Chapter 1
Elliptic problems

An effective method for solving boundary value problems for the Laplace and Helmoltz equations (in domains possessing a definite symmetry) is the *method of separation of variables*. The general idea of this method is to find a set of solutions of the homogeneous partial differential equation in question that satisfy certain boundary conditions. These solutions then serve as "atoms", from which, based on the linear superposition principle, one constructs the "general" solution. Since each of these "atoms" is a solution of the corresponding homogeneous equation, their linear combination is also a solution of the same equation. The solution of our problem is given by a series $\sum_{n=1}^{\infty} c_n u_n(x)$ (where $u_n(x)$ are the atom solutions, $x = (x_1, \dots, x_N)$ is the current point of the domain of space under consideration, and c_n are arbitrary constants). It remains to find constants c_n such that the boundary conditions are satisfied.

1.1. The Dirichlet problem for the Laplace equation in an annulus

Suppose that we are required to solve the Dirichlet problem for the Laplace equation $\Delta u = 0$ in the domain bounded by two concentric circles L_1 and L_2 centered at the origin, of radii R_1 and R_2:

$$\begin{cases} u_{xx} + u_{yy} = 0, \qquad R_1^2 < x^2 + y^2 < R_2^2, \\ u|_{L_1} = f_1, \qquad u|_{L_2} = f_2. \end{cases}$$

Introducing polar coordinates (ρ, φ), this Dirichlet problem can be recast as

$$\begin{cases} \rho^2 u_{\rho\rho} + \rho u_\rho + u_{\varphi\varphi} = 0, \qquad R_1 < \rho < R_2, \qquad 0 \le \varphi < 2\pi, \\ u(R_1, \varphi) = f_1(\varphi), \\ u(R_2, \varphi) = f_2(\varphi), \end{cases} \quad 0 \le \varphi < 2\pi. \tag{1.1}$$

The boundary functions $f_1(\varphi)$ and $f_2(\varphi)$ will be assumed to be 2π-periodic.

To solve the problem we will apply Fourier's method. Namely, we will seek the solution in the form $u(\rho, \varphi) = R(\rho)\Phi(\varphi)$. Substituting this expression in equation (1.1), we obtain

$$\Phi\rho^2 R'' + \Phi\rho R' + R\Phi'' = 0.$$

Next, dividing both sides of this equation by $R\Phi$ we get

$$\frac{\rho^2 R'' + \rho R'}{R} = -\frac{\Phi''}{\Phi}\,. \tag{1.2}$$

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One says that in equation (1.2) the *variables are separated*, since the left- [resp., right-] hand side of the equation depends only on ρ [resp., φ]. Since the variables ρ and φ do not depend of one another, each of the two sides of equation (1.2) must be a constant. Let us denote this constant by λ. Then

$$\frac{\rho^2 R'' + \rho R'}{R} = -\frac{\Phi''}{\Phi} = \lambda. \tag{1.3}$$

It is clear that when the angle φ varies by 2π the single-valued function $u(\rho,\varphi)$ must return to the initial value, i.e., $u(\rho,\varphi) = u(\rho,\varphi+2\pi)$. Consequently, $R(\rho)\Phi(\varphi) = R(\rho)\Phi(\varphi+2\pi)$, whence $\Phi(\varphi) = \Phi(\varphi+2\pi)$, i.e., the function $\Phi(\varphi)$ is 2π-periodic. From the equation $\Phi'' + \lambda\Phi = 0$ it follows that $\Phi(\varphi) = A\cos(\sqrt{\lambda}\varphi) + B\sin(\sqrt{\lambda}\varphi)$ (with A and B arbitrary constants), and in view of the periodicity of $\Phi(\varphi)$ we necessarily have $\lambda = n^2$, where $n \geq 0$ is an integer.

Indeed, the equality

$$A\cos(\sqrt{\lambda}\varphi) + B\sin(\sqrt{\lambda}\varphi) = A\cos[\sqrt{\lambda}(\varphi+2\pi)] + B\sin[\sqrt{\lambda}(\varphi+2\pi)]$$

implies that

$$\sin(\alpha + \sqrt{\lambda}\varphi) = \sin(\alpha + \sqrt{\lambda}\varphi + 2\pi\sqrt{\lambda}),$$

where we denote

$$\sin\alpha = \frac{A}{\sqrt{A^2+B^2}}, \qquad \cos\alpha = \frac{B}{\sqrt{A^2+B^2}}.$$

Therefore, $\sin(\pi\sqrt{\lambda})\cos(\alpha + \sqrt{\lambda}\varphi + \pi\sqrt{\lambda}) = 0$, i.e., $\pi\sqrt{\lambda} = \pi n$, or $\lambda = n^2$, where $n \geq 0$ is an integer. Now equation (1.3) yields

$$\rho^2 R'' + \rho R' - n^2 R = 0. \tag{1.4}$$

If $n \neq 0$, then we seek the solution of this equation in the form $R(\rho) = \rho^\mu$. Substituting this expression in equation (1.4) and simplifying by ρ^μ, we get

$$\mu^2 = n^2, \quad \text{or} \quad \mu = \pm n \qquad (n > 0).$$

For $n = 0$ equation (1.4) has two solutions: 1 and $\ln\rho$. Thus, we now have an infinite set of functions ("atom" solutions)

$$\begin{gathered} 1, \qquad \ln\rho, \qquad \rho^n\cos(n\varphi), \qquad \rho^n\sin(n\varphi), \\ \rho^{-n}\cos(n\varphi), \qquad \rho^{-n}\sin(n\varphi), \qquad n = 1, 2\dots, \end{gathered}$$

which satisfy the given partial differential equation. Since a sum of such solutions is also a solution, we conclude that in our case the "general" solution of the Laplace equation has the form

$$\begin{aligned} u(\rho,\varphi) &= a_0 + b_0\ln\rho + \\ &+ \sum_{n=1}^{\infty}\left[\left(a_n\rho^n + b_n\rho^{-n}\right)\cos(n\varphi) + \left(c_n\rho^n + d_n\rho^{-n}\right)\sin(n\varphi)\right]. \end{aligned} \tag{1.5}$$

It remains only to find all the coefficients in the sum (1.5) so that the boundary conditions $u(R,\varphi) = f_1(\varphi)$, $u(R_2,\varphi) = f_2(\varphi)$ will be satisfied. Setting $\rho = R_1$ and then $\rho = R_2$ in (1.5) we obtain

$$u(R_1,\varphi) = \sum_{n=1}^{\infty} \left[\left(a_n R_1^n + b_n R_1^{-n} \right) \cos(n\varphi) + \right.$$
$$\left. + \left(c_n R_1^n + d_n R_1^{-n} \right) \sin(n\varphi) \right] + a_0 + b_0 \ln R_1,$$

$$u(R_2,\varphi) = \sum_{n=1}^{\infty} \left[\left(a_n R_2^n + b_n R_2^{-n} \right) \cos(n\varphi) + \right.$$
$$\left. + \left(c_n R_2^n + d_n R_2^{-n} \right) \sin(n\varphi) \right] + a_0 + b_0 \ln R_2.$$

Recalling the expressions for the Fourier coefficients of a trigonometric series, we arrive at the following systems of equations:

$$\begin{cases} a_0 + b_0 \ln R_1 = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f_1(s)\, ds, \\ a_0 + b_0 \ln R_2 = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f_2(s)\, ds, \end{cases} \tag{1.6$_1$}$$

(to be solved for a_0 and b_0);

$$\begin{cases} a_n R_1^n + b_n R_1^{-n} = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} f_1(s) \cos(ns)\, ds, \\ a_n R_2^n + b_n R_2^{-n} = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} f_2(s) \cos(ns)\, ds, \end{cases} \tag{1.6$_2$}$$

(to be solved for a_n and b_n); and

$$\begin{cases} c_n R_1^n + d_n R_1^{-n} = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} f_1(s) \sin(ns)\, ds, \\ c_n R_2^n + b d_n R_2^{-n} = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} f_2(s) \sin(ns)\, ds, \end{cases} \tag{1.6$_3$}$$

(to be solved for c_n and d_n).

Thus, from these systems one can find all the unknown coefficients a_0, b_0, a_n, b_n, c_n, d_n. Now the problem (1.1) is completely solved. The solution is given by the expression (1.5), in which the coefficients are obtained from the systems (1.6).

1.2. Examples of Dirichlet problem in an annulus

Example 1. Let us assume that the potential is equal to zero on the inner circle, and is equal to $\cos\varphi$ on the outer circle. Find the potential in the annulus.

We have to solve the problem

$$\begin{cases} \Delta u = 0, \qquad 1 < \rho < 2, \quad 0 \le \varphi < 2\pi, \\ u(1,\varphi) = 0, \qquad u(2,\varphi) = \cos\varphi, \qquad 0 \le \varphi \le 2\pi, \end{cases}$$

in order to find determine the potential $u(\rho,\varphi)$ in the annulus.

Generally speaking, to solve this problem we have to calculate all the integrals in the formulas (1.6), and then solve the corresponding systems of equations to find the coefficients a_0, b_0, a_n, b_n, c_n, d_n. However, in the present case it is simpler to try to choose particular solutions such that a linear combination of them will satisfy the boundary conditions. Here such a role is played by the linear combination $u(\rho,\varphi) = a_1\rho\cos\varphi + b_1\rho^{-1}\cos\varphi$. The boundary conditions yield the system of equations

$$\begin{cases} a_1 + b_1 = 0, \\ 2a_1 + \dfrac{b_1}{2} = 1, \end{cases}$$

from which we find $a_1 = 2/3$, $b_1 = -2/3$. Therefore, the solution is

$$u(\rho,\theta) = \frac{2}{3}\left(\rho - \rho^{-1}\right)\cos\varphi.$$

Example 2. Let us consider the following problem with constant potentials on the boundaries of the annulus:

$$\begin{cases} \Delta u = 0, \qquad 1 < \rho < 2, \quad 0 \le \varphi < 2\pi, \\ u(1,\varphi) = 2, \qquad u(2,\varphi) = 1, \qquad 0 \le \varphi < 2\pi. \end{cases}$$

In this case we will seek the solution as a function that does not depend on φ, i.e., $u(\rho) = a_0 + b_0\ln\rho$. Substituting this expression in the boundary conditions we obtain the system of equations

$$\begin{cases} a_0 + b_0\ln 1 = 2, \\ a_0 + b_0\ln 2 = 1, \end{cases}$$

which yields $a_0 = 2$, $b_0 = -\log_2 e$. Therefore, the sought solution is the function

$$u(\rho) = 2 - \frac{\ln\rho}{\ln 2}.$$

Example 3. Let us solve the following Dirichlet problem

$$\begin{cases} \Delta u = 0, \qquad 1 < \rho < 2, \quad 0 \le \varphi < 2\pi, \\ u(1,\varphi) = \cos\varphi, \qquad u(2,\varphi) = \sin\varphi, \qquad 0 \le \varphi \le 2\pi. \end{cases}$$

One can verify that here all the coefficients a_0, b_0, a_n, b_n, c_n, d_n with $n > 1$ are equal to zero, while the coefficients a_1, b_1, c_1, d_1 are determined from the systems of equations

$$\begin{cases} a_1 + b_1 = 1, \\ 2a_1 + \dfrac{b_1}{2} = 0, \end{cases} \qquad \begin{cases} c_1 + d_1 = 0, \\ 2c_1 + \dfrac{d_1}{2} = 1. \end{cases}$$

Solving these systems we obtain

$$a_1 = -\frac{1}{3}, \qquad b_1 = \frac{4}{3}, \qquad c_1 = \frac{2}{3}, \qquad d_1 = -\frac{2}{3}.$$

Thus, the solution of our problem is the function

$$u(\rho, \varphi) = \left(-\frac{1}{3}\rho + \frac{4}{3\rho} \right) \cos\varphi + \frac{2}{3}\left(\rho - \frac{1}{\rho} \right) \sin\varphi.$$

Since the Dirichlet problem for the Laplace equation in a bounded domain has a unique solution, in examples 1–3 there are no other solutions besides the ones found.

1.3. The interior and exterior Dirichlet problems

Let us consider the two very important cases in which the annulus becomes a disc or the exterior of a disc. The *interior* Dirichlet problem ($R_1 = 0$, $R_2 = R$)

$$\begin{cases} \rho^2 u_{\rho\rho} + \rho u_\rho + u_{\varphi\varphi} = 0, \qquad 0 \le \rho < R, \quad 0 \le \varphi < 2\pi, \\ u(R, \varphi) = f(\varphi), \qquad 0 \le \varphi \le 2\pi, \end{cases}$$

is solved in exactly the same manner as the Dirichlet problem for the annulus, with the only difference that now we must discard the solution "atoms" that are not bounded when ρ approaches 0:

$$\ln\rho, \qquad \rho^{-n}\cos(n\varphi), \qquad \rho^{-n}\sin(n\varphi), \qquad n = 1, 2, \ldots$$

Hence, the solution is given by the remaining terms, i.e.,

$$u(\rho, \varphi) = \sum_{n=0}^{\infty} \left(\frac{\rho}{R} \right)^n \left[a_n \cos(n\varphi) + b_n \sin(n\varphi) \right],$$

where the coefficients a_n and b_n are calculated by means of the formulas

$$\left. \begin{aligned} a_0 &= \frac{1}{2\pi} \int_0^{2\pi} f(\varphi)\, d\varphi, \\ a_n &= \frac{1}{\pi} \int_0^{2\pi} f(\varphi) \cos(n\varphi)\, d\varphi, \qquad n > 0, \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} f(\varphi) \sin(n\varphi)\, d\varphi, \qquad n > 0. \end{aligned} \right\} \tag{1.7}$$

In other words, we simply expand the function $f(\varphi)$ in a Fourier series

$$f(\varphi)=\sum_{n=0}^{\infty}\left[a_n\cos(n\varphi)+b_n\sin(n\varphi)\right],$$

and then multiply each term of the series by the factor $\left(\frac{\rho}{R}\right)^n$. For example, the interior problem

$$\begin{cases}\Delta u=0, \qquad 0\le\rho<1, \quad 0\le\varphi<2\pi,\\ u(1,\varphi)=\cos^2\varphi, \qquad 0\le\varphi<2\pi,\end{cases}$$

has the solution

$$u(\rho,\varphi)=\frac{1}{2}+\frac{1}{2}\rho^2\cos(2\varphi).$$

The *exterior* Dirichlet problem ($R_1=R$, $R_2=\infty$)

$$\begin{cases}\rho^2u_{\rho\rho}+\rho u_\rho+u_{\varphi\varphi}=0, \qquad R\le\rho<\infty, \quad 0\le\varphi<2\pi,\\ u(R,\varphi)=f(\varphi), \qquad 0\le\varphi\le2\pi,\end{cases}$$

is solved in much the same way as the preceding problem, with the difference than now we discard the solution “atoms” that are not bounded when ρ goes to infinity:

$$\ln\rho, \qquad \rho^n\cos(n\varphi), \qquad \rho^n\sin(n\varphi), \qquad n=1,2,\ldots$$

Accordingly, the solution is taken in the form

$$u(\rho,\varphi)=\sum_{n=0}^{\infty}\left(\frac{\rho}{R}\right)^{-n}\left[a_n\cos(n\varphi)+b_n\sin(n\varphi)\right],$$

where the coefficients a_n and b_n are calculated by means of formulas (1.7). For example, the exterior problem

$$\begin{cases}\Delta u=0, \qquad 1\le\rho<\infty, \quad 0\le\varphi<2\pi,\\ u(1,\varphi)=\sin^3\varphi, \qquad 0\le\varphi\le2\pi\end{cases}$$

has the solution

$$u(\rho,\varphi)=\frac{3}{4}\cdot\frac{1}{\rho}\sin\varphi-\frac{1}{4}\cdot\frac{1}{\rho^3}\sin3\varphi.$$

Let us note that the Dirichlet problem for the Laplace equations in an unbounded two-dimensional domain has only one bounded solution.

We conclude this section by examining another example, one exercise (the Poisson integral), and a problem connected with the Poisson integral.

Example [1]. Find the steady temperature distribution in a homogeneous sector $0 \le \rho \le a$, $0 \le \varphi \le \alpha$, which satisfies the boundary conditions $u(\rho, 0) = u(\rho, \alpha) = 0$, $u(a, \varphi) = A\varphi$, where A is a constant (see Figure 1.1).

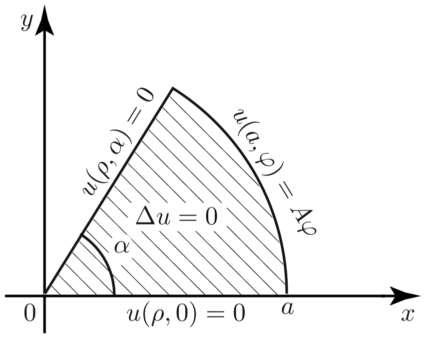

FIGURE 1.1.

Solution. Finding the steady temperature distribution reduces to solving the Dirichlet problem

$$\begin{cases} \rho^2 u_{\rho\rho} + \rho u_\rho + u_{\varphi\varphi} = 0, & 0 \le \rho < a, \quad 0 < \varphi < \alpha < 2\pi, \\ u(\rho, 0) = u(\rho, \alpha) = 0, & 0 \le \rho \le a, \\ u(a, \varphi) = A\varphi, & 0 \le \varphi \le \alpha. \end{cases}$$

Setting $u(\rho, \varphi) = R(\rho)\Phi(\varphi)$ and separating variables, we obtain two ordinary differential equations:

$$\begin{aligned} \rho^2 R'' + \rho R' - \lambda R = 0, \\ \Phi'' + \lambda \Phi = 0. \end{aligned} \tag{1.8}$$

The conditions $0 = u(\rho, 0) = R(\rho)\Phi(0)$ and $0 = u(\rho, \alpha) = R(\rho)\Phi(\alpha)$ yield $\Phi(0) = \Phi(\alpha) = 0$. The separation constant λ is determined by solving the Sturm-Liouville

$$\begin{cases} \Phi'' + \lambda \Phi = 0, & 0 < \varphi < \alpha, \\ \Phi(0) = \Phi(\alpha) = 0. \end{cases}$$

We get $\lambda_n = \left(\frac{n\pi}{\alpha}\right)^2$ and

$$\mu(\mu - 1) + \mu - \left(\frac{n\pi}{\alpha}\right)^2 = 0,$$

whence

$$\mu = \pm \frac{n\pi}{\alpha}\,.$$

Using the fact that the function $R(\rho)$ is bounded (according to the meaning of the problem at hand), we write $R_n(\rho) = \rho^{n\pi/\alpha}$. The atoms from which our solution is built are the functions

$$u_n(\rho, \varphi) = \rho^{n\pi/\alpha} \sin\left(\frac{n\pi}{\alpha}\varphi\right), \qquad n = 1, 2, \dots$$

Thus, the solution itself is

$$u(\rho, \varphi) = \sum_{n=1}^{\infty} c_n \rho^{n\pi/\alpha} \sin\left(\frac{n\pi}{\alpha}\varphi\right).$$

The constants c_n $(n = 1, 2, \dots)$ are found from the condition $u(a, \varphi) = A\varphi$. Since

$$u(a, \varphi) = \sum_{n=1}^{\infty} c_n a^{n\pi/\alpha} \sin\left(\frac{n\pi}{\alpha}\varphi\right).$$

it follows that

$$c_n a^{n\pi/\alpha} = \frac{2}{\alpha}\int_0^{\alpha} A\varphi \sin\left(\frac{n\pi}{\alpha}\varphi\right) d\varphi,$$

and so

$$c_n = \frac{2A}{\alpha a^{n\pi/\alpha}} \int_0^{\alpha} \varphi \sin\left(\frac{n\pi}{\alpha}\varphi\right) d\varphi = (-1)^{n+1}\frac{2\alpha A}{n\pi}.$$

Finally, the solution of our problem is written in the form

$$u(\rho, \varphi) = \frac{2\alpha A}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{\rho}{\alpha}\right)^{n\pi/\alpha} \frac{\sin\left(\frac{n\pi}{\alpha}\right)\varphi}{n}.$$

Notice that the solution has a singularity in the boundary point $\rho = a$, $\varphi = \alpha$ because of the incompatibility of the boundary values.

1.4. The Poisson integral for the disc. Complex form. Solution of the Dirichlet problem when the boundary condition is a rational function $R(\sin\varphi, \cos\varphi)$

Recall that the solution of the interior and exterior Dirichlet problem is can be presented in integral form (the Poisson integral):

$$u(\rho, \varphi) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - \rho^2}{R^2 - 2\rho R\cos(\varphi - \alpha) + \rho^2} f(\alpha)\, d\alpha, \quad \rho < R,$$

$$u(\rho, \varphi) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - R^2}{R^2 - 2\rho R\cos(\varphi - \alpha) + \rho^2} f(\alpha)\, d\alpha, \quad \rho > R.$$

Let us show that these formulas are a consequence of the general superposition method.

For the sake of definiteness we shall consider the interior problem, and then write the result for the exterior problem by analogy.

Substituting the expression for the Fourier coefficients in the formula

$$u(\rho,\varphi) = \sum_{n=0}^{\infty} \left(\frac{\rho}{R}\right)^n \left[a_n \cos(n\varphi) + b_n \sin(n\varphi)\right],$$

we obtain

$$u(\rho,\varphi) = \frac{1}{\pi}\int_0^{2\pi} f(\alpha)\left[\frac{1}{2} + \sum_{n=0}^{\infty}\left(\frac{\rho}{R}\right)^n (\cos(n\varphi)\cos(n\alpha) + \sin(n\varphi)\sin(n\alpha))\right] =$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(\alpha)\left[\frac{1}{2} + \sum_{n=0}^{\infty}\left(\frac{\rho}{R}\right)^n \cos(n(\varphi-\alpha))\right] d\alpha.$$

Further, using the relation $\cos(n(\varphi-\alpha)) = \frac{1}{2}\left(e^{in(\varphi-\alpha)} + e^{-in(\varphi-\alpha)}\right)$, the fact that $q = \rho/R < 1$ and the formula for the sum of an infinite decreasing geometric progresion, we get

$$\frac{1}{2} + \sum_{n=1}^{\infty} q^n \cos(n(\varphi-\alpha)) = \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty} q^n \left[e^{in(\varphi-\alpha)} + e^{-in(\varphi-\alpha)}\right] =$$

$$= \frac{1}{2}\left[1 + \sum_{n=1}^{\infty}\left[\left(qe^{in(\varphi-\alpha)}\right)^n + \left(qe^{-in(\varphi-\alpha)}\right)^n\right]\right] =$$

$$= \frac{1}{2}\left[1 + \frac{qe^{in(\varphi-\alpha)}}{1-qe^{in(\varphi-\alpha)}} + \frac{qe^{-in(\varphi-\alpha)}}{1-qe^{-in(\varphi-\alpha)}}\right] =$$

$$= \frac{1}{2}\cdot\frac{1-q^2}{1-2q\cos(\varphi-\alpha)+q^2} = \frac{1}{2}\cdot\frac{R^2-\rho^2}{R^2-2R\rho\cos(\varphi-\alpha)+\rho^2}.$$

Therefore,

$$u(\rho,\varphi) = \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-\rho^2}{R^2-2R\rho\cos(\varphi-\alpha)+\rho^2} f(\alpha)\,d\alpha, \qquad \rho < R.$$

Let us recast the Poisson formula in a different form (complex notation). Note that

$$\frac{R^2-\rho^2}{R^2-2R\rho\cos(\varphi-\alpha)+\rho^2} = \frac{R^2-|z|^2}{|Re^{i\alpha}-z|^2} = \mathrm{Re}\frac{Re^{i\alpha}+z}{Re^{i\alpha}-z}$$

because

$$\operatorname{Re}\frac{Re^{i\alpha}+z}{Re^{i\alpha}-z} = \operatorname{Re}\frac{\left(Re^{i\alpha}+\rho e^{i\varphi}\right)\left(\overline{Re^{i\alpha}}-\overline{\rho e^{i\varphi}}\right)}{\left(Re^{i\alpha}-\rho e^{i\varphi}\right)\left(\overline{Re^{i\alpha}}-\overline{\rho e^{i\varphi}}\right)}$$
$$= \operatorname{Re}\frac{R^2-|z|^2+\rho R\left[e^{i(\varphi-\alpha)}-e^{i(\varphi-\alpha)}\right]}{|Re^{i\alpha}-z|^2} = \frac{R^2-|z|^2}{|Re^{i\alpha}-z|^2}.$$

It follows that the Poisson integral can be written in the form

$$u(z) = \operatorname{Re}\frac{1}{2\pi}\int_0^{2\pi}\frac{Re^{i\alpha}+z}{Re^{i\alpha}-z}\,f(\alpha)\,d\alpha.$$

If in this integral we set $\zeta = Re^{i\alpha}$ and, accordingly, $d\alpha = d\zeta/i\zeta$, we finally obtain

$$u(z) = \operatorname{Re}\frac{1}{2\pi i}\int_0^{2\pi}\frac{\zeta+z}{\zeta-z}\,f(\zeta)\,\frac{d\zeta}{\zeta}, \qquad |z|<R. \tag{1.9}$$

If the boundary function $f(\zeta)$ is a rational function of $\sin\varphi$ and $\cos\varphi$, then the integral in formula (1.9) can be calculated by means of residues.

Example. Solve the Dirichlet problem

$$\begin{cases} \Delta u = 0, & |z|<2, \\ u\big|_{|z|=2} = \dfrac{2\sin\varphi}{5+3\cos\varphi}. \end{cases}$$

Solution. We shall use formula (1.9). Let $\zeta = 2e^{i\alpha}$; then

$$\sin\alpha = \frac{1}{2i}\left(\frac{\zeta}{2}-\frac{2}{\zeta}\right) \qquad \cos\alpha = \frac{1}{2}\left(\frac{\zeta}{2}+\frac{2}{\zeta}\right)$$

and the boundary function becomes

$$u(\zeta) = \frac{2\sin\alpha}{5+3\cos\alpha} = \frac{2\cdot\dfrac{1}{2i}\cdot\dfrac{\zeta^2-4}{2\zeta}}{5+\dfrac{3}{2}\left(\dfrac{\zeta}{2}+\dfrac{2}{\zeta}\right)} =$$

$$= \frac{2}{i}\cdot\frac{\zeta^2-4}{3\zeta^2+20\zeta+12} = \frac{2}{i}\cdot\frac{\zeta^2-4}{3(\zeta+6)\left(\zeta+\dfrac{2}{3}\right)}.$$

Let us compute the integral

$$J = \frac{1}{2\pi i}\int_{|\zeta|=2}\frac{2(\zeta^2-4)(\zeta+z)}{i\cdot 3(\zeta+6)(\zeta+\frac{2}{3})(\zeta-z)\zeta}\,d\zeta$$

where the circle $|\zeta| = 2$ is oriented counter- clockwise. In our case the integrand $F(\zeta)$ has in the domain $|\zeta| > 2$ only one finite singular point $\zeta = -6$ – a pole of order one – and the removable singular point $\zeta = \infty$. By the Cauchy residue theorem,

$$J = -\operatorname{res}[F(\zeta)]_{\zeta=-6} - \operatorname{res}[F(\zeta)]_{\zeta=\infty}.$$

First let us find the residue at the point $\zeta = -6$;

$$\operatorname{res}[F(\zeta)]_{\zeta=-6} = \frac{2}{3i} \cdot \frac{32}{\left(-\frac{16}{3}\right)} \cdot \frac{z-6}{(z+6)\cdot 6} = -\frac{4}{i} \cdot \frac{z-6}{(z+6)\cdot 6} = \frac{2}{3i} \cdot \frac{6-z}{6+z}.$$

Next let us expand $F(\zeta)$ in a series in the neighborhood of the point $\zeta = \infty$:

$$F(\zeta) = \frac{2}{3i} \cdot \frac{\left(1 - \dfrac{4}{\zeta^2}\right)\left(1 + \dfrac{z}{\zeta}\right)}{\left(1 + \dfrac{6}{\zeta}\right)\left(1 + \dfrac{2}{3\zeta}\right)} \cdot \frac{1}{1 - \dfrac{z}{\zeta}} \cdot \frac{1}{\zeta} = \frac{2}{3i} \cdot \frac{1}{\zeta} + \dots ,$$

whence

$$\operatorname{res}[F(\zeta)]_{\zeta=\infty} = -\frac{2}{3i}.$$

Therefore,

$$J = \frac{2}{3i} \cdot \frac{z-6}{z+6} + \frac{2}{3i} = \frac{2}{3i} \cdot \frac{2z}{z+6} = \frac{4z}{3i(z+6)} =$$

$$= \frac{4}{3i} \cdot \frac{x+iy}{6+x+iy} = \frac{4}{3i} \cdot \frac{(x+iy)(6+x-iy)}{(6+x)^2+y^2},$$

which yields

$$\operatorname{Re} J = \frac{8y}{36 + 12x + x^2 + y^2},$$

or

$$\operatorname{Re} J = \frac{8\rho \sin\varphi}{36 + 12\rho\cos\varphi + \rho^2}.$$

We conclude that the solution of our Dirichlet problem is given by the expression

$$u(\rho, \varphi) = \frac{8\rho \sin\varphi}{36 + 12\rho\cos\varphi + \rho^2}.$$

1.5. The interior and exterior Neumann problems for a disc

It is clear that in the case of a disc of radius R centered at the origin the exterior normal derivative is $\partial u/\partial n|_{\rho=R} = \partial u/\partial\rho|_{\rho=R}$. Accordingly, the solution of the interior Neumann problem is sought in the form of a series

$$u(\rho,\varphi) = \sum_{n=0}^{\infty} \left(\frac{\rho}{R}\right)^n [a_n \cos(n\varphi) + b_n \sin(n\varphi)] .$$

The coefficients a_n and b_n of this series are determined from the boundary condition $\partial u/\partial\rho|_{\rho=R} = f(\varphi)$, i.e., we have

$$\begin{aligned} a_n &= \frac{R}{n\pi}\int_0^{2\pi} f(\varphi)\cos(n\varphi)\,d\varphi, \\ b_n &= \frac{R}{n\pi}\int_0^{2\pi} f(\varphi)\sin(n\varphi)\,d\varphi, \end{aligned} \qquad n = 1, 2, \dots \tag{1.10}$$

Similarly, the solution of the exterior Neumann problem is sought in the form of a series

$$u(\rho,\varphi) = \sum_{n=0}^{\infty} \left(\frac{\rho}{R}\right)^{-n} [a_n \cos(n\varphi) + b_n \sin(n\varphi)] .$$

whose coefficients a_n and b_n, determined from the boundary condition $\partial u/\partial\rho|_{\rho=R}=$ $f(\varphi)$, are calculated by means of the same formulas (1.10) (here we use the fact that $\partial u/\partial n|_{\rho=R} = -\partial u/\partial\rho|_{\rho=R}$).

Example. Find the steady temperature inside of an unbounded cylinder of radius R if on the lateral surface S there is given the heat flux $\partial u/\partial n|_S = \cos^3\varphi$.

Solution. We have to solve the interior Neumann problem

$$\begin{cases} \Delta u = 0, & 0 < \rho < R, \quad 0 \le \varphi < 2\pi, \\ \left.\dfrac{\partial u}{\partial\rho}\right|_{\rho=R} = \cos^3\varphi, & 0 \le \varphi \le 2\pi. \end{cases}$$

First of all we need to verify that the condition for the solvability of the Neumann problem is satisfied, i.e., that $\int_C \frac{\partial u}{\partial n}\,ds = 0$, where C is the circle bounding our disc.

Indeed, we have

$$\int_C \frac{\partial u}{\partial n}\,ds = \int_0^{2\pi} \cos^3\varphi \cdot R\,d\varphi =$$

$$= \frac{R}{2}\int_0^{2\pi} \cos\varphi\,d\varphi + \frac{R}{4}\int_0^{2\pi} [\cos(3\varphi) + \cos\varphi]\,d\varphi = 0.$$

Next, since $\cos^3\varphi = \frac{3}{4}\cos\varphi + \frac{1}{4}\cos(3\varphi)$, it follows that $a_1 = \frac{3}{4}R$, $a_3 = \frac{1}{12}R$, and all the remaining coefficients in the series giving the solution of the interior Neumann problem are equal to zero. Hence, the solution has the form

$$u(\rho,\varphi) = C + \frac{3\rho}{4}\cos\varphi + \frac{\rho^3}{12R^2}\cos(3\varphi),$$

where C is an arbitrary constant.

Remark. The Neumann problem can also be solved for an annulus. In this case the boundary conditions specify the exterior normal derivative:

$$-\frac{\partial u}{\partial\rho}(R_1,\varphi) = f_1(\varphi), \qquad \frac{\partial u}{\partial\rho}(R_2,\varphi) = f_2(\varphi).$$

Here the solution exists only if the condition

$$R_1\int_0^{2\pi} -f_1(\varphi)\,d\varphi = R_2\int_0^{2\pi} f_2(\varphi)\,d\varphi$$

is satisfied, and is uniquely determined up to an arbitrary constant.

1.6. Boundary value problems for the Poisson equation in a disc and in an annulus

When we solve the Dirichlet or Neumann problem (or a problem of mixed type) we need first to find some particular solution u_1 of the Poisson equation $\Delta u = f(x,y)$ and then use the change of dependent variables $u = u_1 + v$ to reduce the task to that of solving the corresponding boundary value problem for the Laplace equation $\Delta v = 0$.

Example 1 [18]. Solve the Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -xy$$

in the disc of radius R centered at the origin, under the condition $u(R,\varphi) = 0$.

Solution. Passing to polar coordinates we obtain the problem

$$\begin{cases} \rho^2 u_{\rho\rho} + \rho u_\rho + u_{\varphi\varphi} = -\dfrac{1}{2}\rho^4\sin(2\varphi), \qquad 0 \le \rho < R, \quad 0 \le \varphi < 2\pi, \\ u(R,\varphi) = 0, \qquad 0 \le \varphi \le 2\pi. \end{cases} \tag{1.11}$$

We shall seek a particular solution in the form

$$u_1(\rho,\varphi) = w(\rho)\sin(2\varphi).$$

Substituting this expression in equation (1.11) and simplifying by $\sin(2\varphi)$ we obtain the equation

$$\rho^2 w'' + \rho w' - 4w = -\frac{1}{2}\,\rho^4. \tag{1.12}$$

The substitution $\rho = e^t$ transforms (1.12) into the equation with constant coefficients

$$\ddot{w} - 4w = -\frac{1}{2}\,e^{4t}, \tag{1.13}$$

where the dot denotes differrentiation with respect to t. A particular solution of equation (1.13) is $w(t) = -\frac{1}{24}\,e^{4t}$. Hence, $w(\rho) = -\frac{1}{24}\,\rho^4$ is a particular solution of equation (1.12). Therefore, we can choose $u_1(\rho,\varphi) = -\frac{1}{24}\,\rho^4\sin(2\varphi)$.

Now let us introduce the function $v(\rho,\varphi) = u(\rho,\varphi) - u_1(\rho,\varphi)$. Clearly, to determine the function $v(\rho,\varphi)$ we must solve the following Dirichlet problem for the Laplace equation:

$$\begin{cases} \rho^2 u_{\rho\rho} + \rho v_\rho + v_{\varphi\varphi} = 0, & 0 < \rho < R, \quad 0 \le \varphi < 2\pi, \\ v(R,\varphi) = \dfrac{1}{24}\,R^4\sin(2\varphi), & 0 \le \varphi \le 2\pi. \end{cases}$$

But we already know the solution of this equation:

$$v(\rho,\varphi) = \left(\frac{\rho}{R}\right)^2 \cdot \frac{1}{24}\,R^4\sin(2\varphi) = \frac{1}{24}\,\rho^2 R^4\sin(2\varphi).$$

Therefore, the solution of our problem is given by

$$u(\rho,\varphi) = \frac{1}{24}\,\rho^2(R^4 - \rho^2)\sin(2\varphi).$$

Example 2. Find the distribution of the electric potential in the annulus $a < \rho < b$ if in its interior there are electrical charges with density $\gamma(x,y) = A(x^2 - y^2)$, the inner circle is maintained at the potential 1 and the intensity of the electric field on the outer circle is 0.

Solution. The problem reduces to that of solving the Poisson equation $\Delta u = A(x^2 - y^2)$ in the annulus $a < \rho < b$ with the boundary conditions $u|_{\rho=a} = 1$, $\partial u/\partial\rho|_{\rho=b} = 0$. Passing to polar coordinates we obtain the problem

$$\begin{cases} \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial u}{\partial\rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial\varphi^2} = A\rho^2\cos(2\varphi) & a < \rho < b, \quad 0 \le \varphi < 2\pi, \\ u(a,\varphi) = 1, \qquad \dfrac{\partial u}{\partial\rho}(b,\varphi) = 0, & 0 \le \varphi \le 2\pi. \end{cases}$$

Let us seek the solution of this problem in the form $u(\rho,\varphi) = v(\rho,\varphi) + w(\rho)$, where the function $w(\rho)$ is a solution of the auxiliary problem

$$\begin{cases} \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial w}{\partial\rho}\right) = 0, & a < \rho < b, \\ w(a) = 1, \quad w'(b) = 0, \end{cases} \tag{1.14}$$

and the function $v(\rho, \varphi)$ is a solution of the problem

$$\begin{cases} \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\rho \dfrac{\partial v}{\partial \rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 v}{\partial \varphi^2} = A\rho^2 \cos(2\varphi) \qquad a < \rho < b, \quad 0 \le \varphi < 2\pi, \\ v(a, \varphi) = 0, \qquad \dfrac{\partial v}{\partial \rho}(b, \varphi) = 0, \qquad 0 \le \varphi \le 2\pi. \end{cases} \tag{1.15}$$

Obviously, the solution of problem (1.14) is $w(\rho) \equiv 1$. We will seek the solution of problem (1.15) in the form $v(\rho, \varphi) = R(\rho)\cos(2\varphi)$. Substituting this expression for $v(\rho, \varphi)$ in the equation (1.15) we obtain

$$\cos(2\varphi)\frac{1}{\rho}\frac{d}{d\rho}(\rho R') - \frac{4}{\rho^4} R\cos(2\varphi) = A\rho^2\cos(2\varphi),$$

or, simplifying by $\cos(2\varphi)$,

$$\rho^2 R'' + \rho R' - 4R = A\rho^4,$$

with the additional conditions $R(a) = 0$, $R'(b) = 0$. The substitution $\rho = e^t$ transform this equation into the equation with constant coefficients

$$\ddot{R} - 4R = Ae^4 t,$$

where the dot denotes differentiation with respect to t. The general solution of this last equation is $R(t) = C_1 e^{2t} + C_2 e^{-2t} + \frac{1}{12} Ae^{4t}$. Back to the variable ρ we have

$$R(\rho) = C_1\rho^2 + \frac{C_2}{\rho^2} + \frac{1}{12} Ae^{4t}.$$

The constants C_1 and C_2 are found from the conditions $R(a) = 0$, $R'(b) = 0$, namely

$$C_1 = \frac{-A(a^6 + 2b^6)}{12(a^4 + b^4)}, \qquad C_2 = \frac{Aa^4 b^4(2b^2 - a^2)}{6(a^4 + b^4)}$$

Hence, the sought solution is

$$u(\rho, \varphi) = 1 + \left[-\frac{A(a^6 + 2b^6)}{12(a^4 + b^4)}\rho^2 + \frac{Aa^4 b^4(2b^2 - a^2)}{6(a^4 + b^4)}\frac{1}{\rho^2} + \frac{A}{12}\rho^4\right]\cos(2\varphi).$$

1.7. Boundary value problems for the Laplace and Poisson equations in a rectangle

Example 1 [18]. Find the distribution of the electrostatic field $u(x, y)$ inside the rectangle $OACB$ for which the potential along the side B is equal to V, while the three other sides and grounded. There are no electric charges inside the rectangle (Figure 1.2).

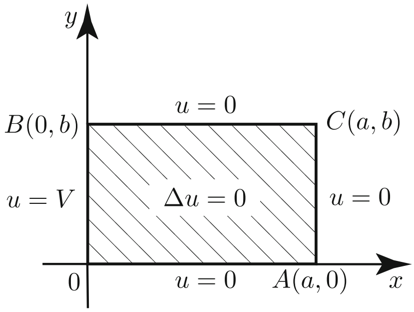

FIGURE 1.2.

Solution. The problem reduces to that of solving the Laplace equation $u_{xx}+u_{yy} = 0$ in the interior of the rectangle with the boundary conditions

$$u(0, y) = V, \quad u(a, y) = 0, \quad u(x, 0) = 0, \quad u(x, b) = 0.$$

First we will seek nontrivial particular solutions of the Laplace equation which satisfy only the boundary conditions

$$u(x, 0) = u(x, b) = b$$

in the form $u(x, y) = X(x)Y(y)$. Substituting this expression in the equation $u_{xx} + u_{yy} = 0$ we get $X''Y + XY'' = 0$, which upon dividing by XY gives

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda^2.$$

Using the fact that $Y(0) = Y(b) = 0$, we obtain the Sturm-Liouville problem

$$\begin{cases} Y'' + \lambda^2 Y = 0, & 0 < y < b, \\ Y(0) = Y(b) = 0, \end{cases}$$

which yields the eigenvalues and eigenfunctions of our problem. We have

$$\lambda_n^2 = \left(\frac{n\pi}{b}\right)^2, \qquad Y_n(y) = \sin\left(\frac{n\pi}{b}\, y\right), \qquad n = 1, 2, \ldots$$

The corresponding functions $X_n(x)$ are solutions of the equation $X'' - \lambda^2 X = 0$, and so

$$X_n(x) = a_n\cosh\left(\frac{n\pi}{b}\,x\right) + b_n\sinh\left(\frac{n\pi}{b}\,x\right)$$

where a_n and b_n are arbitrary constants. It follows that the notrivial particular solutions (“atoms”) have the form

$$u_n(x,y) = \left[a_n\cosh\left(\frac{n\pi}{b}\,x\right) + b_n\sinh\left(\frac{n\pi}{b}\,x\right)\right]\sin\left(\frac{n\pi}{b}\,y\right), \qquad n = 1,2,\dots$$

Now for the sought solution of our problem we take the series

$$u(x,y) = \sum_{n=0}^{\infty}\left[a_n\cosh\left(\frac{n\pi}{b}\,x\right) + b_n\sinh\left(\frac{n\pi}{b}\,x\right)\right]\sin\left(\frac{n\pi}{b}\,y\right). \tag{1.16}$$

The constants a_n and b_n $(n = 1,2,\dots)$ are found from the conditions $u(0,y) = V$, $u(a,y) = 0$. Setting $x = a$ in (1.16) we obtain

$$0 = \sum_{n=0}^{\infty}\left[a_n\cosh\left(\frac{n\pi}{b}\,a\right) + b_n\sinh\left(\frac{n\pi}{b}\,a\right)\right]\sin\left(\frac{n\pi}{b}\,y\right),$$

whence

$$a_n\cosh\left(\frac{n\pi}{b}\,a\right) + b_n\sinh\left(\frac{n\pi}{b}\,a\right) = 0, \qquad n = 1,2,\dots$$

Next, setting $x = 0$ in (1.16) we obtain

$$V = \sum_{n=0}^{\infty} a_n\sin\left(\frac{n\pi}{b}\,y\right),$$

which gives

$$a_n = \frac{2}{b}\int_0^b V\sin\left(\frac{n\pi}{b}\,y\right)\,dy, \qquad \text{or} \qquad a_n = \begin{cases} 0, & \text{if } n \text{ is even},\\ \frac{4V}{n\pi}, & \text{if } n \text{ is odd}.\end{cases}$$

Therefore, the solution has the form

$$u(x,y) = \frac{4V}{\pi}\sum_{k=0}^{\infty}\frac{\sinh\left[\dfrac{(2k+1)(a-x)\pi}{b}\right]\sin\left[\dfrac{(2k+1)\pi y}{b}\right]}{(2k+1)\sinh\left[\dfrac{(2k+1)\pi a}{b}\right]}.$$

Example 2 [18]. Suppose that two sides, AC and BC, of a rectangular homogeneous plate (see Figure 1.2) are covered with a heat insulation, and the other two sides are maintained at temperature zero. Find the stationary temeperature distribution in the plate under the assumption that a quantity of heat $Q = \text{const}$ is extracted it.

Solution. We are dealing with a boundary value problem for the Poisson equation with boundary conditions of mixed type;

$$\begin{cases} u_{xx} + u_{yy} = -\dfrac{Q}{k}, & 0 < x < a, \quad 0 < y < b, \\ u(0,y) = 0, \qquad u_x(a,y) = 0, & 0 \le y \le b, \\ u(x,0) = 0, \qquad u_y(x,b) = 0, & 0 \le x \le a \end{cases} \tag{1.17}$$

(here k is the internal heat conduction coefficient).

The eigenvalues and eigenfunctions of the problem are found by solving the auxiliary boundary value problem (Sturm-Liouville problem)

$$\begin{cases} X'' + \lambda^2 X = 0, & 0 < x < a, \\ X(0) = 0 = X'(a) = 0. \end{cases}$$

We get $\lambda_n^2 = \left[\frac{(2n+1)\pi}{2a}\right]^2$ and $X_n(x) = \sin\left[\frac{(2n+1)\pi}{2a}\,x\right]$, $n = 0, 1, \ldots$ We will seek the solution of the above problem in the form of an expansion in eigenfunctions

$$u(x,y) = \sum_{n=0}^{\infty} Y_n(y) \sin\left[\frac{(2n+1)\pi}{2a}\,x\right],$$

where the functions $Y_n(y)$ are subject to determination. Substituting this expression of the solution in equation (1.17) we obtain

$$-\sum_{n=0}^{\infty} Y_n(y) \frac{(2n+1)^2\pi^2}{4a^2} \sin\left[\frac{(2n+1)\pi}{2a}\,x\right] + \sum_{n=0}^{\infty} Y_n''(y) \sin\left[\frac{(2n+1)\pi}{2a}\,x\right] =$$

$$= \sum_{n=0}^{\infty} \alpha_n \sin\left[\frac{(2n+1)\pi}{2a}\,x\right],$$

where the Fourier coefficients α_n of the function $-Q/k$ are equal to

$$\alpha_n = \frac{2}{a}\int_0^a \left(-\frac{Q}{k}\right) \sin\left[\frac{(2n+1)\pi}{2a}\,x\right] dx = -\frac{4Q}{k\pi(2n+1)}.$$

This yields the following boundary value problem for the determination of the function $Y_n(y)$, $n = 0, 1, 2, \ldots$:

$$\begin{cases} Y_n'' - \dfrac{(2n+1)^2\pi^2}{4a^2} Y_n(y) = -\dfrac{4Q}{k\pi(2n+1)}, & 0 < y < b, \\ Y_n(0) = 0, \qquad Y_n'(b) = 0. \end{cases}$$

Solving this problem, we obtain

$$Y_n(y) = a_n \cosh\left[\frac{(2n+1)\pi}{2a}\,y\right] + b_n \sinh\left[\frac{(2n+1)\pi}{2a}\,y\right] + \frac{16Qa^2}{k\pi^3(2n+1)^3}\,,$$

where

$$a_n = -\frac{16Qa^2}{k\pi^3(2n+1)^3}\,,$$

and

$$b_n = \frac{16Qa^2}{k\pi^3(2n+1)^3}\tanh\left[\frac{(2n+1)\pi b}{2a}\,y\right].$$

The final expression of the solution is

$$u(x,y) = \frac{16Qa^2}{k\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\left(1 - \frac{\cosh\left[\frac{(2n+1)(b-y)\pi}{2a}\right]}{\cosh\left[\frac{(2n+1)\pi b}{2a}\right]}\right)\sin\left[\frac{(2n+1)\pi}{2a}\,x\right].$$

Example 3 [18]. Find the solution of the Laplace equation in the strip $0 \le x \le a$, $0 \le y < \infty$ which satisfies the boundary conditions

$$u(x,0) = 0, \quad u(a,y) = 0, \quad u(x,0) = A\left(1 - \frac{x}{a}\right), \quad u(x,\infty) = 0.$$

Solution. Thus, we need to solve the boundary value problem

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < a, \quad 0 < y < \infty, \\ u(0,y) = u(a,y) = 0, & 0 \le y < \infty, \\ u(x,0) = A\left(1 - \dfrac{x}{a}\right), & u(x,\infty) = 0, \qquad 0 \le x \le a \end{cases} \tag{1.18}$$

Let us begin by finding the solution of the auxiliary problem

$$\begin{cases} v_{xx} + v_{yy} = 0, & 0 < x < a, \quad 0 < y < \infty, \\ v(0,y) = v(a,y) = 0, & 0 \le y < \infty, \end{cases}$$

in the form $v(x,y) = X(x)Y(y)$. We obtain two ordinary differential equations: (1) $X'' + \lambda X = 0$, and (2) $Y'' - \lambda Y = 0$.

From the conditions $v(0,y) = 0$, $v(a,y) = 0$ it follows that $X(0) = X(a) = 0$. Hence, the Sturm-Liouville problem

$$\begin{cases} X'' + \lambda X = 0, & 0 < x < a, \\ X(0) = X(a) = 0 \end{cases}$$

yields $\lambda_n = \left(\frac{n\pi}{a}\right)^2$ and $X_n(x) = \sin\left(\frac{n\pi}{a}\,x\right)$, $n = 1, 2, \ldots$. Then the corresponding solutions of the equation $Y'' - \lambda Y = 0$ are

$$Y_n(y) = A_n e^{-\frac{n\pi}{a}\,y} + B_n e^{\frac{n\pi}{a}\,y}.$$

We conclude that

$$v_n(x,y) = \left[A_n e^{-\frac{n\pi}{a}\,y} + B_n e^{\frac{n\pi}{a}\,y}\right] \sin\left(\frac{n\pi}{a}\,x\right).$$

Therefore, the solution of problem (1.18) is given by a series

$$u(x,y) = \sum_{n=1}^{\infty} \left[A_n e^{-\frac{n\pi}{a}\,y} + B_n e^{\frac{n\pi}{a}\,y}\right] \sin\left(\frac{n\pi}{a}\,x\right). \tag{1.19}$$

From the condition $u(x,\infty)$ it follows that $B_n = 0$, $n = 1, 2, \ldots$. Setting $y = 0$ in (1.19) we get

$$A\left(1 - \frac{x}{a}\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}\,x\right),$$

i.e.,

$$A_n = \frac{2}{a}\int_0^a A\left(1 - \frac{x}{a}\right) \sin\left(\frac{n\pi}{a}\,x\right)\,dx = \frac{2A}{\pi n}.$$

We conclude that

$$u(x,y) = \frac{2A}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\,e^{-\frac{n\pi}{a}\,y} \sin\left(\frac{n\pi}{a}\,x\right).$$

Remark 1. The boundary value problem for the Laplace (Poisson) equation in a rectangular parallelepiped is solved in a similar manner.

Remark 2. Let us assume that the mathematical model of a given physical phenomenon is such that both the equation itself and the boundary conditions are inhomogeneous. Then by using the superposition principle the original boundary value problem can be decomposed into subproblems; one then solves the subproblems and adds their solutions to obtain the solution of the original problem.

For example, the solution of the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in the domain } \Omega, \\ u = \varphi & \text{on the boundary } \partial\Omega, \end{cases}$$

is the sum of the solutions of the following simpler problems:

$$(1) \quad \begin{cases} \Delta u = f & \text{in the domain } \Omega, \\ u = 0 & \text{on the boundary } \partial\Omega, \end{cases} \qquad (2) \quad \begin{cases} \Delta u = 0 & \text{in the domain } \Omega, \\ u = \varphi & \text{on the boundary } \partial\Omega. \end{cases}$$

1.8. Boundary value problems for the Laplace and Poisson equations in a bounded cylinder

To treat the problems mentioned in the title we must resort to special functions, more precisely, to Bessel functions.

First let us consider a boundary value problem for the Laplace equation in a cylinder.

Example 1 [4, Ch. IV, no. 110]. Find the potential of the electrostatic field of a cylindrical wire of section $\rho \leq a$, $0 \leq z \leq l$, such that both bases of the cylinder are grounded and its lateral surface is charged at a potential V_0. Calculate the field intensity on the axis (Figure 1.3).

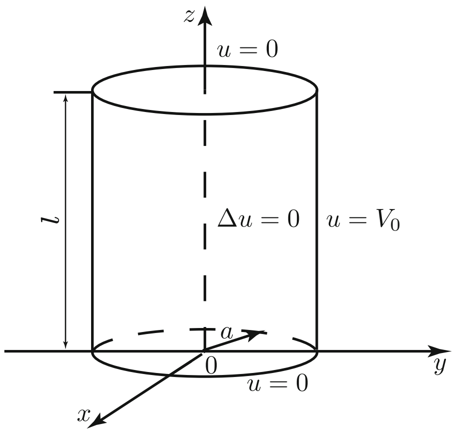

Figure 1.3.

Solution. We need to solve the Laplace equation inside the cylinder with given boundary conditions:

$$\begin{cases} \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\rho \dfrac{\partial u}{\partial \rho}\right) + \dfrac{\partial^2 u}{\partial z^2} = 0, \qquad 0 < \rho < a, \quad 0 < z < l, \\ u(\rho, 0) = u(\rho, l) = 0, \qquad 0 \leq \rho \leq a, \\ u(a, z) = V_0, \qquad 0 \leq z \leq l \end{cases}$$

(the solution $u(\rho, z)$ does not depend on φ since the boundary values are independent of φ). Using the method of separation of variables, we represent the solution in the form $u(\rho, z) = R(\rho)Z(z)$. Substituting this expression in the Laplace equation

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) + \frac{\partial^2 u}{\partial z^2} = 0$$

we get

$$Z\,\rho\frac{\partial}{\partial\rho}(\rho R') + RZ'' = 0$$

whence, upon dividing both sides by RZ,

$$\frac{\rho\,\dfrac{\partial}{\partial\rho}(\rho R')}{R} + \frac{Z''}{Z} = 0,$$

or

$$\frac{\rho\,\dfrac{\partial}{\partial\rho}(\rho R')}{R} = -\frac{Z''}{Z} = \lambda, \tag{1.20}$$

where λ is the separation constant. Clearly, on physical grounds $\lambda > 0$: otherwise the function $Z(z)$, and together with it the potential, would not vanish on the upper and bottom bases of the cylindrical wire.

Equation (1.20) yields two ordinary differential equations:

$$(1) \quad Z'' + \lambda Z = 0,$$

and

$$(2) \quad \frac{1}{\rho}\,\frac{d}{d\rho}(\rho R') - \lambda R = 0.$$

Using the fact that $Z(0) = Z(l) = 0$, we obtain the standard Sturm-Liouville problem:

$$\begin{cases} Z'' + \lambda Z = 0, & 0 < z < l, \\ Z(0) = Z(l) = 0. \end{cases}$$

This problem has the eigenfunctions $Z_n(z) = \sin\left(\frac{n\pi}{l}\,z\right)$, corresponding to the eigenvalues $\lambda_n = \left(\frac{n\pi}{l}\right)^2$, $n = 1, 2, \ldots$. The function $R(\rho)$ is determinded from the equation

$$\frac{1}{\rho}\,\frac{d}{d\rho}(\rho R') - \left(\frac{n\pi}{l}\right)^2 R = 0, \tag{1.21}$$

which is recognized to be the Bessel equation of index zero and imaginary argument. Indeed, from equation (1.21) it follows that

$$\rho^2 R'' + \rho R' - \rho^2\left(\frac{n\pi}{l}\right)^2 R = 0.$$

Passing in this equation to the new independent variable $x = \rho\,\frac{n\pi}{l}$ and using the relations

$$R' = \frac{dR}{dx}\,\frac{n\pi}{l}, \quad R'' = \frac{d^2R}{dx^2}\left(\frac{n\pi}{l}\right)^2,$$

we arrive at the equation

$$x^2 \frac{d^2 R}{dx^2} + x \frac{dR}{dx} - x^2 R = 0.$$

Its general solution is written in the form

$$R(x) = C_1 I_0(x) + C_2 K_0(x),$$

where $I_0(x)$ and $K_0(x)$ are the Bessel functions of index zero and imaginary argument, of the first and second kind, respectively, and C_1 and C_2 are arbitrary constants. Since (the Macdonald) function $K_0(x) \to \infty$ when $x \to 0$, we must set $C_2 = 0$ (otherwise the solution of our problem will be unbounded on the axis of the cylinder). Therefore,

$$R_n(\rho) = C I_0\left(\frac{n\pi}{l}\,\rho\right).$$

The "atoms" from which the solution of the original problem will be constructed are the functions

$$I_0\left(\frac{n\pi}{l}\,\rho\right)\sin\left(\frac{n\pi}{l}\,z\right), \qquad n = 1, 2, \ldots$$

Thus, the solution of our has the series representation

$$u(\rho, z) = \sum_{n=1}^{\infty} c_n I_0\left(\frac{n\pi}{l}\,\rho\right)\sin\left(\frac{n\pi}{l}\,z\right).$$

The constants c_n are found from the boundary condition $u(a, z) = V_0$. We have

$$V_0 = \sum_{n=1}^{\infty} c_n I_0\left(\frac{n\pi}{l}\,a\right)\sin\left(\frac{n\pi}{l}\,z\right),$$

whence

$$c_n I_0\left(\frac{n\pi}{l}\,a\right) = \frac{2}{l}\int_0^l V_0 \sin\left(\frac{n\pi}{l}\,z\right)\,dz = \begin{cases} \dfrac{4V_0}{n\pi}, & n \text{ is odd}, \\ 0, & n \text{ is even}. \end{cases}$$

We conclude that

$$u(z, \rho) = \frac{4V_0}{\pi}\sum_{k=0}^{\infty} \frac{I_0\left[\dfrac{(2k+1)\pi}{l}\,\rho\right]}{I_0\left[\dfrac{(2k+1)\pi}{l}\,a\right]} \cdot \frac{\sin\left[\dfrac{(2k+1)\pi}{l}\,z\right]}{2k+1}.$$

The field on the axis of the cylinder is

$$E_z(0, z) = -\frac{\partial u}{\partial z}(0, z) = -\frac{4V_0}{l}\sum_{k=0}^{\infty} \frac{\cos\left[\dfrac{(2k+1)\pi}{l}\,z\right]}{I_0\left[\dfrac{(2k+1)\pi}{l}\,a\right]}.$$

Example 2 [18]. Consider a cylinder with base of radius R and height h. Assume that the temperature of the lower base and of the lateral surface is equal to zero, while the temperature of the upper base is a given function of ρ. Find the steady temperature distribution in the interior of the cylinder.

Solution. The mathematical formulation of the problems is as follows:

$$\begin{cases} \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial u}{\partial\rho}\right)+\dfrac{\partial^2 u}{\partial z^2}=0, & 0<\rho<R, \quad 0<z<h, \\ u(\rho,0)=0, \quad u(\rho,h)=f(\rho), & 0\le\rho\le R, \\ u(R,z)=0, \qquad 0\le z\le h. \end{cases}$$

Setting, as before, $u(\rho,z)=r(\rho)Z(z)$ and substituting this expression in the Laplace equation, we obtain two ordinary differential equations:

$$\begin{aligned} &(1) \quad \frac{1}{\rho}\frac{d}{d\rho}(\rho r')+\lambda r=0; \\ &(2) \quad Z''-\lambda Z=0. \end{aligned} \tag{1.22}$$

We note that here $\lambda>0$ (this will be clear once we find the solution). The boundary condition $u(R,z)=0$ implies $r(R)=0$. Equation (1.22) can be rewritten as

$$\rho^2 r''+\rho r'+\lambda\rho^2 r=0. \tag{1.23}$$

Passing to the new independent variable $x=\sqrt{\lambda}\rho$ we obtain the Bessel equation of order zero

$$x^2\frac{d^2 r}{dx^2}+x\frac{dr}{dx}+x^2 r=0,$$

whose general solution has the form

$$r(x)=C_1 J_0(x)+C_2 B_0(x),$$

where $J_0(x)$ and $B_0(x)$ are the Bessel function of order zero of first and second kind, respectively, and C_1, C_2 are arbitrary constants.

Returning to the old variable ρ we have

$$r(\rho)=C_1 J_0(\sqrt{\lambda}\rho)+C_2 B_0(\sqrt{\lambda}\rho).$$

Thus, in the present case solving the Sturm-Liouville problem

$$\begin{cases} \rho^2 r''+\rho r'+\lambda\rho^2 r=0, & 0<\rho<R, \\ |r(0)|<\infty, \quad r(R)=0 \end{cases}$$

reduces to the solution of the Bessel equation with the indicated boundary conditions. Since $B_0(\sqrt{\lambda}\rho)\to\infty$ as $\rho\to 0$, we must set $C_2=0$, and so $r(\rho)=$

$CJ_0(\sqrt{\lambda}\rho)$. From the condition $r(R) = 0$ it follows that $J_0(\sqrt{\lambda}R) = 0$. Denoting by $\mu_1, \mu_2, \dots, \mu_n, \dots$ the positive roots of the Bessel function $J_0(x)$ (Figure 1.4), we obtain the eigenvalues $\lambda_n = \left(\frac{\mu_n}{R}\right)^2$ and the corresponding eigenfunctions $J_0\left(\frac{\mu_n}{R}\,\rho\right)$, $n = 1, 2, \dots$. Further, from the equation (2) in (1.22) with $\lambda = \lambda_n = \left(\frac{\mu_n}{R}\right)^2$ we obtain

$$Z_n(x) = A_n\cosh\left(\frac{\mu_n}{R}\,z\right) + B_n\sinh\left(\frac{\mu_n}{R}\,z\right),$$

where A_n and B_n are arbitrary constants. From the boundary condition $u(\rho, 0) = 0$ it follows that $Z(0) = 0$, i.e., $A_n = 0$ for all n. Therefore, the "atoms" of the sought solution are the functions

$$J_0\left(\frac{\mu_n}{R}\,\rho\right)\sinh\left(\frac{\mu_n}{R}\,z\right), \qquad n = 1, 2 \dots .$$

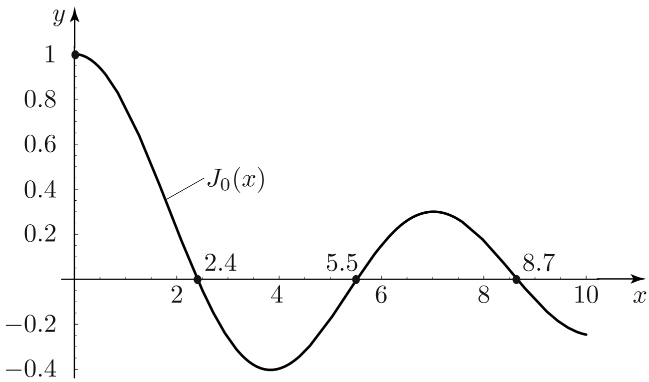

FIGURE 1.4.

The solution of our problem is given by a series

$$u(\rho, z) = \sum_{n=1}^{\infty} B_n J_0\left(\frac{\mu_n}{R}\,\rho\right)\sinh\left(\frac{\mu_n}{R}\,z\right).$$

The constants B_n are found from the boundary condition $u(\rho, h) = f(\rho)$. Indeed, we have

$$u(\rho, h) = \sum_{n=1}^{\infty} B_n J_0\left(\frac{\mu_n}{R}\,\rho\right)\sinh\left(\frac{\mu_n}{R}\,h\right),$$

or

$$f(\rho) = \sum_{n=1}^{\infty} B_n J_0\left(\frac{\mu_n}{R}\,\rho\right)\sinh\left(\frac{\mu_n}{R}\,h\right).$$

Multiplying both sides of this equality by $\rho J_0\left(\frac{\mu_m}{R}\rho\right)$ and integrating the result over the segment $[0, R]$ we get

$$\int_0^R \rho f(\rho) J_0\left(\frac{\mu_m}{R}\rho\right) d\rho = B_m \sinh\left(\frac{\mu_m}{R}h\right) \int_0^R \rho J_0^2\left(\frac{\mu_m}{R}\rho\right) d\rho.$$

But

$$\int_0^R \rho J_0^2\left(\frac{\mu_m}{R}\rho\right) d\rho = \frac{R^2}{2} J_1^2(\mu_m),$$

where $J_1(x)$ is the Bessel function of first kind and order one. Therefore, the solution of the problem has the form

$$u(\rho, z) = \frac{2}{R^2} \sum_{n=1}^{\infty} \frac{\sinh\left(\frac{\mu_n}{R}z\right)}{\sinh\left(\frac{\mu_n}{R}h\right)} \frac{J_0\left(\frac{\mu_n}{R}\rho\right)}{J_1^2(\mu_n)} \int_0^R \rho f(\rho) J_0\left(\frac{\mu_n}{R}\rho\right) d\rho.$$

Example 3. Find the potential in the interior points of a grounded cylinder of height h and with base of radius R, given that in the cylinder there is a charge distribution with density $\gamma = AzJ_0\left(\frac{\mu_3}{R}\rho\right)$ (where A is a constant).

Solution. We must solve the Poisson equation with null boundary conditions:

$$\begin{cases} \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial u}{\partial\rho}\right) + \dfrac{\partial^2 u}{\partial z^2} = -4\pi AzJ_0\left(\dfrac{\mu_3}{R}\rho\right), \\ 0 < \rho < R, \quad 0 < z < h, \\ u(\rho, 0) = u(\rho, h) = 0, \qquad 0 \le \rho \le R, \\ u(R, z) = 0, \qquad 0 \le z \le h. \end{cases} \tag{1.24}$$

Let us seek the solution in the form $u(\rho, z) = J_0\left(\frac{\mu_3}{R}\rho\right) f(z)$, where the function $f(z)$ is subject to determination. Substituting this expression of $u(\rho, z)$ in equation (1.24) we get

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J_0\left(\frac{\mu_3}{R}\rho\right)\right] f(z) + J_0\left(\frac{\mu_3}{R}\rho\right) f''(z) = -4\pi AzJ_0\left(\frac{\mu_3}{R}\rho\right). \tag{1.25}$$

Now let us observe that the function $J_0\left(\frac{\mu_3}{R}\rho\right)$ is an eigenfunction of the Bessel equation, i.e.,

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J_0\left(\frac{\mu_3}{R}\rho\right)\right] + \frac{\mu_3^2}{R^2} J_0\left(\frac{\mu_3}{R}\rho\right) = 0.$$

Consequently, (1.25) gives

$$-\left(\frac{\mu_3}{R}\right)^2 J_0\left(\frac{\mu_3}{R}\rho\right) f(z) + J_0\left(\frac{\mu_3}{R}\rho\right) f''(z) = -4\pi AzJ_0\left(\frac{\mu_3}{R}\rho\right),$$

which in turn yields the following ordinary differential equation for the determination of $f(z)$:

$$f'' - \left(\frac{\mu_3}{R}\right)^3 f = -4\pi A z, \qquad 0 < z < h,$$

with $f(0) = f(h) = 0$. Solving this boundary value problem we find that

$$f(z) = -\frac{4\pi A R^2 h}{\mu_3^2} \cdot \frac{\sinh\left(\frac{\mu_3}{R}\, z\right)}{\sinh\left(\frac{\mu_3}{R}\, h\right)} + \frac{4\pi A R^2}{\mu_3^2}\, z.$$

Thus, the solution of our problem is given by the expression

$$u(\rho, z) = J_0\left(\frac{\mu_3}{R}\,\rho\right) \frac{4\pi A R^2}{\mu_3^2} \left[h\, \frac{\sinh\left(\frac{\mu_3}{R}\, z\right)}{\sinh\left(\frac{\mu_3}{R}\, h\right)} - z \right].$$

1.9. Boundary value problems for the Laplace and Poisson equations in a ball

To deal with the problem mentioned in the title we need to use spherical functions and solid spherical harmonics.

Recall that the general solution of the Laplace equation has the following form (in spherical coordinates (ρ, θ, φ)):

(1) $u(\rho, \theta, \varphi) = \sum_{n=0}^{\infty} \left(\frac{\rho}{a}\right)^n Y_n(\theta, \varphi)$ in the interior the sphere of radius a;

(2) $u(\rho, \theta, \varphi) = \sum_{n=0}^{\infty} \left(\frac{\rho}{a}\right)^{(n+1)} Y_n(\theta, \varphi)$ in the exterior of the sphere of radius a;

(3) $u(\rho, \theta, \varphi) = \sum_{n=0}^{\infty} \left(A_n \rho^n + \frac{B_n}{\rho^{n+1}}\right) Y_n(\theta, \varphi)$ in a spherical layer.

Here

$$Y_n(\theta, \varphi) = \sum_{m=0}^{n} [A_{nm} \cos(m\varphi) + B_{nm} \sin(m\varphi)] P_n^{(m)}(\cos\theta),$$

where $P_n^{(m)}(x)$ are the so-called associated Legendre functions.

Example 1. Find the solution $u(\rho, \theta, \varphi)$ of the interior Dirichlet problem for the Laplac equation with the boundary condition $u(a, \theta, \varphi) = \sin(3\theta)\cos\varphi$.

Solution. In spherical coordinates the problem is written as follows:

$$\begin{cases} \dfrac{1}{\rho^2} \dfrac{\partial}{\partial \rho} \left(\rho^2 \dfrac{\partial u}{\partial \rho} \right) + \dfrac{1}{\rho^2 \sin\theta} \dfrac{\partial}{\partial \theta} \left(\sin\theta \dfrac{\partial u}{\partial \theta} \right) + \dfrac{1}{\rho^2 \sin^2\theta} \dfrac{\partial^2 u}{\partial \varphi^2} = 0, \\ 0 < \rho < a, \quad 0 < \theta < \pi, \quad 0 \le \varphi < 2\pi, \\ u(a, \theta, \varphi) = \sin(3\theta)\cos\varphi, \qquad 0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi. \end{cases} \tag{1.26}$$

Setting $u(\rho,\theta,\varphi) = R(\rho)Y(\theta,\varphi)$ and substituting this expression in equation (1.26), we obtain

$$Y\frac{d}{d\rho}(\rho^2R') + R\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}\right] = 0,$$

which upon dividing both sides by RY yields

$$\frac{\dfrac{d}{d\rho}(\rho^2R')}{R} + \frac{\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial Y}{\partial\theta}\right) + \dfrac{1}{\sin^2\theta}\dfrac{\partial^2 Y}{\partial\varphi^2}}{Y} = 0,$$

or

$$\frac{\dfrac{d}{d\rho}(\rho^2R')}{R} = -\frac{\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial Y}{\partial\theta}\right) + \dfrac{1}{\sin^2\theta}\dfrac{\partial^2 Y}{\partial\varphi^2}}{Y} = \lambda,$$

where λ is the separation constant. This yields two equations:

$$\begin{aligned} &(1)\quad \rho^2R'' + 2\rho R' - \lambda R = 0,\\ &(2)\quad \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2} + \lambda Y = 0; \end{aligned} \tag{1.27}$$

here the function $Y(\theta,\varphi)$ must be restricted to the sphere.

Moreover, the function $Y(\theta,\varphi)$ satisfies the conditions

$$\begin{cases} Y(\theta,\varphi) = Y(\theta,\varphi+2\pi),\\ |Y(0,\varphi)| < \infty, \quad |Y(\pi,\varphi)| < \infty. \end{cases} \tag{1.28}$$

As is known, the bounded solutions of equation (1.27) that have continuous derivatives up to and including order two are called *spherical functions*.

The solution of problem (1.27),(1.28) for $Y(\theta,\varphi)$ will also be sought via separation of variables, setting $Y(\theta,\varphi) = T(\theta)\Phi(\varphi)$. Susbtituting this expression in equation (1.27), we get

$$\Phi\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta T') + \frac{1}{\sin^2\theta}T\Phi'' + \lambda T\Phi = 0,$$

whence

$$\frac{\sin\theta\dfrac{d}{d\theta}(\sin\theta T')}{T} + \lambda\sin^2\theta = -\frac{\Phi''}{\Phi} = \mu.$$

Thus, the function $\Phi(\varphi)$ is found by solving the problem

$$\begin{cases} \Phi'' + \mu\Phi = 0,\\ \Phi(\varphi) = \Phi(\varphi+2\pi). \end{cases}$$

We have already solved such a problem when we considered the Laplace equation in a disc, and found that $\mu = m^2$ and $\Phi_m(\varphi) = C_1 \cos(m\varphi) + C_2 \sin(m\varphi)$, where C_1 and C_2 are arbitrary constants and $m = 0, 1, \ldots$

The function $T(\theta)$ is found from the equation

$$\frac{1}{\sin\theta} \frac{d}{d\theta}(\sin\theta T') + \left(\lambda - \frac{m^2}{\sin^2\theta}\right) T = 0 \tag{1.29}$$

and the conditions that T be bounded at $\theta = 0$ and $\theta = \pi$. Introducing the new variable $x = \cos\theta$ and observing that

$$T' = \frac{dT}{dx}\frac{dx}{d\theta} = \frac{dT}{dx}(-\sin\theta),$$

$$T'' = \frac{d^2T}{dx^2}\sin^2\theta - \frac{dT}{dx}\cos\theta,$$

equation (1.29) yields the following boundary value problem for eigenvalues and eigenfunctions:

$$\begin{cases} (1-x^2)\dfrac{d^2T}{dx^2} - 2x\dfrac{dT}{dx} + \left(\lambda - \dfrac{m^2}{1-x^2}\right) T = 0, \quad -1 < x < 1, \\ |T(-1)| < \infty, \quad |T(+1)| < \infty. \end{cases}$$

The eigenfunctions of this problem,

$$T_n^{(m)}(x) = P_n^{(m)}(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x),$$

are the associated Legendre functions. Hence, the solutions of equation (1.29) are the functions $T_n^{(m)}(x) = P_n^{(m)}(\cos\theta)$.

Combining the solutions of equation (1.29) with the solutions of the equation $\Phi'' + \mu\Phi = 0$, we obtain the $2n+1$ spherical functions

$$\begin{gathered} P_n(\cos\theta), \quad P_n^{(m)}(\cos\theta)\cos(m\varphi), \quad P_n^{(m)}(\cos\theta)\sin(m\varphi), \\ n = 0, 1, \ldots; \quad m = 1, 2, \ldots \end{gathered}$$

The general solution of equation (1.27) for $\lambda = n(n+1)$ is written in the form

$$Y_n(\theta,\varphi) = \sum_{m=0}^{n} \left[A_{nm}\cos(m\varphi) + B_{nm}\sin(m\varphi)\right] P_n^{(m)}(\cos\theta).$$

Now let us return to the search for the function $R(\rho)$. Setting $R(\rho) = \rho^\sigma$ and substituting this expression in the equation $\rho^2 R'' + 2\rho R' - \lambda R = 0$, we obtain

$\sigma(\sigma+1)-n(n+1)=0$, whence $\sigma_1=n$, $\sigma_2=-(n+1)$. Thus, the solution "atoms" are the functions

$$\begin{array}{cc} \rho^n P_n^{(m)}(\cos\theta)\cos(m\varphi), & \rho^n P_n^{(m)}(\cos\theta)\sin(m\varphi), \\ \rho^{-(n+1)} P_n^{(m)}(\cos\theta)\cos(m\varphi), & \rho^{-(n+1)} P_n^{(m)}(\cos\theta)\sin(m\varphi). \end{array}$$

However, the solutions $\rho^{-(n+1)}P_n^{(m)}(\cos\theta)\cos(m\varphi)$, $\rho^{-(n+1)}P_n^{(m)}(\cos\theta)\sin(m\varphi)$ must be discarded because they are not bounded when $\rho\to 0$. Hence, the solution of our problem is given by a series

$$u(\rho,\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\rho^n\left[A_{nm}\cos(m\varphi)+B_{nm}\sin m\varphi\right]P_n^{(m)}(\cos\theta).$$

It remains to choose the constants A_{nm} and B_{nm} so that the boundary condition

$$u(a,\theta,\varphi)=\sin(3\theta)\cos\varphi$$

will be satisfied. We have

$$u(a,\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}a^n\left[A_{nm}\cos(m\varphi)+B_{nm}\sin m\varphi\right]P_n^{(m)}(\cos\theta),$$

i.e., we must satisfy the equality

$$\sin(3\theta)\cos\varphi=\sum_{n=0}^{\infty}\sum_{m=0}^{n}a^n\left[A_{nm}\cos(m\varphi)+B_{nm}\sin m\varphi\right]P_n^{(m)}(\cos\theta).$$

It follows that in the sum $\sum_{m=0}^{n}\cdots$ we must retain only the term corresponding to $m=1$. This yields

$$\sin(3\theta)=\sum_{n=1}^{\infty}a^nA_{n1}P_n^{(1)}(\cos\theta).$$

The coefficients A_{n1} can be found from the general formula: if

$$f(\theta)=\sum_{n=1}^{\infty}b_nP_n^{(1)}(\cos\theta),$$

then

$$b_n=\frac{2n+1}{2}\cdot\frac{(n-1)!}{(n+1)!}\int_0^{\pi}f(\theta)P_n^{(1)}(\cos\theta)\sin\theta\,d\theta.$$

However, it is more convenient to proceed as follows: we have

$$\sin(3\theta) = \sin\theta(4\cos^2\theta - 1), \quad P_n^{(1)}(\cos\theta) = \sin\theta\,\frac{dP_n(\cos\theta)}{d(\cos\theta)},$$

$$P_1(x) = x, \quad P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

Therefore,

$$(4\cos^2\theta - 1)\sin\theta = \sin\theta\left[a\cdot A_{11}\cdot 1 + a^3\cdot A_{31}\cdot\frac{1}{2}(15\cos^2\theta - 3)\right],$$

which gives

$$A_{11} = -\frac{1}{5a}, \quad A_{31} = \frac{8}{15a^3}, \quad A_{n1} = 0, \quad n = 2, 4, 5, \ldots$$

We conclude that the solution of our problem has the form

$$u(\rho,\theta,\varphi) = \Big(-\frac{1}{5}\Big)\frac{\rho}{a}P_1^{(1)}(\cos\theta)\cos\varphi + \frac{8}{15}\Big(\frac{\rho}{a}\Big)^3 P_3^{(1)}(\cos\theta)\cos\varphi.$$

Example 2. Find a function u, harmonic inside the spherical layer $R_1 < \rho < R_2$, and such that

$$u|_{\rho=R_1} = P_2^{(1)}(\cos\theta)\sin\varphi, \qquad u|_{\rho=R_2} = P_5^{(3)}(\cos\theta)\cos(3\varphi).$$

Solution. The mathematical formulation of the problem is

$$\begin{cases} \Delta u = 0, \qquad R_1 < \rho < R_2, \quad 0 < \theta < \pi, \quad 0 < \varphi < 2\pi, \\ u(R_1,\theta,\varphi) = P_2^{(1)}(\cos\theta)\sin\varphi, \\ u(R_2,\theta,\varphi) = P_5^{(3)}(\cos\theta)\cos(3\varphi), \end{cases} \qquad 0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi$$

(see Figure 1.5).

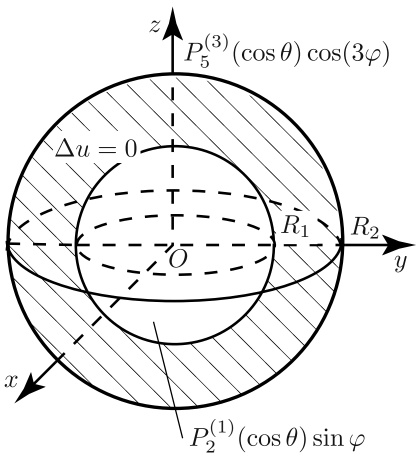

Figure 1.5.

The solution of this problem is written in the form

$$u(\rho,\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[\left(A_{nm}\rho^n+\frac{B_{nm}}{\rho^{n+1}}\right)\cos(m\varphi)+\right.$$

$$\left.+\left(C_{nm}\rho^n+\frac{D_{nm}}{\rho^{n+1}}\right)\sin(m\varphi)\right]P_n^{(m)}(\cos\theta),$$

where the numbers A_{nm}, B_{nm}, C_{nm} and D_{nm} are subject to determination. The boundary conditions yield the following systems of equations for the coefficients of the expansion:

$$(1)\quad\begin{cases} C_{21}R_1^2+\dfrac{D_{21}}{R_1^3}=1,\\ A_{21}R_1^2+\dfrac{B_{21}}{R_1^3}=0,\\ C_{21}R_2^2+\dfrac{D_{21}}{R_2^3}=0,\\ A_{21}R_2^2+\dfrac{B_{21}}{R_2^3}=0,\end{cases}\qquad (2)\quad\begin{cases} A_{53}R_1^5+\dfrac{B_{53}}{R_1^6}=0,\\ C_{53}R_1^5+\dfrac{D_{53}}{R_1^6}=0,\\ A_{53}R_2^5+\dfrac{B_{53}}{R_2^6}=1,\\ C_{53}R_2^5+\dfrac{D_{53}}{R_2^6}=0,\end{cases}$$

All the remaining coefficients are equal to zero. Solving the above systems we obtain

$$A_{21}=B_{21}=0,\qquad C_{53}=D_{53}=0,\qquad C_{21}=-\frac{R_1^3}{R_2^2(R_2^5-R_1^5)},$$

$$D_{21}=\frac{(R_1R_2)^3}{R_2^5-R_1^5},\qquad A_{53}=-\frac{R_2^6}{R_1^5(R_2^{11}-R_1^{11})},\qquad B_{53}=\frac{(R_1R_2)^6}{R_2^{11}-R_1^{11}}.$$

Therefore, the harmonic function sought has the form

$$u(\rho,\theta,\varphi)=\left(C_{21}\rho+\frac{D_{21}}{\rho^2}\right)P_2^{(1)}(\cos\theta)\sin\varphi+$$

$$+\left(A_{53}\rho^5+\frac{B_{53}}{\rho^6}\right)P_5^{(3)}(\cos\theta)\cos(3\varphi).$$

Example 3 [6, 16.25(1)]. Find a function u, harmonic inside the spherical layer $1<\rho<2$, such that

$$\left.\left(3u+\frac{\partial u}{\partial\rho}\right)\right|_{\rho=1}=5\sin^2\theta\sin(2\varphi)\qquad\text{and}\qquad u|_{\rho=2}=-\cos\theta.$$

Solution. The problem is formulated mathematically as follows:

$$\begin{cases} \Delta u = 0, \quad 1 < \rho < 2, \quad 0 < \theta < \pi, \quad 0 \le \varphi < 2\pi, \\ \left(3u + \dfrac{\partial u}{\partial \rho}\right)\Big|_{\rho=1} = 5\sin^2\theta \sin(2\varphi), \qquad 0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi, \\ u|_{\rho=2} = -\cos\theta, \qquad 0 \le \theta \le \pi, \quad 0 \le \varphi < 2\pi. \end{cases}$$

We have

$$u(\rho,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\left(A_{nm}\rho^n + \frac{B_{nm}}{\rho^{n+1}}\right)\cos(m\varphi) + \right.$$

$$\left. + \left(C_{nm}\rho^n + \frac{D_{nm}}{\rho^{n+1}}\right)\sin(m\varphi)\right] P_n^{(m)}(\cos\theta).$$

From the boundary conditions it follows that in this sum we must retain only the terms with the indices $n = 2$, $m = 2$ and $n = 1$, $m = 0$. In other words, it is convenient to seek the solution in the form

$$u(\rho,\theta,\varphi) = \left(a\rho + \frac{b}{\rho^2}\right)\cos\theta + \left(c\rho^2 - \frac{d}{\rho^3}\right)\sin^2\theta\sin(2\varphi).$$

Using the boundary conditions we obtain the following system of equations for the determination of the coefficients a, b, c, d:

$$\begin{cases} 4a + b = 0, \\ 5c = 5, \\ 2a + b/4 == -1, \\ 4c - d/8 = 0. \end{cases}$$

Solving this system, we obtain $a = -1$, $b = 4$, $c = 1$, $d = 32$. Hence, the solution has the expression

$$u(\rho,\theta,\varphi) = \left(-\rho + \frac{4}{\rho^2}\right)\cos\theta + \left(\rho^2 - \frac{32}{\rho^3}\right)\sin^2\theta\sin(2\varphi).$$

Example 4 [4, Ch. IV, no. 125]. Find the solution of the Neumann problem for the Laplace equation in the interior of the sphere of radius a with the condition

$$\frac{\partial u}{\partial n}(a,\theta,\varphi) = A\cos\theta \qquad (A = \text{ const}).$$

Solution. We are dealing with the case of an axially-symmetric solution of the Neumann problem for the Laplace equation, since the boundary condition does not depend on φ, and consequently the solution also does not depend of φ: $u = u(\rho,\theta)$.

First of all, it is readily verified that the necessary condition for the solvability of our problem is satisfied. Indeed

$$\int_0^{2\pi}\int_0^{\pi}\frac{\partial u}{\partial n}\,ds=0, \qquad \text{or} \qquad \int_0^{2\pi}d\varphi\int_0^{\pi}A\cos\theta\sin\theta a^2\,d\theta=0.$$

In the present case the Laplace equation has the form

$$\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)=0, \qquad 0\le\rho<a, \quad 0\le\theta\le\pi.$$

Setting $u(\rho,\theta)=R(\rho)T(\theta)$ and substituting this expression in the equation, we obtain, after separation of variables, two ordinary differential equations:

$$\rho^2R''+2\rho R'-\lambda R=0, \tag{1.30}$$

and

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\cdot T')+\lambda T=0. \tag{1.31}$$

If in the equation (1.31) we pass to the new variable $x=\cos\theta$ we arrive at the Legendre equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dT}{dx}\right]+\lambda T=0, \qquad -1<x<1, \tag{1.32}$$

under the condition $|T(\pm1)|<\infty$. The bounded solutions of the Legendre equation (1.32) on the interval $(-1,1)$ are the Legendre polynomials $P_n(x)$ for $\lambda_n=n(n+1)$. Hence, the bounded solutions of equation (1.31) on the interval $(0,\pi)$ are the functions $P_n(\cos\theta)$. The bounded solutions of equation (1.30) are the functions $R_n(\rho)=\rho^n$ $(n=0,1,2,\dots)$. It follows that

$$u(\rho,\theta)=\sum_{n=0}^{\infty}C_n\rho^nP_n(\cos\theta),$$

where the constants C_n are to be determined from the boundary condition $\partial u/\partial\rho = A\cos\theta$. We have

$$\frac{\partial u}{\partial\rho}(\rho,\theta)=\sum_{n=0}^{\infty}nC_na^{n-1}P_n(\cos\theta),$$

or, setting $\rho=a$,

$$A\cos\theta=\sum_{n=0}^{\infty}nC_na^{n-1}P_n(\cos\theta)$$

whence, upon applying the formula

$$C_n=\frac{2n+1}{2na^{n-1}}\int_0^{\pi}A\cos\theta P_n(\cos\theta)\sin\theta\,d\theta,$$

we find that $C_1 = 1$ and $C_n = 0$ for $n = 2, 3, \ldots$. We conclude that

$$u(\rho, \theta) = C + A\rho\cos\theta,$$

where C is an arbitrary constant.

Example 3. Solve the following Dirichlet problem for the Poisson equation in a ball of radius a centered at the origin:

$$\begin{cases} \Delta u = xz & \text{in the interior of the ball,} \\ u|_{\rho=a} = 1. \end{cases}$$

Solution. Passing to spherical coordinates, we will seek the solution as a sum

$$u(\rho, \theta, \varphi) = v(\rho, \theta, \varphi) + w(\rho),$$

where the function $v(\rho, \theta, \varphi)$ is defined as the solution of the equation

$$\begin{cases} \dfrac{1}{\rho^2}\dfrac{\partial}{\partial\rho}\left(\rho^2\dfrac{\partial v}{\partial\rho}\right) + \dfrac{1}{\rho^2}\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial v}{\partial\theta}\right) + \\ \qquad\qquad + \dfrac{1}{\rho^2}\dfrac{1}{\sin^2\theta}\dfrac{\partial^2 v}{\partial\varphi^2} = \dfrac{\rho^2}{2}\cos\varphi\,\sin(2\theta), \\ 0 < \rho < a, \quad 0 < \theta < a, \quad 0 \le \varphi < 2\pi, \\ v(a, \theta, \varphi) = 0, \end{cases} \tag{1.33}$$

and the function $w(\rho)$ is defined as the solution of the problem

$$\begin{cases} \dfrac{1}{\rho^2}\dfrac{d}{d\rho}(\rho^2 w') = 0, \qquad 0 < \rho < a, \\ w(a) = 1, \quad |w(0)| < \infty. \end{cases} \tag{1.34}$$

Let us solve first problem (1.33), seeking the solution in the form

$$v(\rho, \theta, \varphi) = R(\rho)P_2^{(1)}(\cos\theta)\cos\varphi,$$

where $P_2^{(1)}(x)$ is the associated Legendre function with indices $n = 2$, $m = 1$. Substituting this expression of $v(\rho, \theta, \varphi)$ in the equation of problem (1.33) and denoting $P_2^{(1)}(\cos\theta)\cos\varphi = Y_2^{(1)}(\theta, \varphi)$ we get the equation

$$Y_2^{(1)}\frac{d}{d\rho}(\rho^2 R') + R\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_2^{(1)}}{\partial\theta}\right) + R\frac{1}{\sin^2\theta}\frac{\partial^2 Y_2^{(1)}}{\partial\varphi^2} = \frac{\rho^4}{6}Y_2^{(1)}(\theta, \varphi).$$

But by the definition of the spherical function $Y_2^{(1)}(\theta, \varphi)$ one has the identity

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_2^{(1)}}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_2^{(1)}}{\partial\varphi^2} + 6Y_2^{(1)} = 0, \quad 0 < \theta < \pi, \quad 0 < \varphi < 2\pi.$$

Therefore,

$$\frac{d}{d\rho}(\rho^2 R')Y_2^{(1)} - 6RY_2^{(1)} = \frac{\rho^4}{6}\,Y_2^{(1)},$$

which yields the equation

$$\frac{d}{d\rho}(\rho^2 R') - 6R = \frac{\rho^4}{6}, \qquad 0 < \rho < a,$$

together with the boundary conditions $|R(0)| < \infty$, $R(a) = 0$. Therefore, the function $R(\rho)$ is determined by solving the problem

$$\begin{cases} \rho^2 R'' + 2\rho R' - 6R = \dfrac{\rho^4}{6}, & 0 < \rho < a, \\ |R(0)| < \infty, \quad R(a) = 0. \end{cases}$$

Its solution is $R(\rho) = \frac{1}{84}\rho^2(\rho^2 - a^2)$. The solution of problem (1.34) is $w(\rho) = 1$. We conclude that

$$w(\rho, \theta, \varphi) = 1 + \frac{1}{84}\rho^2(\rho^2 - a^2)P_2^{(1)}(\cos\theta)\cos\varphi.$$

Remark 1. In the general case, when one solves the interior Dirichlet problem for the Laplace equation with the condition $u|_{\partial\Omega} = f(\theta, \varphi)$ (where Ω is the ball of radius a centered at the origin and $\partial\Omega$ is its boundary), one can write

$$f(\theta, \varphi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} a^n\left[A_{nm}\cos(m\varphi) + B_{nm}\sin(m\varphi)\right]P_n^{(m)}(\cos\theta),$$

where the coefficients A_{nm} and B_{nm} are given by the formulas

$$A_{nm} = \frac{\displaystyle\int_0^{2\pi}\int_0^{\pi} f(\theta, \varphi)P_n^{(m)}(\cos\theta)\cos(m\varphi)\sin\theta\,d\theta\,d\varphi}{\|Y_n^{(m)}\|^2 a^n}$$

and

$$B_{nm} = \frac{\displaystyle\int_0^{2\pi}\int_0^{\pi} f(\theta, \varphi)P_n^{(m)}(\cos\theta)\sin(m\varphi)\sin\theta\,d\theta\,d\varphi}{\|Y_n^{(m)}\|^2 a^n};$$

also,

$$\|Y_n^{(m)}\|^2 = \frac{2\pi\varepsilon_m}{2n+1}\,\frac{(n+m)!}{(n-m)!}, \quad \text{where} \quad \varepsilon_m = \begin{cases} 2, & \text{if } m = 0, \\ 1, & \text{if } m > 0. \end{cases}$$

Remark 2. The solution of the aforementioned interior Dirichlet problem for the Laplace equation at a point $(\rho_0, \theta_0, \varphi_0)$ admits the integral representation (Poisson integral)

$$u(\rho_0, \theta_0, \varphi_0) = \frac{a}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(\theta, \varphi)\,\frac{a^2 - \rho^2}{(a^2 - 2a\rho_0\cos\gamma + \rho_0^2)^{3/2}}\,\sin\theta\,d\theta\,d\varphi,$$

where $\cos\gamma = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\varphi - \varphi_0)$.

1.10. Boundary value problems for the Helmholtz equations

The Helmholtz equations $\Delta u + k^2 u = f$ and $\Delta u - k^2 u = f$, alongside with the Laplace and Poisson equations, are an important form of second-order elliptic equation. The homogeneous equation ($f = 0$), for example, arises naturally (in the multi-dimensional case) when the method of separation of variables is applied to hyperbolic and parabolic problems. Finding eigenvalues and eigenfunctions reduced to the solvability of the corresponding boundary value problem for a Helmholtz equation with $f \equiv 0$.

Example 1 [4, Ch. VII, no. 29(a)]. Find the natural oscillations of a membrane that has the shape of a annular sector ($a \le \rho \le b$, $0 \le \varphi \le \varphi_0$), with free boundary.

Solution. The problem is formulated mathematically as follows:

$$\begin{cases} \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\dfrac{\partial u}{\partial\rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial\varphi^2} + \lambda u = 0, \qquad a < \rho < b, \quad 0 < \varphi < \varphi_0, \\ \dfrac{\partial u}{\partial\rho}(a,\varphi) = \dfrac{\partial u}{\partial\rho}(b,\varphi) = 0, \qquad 0 \le \varphi \le \varphi_0, \\ \dfrac{\partial u}{\partial\varphi}(\rho,0) = \dfrac{\partial u}{\partial\varphi}(\rho,\varphi_0) = 0, \qquad a \le \rho \le b. \end{cases} \tag{1.35}$$

We will seek the solution of this problem in the form

$$u(\rho,\varphi) = R(\rho)\Phi(\varphi).$$

Inserting this expression in equation (1.35) and separating the variables we obtain two ordinary differential equations:

$$(1) \quad \Phi'' + \nu\Phi = 0,$$

and

$$(2) \quad \rho\frac{d}{d\rho}(\rho R') + (\lambda\rho^2 - \nu)R = 0.$$

To determine ν we have the Sturm-Liouville problem

$$\begin{cases} \Phi'' + \nu\Phi = 0, \quad 0 < \varphi < \varphi_0, \\ \Phi'(0) = 0, \quad \Phi'(\varphi_0) = 0. \end{cases}$$

This yields $\nu_n = \left(\frac{\pi n}{\varphi_0}\right)^2$, $n = 0, 1, \dots$, and $\Phi_n(\varphi) = \cos\left(\frac{\pi n}{\varphi_0}\varphi\right)$. The function $R(\rho)$ is obtained from the following boundary value problem for the Bessel equation

$$\begin{cases} \rho\dfrac{d}{d\rho}(\rho R') + (\lambda\rho^2 - \nu R) = 0, \qquad a < \rho < b, \\ R'(a) = R'(b) = 0. \end{cases} \tag{1.36}$$

The general solution of equation (1.36) has the form

$$R(\rho) = C_1 J_{\frac{\pi n}{\varphi_0}}(\sqrt{\lambda}\rho) + C_2 N_{\frac{\pi n}{\varphi_0}}(\sqrt{\lambda}\rho),$$

where C_1 and C_2 are arbitrary constants and $N_{\frac{\pi n}{\varphi_0}}(\sqrt{\lambda}\rho)$ is the Bessel function of second kind. The values of λ are determined by means of the boundary conditions in (1.36); namely, they provide the system of equations

$$\begin{cases} C_1 J'_{\frac{\pi n}{\varphi_0}}(\sqrt{\lambda}a) + C_2 N'_{\frac{\pi n}{\varphi_0}}(\sqrt{\lambda}a) = 0, \\ C_1 J'_{\frac{\pi n}{\varphi_0}}(\sqrt{\lambda}b) + C_2 N'_{\frac{\pi n}{\varphi_0}}(\sqrt{\lambda}b) = 0. \end{cases}$$

This system has a nontrivial solution if and only if its determinant

$$\begin{vmatrix} J'_{\frac{\pi n}{\varphi_0}}(\sqrt{\lambda}a) & N'_{\frac{\pi n}{\varphi_0}}(\sqrt{\lambda}a) \\ J'_{\frac{\pi n}{\varphi_0}}(\sqrt{\lambda}b) & N'_{\frac{\pi n}{\varphi_0}}(\sqrt{\lambda}b) \end{vmatrix}$$

is equal to zero. In other words, $\lambda_{m,n} = \left[\mu_m^{(n)}\right]^2$, where $\mu_m^{(n)}$ are the roots of the equation

$$\frac{J'_{\frac{\pi n}{\varphi_0}}(\sqrt{\lambda}a)}{J'_{\frac{\pi n}{\varphi_0}}(\sqrt{\lambda}b)} = \frac{N'_{\frac{\pi n}{\varphi_0}}(\sqrt{\lambda}a)}{N'_{\frac{\pi n}{\varphi_0}}(\sqrt{\lambda}b)}.$$

We see that the radial function has the form

$$R_{m,n}(\rho) = J_{\frac{\pi n}{\varphi_0}}(\mu_m^{(n)}\rho) N'_{\frac{\pi n}{\varphi_0}}(\mu_m^{(n)}a) - J'_{\frac{\pi n}{\varphi_0}}(\mu_m^{(n)}a) N_{\frac{\pi n}{\varphi_0}}(\mu_m^{(n)}\rho).$$

Thus, the natural oscillations of our plate are described by the functions

$$u_{m,n}(\rho,\varphi) = R_{m,n}(\rho)\Phi_n(\varphi) =$$

$$= \left[J_{\frac{\pi n}{\varphi_0}}(\mu_m^{(n)}\rho) N'_{\frac{\pi n}{\varphi_0}}(\mu_m^{(n)}a) - J'_{\frac{\pi n}{\varphi_0}}(\mu_m^{(n)}a) N_{\frac{\pi n}{\varphi_0}}(\mu_m^{(n)}\rho)\right] \cos\left(\frac{\pi n}{\varphi_0}\varphi\right).$$

1.11. Boundary value problem for the Helmoltz equation in a cylinder

Example [4, Ch. VII, no. 10]. Find the steady distribution of the concentration of an unstable gas inside an infinite cylinder of circular section assuming that a constant concentration u_0 is maintained on the surface of the cylinder.

Solution. It is know that the problem of diffusion of an unstable gas that decomposes during the diffusion proces is described by the equation

$$\Delta u - \varkappa^2 u = 0 \qquad (\varkappa > 0).$$

Hence, in polar coordinates the problem is formulated as

$$\begin{cases} \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\dfrac{\partial u}{\partial \rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial \varphi^2} - \varkappa^2 u = 0, \qquad 0 < \rho < a, \quad 0 < \varphi < 2\pi, \\ u(a, \varphi) = u_0, \qquad 0 \le \varphi \le 2\pi, \end{cases} \tag{1.37}$$

where a denotes the radius of the cylinder.

Let us seek the solution in the form $u(\rho, \varphi) = R(\rho)\Phi(\varphi)$. Substituting this expression in equation (1.37) we obtain

$$\frac{1}{\rho}\frac{d}{d\rho}(\rho R')\Phi + \frac{R}{\rho^2}\Phi'' - \varkappa^2 R\Phi = 0,$$

or

$$\frac{\rho \dfrac{1}{\rho}\dfrac{d}{d\rho}(\rho R')}{R} - \varkappa^2\rho^2 = -\frac{\Phi''}{\Phi} = \lambda.$$

This yields two ordinary differrential equations:

$$(1) \quad \Phi'' + \lambda\Phi = 0,$$

and

$$(2) \quad \rho\frac{d}{d\rho}(\rho R') - (\varkappa^2\rho^2 + \lambda)R = 0.$$

From equation (1), by using the fact that $\Phi(\varphi) = \Phi(\varphi + \pi)$, we obtain $\lambda = n^2$ $(n = 0, 1, 2, \dots)$ and $\Phi_n(\varphi) = A_n \cos(n\varphi) + B_n \sin(n\varphi)$, where A_n and B_n are arbitrary constants.

Further, equation (2) yields

$$\rho^2 R'' + \rho R' - (\varkappa^2\rho^2 + n^2)R = 0.$$

After the change of variables $x = \varkappa\rho$ we obtain the equation

$$x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} - (x^2 + n^2)R = 0.$$

This is recognized as being the Bessel equation of imaginary argument of order n. Its general solution has the form

$$R(x) = C_1 I_n(x) + C_2 K_n(x),$$

where $I_n(x)$ and $K_n(x)$ are the cylindrical functions of imaginary argument of first and second kind, respectively. Clearly, we must put $C_2 = 0$ because the solution is required to be bounded on the axis of the cylinder ($K_n(x)$ has a logarithmic singularity as $x \to 0$). Returning to the original variable we write

$$R(\rho) = C I_n(\varkappa\rho),$$

where C is an arbitrary constant.

Thus,

$$u(\rho, \varphi) = \sum_{n=0}^{\infty} \left[A_n \cos(n\varphi) + B_n \sin(n\varphi)\right] I_n(\varkappa\rho),$$

where the constants A_n are determined from the boundary condition. Namely, we have

$$u(a, \varphi) = \sum_{n=0}^{\infty} \left[A_n \cos(n\varphi) + B_n \sin(n\varphi)\right] I_n(\varkappa\rho),$$

and since $u(a, \varphi) = u_0$, we see that $A_0 = u_0/I_0(\varkappa a)$, while all the remaining terms of the series are equal to zero. Hence, the solution is

$$u(\rho, \varphi) = u_0 \frac{I_0(\varkappa\varphi)}{I_0(\varkappa a)}.$$

1.12. Boundary value problems for the Helmoltz equation in a disc

Example 1. Solve the following boundary value problem for the Helmholtz equation in a disc:

$$\begin{cases} \Delta u + k^2 u = 0, & 0 \le \varphi < 2\pi, \quad 0 < \rho < a, \\ u(a, \varphi) = f(\varphi), & 0 \le \varphi \le 2\pi; \end{cases}$$

here one assumes that k^2 is not equal to any of the eigenvalues λ of the homogeneous Dirichlet problem for the equation $\Delta u + \lambda u = 0$.

Solution. Using again separation of variables, we write $u(\rho, \varphi) = R(\rho)\Phi(\varphi)$, which upon substitution in the Helmoltz equation yields

$$\frac{1}{\rho} \frac{d}{d\rho}(\rho R') \cdot \Phi + R \frac{1}{\rho^2} \Phi'' + k^2 R \Phi = 0.$$

Hence,

$$\frac{\rho \dfrac{d}{d\rho}(\rho R')}{R} + k^2 \rho^2 = -\frac{\Phi''}{\Phi} = \lambda,$$

where λ is the separation constant.

The eigenvalues and corresponding eigenfunctions are obtained as the solutions of the already familiar problem

$$\begin{cases} \Phi'' + \lambda \Phi = 0, & -\infty < \varphi < \infty, \\ \Phi(\varphi) = \Phi(\varphi + 2\pi). \end{cases}$$

Hence, $\lambda = n^2$ and $\Phi_n(\varphi) = C_1 \cos(n\varphi) + C_2 \sin(n\varphi)$, $n = 0, 1, 2, \ldots$. Since

$$\frac{\rho \dfrac{d}{d\rho}(\rho R')}{R} + k^2\rho^2 = \lambda,$$

we obtain the following equation for the determination of $R(\rho)$:

$$\rho \frac{d}{d\rho}(\rho R') + (k^2\rho^2 - n^2)R = 0. \tag{1.38}$$

Denoting $x = k\rho$, we rewrite (1.38) in the form

$$x^2 \frac{d^2 R}{dx^2} + x\,\frac{dR}{dx} + (x^2 - n^2)R = 0.$$

This is the Bessel equation of order n and has the general solution

$$R(x) = C_1 J_n(x) + C_2 Y_n(x),$$

where $J_n(x)$ and $Y_n(x)$ are the nth order Bessel functions of the first and second kind, respectively, and C_1, C_2 are arbitrary constants.

Therefore, the solution of equation (1.38) has the form

$$R(\rho) = C_1 J_n(k\rho) + C_2 Y_n(k\rho).$$

Since $Y_n(k\rho) \to \infty$ as $\rho \to 0$ and we are interested in bounded solutions, we mus take $C_2 = 0$. Thus, $R_n(\rho) = J_n(k\rho)$ and the solution of our problem is represented as a series

$$u(\rho, \varphi) = \sum_{n=0}^{\infty} \left[A_n \cos(n\varphi) + B_n \sin(n\varphi)\right] J_n(k\rho). \tag{1.39}$$

The constants A_n and B_n are found from the boundary conditions. Setting $\rho = a$ in (1.39), we obtain

$$f(\varphi) = \sum_{n=0}^{\infty} \left[A_n \cos(n\varphi) + B_n \sin(n\varphi)\right] J_n(ka),$$

whence

$$A_n = \frac{1}{2\pi J_n(ka)} \int_0^{2\pi} f(\varphi)\cos(n\varphi)\, d\varphi, \qquad n = 0, 1, \ldots\,,$$

and

$$B_n = \frac{1}{2\pi J_n(ka)} \int_0^{2\pi} f(\varphi)\sin(n\varphi)\, d\varphi, \qquad n = 1, 2, \ldots\,.$$

In particular, if $f(\varphi) = A\sin(3\varphi)$, we have

$$B_3 = \frac{A}{J_3(ka)}, \quad A_n = 0, \quad n = 0, 1, \dots; \quad B_n = 0, \quad n \neq 3,$$

and the solution has the expression

$$u(\rho, \varphi) = \frac{A}{J_3(ka)} J_3(k\rho)\sin(3\varphi).$$

Problem 2. Solve the following Dirichlet problem for the Helmoltz equation:

$$\begin{cases} \Delta u + k^2 u = 0, & 0 \leq \varphi < 2\pi, \quad \rho > a, \\ u|_{\rho=a} = f(\varphi), & 0 \leq \varphi \leq 2\pi, \\ u_\rho + ik\rho = o(\rho^{-1/2}) & \text{as} \quad \rho \to \infty. \end{cases}$$

Solution. Here, as in the preceding example, we will use separation of variables to find the solution. The only difference is that in the present case, in order to make the solution unique, we must impose for $n = 2$ the radiation condition (Sommerfeld condition)

$$\frac{\partial u}{\partial \rho} + ik\rho = o(\rho^{-1/2}), \quad \rho \to \infty.$$

The solution of problem (1.38) takes now . the form

$$R(\rho) = C_1 H_n^{(1)}(k\rho) + C_2 H_n^{(2)}(k\rho),$$

where $H_n^{(1)}(x)$ and $H_n^{(2)}(x)$ are the Hankel functions of index n of the first and second kind, respectively. The behavior of the Hankel functions at infinity $\rho \to \infty$) is given by the asymptotic formulas

$$H_n^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}}\, e^{i\left(x - \frac{\pi n}{2} - \frac{\pi}{4}\right)} \left[1 + O\left(\frac{1}{x}\right)\right],$$

and

$$H_n^{(2)}(x) \sim \sqrt{\frac{2}{\pi x}}\, e^{-i\left(x - \frac{\pi n}{2} - \frac{\pi}{4}\right)} \left[1 + O\left(\frac{1}{x}\right)\right].$$

It readily checked directly that the radiation condition is satsified by the function $H_n^{(2)}(k\rho)$.

We see that the solution of the above exterior Dirichlet problem for the the Helmholtz equation is given by the series

$$u(\rho, \varphi) = \sum_{n=0}^{\infty} \left[A_n \cos(n\varphi) + B_n \sin(\varphi)\right] H_n^{(2)}(k\rho),$$

where the coefficients A_n and B_n are given by the formulas

$$A_n = \frac{1}{2\pi H_n^{(2)}(ka)} \int_0^{2\pi} f(\varphi)\cos(n\varphi)\, d\varphi, \quad n = 0, 1, \dots$$

and

$$A_n = \frac{1}{2\pi H_n^{(2)}(ka)} \int_0^{2\pi} f(\varphi)\sin(n\varphi)\, d\varphi, \quad n = 1, 2, \dots$$

1.13. Boundary value problems for the Helmoltz equation in a ball

Let us consider several examples of solutions for the interior and exterior Dirichlet and Neumann boundary value problems in a ball.

Example 1 [4, Ch. VII, no. 12]. Find the steady distribution of the concentration of an unstable gas inside a sphere of radius a if on the surface of the sphere one maintains the concentration $u|_{\partial\Omega} = u_0 \cos\theta$ ($u_0 = \text{const}$).

Solution. The problem is formulated mathematically as follows:

$$\begin{cases} \dfrac{1}{\rho^2}\dfrac{\partial}{\partial\rho}\left(\rho^2\dfrac{\partial u}{\partial\rho}\right) + \dfrac{1}{\rho^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial u}{\partial\theta}\right) - \varkappa^2 u = 0, \\ 0 < \rho < a, \quad 0 < \theta < \pi, \\ u(a,\theta) = u_0\cos\theta, \quad 0 \le \theta \le \pi. \end{cases} \tag{1.40}$$

As before, let us seek the solution in the form

$$u(\rho,\theta) = R(\rho)T(\theta).$$

Substituting this expression in equation (1.40) we obtain

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2 R')\cdot T + \frac{1}{\rho^2}\frac{1}{\sin\theta}R\cdot\frac{\partial}{\partial\theta}(\sin\theta\cdot T') - \varkappa^2 RT = 0$$

whence, upon dividing both sides by RT,

$$\frac{\dfrac{\partial}{\partial\rho}(\rho^2 R')}{R} - \varkappa^2\rho^2 = -\frac{\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}(\sin\theta\cdot T')}{T} = \lambda.$$

This yields two ordinary differential equations:

$$(1) \qquad \frac{d}{d\rho}(\rho^2 R') - (\varkappa^2\rho^2 + \lambda)R = 0,$$

and

$$(2) \qquad \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cdot T') + \lambda T = 0.$$

Performing the change of variables $x = \cos\theta$ in equation (2) (and using the conditions $|T(0)| < \infty$, $|T(\pi)| < \infty$), we find the eigenvalues and eigenfunctions

$$\lambda_n = n(n+1), \qquad T_n(\theta) = P_n(\cos\theta), \quad n = 0, 1\dots,$$

where $P_n(x)$ are the Legendre polynomials.

Equation (1) is readily reduced, via the substitution $v(\rho) = \sqrt{\rho}R(\rho)$, to the form (for each n)

$$\rho^2 v'' + \rho v' - \left[(\varkappa\rho)^2 + \left(n + \frac{1}{2}\right)^2\right] v = 0.$$

The corresponding bounded solutions of this equations are

$$v_n(\rho) = C I_{n+1/2}(\varkappa\rho),$$

where $I_{n+1/2}(x)$ are the Bessel functions of half-integer order and imaginary argument. Then

$$R_n(\rho) = \frac{I_{n+1/2}(\varkappa\rho)}{\sqrt{\rho}}.$$

Therefore, the solution of our problem is given by the series

$$u(\rho, \theta) = \sum_{n=0}^{\infty} C_n \frac{I_{n+1/2}(\varkappa\rho)}{\sqrt{\rho}} P_n(\cos\theta),$$

where the constants C_n are determined from the boundary conditions. Specifically,

$$u_0 \cos\theta = \sum_{n=0}^{\infty} C_n \frac{I_{n+1/2}(\varkappa a)}{\sqrt{a}} P_n(\cos\theta).$$

This yields $C_1 = u_0\sqrt{a}/I_{3/2}(\varkappa a)$ (the remaining coefficients are equal to zero). Finally,

$$u(\rho, \theta) = u_0 \frac{\sqrt{a}}{\sqrt{\rho}} \frac{I_{3/2}(\varkappa\rho)}{I_{3/2}(\varkappa a)} \cos\theta.$$

Example 2 [6, Ch. IV, 18.51]. Solve the Neumann problem for the equation $\Delta u + k^2 u = 0$ in the interior as well as in the exterior of the sphere $\rho = R$, under the condition $\partial u/\partial n|_{\rho=R} = A$, where A is a constant.

Solution. (a) The interior Neumann problem can be written as follows:

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right) + k^2 u = 0, \quad 0 < \rho < R, \quad 0 < \theta < \pi, \quad 0 \le \varphi < 2\pi, \tag{1.41}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\rho=R} = A, \qquad 0 \le \theta \le \pi, \quad 0 \le \varphi < 2\pi. \tag{1.42}$$

Since $\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right) = (\rho u)''$, equation (1.41) can be recast as

$$v'' + k^2 v = 0, \quad v(\rho) = \rho u(\rho).$$

The general solution of this equation is

$$v(\rho) = C_1 \cos(k\rho) + C_2 \sin(k\rho),$$

and consequently

$$u(\rho) = C_1 \frac{\cos(k\rho)}{\rho} + C_2 \frac{\sin(k\rho)}{\rho}\,.$$

Since the solution must be bounded at the center of the ball, we must put $C_1 = 0$, and so

$$u(\rho) = C \frac{\sin(k\rho)}{\rho}\,.$$

Now let us calculate the normal derivative:

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial \rho} = C \frac{k\cos(k\rho)\cdot \rho - \sin(k\rho)}{\rho^2}.$$

Further, using the boundary condition (1.42) we obtain

$$C \frac{Rk\cos(kR) - \sin(kR)}{R^2} = A,$$

whence

$$C = \frac{AR^2}{kR\cos(kR) - \sin(kR)}.$$

We conclude that the solution of the interior problem has the form

$$u(\rho) = \frac{AR^2}{kR\cos(kR) - \sin(kR)} \cdot \frac{\sin(k\rho)}{\rho}\,.$$

(b) The exterior Neumann problem reads:

$$\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) + k^2 u = 0, \quad \rho > R, \quad 0 \le \theta \le \pi, \quad 0 \le \varphi < 2\pi, \tag{1.43}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\rho = R} = A, \qquad 0 \le \theta \le \pi, \quad 0 \le \varphi < 2\pi, \tag{1.44}$$

$$u_\rho - iku = o(\rho^{-1}) \quad \text{as } \rho \to \infty. \tag{1.45}$$

As in item (a), equation (1.43) can be recast as

$$v'' + k^2 v = 0, \quad v(\rho) = \rho u(\rho).$$

The general solution of this equation is

$$v(\rho) = C_1 e^{ik\rho} + C_2 e^{-ik\rho}, \qquad \rho \to \infty.$$

Therefore,

$$u(\rho) = C_1 \frac{e^{ik\rho}}{\rho} + C_2 \frac{e^{-ik\rho}}{\rho}.$$

Let us verify that the function $u_1(\rho) = \frac{e^{ik\rho}}{\rho}$ satisfies the Sommerfeld condition

$$\frac{\partial u_1}{\partial \rho} - iku_1 = o(\rho^{-1}) \qquad (\rho \to \infty),$$

i.e., that

$$\lim_{\rho\to\infty} \left[\rho \left(\frac{\partial u}{\partial \rho} - iku_1\right)\right] = 0.$$

Indeed,

$$\rho\left(\frac{\partial u}{\partial \rho} - iku_1\right) = \rho\left(iku_1 - \frac{1}{\rho}u_1 - iku_1\right) = -\frac{e^{ik\rho}}{\rho} \quad \text{and} \quad \left|\frac{e^{ik\rho}}{\rho}\right| \leq \frac{1}{\rho}.$$

It follows that to pick a unique solution we must set $C_2 = 0$, and then $u(\rho) = C\frac{e^{ik\rho}}{\rho}$.

Now let us calculate the normal derivative:

$$\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial \rho} = -\frac{\partial}{\partial \rho}\left(C\frac{e^{ik\rho}}{\rho}\right).$$

We have

$$\frac{\partial u}{\partial n} = \frac{Ce^{ik\rho}}{\rho^2}(1 - ik\rho),$$

and the boundary condition (1.44) yields

$$C = \frac{AR^2}{e^{ikR}(1 - ikR)}.$$

Thus, the solution of the exterior Neumann problem is given by the formula

$$u(\rho) = \frac{AR^2}{e^{ikR}(1 - ikR)} \frac{e^{ik\rho}}{\rho}.$$

Example 3 [6, Ch. V, 18.53]. Solve the Dirichlet problem for the equation $\Delta u - k^2 u = 0$ in the interior and in the exterior of the sphere of radius $\rho = R$ with the condition $u|_{\rho=R} = A$, where A is a constant.

Solution. (a) First let us solve the interior Dirichlet problem

$$\begin{cases} \Delta u - k^2 u = 0, \qquad 0 < \rho < R, \quad 0 < \theta < \pi, \quad 0 \leq \varphi < 2\pi, \\ u|_{\rho=R} = A, \qquad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi < 2\pi. \end{cases}$$

By analogy with Example 2, we must solve the equation $v'' - k^2 v = 0$, where $v(\rho) = \rho u(\rho)$, which has the general solution

$$v(\rho) = C_1 \sinh(k\rho) + C_2 \cosh(k\rho).$$

Therefore,

$$u(\rho) = C_1 \frac{\sinh(k\rho)}{\rho} + C_2 \frac{\cosh(k\rho)}{\rho}\,.$$

where C_1 and C_2 are arbitrary constants.

Note that $\cosh(k\rho)/\rho \to \infty$ as $\rho \to 0$. Hence, we must put $C_2 = 0$, and the solution has the expression

$$u(\rho) = C\,\frac{\sinh(k\rho)}{\rho}\,.$$

The constant C is determined from the boundary condition $u(R) = A$, i.e., $C \sinh(kR)/R = A$, which yields $C = A\,R/\sinh(kR)$. We conclude that the solution of our problem is

$$u(\rho) = A\,\frac{R}{\rho}\,\frac{\sinh(k\rho)}{\sinh(kR)}\,.$$

(b) Now let us solve the exterior Dirichlet problem

$$\begin{cases} \Delta u - k^2 u = 0, \qquad \rho > R, \quad 0 < \theta < \pi, \quad 0 \le \varphi < 2\pi, \\ u|_{\rho=R} = A, \qquad 0 \le \theta \le \pi, \quad 0 \le \varphi < 2\pi, \\ u(\rho) \to 0 \quad \text{as } \rho \to \infty. \end{cases}$$

In this case

$$u(\rho) = C_1 \frac{e^{k\rho}}{\rho} + C_2 \frac{e^{-k\rho}}{\rho}.$$

Since the solution of the exterior problem must satisfy $u(\rho) \to 0$ when $\rho \to \infty$, we must put $C_1 = 0$. Therefore,

$$u(\rho) = C\frac{e^{-k\rho}}{\rho}.$$

The boundary condition $u|_{\rho=R} = A$ yields $C = AR/e^{-kR}$. We conclude that the solution of our problem is

$$u(\rho) = A\,\frac{R}{\rho}\,\frac{e^{-k\rho}}{e^{-kR}}\,.$$

1.14. Guided electromagnetic waves

In this section we will conisder problem connected with steady processes of propagation of electromagnetic waves in systems that have the property of producing conditions under which waves propagate essentially in a given direction. Such waves are know as *guided waves*, and the systems that guide them are called *waveguides*.

The basic tool that we will use to simplify the analysis of such problems is the representation of elecctromagnetic waves as a superposition of waves of several types.

Let us assume that the x_3-axis coincides with the direction of wave propagation. The electromagnetic field of the wave is described by six components, E_1, E_2, E_3, H_1, H_2, H_3, of the electric and magnetic field vectors. Let us write them in matrix form:

$$A = \begin{bmatrix} 0 & E_2 & 0 \\ H_1 & 0 & H_3 \end{bmatrix}, \qquad B = \begin{bmatrix} E_1 & 0 & E_3 \\ 0 & H_2 & 0 \end{bmatrix}.$$

It is clear that the electric field vector $(0, E_2, 0)$ is orthogonal to the direction of propagation of the wave, whereas the magnetic field vector $(H_1, 0, H_2)$ has a nonzero component along that direction. In the matrix B the vector $(E_1, 0, E_3)$ has a nonzero component along the x_3-axis, whereas the vector $(0, H_2, 0)$ is orthogonal to the x_3-axis. In connection with this circumstance the waves characterized by the matrix A are referred to as *transversally-electric* (TE-waves), while those characterized by the matrix B are referred to as *transversally-magnetic* (TM-waves).

It is convenient to consider that an electromagnetic wave is a TE-wave [resp., TM-wave] if $E_3 = 0$ [resp., $H_3 = 0$].

There exists also waves of a third type, characterized by the matrix

$$C = \begin{bmatrix} E_1 & E_2 & 0 \\ H_1 & H_2 & 0 \end{bmatrix}.$$

These are called *transversally- electromagnetic* waves, or TEM-waves.

Example. TM-waves in a waveguide of circular cross section.

Let us consider the propagation of TM-waves in an infinitely long cylinder of radius a. It is known that this problem is connected with the solvability of the following Dirichlet problem for the Helmholtz equation:

$$\begin{cases} \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\,\dfrac{\partial E_3}{\partial\rho}\right) + \dfrac{1}{\rho^2}\,\dfrac{\partial^2 E_3}{\partial\varphi^2} + \delta^2 E_3 = 0, \qquad 0 < \rho < a, \quad 0 \le \varphi < 2\pi, \\ E_3(a,\varphi) = 0, \qquad 0 \le \varphi \le 2\pi, \end{cases}$$

where δ^2 is a real constant.

Separating the variables by means of the substitution $E_3 = R(\rho)\Phi(\varphi)$, we arrive at the equations

$$\begin{cases} \Phi'' + \lambda\Phi = 0, \\ \rho^2 R'' + \rho R' + (\delta^2\rho^2 - \lambda)R = 0, \end{cases} \tag{1.46}$$

where λ is the separation constant. Since $\Phi(\varphi) = \Phi(\varphi+2\pi)$, it follows that $\lambda = n^2$, $n = 0, 1, 2, \ldots$

The change of variables $x = \delta\rho$ takes the second equation in(1.46) into the Bessel equation of order n in the new variable x. Since $|R(0)| < \infty$ and $R(a) = 0$, we have

$$R(\rho) = J_n(\delta_{nm}\rho).$$

Here $\delta_{nm} = \mu_{nm}/a$, where μ_{nm} is the mth positive root of the nth order Bessel function $J_n(x)$.

Therefore, our problem admits the following particular solutions:

$$E_{3,nm} = J_n(\delta_{nm}\rho)\left[A_{nm}\cos(n\varphi) + B_{nm}\sin(n\varphi)\right],$$

where A_{nm} and B_{nm} are arbitrary constants. Each of these solutions corresponds to a certain TM-wave, which can propagate without damping in the given waveguide.

Remark 1. The propagation of a TE-wave in an infinitely long cylinder is associated with the solvability of the Neumann problem for the Helmoltz equation:

$$\begin{cases} \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial H_3}{\partial\rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 H_3}{\partial\varphi^2} + \delta^2 H_3 = 0, \qquad 0 < \rho < a, \quad 0 \le \varphi < 2\pi, \\ \dfrac{\partial H_3}{\partial\vec{n}}(a,\varphi) = 0, \qquad 0 \le \varphi \le 2\pi; \end{cases}$$

here $\vec{n}$ is the unit outer normal of the cylindrical waveguide.

By analogy with the preceding example, we obtain

$$H_{3,nm} = J_n(\delta_{nm}\rho)\left[A_{nm}\cos(n\varphi) + B_{nm}\sin(n\varphi)\right],$$

where now $\delta_{nm} = \lambda nm/a$, with λ_{nm} being the mth positive root of the equation $dJ_n(x)/dx = 0$, $n = 0, 1, 2, \ldots$

Remark. 2 If the component E_3 (or H_3) is known, then the other components of the electric and magnetic field vectors can be found by only one differentiation (this follows from the Maxwell equations for the electromagnetic field).

1.15. The method of conformal mappings (for the solution of boundary value problems in the plane)

Methods of the theory of functions of a complex variables found wide and effective application in solving a large number of mathematical problems that arise in various fields of science. In particular, in many cases the application of complex functions yields simple methods for solving boundary value problems for the Laplace equation. This is the result of the intimate connection between analytic functions of a complex variable and harmonic functions of two real variables, and also of the invariance of the Laplace equation under conformal mappings.

Suppose one wants to solve the Laplace equation $u_{xx} + u_{yy} = 0$ with some boundary condition in a domain of complicated shape in the plane of the variables x, y. This boundary value problem can be transformed into a new boundary value problem, in which one is required to solve the Laplace quation $\widetilde{u}_{\xi\xi} + \widetilde{u}_{\eta\eta} = 0$ in a simpler domain of the variables ξ, η, and such that the second domain is obtained from the first one via a comformal mapping $\zeta = f(z)$, where $z = x+iy$, $\zeta = \xi + i\eta$, and $\widetilde{u}(\zeta) = u(z)$ for $\zeta = f(z)$ (Figure 1.6).

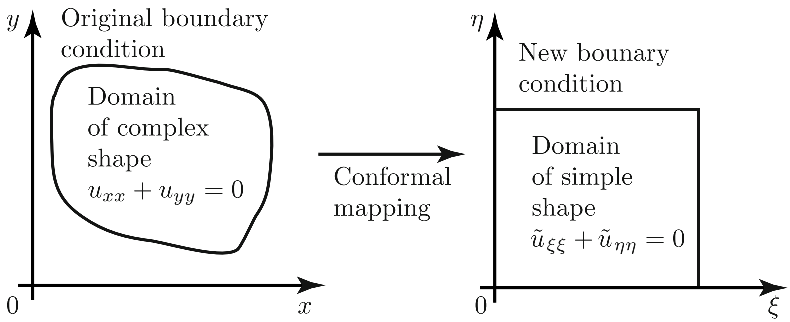

Figure 1.6.

Once the solution $\widetilde{u}(\xi, \eta)$ of the Laplace equation in a simple domain (disc, half-space, rectangle) is found, it suffices to substitute in that solution the expressions $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ in order to obtain the solution $u(x, y)$ of the original problem, expressed in the original variables.

Let us give several examples to show how to solve boundary value problems for the Laplace equation (in the plane) by means of conformal mappings.

Example 1 [6, Ch. V, 17.13(4)]. Find the solution of the equation $\Delta u = 0$ in the first quadrant $x > 0, y > 0$, with the boundary conditions $u|_{x=0} = 0$, $u|_{y=0} = \theta(x-1)$, where $\theta(x) = 1$ if $x > 0$, $\theta(x) = 0$ if $x \le 0$ is the Heaviside function.

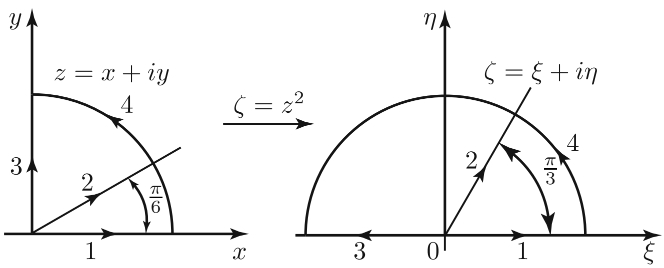

Figure 1.7.

Solution. Clearly, the function $\zeta = z^2$, defined in the first quadrant of the complex z-plane, maps this domain into the entire half-plane $\eta > 0$ of the complex ζ-plane (Figure 1.7), in such a manner that:

the positive x-semiaxis is mapped into the positive real ξ-semiaxis;
the positive y-semiaxis is mapped into the negative real ξ-semiaxis.

Thus, we arrive at the following conclusion:

Boundary value problem in the plane (x, y) $\longrightarrow$ Boundary value problem in the plane (ξ, η)

$$\begin{cases} \Delta u = 0, \quad x > 0, y > 0, \\ u|_{x=0} = 0, \quad y \geq 0, \\ u|_{y=0} = \theta(x-1), \quad x \geq 0, \end{cases} \longrightarrow \begin{cases} \Delta \widetilde{u} = 0, \quad \eta > 0, \\ \widetilde{u}|_{\eta=0} = \begin{cases} 1 & \text{if } \xi > 1, \\ 0 & \text{if } \xi < 1. \end{cases} \end{cases}$$

Notice also that from the equality $\zeta = z^2$, i.e., $\xi + i\eta = (x+iy)^2$, it follows that $\xi = x^2 - y^2$ and $\eta = 2xy$.

The solution of the Dirichlet problem in the (ξ, η)- plane is given by the Poisson integral

$$\widetilde{u}(\xi, \eta) = \frac{\eta}{\pi} \int_{-\infty}^{\infty} \frac{\widetilde{u}(t, 0)\, dt}{(t-\xi)^2 + \eta^2}\,.$$

Imposing the boundary condition on $\widetilde{u}(\xi, 0)$, we obtain

$$\widetilde{u}(\xi, \eta) = \frac{\eta}{\pi} \int_{-\infty}^{\infty} \frac{dt}{(t-\xi)^2 + \eta^2} = \frac{1}{\pi} \text{arctan}\, \frac{t-\xi}{\eta}\bigg|_1^{\infty} =$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - \text{arctan}\frac{1-\xi}{\eta}\right) = \frac{1}{2} - \frac{1}{\pi}\text{arctan}\frac{1-\xi}{\eta}\,.$$

If we now write $x^2 - y^2$ instead of ξ and $2xy$ instead of η, we obtain the solution of the original problem in the form

$$u(x, y) = \frac{1}{2} - \frac{1}{\pi}\text{arctan}\frac{y^2 - x^2 + 1}{2xy}\,.$$

Example 2 [6, Ch. V, 17.14(2)]. Find the solution of the Dirichlet problem for the equation $\Delta u = 0$ in the strip $0 < y < \pi$, with the boundary conditions $u|_{y=0} = \theta(x)$, $u|_{y=\pi} = 0$.

Solution. The complex function $\zeta = e^z$, defined in the strip $0 < y < \pi$, maps this strip into the entire half-plane $\eta > 0$ of the complex ζ-plane (Figure 1.8), in such a manner that:

the positive x-semiaxis is mapped into the positive ξ-semiaxis $[1, \infty)$;
the negative x-semiaxis is mapper into the interval $(0, 1)$;
the line $y = \pi$ is mapped into the negative ξ-semiaxis.

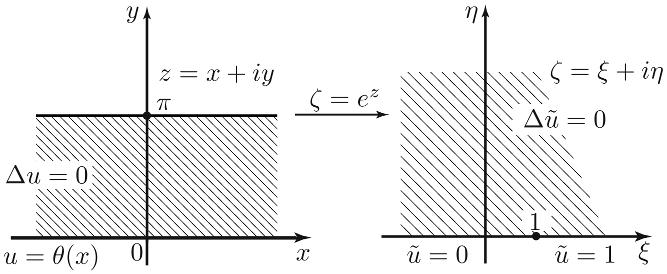

FIGURE 1.8.

Thus, we arrive at the following conclusion:

Boundary value problem in the plane (x, y)

$$\begin{cases} \Delta u = 0, \quad -\infty < x < \infty,\ 0 < y < \pi, \\ u|_{y=0} = \theta(x), \quad -\infty < x < \infty, \\ u|_{y=\pi} = 0, \quad -\infty < x < \infty, \end{cases} \longrightarrow$$

Boundary value problem in the plane (ξ, η)

$$\begin{cases} \Delta\widetilde{u} = 0, \quad \eta > 0, \\ \widetilde{u}|_{\eta=0} = \begin{cases} 1 & \text{if } \xi > 1, \\ 0 & \text{if } \xi < 1. \end{cases} \end{cases}$$

Notice also that $\xi = e^x \cos y$ and $\eta = e^x \sin y$. As in Example 1, we have

$$\widetilde{u}(\xi, \eta) = \frac{1}{2} - \frac{1}{\pi}\arctan\frac{1-\xi}{\eta},$$

and so the solution has the form

$$u(x, y) = \frac{1}{2} - \frac{1}{\pi}\arctan\frac{e^{-x} - \cos y}{\sin y}.$$

Example 3 [6, Ch. V, 17.18]. Find the solution of the Dirichlet problem

$$\begin{cases} \Delta u = 0, \quad \operatorname{Re} z > 0, \quad |z-5| > 3, \\ u|_{\operatorname{Re} z=0} = 0, \quad u|_{|z-5|=3} = 1. \end{cases}$$

Solution. First let us draw the domain D where we must solve the Dirichlet problem (Figure 1.9). It can be regarded as a eccentric annulus (indeed, a line is a circle of infinite radius). Let us find a conformal mapping of D onto a concentric annulus. To this purpose let us find two points that are simultaneously symmetric with respect to the line $\operatorname{Re} z = 0$ and with respect to the circle $|z-5| = 3$. Clearly, such points must lie on the common perpendicular to the line and the circle, i.e., on the real axis. From the symmetry with respect to the line $\operatorname{Re} z = 0$ it follows that these are precisely the points $x_1 = a$ and $x_2 = -a$ with $a > 0$. The symmetry with

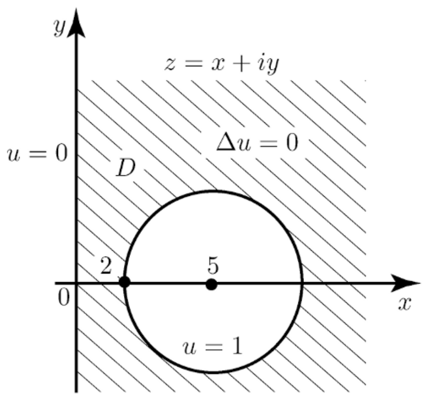

FIGURE 1.9.

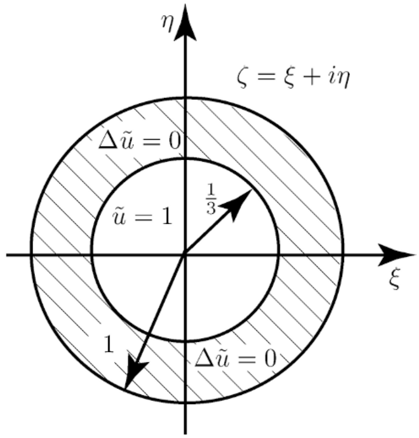

FIGURE 1.10

respect to the circle $|z-5|=3$ translates into the equation $(5+a)(5-a)=9$, which yields $a=4$.

Let us show that the conformal transformation we are seeking is given by the linear-fractional function

$$\zeta = \frac{z-4}{z+4}\,. \tag{1.47}$$

Indeed, this mapping takes the line $\operatorname{Re} z = 0$ into a circle γ. Since symmetry must be preserved, the points $z_1 = 4$ and $z_2 = -4$ are taken into the points $\zeta = 0$ and $\zeta = \infty$, respectively, which are symmetric with respect to the circle γ. Hence, $\zeta = 0$ is the center of γ. Further, since the point $z = 0$ is taken into the point $\zeta = 1$, γ is the circle $|\zeta| = 1$ (Figure 1.10).

Now let us show than under the above mapping the circle $|z-5| = 3$ goes into the circle $|\zeta| = 1/3$. Indeed, the linear-fractional-transformation (1.47) takes the circle $|z-5| = 3$ into a circle, of radius $|\zeta| = |(2-4)/(2+4)| = 1/3$. We see that (1.47) maps the domain D conformally onto the concentric annulus $\frac{1}{3} < |\zeta| < 1$. We conclude that the given boundary value problem in the plane (x, y)

$$\begin{cases} \Delta u = 0, \quad \operatorname{Re} z > 0, \quad |z-5| = 3, \\ u|_{\operatorname{Re} z = 0} = 0, \\ u|_{|z-5|=3} = 1, \end{cases}$$

is transformed into the following boundary value problem in the plane (ξ, η):

$$\Delta \widetilde{u} = 0, \quad \frac{1}{3} < |\zeta| < 1, \tag{1.48}$$

$$\widetilde{u}|_{|\zeta|=1} = 0, \quad \widetilde{u}|_{|\zeta|=\frac{1}{3}} = 1. \tag{1.49}$$

Let us solve the problem in the annulus $1/3 < |\zeta| < 1$ (in the plane (ξ, η)). Since the boundary conditions (1.49) do not depend on the polar angle φ, it is natural to assume that the solution $\widetilde{u}(\zeta)$ depends only on the variable ρ (here $\xi = \rho\cos\varphi$, $\eta = \rho\sin\varphi$). To find this solution, we rewrite the equation $\Delta\widetilde{u}$ in the form $\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial\widetilde{u}}{\partial\rho}\right) = 0$. The general solution of this equation is

$$\widetilde{u}(\zeta) = c_1 + c_2 \ln\rho,$$

where c_1 and c_2 are arbitrary constants. Imposing the conditions (1.49), we obtain $c_1 = 0$, $c_2 = -1/\ln 3$. Therefore,

$$\widetilde{u}(\zeta) = -\frac{1}{\ln 3}\ln|\zeta|, \quad \text{because } \rho = |\zeta|.$$

To find the solution of the original problem it suffices to return to the variable z, using (1.47), which finally yields

$$u(z) = \frac{1}{\ln 3}\,\ln\left|\frac{z+4}{z-4}\right|.$$

1.16. The Green function method

Definition of the Green function. Let us consider the boundary value problem

$$\begin{cases} \Delta u = f \quad \text{in the domain } \Omega, \\ \left(\alpha_1 u + \alpha_2 \dfrac{\partial u}{\partial n}\right) = g \quad \text{on the boundary } \partial\Omega. \end{cases} \tag{1.50}$$

We shall assume that the function $u(x)$ is continuous together with it first-order derivatives in the closed domain $\overline{\Omega} \subset \mathbf{R}^n$, bounded by a sufficiently smooth hypersurface $\partial\Omega$, and has second-order derivatives that are square integrable in Ω. Here $\vec{n}$ is the outward unit normal to $\partial\Omega$ and α_1, α_2 are given real numbers satisfying $\alpha_1^2 + \alpha_2^2 \neq 0$; $x = (x_1, \dots, x_n)$.

The Green function method for solving such problems consists in the following. First we solve the auxiliary problem (see [1])

$$\begin{cases} \Delta G = -\delta(x, x_0), \quad x_0 \in \Omega, \\ \left(\alpha_1 G + \alpha_2 \dfrac{\partial G}{\partial n}\right)\Big|_{\partial\Omega} = 0, \end{cases} \tag{1.51}$$

where $\delta = \delta(x, x_0)$ is the δ-function, which can formally be defined by the relations

$$\delta(x, x_0) = \begin{cases} 0, & \text{if } x \neq x_0, \\ \infty, & \text{if } x = x_0, \end{cases} \quad , \qquad \int_\Omega \delta(x, x_0)dx = \begin{cases} 1, & \text{if } x_0 \in \Omega, \\ 0, & \text{if } x_0 \notin \Omega, \end{cases}$$

where $x_0 = (x_{01}, \dots, x_{n0})$ (the notation dx is obvious). The main property of the δ-function is expressed by the equality

$$\int_\Omega \delta(x, x_0) f(x) dx = \begin{cases} f(x_0), & \text{if } x_0 \in \Omega, \\ 0, & \text{if } x_0 \notin \Omega, \end{cases}$$

where $f(x)$ is an arbitrary continuous function of the point x.

Definition. The solution of problem (1.51) is called the *Green function* of problem (1.50).

We will require that the Green function $G(x, x_0)$ be continuous (together with its first-order partial derivatives) everywhere in the closed domain $\overline{\Omega}$, except for the point x_0, at which $G(x, x_0)$ may have a singularity.

Once the function $G(x, x_0)$ is found, one can use it to easily find the solution of the original problem (1.50). To that end we will use the *second Green formula*

$$\int_\Omega (v\Delta u - u\Delta v) dx = \int_{\partial\Omega} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds. \tag{1.52}$$

This formula is readily obtained from the Gauss-Ostrogradskiĭ formula

$$\int_{\partial\Omega} (\vec{a}, \vec{n}) ds = \int_\Omega \operatorname{div} \vec{a}\, dx$$

(where $\vec{a}$ is a vector field and $(\vec{a}, \vec{n})$ denotes the scalar product of the vectors $\vec{a}$ and $\vec{n}$) if one puts successively $\vec{a} = v\nabla u$ and $\vec{a} = u\nabla v$ and subtract the results from one another. Indeed, we have

$$\int_{\partial\Omega} v(\nabla u, \vec{n}) ds = \int_\Omega \operatorname{div}(v\nabla u) dx, \tag{1.53}$$

and

$$\int_{\partial\Omega} u(\nabla v, \vec{n}) ds = \int_\Omega \operatorname{div}(u\nabla v) dx. \tag{1.54}$$

Since $(\nabla u, \vec{n}) = \partial u/\partial n$, $(\nabla v, \vec{n}) = \partial v/\partial n$, $\operatorname{div}(v\nabla u) = (\nabla u, \nabla v) + v\Delta u$ and $\operatorname{div}(u\nabla u) = (\nabla u, \nabla v) + u\Delta v$, subtracting (1.54) from (1.53) we get the second Green formula.

Now let us put $v = G$ in (1.52). Then, since $\Delta u = f(x)$ and $\Delta G = -\delta(x, x_0)$, we obtain

$$\int_\Omega G(x, x_0) f(x) dx + \int_\Omega u(x)\delta(x, x_0) dx = \int_{\partial\Omega} \left(G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n} \right) ds.$$

But, by the main property of the δ-function,

$$\int_\Omega u(x)\delta(x, x_0) dx = u(x_0),$$

and so the last equality yields

$$u(x_0) = \int_{\partial\Omega} \left(G \frac{\partial u}{\partial n} - u \frac{\partial G}{\partial n} \right) ds - \int_{\Omega} G(x, x_0) f(x) dx.$$

From this formula we obtain:

(a) the solution of the Dirichlet problem for

$$\alpha_1 = 1, \quad \alpha_2 = 0, \quad G|_{\partial\Omega} = 0, \quad u|_{\partial\Omega} = g$$

in the form

$$u(x_0) = -\int_{\partial\Omega} g \frac{\partial G}{\partial n} ds - \int_{\Omega} G(x, x_0) f(x) dx;$$

(b) the solution of the Neumann problem for

$$\alpha_1 = 0, \quad \alpha_2 = 1, \quad \left.\frac{\partial G}{\partial n}\right|_{\partial\Omega} = 0, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = g$$

in the form

$$u(x_0) = \int_{\partial\Omega} Gg\, ds - \int_{\partial\Omega} G(x, x_0) f(x) dx.$$

Remark 1. The integral

$$\int_{\partial\Omega} G(x, x_0) f(x) dx$$

admits the following physical interpretation: the right-hand side of the equation is regarded as an external action on the system and is decomposed into a continual contribution of source distributed over the domain Ω. Then one finds the response of the system to each such source and one sums all these responses.

Construction of the Green function. One of the methods for constructing the Green function is the *reflection method*. For example, the Green function for the Poisson equation in the case of the half-space ($z > 0$) has the form

$$G(M, M_0) = \frac{1}{4\pi R_{MM_0}} - \frac{1}{4\pi R_{MM_1}},$$

where R_{AB} denotes the distance between the points A and B, $M_0(x_0, y_0, z_0)$ is a point lying in the uper half-plane $z > 0$, $M_1(x_0, y_0, -z_0)$ is the point symmetric to $M_0(x_0, y_0, z_0)$ with respect to the plane $z = 0$, and $M(x, y, z)$ is an arbitrary point of the half-plane $z > 0$.

Physically the Green function can be interpreted as the potential of the field produced by point-like charges placed at the point M_0 (over the grounded plane $z = 0$) and the point M_1 (Figure 1.11).

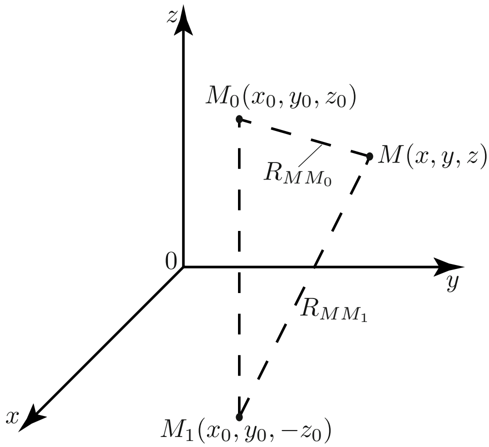

FIGURE 1.11. The potential at the point $M(x, y, z)$ equals $G(M, M_0) = \frac{1}{4\pi R_{MM_0}} - \frac{1}{4\pi R_{MM_1}}$ ($z = 0$ is a grounded conducting plane)

Notice that in the case of a half-plane ($y > 0$) the Green function has the form (Figure 1.12)

$$G(M, M_0) = \frac{1}{2\pi} \ln \frac{1}{R_{MM_0}} - \frac{1}{2\pi} \ln \frac{1}{R_{MM_1}} .$$

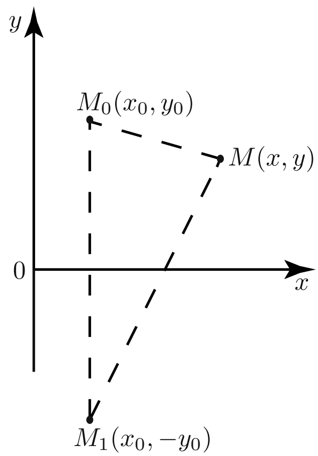

FIGURE 1.12. The potential at the point $M(x, y)$ equals $G(M, M_0) = \frac{1}{2\pi} \ln \frac{1}{R_{MM_0}} - \frac{1}{2\pi} \ln \frac{1}{R_{MM_1}}$

Examples of problems solved by means of the Green function. Suppose we want to solve the Dirichlet problem for the Laplace equation in a half-plane

$$\begin{cases} \Delta u = 0, \quad -\infty < x < \infty, \quad y > 0, \\ u(x,0) = f(x), \quad -\infty < x < \infty. \end{cases}$$

The solution of this problem is

$$u(x,y) = -\frac{y}{\pi} \int_{-\infty}^{\infty} f(s) \left. \frac{\partial G}{\partial t} \right|_{t=0} ds$$

(we put $M_0 = M_0(x,y)$, $M = M(s,t)$), where

$$G(x,y;s,t) = \frac{1}{2\pi} \frac{1}{\sqrt{(x-s)^2 + (y-t)^2}} - \frac{1}{2\pi} \frac{1}{\sqrt{(x-s)^2 + (y+t)^2}} .$$

Calculating $\partial G/\partial t|_{t=0}$, we obtain

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(s)}{(s-x)^2 + y^2} \, ds. \tag{1.55}$$

Example 1 [3, no. 244]. Find a function $u(x,y)$, harmonic in the half-plane $y > 0$, if it is known that

$$u(x,0) = \frac{x}{x^2+1} .$$

Solution. We must calculate the integral

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{s}{(1+s^2)[(s-x)^2 + y^2]} \, ds.$$

Apparently, the easiest way to do this is to use the method of residues, namely, the following formula:

$$\int_{-\infty}^{\infty} \frac{s}{(1+s^2)[(s-x)^2 + y^2]} \, ds = 2\pi i [\mathrm{res}[f(z)]_{z=i} + \mathrm{res}[f(z)]_{z=x+iy},$$

where $f(z) = z/\big((1+z^2)[(z-x)^2 + y^2]\big)$.

Since

$$\mathrm{res}[f]_{z=i} = \frac{1}{2[(i-x)^2 + y^2]}, \qquad \mathrm{res}[f(z)]_{z=x+iy} = \frac{x+iy}{2iy[1 + (x+iy)^2]},$$

it follows that

$$\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{s}{(1+s^2)[(s-x)^2 + y^2]} \, ds = \frac{iy}{[(i-x)^2 + y^2]} + \frac{x+iy}{[1 + (x+iy)^2]} =$$

$$= \frac{iy}{[(i-x) + iy][(i-x) - iy]} + \frac{x+iy}{(x+iy-1)(x+iy+1)} =$$

$$= \frac{1}{2}\left[\frac{1}{i-x-iy} - \frac{1}{i-x+iy}\right] + \frac{1}{2}\left[\frac{1}{x+iy-1} + \frac{1}{x+iy+1}\right] =$$

$$= \frac{1}{2}\left[\frac{1}{i(1-y)-x} - \frac{1}{i(1+y)-x} + \frac{1}{i(y-1)+x} + \frac{1}{i(1+y)+x}\right] =$$

$$= \frac{1}{2}\left[\frac{1}{i(1+y)+x} - \frac{1}{i(1+y)-x}\right] =$$

$$= \frac{1}{2}\left[\frac{x-i(1+y)}{x^2+(1+y)^2} + \frac{x+i(1+y)}{x^2+(1+y)^2}\right] = \frac{x}{x^2+(1+y)^2}.$$

Therefore, the solution of the problem is given by

$$u(x,y) = \frac{x}{x^2+(1+y)^2}.$$

Remark 2. The solution of the problem considered above,

$$\begin{cases} \Delta u = 0, \quad -\infty < x < \infty, \quad y > 0, \\ u|_{y=0} = \dfrac{x}{x^2+1}, \end{cases}$$

can also be obtained without resorting to the Green function.

Indeed, one can use the fact that the function $u = \frac{1}{2\pi} \ln \frac{1}{\sqrt{x^2+(y+a)^2}}$, where $a \geq 0$, is a solution of the Laplace equation in the upper half-plane $y > 0$, i.e.,

$$\Delta \ln \frac{1}{\sqrt{x^2+(y+a)^2}} = 0.$$

Differentiating this equality with respect to x, we obtain

$$\frac{\partial}{\partial x}\Delta \ln \frac{1}{\sqrt{x^2+(y+a)^2}} = 0, \quad \text{or} \quad \Delta \frac{\partial}{\partial x} \ln \frac{1}{\sqrt{x^2+(y+a)^2}} = 0,$$

i.e., $\Delta(x/r^2) = 0$, where $r = \sqrt{x^2+y^2}$.

Thus, the function $u = x/[x^2+(y+a)^2]$ is harmonic in the upper half-plane. Imposing the boundary condition, we conclude that the solution of our Dirichlet problem is the function

$$u(x,y) = \frac{x}{x^2+(y+1)^2}.$$

Example 2 [6, Ch. V, 17.4(2)]. Find the solution of the Dirichlet problem

$$\begin{cases} \Delta u = 0, \quad -\infty < x, y < \infty, \quad z > 0, \\ u|_{z=0} = \cos x \cos y, \quad -\infty < x, y < \infty. \end{cases}$$

Solution. It is known that the harmonic function we are asked to find is given by formula

$$u(x,y,z) = \frac{z}{2\pi}\iint\limits_{-\infty}^{\infty} \frac{\cos\xi\cos\eta\,d\xi\,d\eta}{[(\xi-x)^2+(\eta-y)^2+z^2]^{3/2}}\,.$$

To calculate this integral we will make change the variables $\xi - x = u$, $\eta - y = v$, the Jacobian of which is 1. We obtain

$$u(x,y,z) = \frac{z}{2\pi}\iint\limits_{-\infty}^{\infty} \frac{\cos(u+x)\cos(v+y)\,du\,dv}{(u^2+v^2+z^2)^{3/2}} =$$

$$= \frac{z}{2\pi}\iint\limits_{-\infty}^{\infty} \frac{(\cos u\cos x - \sin u\sin x)(\cos v\cos y - \sin v\sin y)\,du\,dv}{(u^2+v^2+z^2)^{3/2}} =$$

$$= \frac{z}{2\pi}\cos x\cos y\iint\limits_{-\infty}^{\infty} \frac{\cos u\cos v\,du\,dv}{(u^2+v^2+z^2)^{3/2}}$$

because the other three integrals vanish thanks to the fact that their integrands are odd functions.

Now let us calculate the integral

$$J = \iint\limits_{-\infty}^{\infty} \frac{\cos u\cos v\,du\,dv}{(u^2+v^2+z^2)^{3/2}} =$$

$$= \iint\limits_{-\infty}^{\infty} \frac{[\cos(u+v)+\sin u\sin v]du\,dv}{(u^2+v^2+z^2)^{3/2}} = \iint\limits_{-\infty}^{\infty} \frac{\cos(u+v)\,du\,dv}{(u^2+v^2+z^2)^{3/2}},$$

because the other integral is equal to zero.

Let us make the change of variables

$$p = \frac{1}{\sqrt{2}}(u+v), \quad q = \frac{1}{\sqrt{2}}(u-v),$$

which correspond to a counter-clockwise rotation of the plane by 45°. Then we have

$$\iint\limits_{-\infty}^{\infty} \frac{\cos(\sqrt{2}p)\,dp\,dq}{(p^2+q^2+z^2)^{3/2}} = \int_{-\infty}^{\infty}\cos(\sqrt{2}p)\,dp\int_{-\infty}^{\infty}\frac{dq}{(p^2+q^2+z^2)^{3/2}}\,.$$

But the substitution $q = \sqrt{p^2 + z^2}\tan t$ transforms the integral

$$J_1 = \int_{-\infty}^{\infty} \frac{dq}{(p^2 + q^2 + z^2)^{3/2}}$$

into

$$J_1 = \int_{-\pi/2}^{\pi/2} \frac{|\cos t|}{p^2 + z^2}\,dt = \frac{2}{p^2 + z^2}\,.$$

Finally, the resulting integral

$$J = 2\int_{-\infty}^{\infty} \frac{\cos(\sqrt{2}p)\,dp}{p^2 + z^2}$$

is calculated using the Cauchy residue theorem as follows:

$$J = 2\,\mathrm{Re}\int_{-\infty}^{\infty} \frac{e^{i\sqrt{2}p}dp}{p^2 + z^2} = 4\pi i\,\mathrm{res}\left[\frac{e^{i\sqrt{2}p}}{p^2 + z^2}\right]_{p=zi} =$$

$$= 4\pi i\,\frac{e^{-\sqrt{2}z}}{2zi} = \frac{2\pi}{z}\,e^{-\sqrt{2}z}.$$

Therefore the solution is

$$u(x, y, z) = e^{-\sqrt{2}z}\cos x\cos y.$$

Remark 3. Since $y/[(t-x)^2 + y^2] = \mathrm{Re}[1/(i(t-z))]$, where $z = x + iy$, the Poisson formula (1.55) can be recast as

$$u(z) = \mathrm{Re}\,\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{u(t)\,dt}{t - z}\,. \tag{1.56}$$

Now let us consider the Dirichlet problem for the Laplace equation in the half-plane $\mathrm{Im}\,z > 0$ (i.e., for $y > 0$):

$$\begin{cases} \Delta u = 0, & -\infty < x < \infty, \quad y > 0, \\ u|_{y=0} = R(x), & -\infty < x < \infty, \end{cases}$$

where the rational function $R(z)$ is real, has no poles on the real axis, and $R(z) \to 0$ when $z \to \infty$. By (1.56), the solution of this problem is the function

$$u(z) = \mathrm{Re}\,\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{R(t)\,dt}{t - z}\,.$$

This integral can be calculated by using Cauchy's residue theorem:

$$u(z) = -2\,\mathrm{Re} \sum_{\mathrm{Im}\,\zeta_k<0} \mathrm{res} \left[\frac{R(\zeta)}{\zeta - z}\right]_{\zeta=\zeta_k}, \tag{1.57}$$

where the residues are taken for all poles of the function $R(z)$ in the lower half-plane $\mathrm{Im}\,z < 0$.

Example 3. Solve the Dirichlet problem

$$\begin{cases} \Delta u = 0, & -\infty < x < \infty, \quad y > 0, \\ u|_{y=0} = \dfrac{k}{1+x^2}, & k = \mathrm{const}, \quad -\infty < x < \infty. \end{cases}$$

Solution. Using formula (1.57), we have

$$u(z) = -2\,\mathrm{Re}\,\mathrm{res}\left[\frac{k}{(1+\zeta^2)(\zeta - z)}\right]_{\zeta=-i} = -2\,\mathrm{Re}\,\frac{k}{2i(z+i)} = \frac{k(y+1)}{x^2+(y+1)^2}.$$

1.17. Other methods

In this section we will consider methods for solving boundary value problems for the biharmonic equation and the equations $\Delta^2 u = f$, as well as boundary value problemd for the Laplace and Poisson equations (without employing the Green function).

Biharmonic equation.

Example 1. Solve the following boundary value problem in the disc $\{(\rho, \varphi) : 0 \le \rho \le a, 0 \le \varphi < 2\pi\}$:

$$\begin{cases} \Delta^2 u = 0 & \text{in the disc,} \\ u|_{\rho=a} = 0, \quad \left.\dfrac{\partial u}{\partial n}\right|_{\rho=a} = A\cos\varphi & \text{on the boundary of the disc.} \end{cases}$$

Here $\vec{n}$ is the unit outward normal to the boundary of the disc.

Solution. What we have is the Dirichlet problem for the biharmonic equation. It is known that it has a unique solution, given by the formula

$$u(\rho,\varphi) = \frac{1}{2\pi a}(\rho^2 - a^2)^2 \left[\frac{1}{2}\int_0^{2\pi} \frac{-g\,d\alpha}{\rho^2 + a^2 - 2a\rho\cos(\varphi-\alpha)} + \right.$$

$$\left. + \int_0^{2\pi} \frac{f[a - \rho\cos(\varphi - \alpha)]\,d\alpha}{[\rho^2 + a^2 - 2a\rho\cos(\varphi-\alpha)]^2}\right] \tag{1.58}$$

(here $f = u|_{\rho=a}$ and $g = \partial u/\partial n|_{\rho=a}$.)

In our case $f = 0$, $g = A\cos\varphi$, and so the solution is

$$u(\rho,\varphi) = \frac{1}{2\pi a}\,(\rho^2 - a^2)^2\left(-\frac{1}{2}\right)\int_0^{2\pi}\frac{A\cos\alpha\,d\alpha}{\rho^2 + a^2 - 2a\rho\cos(\varphi-\alpha)} =$$

$$= -\frac{A(\rho^2-a^2)^2}{4\pi a}\int_0^{2\pi}\frac{\cos\alpha\,d\alpha}{\rho^2 + a^2 - 2a\rho\cos(\varphi-\alpha)}\,,$$

or

$$u(\rho,\varphi) = -\frac{A}{2a}(a^2-\rho^2)\cdot\frac{1}{2\pi}\int_0^{2\pi}\frac{(a^2-\rho^2)\cos\alpha\,d\alpha}{\rho^2 + a^2 - 2a\rho\cos(\varphi-\alpha)}\,.$$

To compute the last integral we remark that it yields a solution of the following Dirichlet problem for the Laplace equation in the disc:

$$\begin{cases} \Delta v = 0 & \text{in the disc,} \\ v|_{\rho=a} = \cos\varphi & \text{on the boundary of the disc.} \end{cases}$$

But the solution of this problem is clearly $v = \frac{\rho}{a}\cos\varphi$. Then, by the uniqueness of the solution of the Dirichlet problem for the Laplace equation, we have the identity

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{(a^2-\rho^2)\cos\alpha\,d\alpha}{\rho^2 + a^2 - 2a\rho\cos(\varphi-\alpha)} = \frac{\rho}{a}\cos\varphi\,.$$

Therefore, the solution of our problem is

$$u(\rho,\varphi) = \frac{A\rho(\rho^2-a^2)}{2a^2}\cos\varphi\,.$$

Example 2. Solve the following boundary value problem in the disc $\{(\rho,\varphi) : 0 \le \rho \le a, 0 \le \varphi < 2\pi\}$:

$$\begin{cases} \Delta^2 u = 1 & \text{in the disc,} \\ u|_{\rho=a} = 0, \quad \left.\dfrac{\partial u}{\partial n}\right|_{\rho=a} = 0 & \text{on the boundary of the disc.} \end{cases}$$

Solution. One can consider that the solution of the problem depends only on the variable ρ, i.e., $u = u(\rho)$. Next, let us remark that

$$\Delta^2 u = \left(\frac{\partial^4}{\partial\rho^4} + \frac{2}{\rho}\frac{\partial^3}{\partial\rho^3} - \frac{1}{\rho^2}\frac{\partial^2}{\partial\rho^2}\right)u + \frac{1}{\rho^3}\frac{\partial u}{\partial\rho},$$

and so we obtain a boundary value problem for an ordinary differential equation:

$$\begin{cases} \dfrac{\partial^4 u}{\partial\rho^4} + \dfrac{2}{\rho}\dfrac{\partial^3 u}{\partial\rho^3} - \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial\rho^2} + \dfrac{1}{\rho^3}\dfrac{\partial u}{\partial\rho} = 1, \qquad 0 < \rho < a, & (1.61) \\ u|_{\rho=a} = 0, \qquad \left.\dfrac{du}{d\rho}\right|_{\rho=a} = 0. & (1.62) \end{cases}$$

Equation (1.61) can be rewritten in the form

$$\rho^3 u'''' + 2\rho^2 u''' - \rho u'' + u' = \rho^3.$$

Let us denote $v = \frac{du}{d\rho}$. Then we obtain a third-order equation for the function $v = v(\rho)$:

$$\rho^3 v''' + 2\rho^2 v'' - \rho v' + v = \rho^3,$$

which is recongnized to be the well-known Euler equation. Its general solution is given by the function

$$v(\rho) = C_1\rho^{-1} + C_2\rho\ln\rho + A\rho + \frac{1}{16}\rho^3.$$

We must take $C_1 = 0$ and $C_2 = 0$, because otherwise the function $v'(\rho)$ would become infinite at the center of the disc (i.e., when $\rho \to 0$). Therefore, $u' = A\rho + \frac{1}{16}\rho^3$, and so

$$u(\rho) = \frac{A\rho^2}{2} + \frac{1}{64}\rho^4 + B.$$

The constants A and B are found from the boundary conditions (1.62). We conclude that the solution is

$$u(\rho) = \frac{1}{64}(a^2 - \rho^2)^2,$$

or

$$\dot{u}(\rho) = \frac{a^4}{64}\left[1 - \left(\frac{\rho}{a}\right)^2\right]^2.$$

Example 3. Solve the following boundary value problem in the half-plane $\{(x, y) : -\infty < x < \infty, y > 0\}$:

$$\Delta^2 u = e^{-2y}\sin x \qquad \text{in the half-plane,} \tag{1.63}$$

$$u|_{y=0} = 0, \quad \left.\frac{\partial u}{\partial y}\right|_{y=0} = 0 \qquad \text{on the boundary of the half-plane.} \tag{1.64}$$

Let us rewrite the equation(1.63) in the form

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = e^{-2y}\sin x. \tag{1.65}$$

We shall seek for $u(x, y)$ in the form $u(x, y) = f(y)\sin x$, where the function $f(y)$ is subject to determination. Substituting this expression in equation (1.65) we obtain

$$f(y)\sin x + 2f''(y)(-\sin x) + f^{(\text{iv})}(y)\sin x = e^{-2y}\sin x,$$

whence

$$f^{(\mathrm{iv})} - 2f'' + f = e^{-2y}. \tag{1.66}$$

The general solution of equation (1.66) has the form

$$f(y) = C_1 e^y + C_2 y e^y + C_3 e^{-y} + C_4 y e^{-y} + \frac{1}{9}\, e^{-2y}.$$

The constants C_1 and C_2 are equal to zero: otherwise, $f(y) \to \infty$ as $y \to \infty$. Hence,

$$f(y) = C_3 e^{-y} + C_4 y e^{-y} + \frac{1}{9}\, e^{-2y}.$$

The constants C_3 and C_4 are found from the boundary conditions (1.64), which translate into $f(0) = 0$ and $f'(0) = 0$. We have

$$f(y) = -\frac{1}{9}\, e^{-y} + \frac{1}{9}\, y e^{-y} + \frac{1}{9}\, e^{-2y}.$$

Thus, the solution of our problem is

$$u(x, y) = \frac{1}{9}(e^{-2y} - e^{-y} + y e^{-y}) \sin x.$$

The Laplace and Poisson equations.

Example 4. Solve the following boundary value problem in the half-space $\{(x, y, z) : -\infty < x, y < \infty, z > 0\}$:

$$\begin{cases} \Delta u = z e^{-z} \sin x \sin y & \text{in the half-space,} \\ u|_{z=0} = 0. \end{cases}$$

Solution. We will seek the function $u = u(x, y, z)$ in the form

$$u = f(z) \sin x \sin y,$$

where the function $f(z)$ needs to be determined. Then we get

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} = -f \sin x \sin y - f \sin x \sin y + f'' \sin x \sin y,$$

and so our equation becomes

$$-2f \sin x \sin y + f'' \sin x \sin y = z e^{-z} \sin x \sin y.$$

Hence, to find $f(z)$ we must solve the ordinary differential equation

$$f'' - 2f = z e^{-z}.$$

Its general solution is

$$f(z) = C_1 e^{\sqrt{2}z} + C_2 e^{-\sqrt{2}z} + e^{-z}(2 - z).$$

The constants C_1 and C_2 are found from the boundary conditions. First notice that $C_1 = 0$, because otherwise $f(z) \to \infty$ when $z \to \infty$. Therefore,

$$f(z) = C_2 e^{-\sqrt{2}z} + e^{-z}(2 - z).$$

Putting here $z = 0$ we find $f(0) = C_2 + 2$, and since $f(0) = 0$, it follows that $C_2 = -2$.

Therefore, the solution of our problem is

$$u(x, y, z) = [e^{-z}(2 - z) - 2e^{-\sqrt{2}z}] \sin x \sin y.$$

Example 3. Solve the boundary problem in the half-space $\{(x, y, z) : -\infty < x, y < \infty, z > 0\}$

$$\begin{cases} \Delta u = 0 & \text{in the half-space,} \\ u|_{z=0} = \dfrac{x^2 + y^2 - 2}{(1 + x^2 + y^2)^{5/2}}. \end{cases}$$

Solution. Notice that the function

$$u(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + (z + 1)^2}}$$

satisfies the Laplace equation in the whole half-space $z > 0$ (is a fundamental solution), i.e.,

$$\Delta \frac{1}{\sqrt{x^2 + y^2 + (z + 1)^2}} = 0.$$

Now let us differentiate both sides of this equality with respect to z. We get

$$\Delta \frac{z + 1}{[x^2 + y^2 + (z + 1)^2]^{3/2}} = 0.$$

Differentiating one more time with respect to z we have

$$\Delta \frac{x^2 + y^2 - 2(z + 1)^2}{[x^2 + y^2 + (z + 1)^2]^{5/2}} = 0.$$

This suggests to consider the function

$$u(x, y, z) = \frac{x^2 + y^2 - 2(z + 1)^2}{[x^2 + y^2 + (z + 1)^2]^{5/2}}.$$

This function is harmonic in the whole half-space $z > 0$ (since $\Delta u = 0$, as we just showed), and for $z = 0$ we have

$$u|_{z=0} = \frac{x^2 + y^2 - 2}{(1 + x^2 + y^2)^{5/2}},$$

which proves that $u(x, y, z)$ is the sought solution.

1.18. Problems for independent study

1. Find the distribution of the potential of an infinitely long ($-\infty < z < \infty$) long cylindrical capacitor if its interior plate $\rho = a$ [resp., exterior plate $\rho = b$] is charged at the potential u_1 [resp., u_2].

2. Find the distribution of the potential inside a spherical capacitor if the sphere $\rho = a$ [resp., $\rho = b$] is maintained at the potential u_1 [resp., u_2].

3. One side of a right-angle parallelepiped is subject to a potential V, while the remaining sides are grounded. Find the distribution of the potential inside the parallelepiped.

4. An infinite ($-\infty < z < \infty$) conducting cylinder is charged at the potential

$$V = \begin{cases} 1, & \text{if } 0 < \varphi < \pi, \\ 0, & \text{if } \pi < \varphi < 2\pi. \end{cases}$$

Find the distribution of the potential inside the cylindrical cavity.

5. Find the temperature distribution in an infinitely long ($-\infty < z < \infty$) circular cylinder if the a heat flux $Q = q\cos\varphi$ per unit of length is given on its surface.

6. A constant current J passes through an infinite ($-\infty < z < \infty$) coaxial cyclindrical cable ($a < \rho < b$). Find the temperature distribution inside the cable if its inner surface $\rho = a$ is kept at temperature zero and the outer surface is thermally insulated.

7. Find the distribution of the potential in a thin plate shaped as a half-disc when the diameter of the half-disc is charged at potential V_1, while the remaining part of the boundary is charged at potential V_2.

8. Find the temperature distribution inside a thin rectangular plate if a constant heat flux Q is introduced through one of its sides, whereas the other three sides are kept at temperature zero.

9. Find the temperature distribution inside an infinite ($-\infty < z < \infty$) circular cylinder if its surface is mantained at the temperature $A\cos\varphi + B\sin\varphi$, where A and B are constants.

10. Find the distribution of the potential inside an empty cylinder of radius R and height h whose two bases are grounded, whereas the lateral surface has the potential V.

11. Determine the steady temperature distribution inside a circular cylinder of finite length is a constant heat flux q is introduced through the lower base $z = 0$, whereas the lateral surface $\rho = a$ and the upper base are maintained at temperature zero.

12. Find the steady temperature distribution inside a homogeneous and isotropic ball if its surface is maintained at the temperature $A \sin^2 \theta$ ($A =$const).

13. Find the distribution of the potential in a spherical capacitor $1 < r < 2$ if the inner and outer plates have the potential $V_1 = \cos^2 \theta$ and $V_2 = \frac{1}{8}(\cos^2 \theta + 1)$, respectively.

14. Find the temperature distribution inside a spherical layer $1 < r < 2$ if the inner sphere is maintained at the temperature $T_1 = \sin \theta \sin \varphi$, whereas the outer sphere is maintained at the temperature of melting ice.

15. Solve the Dirichlet problem for the Poisson equation $\Delta u = e^y \sin x$ in the square $0 \leq x \leq \pi$, $0 \leq y \leq \pi$, with null boundary condition.

16. Solve the Dirichlet problem for the Poisson equation $\Delta u = x^4 - y^4$ in the disc of radius one, with null boundary condition.

17. Solve the Dirichlet problem for the Poisson equation $\Delta u = z$ in the ball of radius one, with null boundary condition.

18. Solve the Dirichlet problem for the Poisson equation $\Delta u = J_0 \left(\frac{\mu_1}{R} \rho \right)$ in a cylinder of radius R and height h, with null boundary conditions.

19. Find the eigenoscillations of a rectangular membrane when two opposite edges are clamped and the other two are free.

20. Find the eigenoscillations of a circular cylinder under null boundary conditions of the first kind.

21. Find the steady concentration distribution of an unstable gas inside a sphere of radius a if a constant concentration u_0 is maintained at the surface of the sphere.

22. Solve the Dirichlet problem for the equation $\Delta u + k^2 u = 0$ in the interior and in the exterior of the sphere $\rho = R$ under the condition $u|_{\rho=R} = A$ ($A =$const).

23. Solve the Neumann problem for the equation $\Delta u - k^2 u = 0$ in the interior and in the exterior of the sphere $\rho = R$ under the condition $\partial u / \partial n|_{\rho=R} = A$ ($A =$const).

24. Find the steady distribution of potential in the first quadrant $x > 0$, $y > 0$ if the half-line $y = 0$ is grounded while the half-line $x = 0$ is maintained at the potential V.

25. Find the steady temperature distribution in the strip $0 < y < \pi$ if the temperature on the lower boundary $y = 0$ equals $A \cos x$ while the upper boundary is kept at the temperature of melting ice ($A =$const).

26. Find the steady distribution of potential in the strip $0 < y < \pi$, $x > 0$ if the horizontal sides of the strip are grounded and the vertical side has the potential V.

27. Find the distribution of potential in an infinitely long eccentric cylindrical capacitor if the inner plate $|z+1| = 9$ has the potential 1 while the outer plate $|z+6| = 16$ is grounded.

28. Find the solution of the Dirichlet problem $\Delta u = 0$ in the domain $\operatorname{Im} z < 0$, $|z+5i| > 3$ if

$$u|_{\operatorname{Im} z=0} = 0, \qquad u|_{|z+5i|=3} = 1.$$

29. Find the temperature distribution in the lower half-plane $y < 0$ if its boundary $y = 0$ is maintained at the temperature $A \sin x$ ($A = \text{const}$).

30. Find the temperature distribution in the upper half-plane $y > 0$ if its boundary $y = 0$ is maintained at the temperature $\theta(-x)$, where $\theta(x)$ is Heaviside function.

31. Find the distribution of potential in the upper half-space $z > 0$ if its boundary $z = 0$ has the potential $(1 + x^2 + y^2)^{-3/2}$.

32. Solve the Dirichlet problem for the Poisson equation $\Delta u = -e^{-z} \sin x \cos y$ in the half-space $z > 0$ with null boundary condition.

33. Find the steady temperature distribution in the exterior of a bounded circular cylinder ($\rho > 1$, $-\infty < z < \infty$) if the lateral surface ($\rho = 1$) is maintained at the temperature $u|_{\rho=1} = A\cos(2\varphi) + B\cos(5\varphi) + C\cos(10\varphi)$, where A, B, C are constants.

34. Solve the Dirichlet problem

$$\begin{cases} \Delta u = 0, \qquad 0 < \rho < 1, \quad 0 \le \varphi \le 2\pi, \\ u|_{\rho=1} = \dfrac{\sin\varphi}{5 + 4\cos\varphi} \qquad 0 \le \varphi \le 2\pi. \end{cases}$$

35. Solve the following Neumann problem for the Laplace equation in the spherical layer $1 < \rho < 2$:

$$\begin{cases} \Delta u = 0 \qquad \text{inside the layer}, \\ \left.\dfrac{\partial u}{\partial n}\right|_{\rho=1} = P_2(\cos\theta), \\ \left.\dfrac{\partial u}{\partial n}\right|_{\rho=2} = P_3(\cos\theta). \end{cases}$$

36. Solve the following boundary value problem in the disc $\{0 \le \rho \le a, 0 \le \varphi < 2\pi\}$:

$$\begin{cases} \Delta^2 u = x^2 + y^2 \qquad \text{in the disc}, \\ u|_{\rho=a} = 0, \quad \left.\dfrac{\partial u}{\partial n}\right|_{\rho=a} = 0. \end{cases}$$

37. Solve the following boundary value problem in the disc $\{0 \le \rho \le a, 0 \le \varphi < 2\pi\}$:

$$\begin{cases} \Delta^2 u = 0 & \text{in the disc,} \\ u|_{\rho=a} = 1, & \left.\dfrac{\partial u}{\partial n}\right|_{\rho=a} = \sin^3 \varphi. \end{cases}$$

38. Solve the following boundary value problem in the ball $\{0 \le \rho \le a\}$:

$$\begin{cases} \Delta^2 u = x^2 + y^2 + z^2 & \text{inside the ball,} \\ u|_{\rho=a} = 0, & \left.\dfrac{\partial u}{\partial n}\right|_{\rho=a} = 0. \end{cases}$$

39. Solve the following boundary value problem in the half-space:

$$\begin{cases} \Delta^2 u = e^{-z} \sin x \cos y & -\infty < x, y < \infty, \quad z > 0, \\ u|_{z=0} = 0, & \left.\dfrac{\partial u}{\partial z}\right|_{z=0} = 0. \end{cases}$$

40. Solve the following boundary value problem in the lower half-plane $(y < 0)$:

$$\begin{cases} \Delta u = 0 & -\infty < x < \infty, \quad y < 0, \\ u|_{y=0} = \dfrac{2x}{(1+x^2)^2}. \end{cases}$$

1.19. Answers

1. $u_1 = u_2 + (u_1 - u_2)\dfrac{\ln b/\rho}{\ln b/a}$.

2. $u = u_2 + (u_1 - u_2)\dfrac{1/r - 1/b}{1/a - 1/b}$.

3. $u = \displaystyle\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sinh\left(\pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\,z\right)$, where

$$A_{nm} = \begin{cases} \dfrac{16V}{\pi^2 nm \sinh\left(\pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\,c\right)}, & \text{if } n \text{ and } m \text{ are odd,} \\ 0, & \text{if } n \text{ or } m \text{ is even,} \end{cases}$$

and a, b, c are the sides of the parallelepiped.

4. $u = \dfrac{1}{2} + \dfrac{1}{\pi}\arctan\dfrac{2a\rho\sin\varphi}{a^2-\rho^2}$, where a is the radius of the cylinder.

5. $u = -\dfrac{q}{k}\,\rho\cos\varphi + c$, where k is the heat conduction coefficient of the cylinder.

6. $u = -\dfrac{q}{4}(\rho^2 - a^2) - \dfrac{qb^2}{2}\,\ln\dfrac{\rho}{a}$, where $q = -q_0/k$, $q_0 = 0.24J^2R$, R is the resistance per unit of length of the conductor, and k is the heat conduction coefficient.

7. $u = V_1 + \dfrac{2}{\pi}(V_2 - V_1)\arctan\dfrac{a\rho\sin\varphi}{\rho^2 - a^2}$.

8. $$u = -\frac{4qa}{k\pi^2}\sum_{m=0}^{\infty}\frac{\sin\left[\frac{(2m+1)\pi}{a}x\right]}{(2m+1)^2}\cdot\frac{\sinh\left[\frac{(2m+1)\pi}{a}y\right]}{\cosh\left[\frac{(2m+1)\pi}{a}b\right]}.$$

9. $u = A\dfrac{\rho}{a}\cos\varphi + B\dfrac{\rho}{a}\sin\varphi$, where a is the radius of the cylinder.

10. $$u = \frac{4V}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left[\frac{(2n+1)\pi}{h}z\right]}{2n+1}\cdot\frac{I_0\left(\frac{(2n+1)\pi}{h}\rho\right)}{I_0\left(\frac{(2n+1)\pi}{h}R\right)}.$$

11. $u = \displaystyle\sum_{m=0}^{\infty} A_m \frac{\sinh\left]\frac{\mu_m}{a}(l-z)\right]}{\cosh\left[\frac{\mu_m}{l}z\right]}\cdot J_0\left(\frac{\mu_m}{a}\rho\right)$, where $A_m = \frac{2aq}{k\mu_m^2 J_1(\mu_m)}$, k is the heat conduction coefficient, and μ_m is the mth positive root of the equation $J_0(x) = 0$.

12. $u = \dfrac{2}{3}A - A\left(\dfrac{r}{a}^2\right)^2\cdot\dfrac{3\cos^2\theta - 1}{3}$, where a is the radius of the ball.

13. $u = \dfrac{1}{3r} + \dfrac{3\cos^2\theta - 1}{3r^3}$.

14. $u = \dfrac{1}{7}\left(-r + \dfrac{8}{r^2}\right)\sin\theta\sin\varphi$.

15. $u = \dfrac{1}{2\sinh\pi}(ye^y\sinh\pi - \pi e^\pi\sinh y)\sin x$.

16. $u = \dfrac{1}{32}\,\rho^2(\rho^4 - 1)\cos(2\varphi)$.

17. $u = \dfrac{1}{10}(r^3 - r)\cos\theta.$

18. $u = \left\{ \dfrac{R^2}{\mu_1^2}\left[\cosh\left(\dfrac{\mu_1}{R}z\right) - 1\right] + \dfrac{R^2}{\mu_1^2}\left[1 - \cosh\left(\dfrac{\mu_1}{R}h\right)\right]\dfrac{\sin\left(\frac{\mu_1}{R}z\right)}{\sin\left(\frac{\mu_1}{R}h\right)}\right\} J_0\left(\dfrac{\mu_1}{R}\rho\right).$

19. $\lambda_{m,n} = \pi^2\left(\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}\right)$, $m = 1, 2 \ldots$, $n = 1, 2, \ldots$, where a and b are the side lengths of the membrane; $u_{m,n} = \sin\left(\dfrac{m\pi}{a}x\right)\cos\left(\dfrac{n\pi}{a}y\right)$.

20. $\lambda_{m,n,k} = \left(\dfrac{k\pi}{h}\right)^2 + \left(\dfrac{\mu_m^{(n)}}{a}\right)^2$, $n = 0, 1, \ldots, m$, $k = 1, 2, \ldots$, where $\mu_m^{(n)}$ is the mth positive root of the equation $J_n(x) = 0$, h is the height of the cylinder, and a is its radius;

$$v_{n,m,k} = \sin\left(\frac{k\pi}{h}z\right) J_n\left(\frac{\mu_m^{(n)}}{a}\rho\right)\begin{cases}\cos(n\varphi)\\ \sin(n\varphi)\end{cases}.$$

21. $u = u_0\dfrac{a}{\rho}\cdot\dfrac{\sinh(k\rho)}{\sinh(ka)}$, where k is taken from the equation $\Delta u - k^2 u = 0$.

22. $u = \dfrac{AR}{\rho}\cdot\dfrac{\sin(k\rho)}{\sin(kR)}$ if $\rho \le R$, and $u = \dfrac{AR}{\rho}\cdot\dfrac{e^{ik\rho}}{e^{ikR}}$ if $\rho \ge R$.

23. $u = \dfrac{AR^2\sinh(k\rho)}{\rho[kR\cosh(kR) - \sinh(kR)]}$ if $\rho \le R$, and $u = -\dfrac{aR^2}{\rho}\cdot\dfrac{e^{k(R-\rho)}}{1 + kR}$ if $\rho \ge R$.

24. $u = \dfrac{2V}{\pi}\arctan\dfrac{y}{x}.$

25. $u = \dfrac{A}{\sinh\pi}\cos x\sinh(\pi - y).$

26. $u = \dfrac{V}{\pi}\arctan\left(\dfrac{2\sinh x\sin y}{\sinh^2 x - \sin^2 y}\right).$

27. $u = \dfrac{1}{\ln(2/3)}\left(\ln 2 + \ln\left|\dfrac{z-2}{z-26}\right|\right).$

28. $u = \dfrac{1}{\ln 3}\ln\left|\dfrac{z+4i}{z-4i}\right|.$

29. $u = Ae^y \sin x$.

30. $u = \dfrac{1}{2} - \dfrac{1}{\pi} \arctan \dfrac{x}{y}$.

31. $u = \dfrac{z+1}{[x^2 + (z+1)^2 + y^2]^{3/2}}$.

32. $u = (e^{-\sqrt{2}z} - e^{-z}) \sin x \cos y$.

33. $u = \dfrac{A}{\rho^2} \cos(2\varphi) + \dfrac{B}{\rho^5} \cos(5\varphi) + \dfrac{C}{\rho^{10}} \cos(10\varphi)$.

34. $u = \dfrac{\rho \sin \varphi}{\rho^2 + 4\rho \cos \varphi + 4}$.

35. $u = \dfrac{1}{31} \left(\dfrac{\rho^2}{2} + \dfrac{32}{2} \cdot \dfrac{1}{\rho^3} \right) P_2(\cos \theta) + \dfrac{1}{47} \left(4\rho^3 + \dfrac{3}{\rho^4} \right) P_3(\cos \theta) + C$,
where C is an arbitrary constant.

36. $u = \dfrac{a^6}{576} \left[\left(\dfrac{\rho}{a} \right)^6 - 3 \left(\dfrac{\rho}{a} \right)^2 + 2 \right]$.

37. $u = 1 - \dfrac{a^2 - \rho^2}{2a} \left[\dfrac{3}{4} \left(\dfrac{\rho}{a} \right) \sin \varphi - \dfrac{1}{4} \left(\dfrac{\rho}{a} \right)^3 \sin(3\varphi) \right]$.

38. $u = \dfrac{a^6}{840} \left[\left(\dfrac{\rho}{a} \right)^6 - 3 \left(\dfrac{\rho}{a} \right)^2 + 2 \right]$.

39. $u = \left[Ae^{-\sqrt{2+\sqrt{2}}z} + (1-A)e^{-\sqrt{2-\sqrt{2}}z} - e^{-z} \right] \sin x \cos y$, where

$$A = \frac{1}{2}\sqrt{1 + \frac{1}{\sqrt{2}}} + \frac{1}{2}\sqrt{1 - \frac{1}{\sqrt{2}}} - \frac{1}{\sqrt{2}}.$$

40. $u = \dfrac{2x(1-y)}{[x^2 + (1-y)^2]^2}$.

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