,
b) 4, 23, 24, 35, 40, 45 are drawn,
c) 44 is one of the numbers drawn.

2.12 What is the probability that among 25 people, at least 2 have their birthday on the same day of the year.

2.13 Among $t = 60$ lottery tickets, $w = 20$ win prizes. We buy $b = 6$. What is the probability that $g = 2$ will be winning? Generalize this to arbitrary numbers t, w, b, g.

∘ **2.14** In a series of 1000 light bulbs, 2% are defective. What is the probability that among 20 bulbs bought, there are 2 faulty ones?

∘ **2.15** From a bridge deck of 52 cards, we draw 13. What is the probability that we have 5 spades in our hand?

2.16 A bridge deck of 52 cards is dealt among the players. Suppose that I have 4 spades and the opponents have shown (by bidding) that they have 8 hearts. What is the probability that my partner has at least 3 spades?

For bridge players. This problem has practical consequences. Suppose that as South you have

$$\spadesuit \text{ AJ84} \quad \heartsuit \text{ 32} \quad \diamondsuit \text{ 54} \quad \clubsuit \text{ KQ952}$$

and the bidding is

N	E	S	W
	1♡	pass	2♡
pass	pass	?	

The risk of takeout double is 3♢ response. The best contract may be 2♠, and the estimation of the probability of a complete misfit (partner holding less than three spades) is important.

2.17 We draw 5 cards out of a deck of 24. What is the probability that we have three of one kind.

2.18 In the game of poker played with the 24-card deck you get AAAKJ.
a) What is the probability of getting one Ace if you discard KJ?
b) What is the probability of getting a King or an Ace if you discard J?

∘ **2.19** In the game of poker played with the 24-card deck you get AAKJ9.
a) What is the probability of getting one Ace or two Aces if you discard KJ9?
b) What is the probability of getting a King or an Ace if you discard J?

2.20 A monkey hits a computer keyboard three times at random. What is the chance of getting a three-letter word with a consonant followed by two vowels? The word does not have to make sense. For simplicity, assume that there are 100 keys.

2.21 From a pack of 52 cards, we draw one-by-one. What is the probability that an Ace will appear at the fifth turn?

2.22 How likely is it that the word ABRACADABRA will show if the letters A, A, A, A, A, B, B, C, D, R, R are shuffled randomly?

2.2 Hints

2.1 It suffices to consider n-element sets of the form $\{1, 2, \ldots, n\}$. Try 1, 2, or 3-element sets. The set $\{1\}$ has 2 subsets $\emptyset$ and $\{1\}$, the set $\{1, 2\}$ has 4 subsets, and $\{1, 2, 3\}$ has 8 subsets. If you add one new element to a finite set, how many new subsets (i.e., those containing the new element) will arise? Now try to guess the general formula.

2.2 A *permutation* of a set A is a one-to-one mapping $f : A \to A$.

If $A = \{1, 2, \ldots, n\}$, then the sequence $f(1), f(2), \ldots, f(n)$ contains all the elements of A. So one can say that a permutation of a finite set is the order in which the elements are arranged.

Try to figure out the general formula by finding all permutations of 1-, 2-, 3-element sets. How many new permutations will arise if you add one new element to a finite set?

2.3 Consider an example. For $A = \{1, 2, 3, 4, 5\}$, count the number of 3-element subsets of A. The first element of a subset can be chosen in 5 ways, the second in 4, and the third in 3 ways, giving in total $5 \times 4 \times 3 = 60$. But this is incorrect! Each subset has been counted 6 times, so we have to divide 60 by 6. To see this, consider, for example, $\{1, 3, 4\}$. The elements of this set could have been picked in any order. So the following sets emerge from this method of counting: $\{1, 3, 4\}$, $\{1, 4, 3\}$, $\{3, 1, 4\}$, $\{3, 4, 1\}$, $\{4, 1, 3\}$, $\{4, 3, 1\}$. But they are all identical!

2.4 Suppose we are counting the number of one-to-one mappings f from the set $\{1, 2, 3\}$ to $\{1, 2, 3, 4, 5\}$. First, $f(1)$ can be chosen in 5 ways, then $f(2)$ in 4 ways, and, finally, $f(3)$ in 3 ways. The total number is $5 \times 4 \times 3$.

2.5 Suppose we are counting the number of mappings f from $\{1, 2, 3\}$ to $\{1, 2, 3, 4, 5\}$. We can choose $f(1)$ in 5 ways, then $f(2)$ in 5 ways, and, finally, $f(3)$ also in 5 ways (the values may coincide). The total number is $5 \times 5 \times 5 = 5^3$.

2.6 Use Problems 2.1 and 2.3.

2.7 Imagine choosing k balls, among which are m white balls and n black balls. You could do that, for example, by choosing i white balls and $k-i$ black balls for some i between 0 and k. Counting the number of choices in this way, you can obtain the right-hand side of Van der Monde's formula. To obtain the left-hand side, refer to Problem 2.3.

2.8 How in practice can we seat 10 people randomly at the table?

1. We number the seats from 1 to 10, prepare a deck of 10 cards with the same numbers on them, shuffle, then deal among the people. The number of outcomes is the same as the number of all permutations of 10 elements. Count the "favorable" outcomes that the two people are seated next to each other.

2. Prepare the cards and the seats as before. Now deal 2 cards to the couple. This can be done in the same number of ways as the number of 2-element subsets of a 10-element set. How many sets are "favorable"?

2.9 Consider the tree in Figure 2.1.

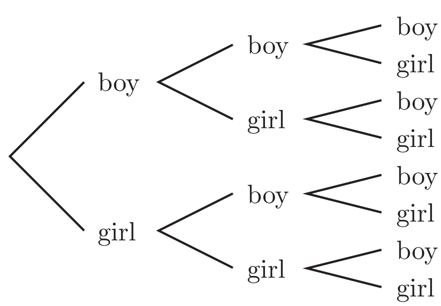

FIGURE 2.1. Three children

All branches are equally likely. How many "favorable" ones are there?

2.10 All possible outcomes are shown in Figure 2.2.

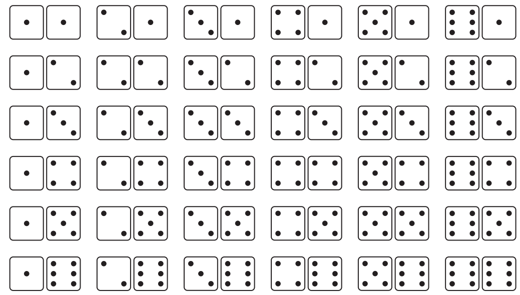

FIGURE 2.2. Two dice

All possible sums are in the table:

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Some sums appear more frequently than others. It would be ill-advised to take $\Omega = \{2, 3, \ldots, 11, 12\}$ with uniform probability measure.

2.11 The order is irrelevant, so the model should be based on selecting 6-element subsets.

2.12 It is best to consider the opposite first: the probability that all 25 people have their birthdays on different days. Disregard leap years for simplicity.

2.13 Buying tickets is like selecting a subset. The same can be said about winning and losing tickets.

2.14 See Problem 2.13, in particular the general formula given in the solution. Here, a bulb is a ticket, a defective bulb is a prize, and buying is the same as drawing at random.

2.15 See Problem 2.13.

2.16 The situation is similar to that in Problem 2.15. It is best to begin with computing the probability that the partner has 0, 1, or 2 spades.

2.17 Drawing 5 cards means choosing a 5-element subset. There are 6 possible triplets: 3 Aces, Kings, Queens, Jacks, Tens, or Nines.

2.18 You draw 1 or 2 out of the remaining 19 cards, so the sample space consists of 1- or 2-element subsets. You have to count the "favorable" outcomes carefully.

2.19 See Problem 2.18.

2.20 The sample space consists of all functions from the set $\{1, 2, 3\}$ to $\{1, 2, \ldots, 100\}$. Count the number of "favorable" ones.

2.21 Since the order is relevant, we consider functions (sequences) rather than sets.

2.22 If all the letters had been different, we would have a 1 in 11! chance. But in our case, there are multiple instances of some letters and certain permutations lead to the same word.

2.3 Solutions

2.1 The answer is 2^n. We shall prove this by induction. We can assume without loss of generality that the n-element set is of the form $\{1, \ldots, n\}$.

The case $n = 1$ was discussed in the hint.

Induction hypothesis: Suppose that for some $n = k$, the k-element set $\{1, 2, \ldots, k\}$ has 2^k subsets. Consider the $(k+1)$-element set $\{1, 2, \ldots, k, k+1\}$. Each subset A of $\{1, 2, \ldots, k, k+1\}$ is either contained in $\{1, 2, \ldots, k\}$ or not depending on whether $k+1$ belongs to A or not. The number of subsets A contained in $\{1, 2, \ldots, k\}$ is 2^k by the induction hypothesis. The number of subsets A contained in $\{1, 2, \ldots, k, k+1\}$ but not in $\{1, 2, \ldots, k\}$ is also 2^k because the mapping $A \mapsto A \cup \{k+1\}$ defines a one-to-one correspondence between these two kinds of subsets. This means that the total number of subsets of the $(k+1)$-element set $\{1, 2, \ldots, k, k+1\}$ is $2^k + 2^k = 2^{k+1}$, completing the proof.

2.2 The answer is $n! = 1 \times 2 \times \cdots \times n$.

To see this, let us count all possible permutations of $\{1, 2, \ldots, n\}$ by putting its elements into numbered cells. Number 1 can be placed in n cells, which leaves $n - 1$ free cells in which to put the next number. This means that there are $n \times (n-1)$ ways in which 1 and 2 can be placed in the cells. Next, we have $n - 2$ free cells in which to put 3, which gives $n \times (n-1) \times (n-2)$ ways of placing 1, 2, and 3. After placing all n numbers in this manner, we see that this can be done in $n \times (n-1) \times \cdots \times 2 \times 1 = n!$ ways, as claimed.

2.3 The answer is

$$\frac{n \times (n-1) \times \cdots \times (n-k+1)}{1 \times 2 \times 3 \times \cdots \times k}.$$

To select a k-element subset, choose the elements one-by-one. The first can be chosen in n ways, the second in $n-1$ ways, and so on. The number of all possibilities is the product in the numerator. This method of counting distinguishes between k-element sequences ordered in a different way. To obtain the number of k-element subsets (where the order of elements does not matter), we have to divide by the number of permutations of a k-element set, i.e., by $k! = 1 \times 2 \times 3 \times \cdots \times k$.

2.4 The answer is $m \times (m-1) \times (m-2) \times \cdots \times (m-n+1)$. Without loss of generality, we can assume that the domain of our mappings is the set $\{1, 2, 3, \ldots, n\}$ and the range is $\{1, 2, 3, \ldots, m\}$. The value $f(1)$ can be selected in m ways, $f(2)$ in $m-1$ ways, and so on. After n steps, we have the total number of possibilities as claimed.

2.5 The answer is m^n. Without loss of generality, we can assume that the domain of our mappings is the set $\{1, 2, 3, \ldots, n\}$ and the range is $\{1, 2, 3, \ldots, m\}$. The value $f(1)$ can be selected in m ways, $f(2)$ also in m ways (it is possible that $f(1) = f(2)$), and so on. After n steps, we get the result.

2.6 The left-hand side of the equality to be verified is the number of all subsets of an n-element set. On the right, we have the sum of the numbers of 0-, 1-, 2-, $\ldots$, n-element sets. Together, they exhaust all possible subsets.

2.7 Consider an $(m+n)$-element set A. The number of k-element subsets in A is $\binom{m+n}{k}$; see Problem 2.3. Now, take an m-element set B and an n-element set C such that $A = B \cup C$ and consider those k-element subsets in A that have i elements in B and $k-i$ elements in C. There are $\binom{m}{i}\binom{n}{k-i}$ such subsets, again by Problem 2.3. The sum from $i = 0$ to k gives the number of all k-element subsets in A, which proves Van der Monde's formula.

2.8 We give 3 solutions depending on the model for seating 10 persons randomly at a round table.

1. Having numbered the seats from 1 to 10, we shuffle and deal a deck of 10 cards with these numbers. Here, Ω is the set of all permutations of a 10-element set, so $\#\Omega = 10!$. A permutation is "favorable" when the couple sit next to each other, i.e., draw $(1, 2)$ or $(2, 3)$ or $\ldots$ or $(9, 10)$ or $(10, 1)$ (the table is round), or the inverted pairs $(2, 1)$ or $(3, 2)$ or $\ldots$ or $(1, 10)$. This gives 20 possibilities. The

remaining 8 people can be seated in any order, so we multiply by 8!. The required probability is

$$\frac{20 \times 8!}{10!} = \frac{2}{9}.$$

2. We consider the cards drawn by the couple as a set (the order does not matter) and we are not bothered with the rest of the people. So, Ω is the set of all 2-element subsets of a 10-element set with $\#\Omega = \binom{10}{2}$. There are 10 "favorable" sets: $\{1,2\}, \{2,3\}, \ldots, \{10,1\}$, so the probability is

$$\frac{10}{\binom{10}{2}} = \frac{10}{\frac{10\times 9}{1\times 2}} = \frac{2}{9}.$$

3. The neatest solution which comes to mind is this. One partner draws a card. There are 9 left with 2 "favorable" ones, so the probability that the couple will sit together is $\frac{2}{9}$, as before.

2.9 There are eight possibilities:

BBB, BBG, BGB, BGG, GGG, GGB, GBG, GBB,

out of which three are "favorable." The probability is $\frac{3}{8}$.

2.10 Counting the "favorable" outcomes in Figure 2.2, we easily get the following answers:

a) $\frac{6}{36} = \frac{1}{6}$, c) $\frac{26}{36} = \frac{13}{18}$,
b) $\frac{2}{36} = \frac{1}{18}$, d) $\frac{18}{36} = \frac{1}{2}$.

2.11 Ω consists of all 6-element subsets of the set $\{1, 2, \ldots, 49\}$. Thus, $\#\Omega = \binom{49}{6}$. In cases a) and b), we have one "favorable" outcome, so the probability is

$$\frac{1}{\binom{49}{6}} = \frac{1}{\frac{49\times 48\times 47\times 46\times 45\times 44}{1\times 2\times 3\times 4\times 5\times 6}} = \frac{1}{\frac{10068347520}{720}} = \frac{1}{13983816}.$$

In case c), the number of selections containing a specific number, 44 in this case, is equal to the number of ways the remaining 5 numbers can be chosen out of 48, which is $\binom{48}{5}$. The probability is equal to

$$\frac{\binom{48}{5}}{\binom{49}{6}} = \frac{\frac{48\times 47\times 46\times 45\times 44}{1\times 2\times 3\times 4\times 5}}{\frac{49\times 48\times 47\times 46\times 45\times 44}{1\times 2\times 3\times 4\times 5\times 6}} = \frac{6}{49}.$$

2.12 The desired probability is equal to $p = 1 - q$, where q is the probability that all 25 people have birthdays on different days.

The sample space is the set of all functions from $\{1, 2, \ldots, 25\}$ to $\{1, 2, \ldots, 365\}$ with $\#\Omega = 365^{25}$ (we disregard leap years for simplicity). The number of assignments with different birthdays is the same as the number of one-to-one mappings: $365 \times 364 \times \cdots \times 341$. Dividing these two, we have

$$q = \frac{365 \times 364 \times \cdots \times 341}{365^{25}} = \frac{365}{365} \times \frac{364}{365} \times \cdots \times \frac{341}{365} \approx 0.40,$$

so $p \approx 0.60$, which is surprisingly large.

2.13 The sample space consists of all 6-element subsets of a 60-element set, so $\#\Omega = \binom{60}{6}$. Our 2 winning tickets are 2-element subsets of the set of 20. To each such a selection, there correspond $\binom{40}{4}$ ways in which the remaining, losing, tickets can be selected. It follows that

$$p = \frac{\binom{20}{2} \times \binom{40}{4}}{\binom{60}{6}} \approx 0.35.$$

The general formula has the form

$$p = \frac{\binom{w}{g} \times \binom{t-w}{b-g}}{\binom{t}{b}}.$$

2.16 The sample space consists of all 13-element subsets of a 31-element set ($52 - 13 - 8$, excluding our hand and the hearts shown), so $\#\Omega = \binom{31}{13}$. The number of hands with 0 spades is $\binom{22}{13}$ (since there are 9 spades among the 31 cards), with 1 spade $9 \times \binom{22}{12}$, and with 2 spades $\binom{9}{2} \times \binom{22}{11}$. So the answer is

$$p = 1 - \frac{\binom{22}{13} + 9 \times \binom{22}{12} + \binom{9}{2} \times \binom{22}{11}}{\binom{31}{13}} \approx 0.85.$$

2.17 The sample space consists of all 5-element subsets of the 24-element set of cards, so $\#\Omega = \binom{24}{5}$. Three aces can be selected in $\binom{4}{3}$ ways and the remaining 2 cards in $\binom{20}{2}$ ways. Multiply this by 6 to get

$$p = \frac{6 \times \binom{4}{3} \times \binom{20}{2}}{\binom{25}{5}} \approx 0.086.$$

2.18 a) We have $\#\Omega = \binom{19}{2} = 171$ and there are 18 "favorable" outcomes: the remaining Ace with any of the other 18 cards. So $p = \frac{18}{171} \approx 0.1052$.

b) We have $\#\Omega = \binom{19}{1} = 19$ and there are 4 "favorable" outcomes: three Kings and the remaining Ace, so $p = \frac{4}{19} \approx 0.2105$.

2.20 Taking Ω to be the set of all functions from $\{1, 2, 3\}$ to $\{1, 2, \ldots, 100\}$, we have $\#\Omega = 100^3$. There are 20 consonants and 6 vowels. The number of "favorable" outcomes is $20 \times 6 \times 6$ and the probability is

$$p = \frac{20 \times 6 \times 6}{100^3} = 0.00072.$$

2.21 Consider 5-element sequences of cards, i.e., one-to-one mappings from $\{1, 2, 3, 4, 5\}$ to $\{1, 2, \ldots, 52\}$ with $\#\Omega = 52 \times 51 \times 50 \times 49 \times 48$. To represent the event in question, we count all sequences of 4 cards out of 48 and multiply by 4, as there are 4 aces in the deck. Hence,

$$p = \frac{48 \times 47 \times 46 \times 45 \times 4}{52 \times 51 \times 50 \times 49 \times 48} \approx 0.05989.$$

2.22 Suppose that we label each letter with a number from 1 to 11. The sample space is the set of permutations of 11 numbers, so $\#\Omega = 11!$. We count the permutations giving the required word: The letters A can be rearranged in $5!$ ways, and B and R in $2!$ ways each, so we obtain $5! \times 2! \times 2!$ "favorable" outcomes and

$$p = \frac{5! \times 2! \times 2!}{11!} = \frac{1}{83160}\,.$$

Probability Through Problems
Capinski, M.; Zastawniak, T.J.
2001, VIII, 260 p., Hardcover
ISBN: 978-0-387-95063-1