Chapter II

Preliminary Results

1 Embedding properties and related facts

1.1 Poincaré inequalities

We consider some basic facts on Sobolev spaces without proof. First we collect several inequalities which compare the L^q-norm of a function u with the L^q-norm of its gradient

$$\nabla u = (D_1 u, \dots, D_n u).$$

Such estimates are called Poincaré estimates. For the proofs we refer to [Nec67], [Agm65], [Ada75], and [Fri69].

1.1.1 Lemma *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$, be any bounded domain, let $1 < q < \infty$, and let*

$$d = d(\Omega) := \sup_{x,y \in \Omega} |x - y|$$

denote the diameter of Ω. Then

$$\|u\|_{L^q(\Omega)} \leq C \|\nabla u\|_{L^q(\Omega)^n} \tag{1.1.1}$$

for all $u \in W_0^{1,q}(\Omega)$ where $C = C(q,d) > 0$ depends only on q and d.

Proof. See [Ada75, VI, 6.26]. $\square$

From (1.1.1) we conclude that the norms $\|u\|_{W^{1,q}(\Omega)}$ and $\|\nabla u\|_{L^q(\Omega)}$ are equivalent on the subspace $W_0^{1,q}(\Omega) \subseteq W^{1,q}(\Omega)$. To get estimates for general functions $u \in W^{1,q}(\Omega)$, we need that Ω is a bounded Lipschitz domain, see Section 3.2, I.

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1.1.2 Lemma *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain with $n \geq 2$, let $\Omega_0 \subseteq \Omega$ be any (nonempty) subdomain, and let $1 < q < \infty$. Then*

$$\|u\|_{L^q(\Omega)} \;\leq\; C\,(\|\nabla u\|_{L^q(\Omega)^n} \;+\; |\int_{\Omega_0} u\,dx|) \tag{1.1.2}$$

for all $u \in W^{1,q}(\Omega)$ where $C = C(q, \Omega, \Omega_0) > 0$ is a constant.

Proof. See [Nec67, Chap. 1, (1.21)]. Inequality (1.1.2) also holds for $n = 1$ where Ω is a bounded open interval. $\square$

From (1.1.2) we conclude that $\|u\|_{W^{1,q}(\Omega)}$ and $\|\nabla u\|_{L^q(\Omega)^n} + |\int_{\Omega_0} u\,dx|$ are equivalent norms on $W^{1,q}(\Omega)$.

The next result yields a bound for $\|u\|_{L^q(\Omega)}$ using the norms $\|\nabla u\|_{W^{-1,q}(\Omega)^n}$ and $\|u\|_{W^{-1,q}(\Omega)}$. We need some preparations.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain with $n \geq 2$ and let $1 < q < \infty$, $q' := \frac{q}{q-1}$.

Consider the spaces $W^{-1,q}(\Omega)^n$ and $W^{-1,q}(\Omega)$, see Section 3.6, I. Then we identify each $u \in L^q(\Omega)$ with the functional

$$< u, \cdot >:\; v \mapsto < u, v > \;=\; \int_\Omega uv\,dx \;\;,\quad v \in W_0^{1,q'}(\Omega),$$

which yields the embedding

$$L^q(\Omega) \subseteq W^{-1,q}(\Omega) \tag{1.1.3}$$

as usual for distributions. We get

$$|< u, v >| \;\leq\; \|u\|_q \|v\|_{q'} \;\leq\; \|u\|_q \|v\|_{1,q'} \;,$$

and this yields

$$\|u\|_{W^{-1,q}(\Omega)} \;\leq\; \|u\|_{L^q(\Omega)} \tag{1.1.4}$$

which shows that the embedding (1.1.3) is continuous.

Further, for each $u \in L^q(\Omega)$ we define the functional $\nabla u = [\nabla u, \cdot]$ by

$$[\nabla u, v] := \;-\; < u,\ \mathrm{div}\ v > = \;-\int_\Omega u\ \mathrm{div}\ v\,dx$$

for all $v = (v_1, \ldots, v_n) \in C_0^\infty(\Omega)^n$. Then we see that

$$\nabla u \in W^{-1,q}(\Omega)^n \;,$$

and we get the estimate

$$|[\nabla u, v]| \;=\; |< u,\; \mathrm{div}\; v >| \;\le\; \|u\|_q \,\|\nabla v\|_{q'} \;\le\; \|u\|_q \,\|v\|_{1,q'}$$

which shows that

$$\|\nabla u\|_{-1,q} \;:=\; \sup_{0\neq v\in C_0^\infty(\Omega)^n} \left(|[\nabla u, v]| \,/\, \|v\|_{1,q'}\right) \;\le\; \|u\|_q \,. \tag{1.1.5}$$

The inequality in the next lemma is basic for the theory of the operators div and ∇ in the next section.

1.1.3 Lemma *Let $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, be a bounded Lipschitz domain and let $1 < q < \infty$. Then*

$$\|u\|_{L^q(\Omega)} \;\le\; C\,(\|\nabla u\|_{W^{-1,q}(\Omega)^n} + \|u\|_{W^{-1,q}(\Omega)}) \tag{1.1.6}$$

for all $u \in L^q(\Omega)$ where $C = C(q, \Omega) > 0$ is a constant.

Proof. See [Nec67, Chap. 3, Lemma 7.1] for $q = 2$ and [Nec67b] for general q. The proof for $q = 2$ can be extended to all $1 < q < \infty$ if we replace the argument based on the Fourier transform by a potential theoretic fact. Here we use this lemma only for $q = 2$. □

Using (1.1.4) and (1.1.5) we see that

$$\|\nabla u\|_{W^{-1,q}(\Omega)^n} + \|u\|_{W^{-1,q}(\Omega)} \;\le\; 2\|u\|_{L^q(\Omega)}. \tag{1.1.7}$$

Therefore, under the assumptions of Lemma 1.1.3 we conclude that

$$\|u\|_{L^q(\Omega)} \quad \text{and} \quad \|\nabla u\|_{W^{-1,q}(\Omega)^n} + \|u\|_{W^{-1,q}(\Omega)}$$

are equivalent norms in $L^q(\Omega)$.

Inequality (1.1.6) can be extended as follows:

Let $k \in \mathbb{N}$ and consider the spaces

$$W^{-k,q}(\Omega) \;\;,\quad W^{-k-1,q}(\Omega)^n \;\;,\quad W^{-k-1,q}(\Omega)$$

which are the dual spaces of

$$W_0^{k,q'}(\Omega) \;\;,\quad W_0^{k+1,q'}(\Omega)^n \;\;,\quad W_0^{k+1,q'}(\Omega),$$

respectively. Let $u : v \mapsto [u, v]$ be any functional from $W^{-k,q}(\Omega)$. Then the inequality

$$|[u, v]| \;\le\; \|u\|_{-k,q}\|v\|_{k,q'} \;\le\; \|u\|_{-k,q}\|v\|_{k+1,q'}$$

shows that

$$\|u\|_{W^{-k-1,q}(\Omega)} \le \|u\|_{W^{-k,q}(\Omega)} .$$

The gradient ∇u is treated as a functional $[\nabla u, \cdots] : v \mapsto [\nabla u, v]$ defined by

$$[\nabla u, v] := -[u, \text{ div } v] \ , \quad v \in C_0^\infty(\Omega)^n \ ,$$

and using

$$\begin{aligned} |[\nabla u, v]| &= |[u, \text{ div } v]| \le \|u\|_{-k,q} \, \|\text{div } v\|_{k,q'} \\ &\le C\|u\|_{-k,q} \, \|v\|_{k+1,q'} \ , \end{aligned}$$

we get $\nabla u \in W^{-k-1,q}(\Omega)^n$ and

$$\|\nabla u\|_{W^{-k-1,q}(\Omega)^n} \le C\|u\|_{W^{-k,q}(\Omega)}$$

with some $C = C(n) > 0$.

1.1.4 Lemma *Let $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, be a bounded Lipschitz domain and let $1 < q < \infty$, $k \in \mathbb{N}$. Then*

$$\|u\|_{W^{-k,q}(\Omega)} \le C \left(\|\nabla u\|_{W^{-k-1,q}(\Omega)^n} + \|u\|_{W^{-k-1,q}(\Omega)}\right) \tag{1.1.8}$$

for all $u \in W^{-k,q}(\Omega)$ where $C = C(q, k, \Omega) > 0$ is a constant.

Proof. See [Nec67, Chap. 3, Lemma 7.1]. Using the estimates above we see that the both sides of (1.1.8) define equivalent norms. Lemma 1.1.3 is obtained by setting $k = 0$. □

The next lemma shows that $u \in L^q_{loc}(\Omega)$, $\nabla u \in L^q(\Omega)^n$ even implies $u \in W^{1,q}(\Omega)$ if Ω is a bounded Lipschitz domain.

1.1.5 Lemma *Let $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, be any Lipschitz domain and let $1 < q < \infty$. Then we have:*

a) *If $u \in L^q_{loc}(\Omega)$ and $\nabla u \in L^q(\Omega)^n$, then*

$$u \in L^q_{loc}(\overline{\Omega}) \ \textit{ and therefore } \ u \in W^{1,q}_{loc}(\overline{\Omega}). \tag{1.1.9}$$

b) *If Ω is a bounded Lipschitz domain and $u \in L^q_{loc}(\Omega)$, $\nabla u \in L^q(\Omega)^n$, then*

$$u \in L^q(\Omega) \ \textit{ and therefore } u \in W^{1,q}(\Omega). \tag{1.1.10}$$

Proof. This result follows by applying [Nec67, Chap. 2, Theorem 7.6] to bounded Lipschitz subdomains of Ω. However, we can argue directly: Indeed, b) is a consequence of a), and a) can be derived using b). It its sufficient to prove the result in a neighbourhood of any $x_0 \in \partial\Omega$. Use a local coordinate system in x_0, see Section 3.2, I, define a translation in the exterior normal direction and apply the estimate of Lemma 1.1.2. This yields the result. □

1.2 Traces and Green's formula

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain with boundary $\partial\Omega$, and let $1 < q < \infty$, $q' = \frac{q}{q-1}$.

Our purpose is to introduce a bounded linear operator

$$\Gamma : u \mapsto \Gamma u \tag{1.2.1}$$

from $W^{1,q}(\Omega)$ onto $W^{1-\frac{1}{q},q}(\partial\Omega)$ so that

$$\Gamma u = u|_{\partial\Omega} \tag{1.2.2}$$

holds for all $u \in C^\infty(\overline{\Omega})$. This means, Γu coincides with the restriction of u to the boundary $\partial\Omega$ if u is smooth. In other words, Γ extends the restriction operator $u \mapsto u|_{\partial\Omega}$ from the smooth function space $C^\infty(\overline{\Omega})$ to the larger space $W^{1,q}(\Omega)$. $W^{1-\frac{1}{q},q}(\partial\Omega)$ will be the right space such that this operator is bounded and even surjective.

Γ is called the **trace operator** of Ω. The existence, boundedness, and surjectivity of such an operator

$$\Gamma : W^{1,q}(\Omega) \to W^{1-\frac{1}{q},q}(\partial\Omega)\,,$$

satisfying (1.2.2) for all $u \in C^\infty(\overline{\Omega})$, follows by combining [Nec67, Chap. 2, Theorem 5.5] with [Nec67, Chap. 2, Theorem 5.7]. See also [Ada75, VII, 7.53].

We use the notation (1.2.2) not only for $u \in C^\infty(\overline{\Omega})$ but for all $u \in W^{1,q}(\Omega)$, and call $\Gamma u = u|_{\partial\Omega}$ the **trace** of $u \in W^{1,q}(\Omega)$. We consider the trace of u as the restriction of u to $\partial\Omega$ in the generalized sense.

The construction of Γ rests on the use of the local coordinate systems, see Section 3.2, I. If the boundedness of Γ is shown on the subspace $C^\infty(\overline{\Omega}) \subseteq W^{1,q}(\Omega)$, the density property

$$\overline{C^\infty(\overline{\Omega})}^{\|\cdot\|_{W^{1,q}(\Omega)}} = W^{1,q}(\Omega)\,, \tag{1.2.3}$$

see [Nec67, Chap. 2, Theorem 3.1], then yields boundedness on $W^{1,q}(\Omega)$.

The boundedness of Γ means that there is a constant $C = C(q,\Omega) > 0$ so that the estimate

$$\|\Gamma u\|_{W^{1-\frac{1}{q},q}(\partial\Omega)} \leq C\|u\|_{W^{1,q}(\Omega)} \tag{1.2.4}$$

holds for all $u \in W^{1,q}(\Omega)$. We will simply write

$$\|\Gamma u\|_{W^{1-\frac{1}{q},q}(\partial\Omega)} = \|u\|_{W^{1-\frac{1}{q},q}(\partial\Omega)} = \|u\|_{1-\frac{1}{q},q,\partial\Omega}$$

if there is no confusion. See Section 3.4, I, for the definition of this norm.

Using the trace $\Gamma u = u|_{\partial\Omega}$, we get a direct characterization of the space $W_0^{1,q}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{1,q}}}$. It holds that

$$W_0^{1,q}(\Omega) = \{u \in W^{1,q}(\Omega);\ \ u|_{\partial\Omega} = 0\} \tag{1.2.5}$$

for our bounded Lipschitz domain Ω, see [Nec67, Chap. 2, Theorem 4.10] or [Ada75, VII, 7.55].

Since Γ is a surjective operator, each given element $g \in W^{1-\frac{1}{q},q}(\partial\Omega)$ is the trace $g = u|_{\partial\Omega}$ of at least one $u \in W^{1,q}(\Omega)$. Moreover, it is even possible to select some $u \in W^{1,q}(\Omega)$ for each $g \in W^{1-\frac{1}{q},q}(\partial\Omega)$ in such a way that the mapping

$$g \mapsto u \quad \text{with} \quad g = u|_{\partial\Omega}$$

is a bounded linear operator from $W^{1-\frac{1}{q},q}(\partial\Omega)$ into $W^{1,q}(\Omega)$.

Thus there exists a bounded linear operator

$$\Gamma_e :\ W^{1-\frac{1}{q},q}(\partial\Omega) \ \to\ W^{1,q}(\Omega) \tag{1.2.6}$$

with the property

$$\Gamma\Gamma_e g = g \tag{1.2.7}$$

for all $g \in W^{1-\frac{1}{q},q}(\partial\Omega)$. We call $u = \Gamma_e g$ an **extension** of g from $\partial\Omega$ to Ω.

Γ_e is called an **extension operator** from $W^{1-\frac{1}{q},q}(\partial\Omega)$ into $W^{1,q}(\Omega)$, see [Nec67, Chap. 2, Theorem 5.7]. The boundedness of Γ_e means that there is a constant $C = C(q,\Omega) > 0$ such that

$$\|\Gamma_e g\|_{W^{1,q}(\Omega)} \ \leq\ C\|g\|_{W^{1-\frac{1}{q},q}(\partial\Omega)} \tag{1.2.8}$$

holds for all $g \in W^{1-\frac{1}{q},q}(\partial\Omega)$.

Green's formula is well known in elementary classical analysis for smooth functions, see [Miz73, Chap. 3, (3.54)] or [Nec67, Chap. 1, (2.9)]. It extends the elementary rule of partial integration from intervals in $\mathbb{R}$ to higher dimensions $n \geq 2$. The following general formulation can be derived from the classical one by using density and closure arguments, see [Nec67, Chap. 3, 1.2].

Let $u \in C^\infty(\overline{\Omega})$, $v \in C^\infty(\overline{\Omega})^n$, and let $\int_{\partial\Omega} \cdots\, dS$ denote the surface integral, see Section 3.4, I. Then we get

$$\text{div}\ (uv) = (\nabla u) \cdot v + u\ \text{div}\ v$$

by an elementary calculation, and Green's formula reads

$$\int_\Omega u\ \text{div}\ v\, dx = \int_{\partial\Omega} uN \cdot v\, dS - \int_\Omega (\nabla u) \cdot v\, dx\ , \tag{1.2.9}$$

where $N : x \mapsto N(x) = (N_1(x), \dots, N_n(x))$ means the **exterior normal vector field** at the boundary $\partial\Omega$, see (3.4.7), I. We can write this formula in the form

$$< u, \text{div } v >_\Omega \; = \; < u, N \cdot v >_{\partial\Omega} \; - \; < \nabla u, v >_\Omega \; , \tag{1.2.10}$$

see (3.4.6), I, for this notation.

Using the density property (1.2.3) and the trace operator Γ above, we can extend Green's formula to all $u \in W^{1,q}(\Omega)$ and $v \in W^{1,q'}(\Omega)^n$. Then $< u, N \cdot v >_{\partial\Omega}$ remains well defined as a surface integral, see (3.4.3), I, with the traces

$$u|_{\partial\Omega} \in W^{1-\frac{1}{q},q}(\partial\Omega) \quad \text{and} \quad N \cdot v|_{\partial\Omega} \in W^{1-\frac{1}{q'},q'}(\partial\Omega) \; ; \tag{1.2.11}$$

we see that $uN \cdot v|_{\partial\Omega} \in L^1(\partial\Omega)$. Note that $|N| \in L^\infty(\partial\Omega)$, see (3.4.9), I. This leads to the following result.

1.2.1 Lemma *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain with boundary $\partial\Omega$, and let $1 < q < \infty$, $q' := \frac{q}{q-1}$. Then for all $u \in W^{1,q}(\Omega)$ and $v \in W^{1,q'}(\Omega)^n$,* (1.2.11) *holds in the trace sense and we get the formula*

$$< u, \text{div } v >_\Omega \; = \; < u, N \cdot v >_{\partial\Omega} \; - \; < \nabla u, v >_\Omega \; , \tag{1.2.12}$$

where N means the exterior normal field at $\partial\Omega$.

Proof. See [Nec67, Chap. 3, Theorem 1.1]. □

Lemma 1.2.3 will give a further extension of Green's formula (1.2.12) to more general functions v. For this purpose we use the more general trace operator Γ_{N}, see the next lemma, for which we need some preparation.

Inserting $u = \Gamma_e \, g \in W^{1,q}(\Omega)$ with $u|_{\partial\Omega} = g \in W^{1-\frac{1}{q},q}(\partial\Omega)$ and $v \in W^{1,q'}(\Omega)^n$ in (1.2.12), we get

$$< \Gamma_e \, g, \; \text{div } v >_\Omega = \; < g, N \cdot v >_{\partial\Omega} - < \nabla\Gamma_e \, g, v >_\Omega \; ,$$

and using (1.2.8) yields the estimate

$$\begin{aligned} |< g, N \cdot v >_{\partial\Omega}| \quad &\leq \quad |< \nabla\Gamma_e \, g, v >_\Omega| \; + \; |< \Gamma_e \, g, \text{div } v >_\Omega| \\ &\leq \quad C \, \|g\|_{W^{1-\frac{1}{q},q}(\partial\Omega)} (\|v\|_{q'} + \| \text{ div } v\|_{q'}), \end{aligned} \tag{1.2.13}$$

with some constant $C = C(q, \Omega) > 0$. This shows that the functional

$$< \cdot, N \cdot v >_{\partial\Omega} : \quad g \mapsto < g, N \cdot v >_{\partial\Omega} \quad , \quad g \in W^{1-\frac{1}{q},q}(\partial\Omega) \tag{1.2.14}$$

is continuous in the norm $\|g\|_{W^{1-\frac{1}{q},q}(\partial\Omega)}$, for each fixed $v \in W^{1,q'}(\Omega)^n$. Therefore, $< \cdot, N \cdot v >_{\partial\Omega}$ belongs to the dual space of $W^{1-\frac{1}{q},q}(\partial\Omega)$, which is the space

$$W^{1-\frac{1}{q},q}(\partial\Omega)' = W^{-(1-\frac{1}{q}),q'}(\partial\Omega) = W^{-\frac{1}{q'},q'}(\partial\Omega),$$

see (3.6.9), I. Thus we get

$$< \cdot, N \cdot v >_{\partial\Omega} \in W^{-\frac{1}{q'},q'}(\partial\Omega) \quad \text{for all} \;\; v \in W^{1,q'}(\Omega)$$

and we may treat the well defined functional (1.2.14) as the trace $N \cdot v|_{\partial\Omega}$ of the normal component of v at $\partial\Omega$ in the generalized sense. Further we get from (1.2.13) that

$$\| < \cdot, N \cdot v >_{\partial\Omega} \|_{W^{-\frac{1}{q'},q'}(\partial\Omega)} \leq C\, (\|v\|_{q'}^{q'} + \|\mathrm{div}\; v\|_{q'}^{q'})^{\frac{1}{q'}} \tag{1.2.15}$$

holds with some constant $C = C(q, \Omega) > 0$.

Let $E_{q'}(\Omega)$ be the Banach space of all $v \in L^{q'}(\Omega)^n$ with div $v \in L^{q'}(\Omega)$ (in the sense of distributions) and norm $\|v\|_{E_{q'}(\Omega)} := (\|v\|_{q'}^{q'} + \|\mathrm{div}\; v\|_{q'}^{q'})^{\frac{1}{q'}}$. The same density argument as in (1.2.3) yields that

$$\overline{C^\infty(\overline{\Omega})^n}^{\|\cdot\|_{E_{q'}(\Omega)}} = E_{q'}(\Omega)\;, \tag{1.2.16}$$

and therefore that

$$\overline{W^{1,q'}(\Omega)^n}^{\|\cdot\|_{E_{q'}(\Omega)}} = E_{q'}(\Omega)\;. \tag{1.2.17}$$

Estimate (1.2.15) means that the operator

$$v \mapsto < \cdot, N \cdot v >_{\partial\Omega} \;\;, \quad v \in W^{1,q'}(\Omega), \tag{1.2.18}$$

from $W^{1,q'}(\Omega)$ to $W^{-\frac{1}{q'},q'}(\partial\Omega)$ is continuous in the norm of $E_{q'}(\Omega)$. Therefore, using (1.2.17) we see that the operator (1.2.18) extends by closure to a bounded linear operator

$$v \mapsto < \cdot, N \cdot v >_{\partial\Omega} \;\;, \quad v \in E_{q'}(\Omega), \tag{1.2.19}$$

from $E_{q'}(\Omega)$ to $W^{-\frac{1}{q'},q'}(\partial\Omega)$. The functional $< \cdot, N \cdot v >_{\partial\Omega}$ is therefore well defined as an element of $W^{-\frac{1}{q'},q'}(\partial\Omega)$ for each $v \in E_{q'}(\Omega)$.

Replacing q' by q, we thus obtain the following general trace lemma.

1.2.2 Lemma *Let* $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, *be a bounded Lipschitz domain with boundary* $\partial\Omega$, *let* $1 < q < \infty$, $q' = \frac{q}{q-1}$, *and let*

$$E_q(\Omega) := \{v \in L^q(\Omega)^n \;;\; \mathrm{div}\; v \in L^q(\Omega)\} \tag{1.2.20}$$

be the Banach space with norm

$$\|v\|_{E_q(\Omega)} := (\|v\|_q^q + \|\mathrm{div}\ v\|_q^q)^{\frac{1}{q}}\ . \tag{1.2.21}$$

Then there exists a bounded linear operator

$$\Gamma_N : v \mapsto \Gamma_N v\ , \quad v \in E_q(\Omega), \tag{1.2.22}$$

from $E_q(\Omega)$ *into* $W^{-\frac{1}{q},q}(\partial\Omega)$ *such that* $\Gamma_N\, v$ *coincides with the functional*

$$g \mapsto < g, N \cdot v >_{\partial\Omega} = \int_{\partial\Omega} g(x) N(x) \cdot v(x)\, dS\ , \quad g \in W^{\frac{1}{q},q'}(\partial\Omega) \tag{1.2.23}$$

if $v \in C^\infty(\overline{\Omega})^n$.

Proof. See [SiSo92, Theorem 5.3] or [Tem77, Chap. I, Theorem 1.2]. □

The operator $\Gamma_N : v \mapsto \Gamma_N\, v$ from $E_q(\Omega)$ to $W^{-\frac{1}{q},q}(\partial\Omega)$ is called the **generalized trace operator** for the normal component. For each $v \in E_q(\Omega)$, the functional $\Gamma_N v \in W^{-\frac{1}{q},q}(\partial\Omega)$ is called the **generalized trace** of the normal component $N \cdot v$ at $\partial\Omega$. We use the notation

$$\Gamma_N v \ = \ < \cdot, N \cdot v >_{\partial\Omega} = \ N \cdot v|_{\partial\Omega} \tag{1.2.24}$$

for all $v \in E_q(\Omega)$, although $N \cdot v|_{\partial\Omega}$ need not exist in the sense of usual traces (unless $v \in W^{1,q}(\Omega)^n$). Note that v itself need not have a well defined trace at $\partial\Omega$ in any sense. We refer to [Tem77, Chap. I, 1.2] and to [SiSo92, (5.1)] concerning the space $E_q(\Omega)$.

The next lemma yields the most general formulation of Green's formula.

1.2.3 Lemma *Let* $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, *be a bounded Lipschitz domain with boundary* $\partial\Omega$, *and let* $1 < q < \infty$, $q' = \frac{q}{q-1}$.

Then for all $u \in W^{1,q}(\Omega)$ *and* $v \in E_{q'}(\Omega)$,

$$< u, \mathrm{div}\ v >_\Omega \ = \ < u, N \cdot v >_{\partial\Omega} - < \nabla u, v >_\Omega \tag{1.2.25}$$

where $< u, N \cdot v >_{\partial\Omega}$ *is well defined in the sense of the generalized trace with*

$$N \cdot v|_{\partial\Omega} \in W^{-\frac{1}{q'},q'}(\partial\Omega)\ \ , \quad u|_{\partial\Omega} \in W^{1-\frac{1}{q},q}(\partial\Omega).$$

Proof. Using (1.2.17) we find a sequence $(v_j)_{j=1}^\infty$ in $W^{1,q'}(\Omega)^n$ with $v = \lim_{j\to\infty} v_j$ in the norm of $E_{q'}(\Omega)$. Then we insert v_j for v in formula (1.2.12) and let $j \to \infty$. The estimate (1.2.15), used with v replaced by $v - v_j$, shows that

$$< u, N \cdot v >_{\partial\Omega} = \lim_{j\to\infty} < u, N \cdot v_j >_{\partial\Omega}\ .$$

This leads to (1.2.25). □

1.3 Embedding properties

The embedding properties below will be used frequently, for example in order to estimate the nonlinear term $u \cdot \nabla u$ of the Navier-Stokes equations. The first lemma contains a special case of Sobolev's embedding theorem. For the proofs we refer to [Nir59], [Fri69], [Nec67], [Ada75].

1.3.1 Lemma *Let $n \in \mathbb{N}$. Then we get:*

a) *If* $1 < r \leq n,\ 1 < q < \infty,\ 1 < \gamma < \infty,\ 0 \leq \beta \leq 1$ *such that*

$$\beta(\frac{1}{r} - \frac{1}{n}) + (1-\beta)\frac{1}{\gamma} \ = \ \frac{1}{q}\ , \tag{1.3.1}$$

then

$$\begin{aligned} \|u\|_{L^q(\mathbb{R}^n)} \ &\leq \ C\|\nabla u\|^{\beta}_{L^r(\mathbb{R}^n)^n}\|u\|^{1-\beta}_{L^\gamma(\mathbb{R}^n)} \\ &\leq \ C\left(\|\nabla u\|_{L^r(\mathbb{R}^n)^n} + \|u\|_{L^\gamma(\mathbb{R}^n)}\right) \end{aligned} \tag{1.3.2}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$ where $C = C(n, r, q, \gamma) > 0$ is a constant.

b) *If $r > n$, then*

$$\sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|u(x) - u(y)|}{|x-y|^{1-\frac{n}{r}}} \leq C\|\nabla u\|_{L^r(\mathbb{R}^n)^n} \tag{1.3.3}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$ where $C = C(n, r) > 0$ is a constant.

Proof. See [Nir59], [Fri69, Part 1, Theorem 9.3]. □

Remarks

a) In the special case $r = n$ we get $(1-\beta)\frac{1}{\gamma} = \frac{1}{q},\ 0 \leq \beta < 1$ ($q = \infty$ is excluded), $1 < \gamma \leq q < \infty,\ \beta = 1 - \frac{\gamma}{q}$, and this leads to

$$\|u\|_{L^q(\mathbb{R}^n)} \ \leq \ C\,\|\nabla u\|^{1-\frac{\gamma}{q}}_{L^n(\mathbb{R}^n)^n}\,\|u\|^{\frac{\gamma}{q}}_{L^\gamma(\mathbb{R}^n)} \tag{1.3.4}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. Note that an inequality of the form $\|u\|_\infty \leq C\|\nabla u\|_n$ is excluded.

b) The second inequality in (1.3.2) follows from the first one by Young's inequality (3.3.8), I.

c) Inequality (1.3.2) leads in the case $1 < r < n,\ \beta = 1,\ r < q,\ n \geq 2,\ \frac{1}{n} + \frac{1}{q} = \frac{1}{r}$ to the estimate

$$\|u\|_{L^q(\mathbb{R}^n)} \ \leq \ C\,\|\nabla u\|_{L^r(\mathbb{R}^n)^n} \tag{1.3.5}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$ with $C = C(n, q) > 0$.

The following lemma yields a restricted result but includes the important case $q = \infty$. It is a consequence of (1.3.3) and the Poincaré inequality (1.1.2).

1.3.2 Lemma *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$, be an arbitrary domain with $\overline{\Omega} \neq \mathbb{R}^n$, and let $B \subseteq \mathbb{R}^n$ be any open ball with $B \cap \Omega \neq \emptyset$. Then we have:*

a) *If $1 < q < \infty$, then*

$$\|u\|_{L^q(B\cap\Omega)} \;\leq\; C\,\|\nabla u\|_{L^q(\Omega)^n} \tag{1.3.6}$$

for all $u \in C_0^\infty(\Omega)$ with $C = C(q,\Omega,B) > 0$.

b) *If $q > n$, then*

$$\|u\|_{L^\infty(B\cap\Omega)} \;\leq\; C\,\|\nabla u\|_{L^q(\Omega)^n} \tag{1.3.7}$$

for all $u \in C_0^\infty(\Omega)$ with $C = C(q,\Omega,B) > 0$.

Proof. Since $\overline{\Omega} \neq \mathbb{R}^n$ we can choose some open ball $B_0 \subseteq \mathbb{R}^n$ with $\overline{B_0} \cap \overline{\Omega} = \emptyset$. To prove a) we use Poincaré's inequality in Lemma 1.1.2 with Ω_0, Ω replaced by $B_0, \widetilde{\Omega}$; $\widetilde{\Omega}$ means any bounded Lipschitz domain containing B_0 and $B \cap \Omega$. Extending each $u \in C_0^\infty(\Omega)$ by zero we get $u \in C_0^\infty(\mathbb{R}^n)$, and since $u = 0$ in B_0 we obtain from (1.1.2) that

$$\|u\|_{L^q(B\cap\Omega)} \;\leq\; \|u\|_{L^q(\widetilde{\Omega})} \;\leq\; C\,\|\nabla u\|_{L^q(\widetilde{\Omega})^n} \;\leq\; C\,\|\nabla u\|_{L^q(\Omega)^n}$$

for all $u \in C_0^\infty(\Omega)$ with some $C = C(q,\Omega,B) > 0$. Indeed, C depends only on q, B_0 and B.

To prove b) we apply the above estimate (1.3.3) to $u \in C_0^\infty(\Omega)$ with r replaced by q. Let y_0 be the center of B_0. Then we get, extending u by zero as above, that

$$\begin{aligned}
\|u\|_{L^\infty(B\cap\Omega)} &= \sup_{x\in B\cap\Omega} |u(x)| \;=\; \sup_{x\in B\cap\Omega} |u(x) - u(y_0)| \\
&\leq \Big(\sup_{x\in B\cap\Omega} |x-y_0|^{1-\frac{n}{q}}\Big) \sup_{x\in B\cap\Omega} \frac{|u(x)-u(y_0)|}{|x-y_0|^{1-\frac{n}{q}}} \\
&\leq C\Big(\sup_{x\in B\cap\Omega} |x-y_0|^{1-\frac{n}{q}}\Big)\|\nabla u\|_{L^q(\Omega)^n}
\end{aligned}$$

with $C = C(n,q) > 0$. This proves the lemma. □

The next two lemmas are special cases of Sobolev's embedding theorem for bounded domains.

1.3.3 Lemma *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded C^1-domain, and let $1 < r \leq n$, $1 < q < \infty$, $1 < \gamma < \infty$, $0 \leq \beta \leq 1$ so that*

$$\beta\Big(\frac{1}{r} - \frac{1}{n}\Big) + (1-\beta)\frac{1}{\gamma} \;=\; \frac{1}{q}. \tag{1.3.8}$$

Then

$$\begin{aligned} \|u\|_{L^q(\Omega)} &\leq C\,\|u\|^{\beta}_{W^{1,r}(\Omega)}\|u\|^{1-\beta}_{L^\gamma(\Omega)} \\ &\leq C\left(\|u\|_{W^{1,r}(\Omega)} + \|u\|_{L^\gamma(\Omega)}\right) \end{aligned} \tag{1.3.9}$$

for all $u \in W^{1,r}(\Omega) \cap L^\gamma(\Omega)$ *where* $C = C(\Omega, q, r, \gamma) > 0$ *is a constant.*

Proof. See [Fri69, Part 1, Theorem 10.1]. Note that the case $n = r$ is not excluded. In this case we have $0 \leq \beta < 1$. □

The next lemma concerns the embedding of continuous functions in certain $W^{m,q}$-spaces for bounded domains.

1.3.4 Lemma *Let* $k \in \mathbb{N}_0,\ m \in \mathbb{N},\ 1 < q < \infty$ *with* $m - \frac{n}{q} > k,\ n \geq 2$, *and let* $\Omega \subseteq \mathbb{R}^n$ *be a bounded* C^m*-domain. Then, after redefinition on a subset of* Ω *of measure zero, each* $u \in W^{m,q}(\Omega)$ *is contained in* $C^k(\overline{\Omega})$ *and*

$$\|u\|_{C^k(\overline{\Omega})} \leq C\,\|u\|_{W^{m,q}(\Omega)} \tag{1.3.10}$$

where $C = C(\Omega, m, q) > 0$ *is a constant.*

Proof. See [Fri69, Part 1, Theorem 11.1]. □

Finally we mention a special embedding result for the two-dimensional case.

1.3.5 Lemma *Let* $\Omega \subseteq \mathbb{R}^2$ *be any two-dimensional domain with* $\overline{\Omega} \neq \mathbb{R}^2$, *let* $B_0, B \subseteq \mathbb{R}^2$ *be open balls with* $\overline{B_0} \cap \overline{\Omega} = \emptyset,\ B \cap \Omega \neq \emptyset$, *and let* $1 < q < \infty$. *Then*

$$\|u\|_{L^q(B\cap\Omega)} \leq C\,\|\nabla u\|_{L^2(\Omega)^2} \tag{1.3.11}$$

for all $u \in C_0^\infty(\Omega)$ *where* $C = C(B_0, B, q) > 0$ *is a constant.*

Proof. Let x_0 be the center of B_0, $R > 0$ the radius, and let $u \in C_0^\infty(\Omega)$. Then we use the inequality

$$\left(\int_\Omega \left(\frac{|u(x)|}{|x - x_0|\ln|x - x_0|/R}\right)^2 dx\right)^{\frac{1}{2}} \leq C\,\|\nabla u\|_{L^2(\Omega)^2} \tag{1.3.12}$$

where $C = C(B_0) > 0$ is a constant. An elementary proof of this inequality can be found in [Lad69, Chap. 1, (14)].

Next we use the above inequality (1.3.9) for B with $n = 2,\ 2 < q < \infty,\ r = \gamma = 2,\ \beta = 1 - \frac{2}{q}$, and get

$$\|u\|_{L^q(B\cap\Omega)} \leq \|u\|_{L^q(B)} \leq C\left(\|\nabla u\|_{L^2(B)^2} + \|u\|_{L^2(B)}\right) \tag{1.3.13}$$

with some $C = C(B, q) > 0$. On the right side, B can be replaced by $B \cap \Omega$.

If $1 < q \le 2$ we get using (1.3.12) that

$$\begin{aligned}\|u\|_{L^q(B\cap\Omega)} &\le C_1\|u\|_{L^2(B\cap\Omega)}\\ &\le C_1\Big(\sup_{x\in B\cap\Omega}(|x-x_0|\ln|x-x_0|/R)\Big)\Big(\int_{B\cap\Omega}\Big(\frac{|u(x)|}{|x-x_0|\ln|x-x_0|/R}\Big)^2 dx\Big)^{\frac12}\\ &\le C_2\|\nabla u\|_{L^2(\Omega)^2}\end{aligned}$$

with constants $C_1 = C_1(B,q) > 0,\ C_2 = C_2(B_0,B,q) > 0$. This yields the result for $1 < q \le 2$. If $q > 2$ we deduce from (1.3.13) and the last inequality for $q = 2$ that

$$\begin{aligned}\|u\|_{L^q(B\cap\Omega)} &\le C\,\|\nabla u\|_{L^2(\Omega)^2} + \|u\|_{L^2(B\cap\Omega)}\\ &\le C\,(\|\nabla u\|_{L^2(\Omega)^2} + C_2\|\nabla u\|_{L^2(B\cap\Omega)^2})\\ &\le C\,(1+C_2)\|\nabla u\|_{L^2(\Omega)^2}.\end{aligned}$$

This proves the lemma. □

1.4 Decomposition of domains

The decomposition property below will be used later on for technical reasons in order to "approximate" an arbitrary unbounded domain Ω by a sequence of bounded Lipschitz subdomains.

We need it, for example, for the existence proof of weak solutions, see the proof of Theorem 3.5.1, III. A similar result as that in the following lemma is contained in [Gal94a, III, proof of Lemma 1.1].

Recall the definition

$$\mathrm{dist}(A,B) := \inf_{x\in A,\, y\in B} |x-y|$$

for arbitrary subsets $A, B \subseteq \mathbb{R}^n$.

1.4.1 Lemma *Let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary domain with $n \ge 2$. Then there exists a sequence $(\Omega_j)_{j=1}^\infty$ of bounded Lipschitz subdomains of Ω and a sequence $(\varepsilon_j)_{j=1}^\infty$ of positive numbers with the following properties:*

a) $\overline{\Omega}_j \subseteq \Omega_{j+1}, \quad j \in \mathbb{N}$,
b) $\mathrm{dist}\,(\partial\Omega_{j+1}, \Omega_j) \ge \varepsilon_{j+1}, \quad j \in \mathbb{N}$,
c) $\lim_{j\to\infty} \varepsilon_j = 0$,
d) $\Omega = \bigcup_{j=1}^\infty \Omega_j$.

Proof. The proof rests on the following elementary considerations. Let

$$B_r(x) := \{y \in \mathbb{R}^n;\ \ |y-x| < r\}$$

be the open ball with center $x \in \mathbb{R}^n$ and radius $r > 0$.

We fix some $x_0 \in \Omega$. Let $\tilde{\Omega}$ be the largest domain concerning inclusions such that

$$\widetilde{\Omega} \subseteq \Omega \cap B_1(x_0) \;\;,\quad x_0 \in \widetilde{\Omega}.$$

The boundary $\partial\widetilde{\Omega}$ of $\widetilde{\Omega}$ is compact and therefore, for a given $\varepsilon > 0$, we can choose finitely many balls $B_\varepsilon(x_j)$ with $x_j \in \partial\tilde{\Omega}$, $j = 1, \ldots, m$, and

$$\partial\widetilde{\Omega} \subseteq \bigcup_{j=1}^{m} B_\varepsilon(x_j).$$

Let $\hat{\Omega} := \widetilde{\Omega} \setminus \bigcup_{j=1}^{m} \overline{B}_\varepsilon(x_j)$. We can choose ε with $0 < \varepsilon < 1$ in such a way that $x_0 \in \hat{\Omega}$. Obviously, $\hat{\Omega}$ is a bounded Lipschitz domain, its boundary consists of parts of the boundaries of balls. We set $\Omega_1 := \hat{\Omega}$ and $\varepsilon_1 := \varepsilon$.

Next we choose $\widetilde{\Omega}$ as the largest domain with

$$\widetilde{\Omega} \subseteq \Omega \cap B_2(x_0) \;\;,\quad x_0 \in \widetilde{\Omega}.$$

Then the domain $\hat{\Omega}$ is constructed in the same way as before with $0 < \varepsilon < \frac{1}{2}$ and $\varepsilon < \frac{1}{2}$ dist $(\partial\widetilde{\Omega}, \Omega_1)$. Now we set $\Omega_2 := \hat{\Omega}$, $\varepsilon_2 := \varepsilon$ and obtain $\overline{\Omega}_1 \subseteq \Omega_2$, dist $(\partial\Omega_2, \Omega_1) > \varepsilon_2$.

Repeating this procedure, we find by induction a sequence $(\Omega_j)_{j=1}^{\infty}$ of Lipschitz subdomains of Ω and a sequence $(\varepsilon_j)_{j=1}^{\infty}$ with $0 < \varepsilon_j < \frac{1}{j}$, $j \in \mathbb{N}$. The properties a), b) and c) are satisfied. In order to prove d) we consider any $x \in \Omega$. Since Ω is a domain, we can choose some $j_0 \in \mathbb{N}$ and some subdomain $\Omega_0 \subseteq \Omega$ such that

$$x \in \Omega_0 \subseteq \Omega \cap B_{j_0}(x_0) \;\;,\quad x_0 \in \Omega_0.$$

Let $d :=$ dist $(\partial\Omega_0, x)$ and choose $j_1 > j_0$ with $\varepsilon_{j_1} < d$. Then the above construction shows that $x \in \Omega_{j_1}$. This proves the lemma. $\square$

1.4.2 Remark The construction above yields the following additional property: To each bounded subdomain $\Omega' \subseteq \Omega$ with $\overline{\Omega'} \subseteq \Omega$ there exists some $j \in \mathbb{N}$ such that $\Omega' \subseteq \Omega_j$.

1.5 Compact embeddings

Such embedding properties are needed later on in the proofs for technical reasons.

Consider a bounded domain $\Omega \subseteq \mathbb{R}^n$ with $n \geq 1$, and let $1 < q < \infty$. Then the natural embedding

$$u \mapsto u \;\;,\quad u \in W_0^{1,q}(\Omega) \tag{1.5.1}$$

defines a bounded linear operator from $W_0^{1,q}(\Omega)$ into $L^q(\Omega)$ since

$$\|u\|_{L^q(\Omega)} \leq \|u\|_{W^{1,q}(\Omega)} \quad , \quad u \in W_0^{1,q}(\Omega). \tag{1.5.2}$$

Hence the embedding $W_0^{1,q}(\Omega) \subseteq L^q(\Omega)$ is continuous. The following lemma shows that the embedding operator (1.5.1) is even a compact operator. This means that each sequence $(u_j)_{j=1}^{\infty}$ in $W_0^{1,q}(\Omega)$, which is bounded in the norm of $W^{1,q}(\Omega)$, contains a subsequence which converges in the norm of $L^q(\Omega)$ to some element $u \in L^q(\Omega)$. Since $\sup_{j \in \mathbb{N}} \|u_j\|_{1,q} < \infty$, it even holds that $u \in W_0^{1,q}(\Omega)$.

1.5.1 Lemma *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$, be any bounded domain, and let $1 < q < \infty$. Then the embedding operator $u \mapsto u$ from $W_0^{1,q}(\Omega)$ into $L^q(\Omega)$ is compact. Therefore, each bounded sequence in $W_0^{1,q}(\Omega)$ contains a subsequence which converges in the norm of $L^q(\Omega)$ to some element of $W_0^{1,q}(\Omega)$.*

Proof. This is a special case of Rellich's theorem [Ada75, VI, Theorem 6.2, Part IV]. See also [Agm65, Sec. 8, Theorem 8.3] or [Tem77, Chap. II, Theorem 1.1]. □

Next we consider the dual space $L^q(\Omega)'$ of $L^q(\Omega), 1 < q < \infty$, consisting of all linear functionals defined on $L^q(\Omega)$ which are continuous in the norm $\|\cdot\|_q$. We know, see [Nec67, Chap. 2, Proposition 2.5], each such functional has the form

$$u \mapsto < f, u > = \int_\Omega f u \, dx \quad , \quad u \in L^q(\Omega) \tag{1.5.3}$$

with some $f \in L^{q'}(\Omega),\ q' = \frac{q}{q-1}$. Thus we get

$$L^{q'}(\Omega) = L^q(\Omega)' \tag{1.5.4}$$

if we identify each $f \in L^{q'}(\Omega)$ with the functional

$$< f, \cdot > : u \mapsto < f, u > \quad , \quad u \in L^q(\Omega).$$

Since $1 < q' < \infty$ we get in the same way that

$$L^q(\Omega)'' = L^{q'}(\Omega)' = L^q(\Omega). \tag{1.5.5}$$

Here $u \in L^q(\Omega)$ is identified with the functional

$$< \cdot, u >: f \mapsto < f, u > \quad , \quad f \in L^{q'}(\Omega).$$

Thus $L^q(\Omega)$ is a reflexive Banach space for $1 < q < \infty$. See Section 3.1 for some explanations.

If $u \in W_0^{1,q}(\Omega)$ we can use the Poincaré inequality (1.1.1) and see that

$$|< f, u >| \leq \|f\|_{q'} \|u\|_q \leq C \, \|f\|_{q'} \|\nabla u\|_q \tag{1.5.6}$$

for all $f \in L^{q'}(\Omega)$ with $C = C(q, \Omega) > 0$.

Consider now the dual space $W^{-1,q'}(\Omega) = W_0^{1,q}(\Omega)'$ of $W_0^{1,q}(\Omega)$, see (3.6.5), I. By (1.5.6) we know that each $f \in L^{q'}(\Omega)$ defines the continuous functional

$$< f, \cdot >: u \mapsto < f, u > \ , \quad u \in W_0^{1,q}(\Omega).$$

Thus, identifying each f with $< f, \cdot >$ we obtain the natural continuous embedding

$$L^{q'}(\Omega) \subseteq W^{-1,q'}(\Omega). \tag{1.5.7}$$

The embedding operator from $L^{q'}(\Omega)$ into $W^{-1,q'}(\Omega)$ can be understood as the dual operator of the embedding operator from $W_0^{1,q}(\Omega)$ into $L^q(\Omega)$. See [Yos80, VII, 1] concerning dual operators. We know, see Schauder's theorem [Yos80, X, 4], that the dual operator of a compact linear operator is again compact. Therefore, (1.5.7) is a compact embedding. Replacing q' by q we thus obtain the following result.

1.5.2 Lemma *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$, be any bounded domain, and let $1 < q < \infty$. Then the embedding*

$$L^q(\Omega) \subseteq W^{-1,q}(\Omega) \tag{1.5.8}$$

is compact. Therefore, each bounded sequence in $L^q(\Omega)$ contains a subsequence which converges in the norm of $W^{-1,q}(\Omega)$ to some element of $L^q(\Omega)$.

Proof. Use Lemma 1.5.1 and apply [Yos80, X, 4]. □

If Ω is a bounded Lipschitz domain, a similar compactness result also holds for the embedding $W^{1,q}(\Omega) \subseteq L^q(\Omega)$.

1.5.3 Lemma *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$, be a bounded Lipschitz domain, and let $1 < q < \infty$. Then the embedding*

$$W^{1,q}(\Omega) \subseteq L^q(\Omega) \tag{1.5.9}$$

is compact. Therefore, each bounded sequence in $W^{1,q}(\Omega)$ contains a subsequence which converges in the norm of $L^q(\Omega)$ to some element of $W^{1,q}(\Omega)$.

Proof. See [Nec67, Chap. 2, Theorem 6.3] □

The compactness of the embedding (1.5.7) can be used to improve the estimate (1.1.6) in Lemma 1.1.3. We can "remove" the second term on the right side of (1.1.6) under an additional condition on u. This leads to the following result.

1.5.4 Lemma *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain, let $\Omega_0 \subseteq \Omega$, $\Omega_0 \neq \emptyset$, be any subdomain, and let $1 < q < \infty$. Then*

$$\|u\|_{L^q(\Omega)} \leq C_1 \|\nabla u\|_{W^{-1,q}(\Omega)^n} \leq C_1 C_2 \|u\|_{L^q(\Omega)} \tag{1.5.10}$$

for all $u \in L^q(\Omega)$ satisfying

$$\int_{\Omega_0} u\,dx \;=\; 0\;; \tag{1.5.11}$$

$C_1 = C_1(q, \Omega, \Omega_0) > 0$ and $C_2 = C_2(n) > 0$ are constants.

Proof. Recall that $\nabla u \in W^{-1,q}(\Omega)^n$ with $u \in L^q(\Omega)$ means the functional

$$[\nabla u,\,\cdot\,] : v \mapsto [\nabla u, v] \;=\; -<u, \text{div } v> = \;-\int_\Omega u \text{ div } v\,dx,$$

$v \in W_0^{1,q'}(\Omega)^n$, $q' = \frac{q}{q-1}$, see the proof of Lemma 1.1.3.

The estimate

$$\begin{aligned} |[\nabla u, v]| \;&=\; |<u,\;\text{div } v>| \;\le\; \|u\|_q\,\|\text{div } v\|_{q'} \\ &\le\; C\,\|u\|_q\,\|v\|_{W^{1,q'}(\Omega)^n} \end{aligned} \tag{1.5.12}$$

for all $v \in W_0^{1,q'}(\Omega)^n$, with $C = C(n) > 0$, proves the second inequality in (1.5.10).

Thus it remains to prove the first inequality in (1.5.10). To prove it we use a contradiction argument. Assume there does not exist a constant $C > 0$ such that

$$\|u\|_q \;\le\; C\,\|\nabla u\|_{-1,q}$$

holds for all $u \in L^q(\Omega)$ with $\int_{\Omega_0} u\,dx = 0$. Then for each $j \in \mathbb{N}$ there is some $u_j \in L^q(\Omega)$ with $\|u_j\|_q > j\|\nabla u_j\|_{-1,q}$, $\int_{\Omega_0} u_j\,dx = 0$. Setting

$$\tilde{u}_j := \|u_j\|_q^{-1} u_j \;\;,\quad j \in \mathbb{N}$$

we obtain a sequence $(\tilde{u}_j)_{j=1}^\infty$ in $L^q(\Omega)$ satisfying

$$\|\tilde{u}_j\|_q = 1 \;\;,\quad \int_{\Omega_0} \tilde{u}_j\,dx = 0 \;\;,\quad \|\nabla \tilde{u}_j\|_{-1,q} < \frac{1}{j}$$

for all $j \in \mathbb{N}$.

Since $L^q(\Omega)$ is reflexive and the sequence $(\tilde{u}_j)_{j=1}^\infty$ is bounded, there exists a subsequence which converges weakly in $L^q(\Omega)$ to some element $u \in L^q(\Omega)$, see Section 3.1. For simplicity we may assume that $(\tilde{u}_j)_{j=1}^\infty$ itself has this property. This means that

$$<u, v> \;=\; \lim_{j\to\infty} <\tilde{u}_j, v>$$

for all $v \in L^{q'}(\Omega)$. In particular, it follows that $\int_{\Omega_0} u\,dx = 0$. Using

$$\lim_{j\to\infty} \|\nabla \tilde{u}_j\|_{-1,q} \;=\; 0 \quad ,$$
$$|[\nabla \tilde{u}_j, v]| \;=\; |< \tilde{u}_j, \mathrm{div}\; v >| \;\leq\; \|\nabla \tilde{u}_j\|_{-1,q}\, \|v\|_{1,q'} \;,$$

and

$$\begin{aligned} |[\nabla u, v]| &= |< u,\ \mathrm{div}\ v >| \;=\; |\lim_{j\to\infty} < \tilde{u}_j,\ \mathrm{div}\ v >| \\ &= \lim_{j\to\infty} |< \tilde{u}_j,\ \mathrm{div}\ v >| = \lim_{j\to\infty}\inf |< \tilde{u}_j,\ \mathrm{div}\ v >| \\ &\leq \lim_{j\to\infty}\inf(\|\nabla \tilde{u}_j\|_{-1,q}\, \|v\|_{1,q'}) \\ &= (\lim_{j\to\infty}\inf \|\nabla \tilde{u}_j\|_{-1,q})\, \|v\|_{1,q'} \\ &= (\lim_{j\to\infty} \|\nabla \tilde{u}_j\|_{-1,q})\, \|v\|_{1,q'} = 0, \end{aligned}$$

$v \in W_0^{1,q'}(\Omega)$, we see that $\|\nabla u\|_{-1,q} = 0$. Therefore, it holds that $\nabla u = 0$ in the sense of distributions and therefore, u is a constant. The mollification method in Section 1.7 will give a proof of this property, see (1.7.18). Since $\int_{\Omega_0} u\,dx = 0$ we conclude that $u = 0$.

On the other hand, applying inequality (1.1.6) to $\tilde{u}_j$ yields

$$\|\tilde{u}_j\|_q \;=\; 1 \;\leq\; C(\|\nabla \tilde{u}_j\|_{-1,q} + \|\tilde{u}_j\|_{-1,q}) \tag{1.5.13}$$

for all $j \in \mathbb{N}$, where $C > 0$ is the constant in (1.1.6). Since $(\tilde{u}_j)_{j=1}^{\infty}$ is bounded in $L^q(\Omega)$ and since the embedding $L^q(\Omega) \subseteq W^{-1,q}(\Omega)$ is compact, see Lemma 1.5.2, there is a subsequence of $(\tilde{u}_j)_{j=1}^{\infty}$ which converges in $W^{-1,q}(\Omega)$ to some $\tilde{u} \in L^q(\Omega)$. It also converges weakly to $\tilde{u} \in L^q(\Omega)$, and therefore we get $\tilde{u} = u = 0$. We may assume that the sequence $(\tilde{u}_j)_{j=1}^{\infty}$ itself converges in $W^{-1,q}(\Omega)$ to $u = 0$. Therefore,

$$\lim_{j\to\infty} \|\tilde{u}_j\|_{-1,q} \;=\; 0.$$

However, from (1.5.13) we get that

$$1 \;\leq\; \lim_{j\to\infty} C\,(\|\nabla \tilde{u}_j\|_{-1,q} + \|\tilde{u}_j\|_{-1,q}) \;=\; 0.$$

This is a contradiction and the lemma is proved. The argument used here is well known, see Peetre's lemma [LiMa72, Chap. 2, Lemma 5.1]. □

1.6 Representation of functionals

In the theory of the Navier-Stokes equations we are interested in the case that the external force $f = (f_1, \dots, f_n)$ has the special form

$$f = \text{div } F \tag{1.6.1}$$

in the sense of distributions. Here $F = (F_{jl})_{j,l=1}^n$ means a matrix and (1.6.1) means by definition that

$$f_l = \text{div } (F_{1l}, \dots, F_{nl}) = \sum_{j=1}^n D_j F_{jl},$$

$l = 1, \dots, n$. Thus the operation div applies to the columns of the matrix F.

Below we consider some conditions which are sufficient for the representation (1.6.1). If Ω is bounded, we may use the Poincaré inequality and get the following easy fact.

1.6.1 Lemma *Let $\Omega \subseteq \mathbb{R}^n$ be any bounded domain with $n \geq 2$, and let $f \in W^{-1,2}(\Omega)^n$.*

Then there exists at least one matrix $F \in L^2(\Omega)^{n^2}$ satisfying

$$f = \text{div } F$$

in the sense of distributions, and

$$\|f\|_{W^{-1,2}(\Omega)^n} \leq \|F\|_{L^2(\Omega)^{n^2}} \leq C\|f\|_{W^{-1,2}(\Omega)^n} \tag{1.6.2}$$

with $C = C(\Omega) > 0$.

Proof. Consider the closed subspace

$$D := \{\nabla v \in L^2(\Omega)^{n^2}; v \in W_0^{1,2}(\Omega)^n\} \subseteq L^2(\Omega)^{n^2} \tag{1.6.3}$$

of all gradients $\nabla v = (D_j v_l)_{j,l=1}^n$ of functions $v = (v_1, \dots, v_n) \in W_0^{1,2}(\Omega)^n$. Let the functional

$$\tilde{f} : \nabla v \mapsto [\tilde{f}, \nabla v] \,, \quad \nabla v \in D$$

be defined by $[\tilde{f}, \nabla v] := [f, v]$ for all $v \in W_0^{1,2}(\Omega)^n$. Then the Poincaré inequality (1.1.1) yields some $C = C(\Omega) > 0$ such that

$$|[\tilde{f}, \nabla v]| = |[f, v]| \leq \|f\|_{-1,2} \|v\|_{1,2} \leq C\|f\|_{-1,2} \|\nabla v\|_2$$

for all $\nabla v \in D$. Therefore, $\tilde{f}$ is a continuous functional defined on the subspace $D \subseteq L^2(\Omega)^{n^2}$.

The Hahn-Banach theorem, see [Yos80, IV, 1], yields a linear extension of $\tilde{f}$ from D to $L^2(\Omega)^{n^2}$ with the same functional norm. Then we may use the Riesz representation theorem, see [Yos80, III, 6], and obtain a matrix $F \in L^2(\Omega)^{n^2}$ satisfying

$$< F, \nabla v > = \sum_{j,l=1}^{n} \int_\Omega F_{jl}(D_j v_l)\, dx = \int_\Omega F \cdot \nabla v \, dx \; = \; [\tilde{f}, \nabla v] \; = \; [f, v]\,,$$

$v = (v_1, \ldots, v_n) \in W_0^{1,2}(\Omega)^n$, and

$$\|F\|_{L^2(\Omega)^{n^2}} \; \leq \; C\,\|f\|_{-1,2}\,.$$

Further we get

$$|[f, v]| = \; |< F, \nabla v >| \; \leq \; \|F\|_2 \,\|\nabla v\|_2 \; \leq \; \|F\|_2(\|v\|_2^2 + \|\nabla v\|_2^2)^{\frac{1}{2}}$$

for all $v \in W_0^{1,2}(\Omega)^n$ which shows that

$$\|f\|_{W^{-1,2}(\Omega)} \; \leq \; \|F\|_2.$$

If $v \in C_0^\infty(\Omega)^n$ we see that

$$\begin{aligned} < F, \nabla v > \; &= \; \sum_{j,l=1}^{n} < F_{jl}, D_j v_l > \; = - \sum_{j,l=1}^{n} < D_j F_{jl}, v_l > \\ &= \; -[\;\mathrm{div}\; F, v] \; = \; [f, v] \end{aligned}$$

holds in the sense of distributions. This yields the representation $\mathrm{div}(-F) = f$ and (1.6.1) holds with F replaced by $-F$. This proves the lemma. □

Consider the bounded domain Ω as in Lemma 1.6.1 and let $f \in L^2(\Omega)^n$. Then we identify f with the functional $< f, \cdot >$ and get

$$f \in W^{-1,2}(\Omega)^n \;\; , \quad \|f\|_{-1,2} \, \leq \, C\,\|f\|_2, \tag{1.6.4}$$

with C from (1.1.1). This yields the continuous embedding

$$L^2(\Omega)^n \; \subseteq \; W^{-1,2}(\Omega)^n. \tag{1.6.5}$$

Using the above lemma we see that for each $f \in L^2(\Omega)^n$ there exists some $F \in L^2(\Omega)^{n^2}$ satisfying

$$f \; = \; \mathrm{div}\; F \tag{1.6.6}$$

in the sense of distributions, and

$$\|F\|_{L^2(\Omega)^{n^2}} \;\le\; C\,\|f\|_{L^2(\Omega)^n} \tag{1.6.7}$$

where $C = C(\Omega) > 0$ is a constant.

If Ω is not bounded, then, in general, $\|\nabla v\|_{L^2(\Omega)^{n^2}}$ and $\|v\|_{W^{1,2}(\Omega)^n}$ are not equivalent norms in $W_0^{1,2}(\Omega)^n$. Therefore, we cannot expect that each $f \in W^{-1,2}(\Omega)^n$ has a representation $f = \operatorname{div} F$ with $F \in L^2(\Omega)^{n^2}$. The following lemma yields a criterion for this property. We have to distinguish the cases $n \ge 3$ and $n = 2$. If $n = 2$ we need an open ball $B_R(x_0)$ with center x_0 and radius R.

1.6.2 Lemma

a) *Let $\Omega \subseteq \mathbb{R}^n$ be any unbounded domain with $n \ge 3$ and let $f \in L^q(\Omega)^n$ with $q = \frac{2n}{n+2}$. Then there exists a matrix function $F \in L^2(\Omega)^{n^2}$ satisfying*

$$f = \operatorname{div} F \tag{1.6.8}$$

in the sense of distributions, and

$$\|f\|_{W^{-1,2}(\Omega)^n} \;\le\; \|F\|_{L^2(\Omega)^{n^2}} \;\le\; C\|f\|_{L^q(\Omega)^n} \tag{1.6.9}$$

with some constant $C = C(n) > 0$.

b) *Let $\Omega \subseteq \mathbb{R}^2$ be any unbounded domain with $\overline{\Omega} \ne \mathbb{R}^2$, let $x_0 \notin \overline{\Omega}$, $R > 0$, $\overline{B}_R(x_0) \cap \overline{\Omega} = \emptyset$, $f \in L^2_{loc}(\overline{\Omega})^2$, and suppose that*

$$\|f\|^2_{\wedge} \;:=\; \int_\Omega |f(x)|^2\,|x-x_0|^2\,(\ln|x-x_0|/R)^2\,dx \;<\; \infty. \tag{1.6.10}$$

Then there exists a matrix function $F \in L^2(\Omega)^4$ satisfying

$$f = \operatorname{div} F \tag{1.6.11}$$

in the sense of distributions, and

$$\|f\|_{W^{-1,2}(\Omega)^n} \;\le\; \|F\|_{L^2(\Omega)^{n^2}} \;\le\; C\,\|f\|_{\wedge} \tag{1.6.12}$$

with some constant $C = C(\Omega) > 0$.

Proof. To prove a) we use Sobolev's inequality (1.3.5) with $q' = \frac{q}{q-1} = \frac{2n}{n-2}$, $\frac{1}{n} + \frac{1}{q'} = \frac{1}{2}$. This yields

$$\|v\|_{q'} \;\le\; C\|\nabla v\|_2 \;\;,\quad v \in C_0^\infty(\Omega)^n, \tag{1.6.13}$$

with $C = C(n,q) > 0$.

Since $\frac{1}{q} + \frac{1}{q'} = \frac{n+2}{2n} + \frac{n-2}{2n} = 1$, we get the estimate

$$|< f, v >| \;\leq\; \|f\|_q \, \|v\|_{q'} \;\leq\; C \, \|f\|_q \, \|\nabla v\|_2. \tag{1.6.14}$$

This shows that the functional defined by $\nabla v \mapsto < f, v >$ is continuous on the subspace $D \subseteq L^2(\Omega)^{n^2}$, see (1.6.3), and the same argument as in the proof of Lemma 1.6.1 yields some F satisfying (1.6.8) and (1.6.9).

To prove b) we may assume that $R = 1$. Then we use the embedding inequality (1.3.12) and obtain

$$\left(\int_\Omega \left(\frac{|v(x)|}{|x - x_0| \ln |x - x_0|} \right)^2 dx \right)^{\frac{1}{2}} \leq\; C \, \|\nabla v\|_{L^2(\Omega)^4}$$

for all $v \in C_0^\infty(\Omega)^n$ with $C = C(\Omega) > 0$. This leads to

$$\begin{aligned} & | < f, v > | \\ = \; & \Big| \int_\Omega (f(x)|x - x_0| \ln|x - x_0|) \cdot (v(x)|x - x_0|^{-1} (\ln|x - x_0|)^{-1}) \Big| \, dx \\ \leq \; & C \|f\|_\wedge \|\nabla v\|_2, \end{aligned}$$

and the assertion in b) follows in the same way as before. $\square$

1.7 Mollification method

This method enables us to approximate L^q-functions by C^∞-functions. It will be used later on in the proofs. See [Ada75, II, 2.17], [Nec67, Chap. 2, 1.3], [Yos80, I, Prop. 8], [Fri69, Part 1, (6.3)], [Miz73, Chap. 1, end of 7, and Chap. 2, Prop. 2.4, (3)], [Agm65, Sec. 1, Def. 1.7].

Let $\Omega \subseteq \mathbb{R}^n$ be a domain with $n \geq 1$ and let $\Omega_0 \subseteq \Omega$, $\Omega_0 \neq \emptyset$, be a bounded subdomain with $\overline{\Omega}_0 \subseteq \Omega$. Let

$$B_r(x) := \{ y \in \mathbb{R}^n ; \; |x - y| < r \} \tag{1.7.1}$$

be the open ball with center x and radius $r > 0$, and let the function $\mathcal{F} \in C_0^\infty(\mathbb{R})^n$ satisfy the following properties:

$$\begin{aligned} \operatorname{supp} \mathcal{F} \;&\subseteq\; B_1(0) \;\; , \quad 0 \leq \mathcal{F} \leq 1 \;\; , \quad \int_{B_1(0)} \mathcal{F} dx = 1, \\ \mathcal{F}(x) \;&=\; \mathcal{F}(-x) \quad \text{for all } \; x \in \mathbb{R}^n. \end{aligned} \tag{1.7.2}$$

Let $\mathcal{F}_\varepsilon \in C_0^\infty(\mathbb{R}^n)$, $\varepsilon > 0$, be defined by

$$\mathcal{F}_\varepsilon(x) := \varepsilon^{-n} \mathcal{F}(\varepsilon^{-1} x) \;\; , \quad x \in \mathbb{R}^n. \tag{1.7.3}$$

Then supp $\mathcal{F}_\varepsilon \subseteq B_\varepsilon(0)$ and the transformation formula for integrals, see [Apo74, Theorem 15.11], yields

$$\int_{\mathbb{R}^n} \mathcal{F}_\varepsilon(x)\,dx \;=\; \int_{\mathbb{R}^n} \mathcal{F}(y)\,dy \;=\; 1 \tag{1.7.4}$$

with $y = \frac{1}{\varepsilon}x,\; dy = \varepsilon^{-n}\,dx$.

Consider any function $u \in L^1_{loc}(\overline{\Omega})$ and set $u(x) := 0$ for all $x \notin \Omega$. Then we get $u \in L^1_{loc}(\mathbb{R}^n)$. Let $u^\varepsilon = \mathcal{F}_\varepsilon \star u$ be defined by

$$u^\varepsilon(x) = (\mathcal{F}_\varepsilon \star u)(x) := \int_{\mathbb{R}^n} \mathcal{F}_\varepsilon(x-y)u(y)\,dy \quad , \quad x \in \mathbb{R}^n. \tag{1.7.5}$$

Using again the transformation formula for integrals we see that

$$u^\varepsilon(x) = \int_{\mathbb{R}^n} \mathcal{F}_\varepsilon(x-y)u(y)\,dy \;=\; \int_{\mathbb{R}^n} \mathcal{F}_\varepsilon(z)u(x-z)\,dz \tag{1.7.6}$$

with $x - y = z,\; dy = dz$, and that

$$u^\varepsilon(x) = \int_{\mathbb{R}^n} \mathcal{F}_\varepsilon(x-y)u(y)\,dy \;=\; \int_{\mathbb{R}^n} \mathcal{F}(z)u(x-\varepsilon z)\,dz \tag{1.7.7}$$

with $\varepsilon^{-1}(x-y) = z,\; y = x - \varepsilon z,\; dy = \varepsilon^n dz$.

If u is continuous in Ω, then

$$\lim_{\varepsilon \to 0} u^\varepsilon(x) = u(x) \quad \text{uniformly for all } \; x \in \Omega_0. \tag{1.7.8}$$

The proof of this fact rests on the representation

$$u^\varepsilon(x) - u(x) = \int_{\mathbb{R}^n} \mathcal{F}(z)(u(x-\varepsilon z) - u(x))\,dz \quad , \quad x \in \Omega_0.$$

Let $u \in L^q(\Omega),\; 1 < q < \infty$, and $q' = \frac{q}{q-1}$. Then by Hölder's inequality and Fubini's theorem, see [Apo74], we get

$$\begin{aligned}
\|\mathcal{F}_\varepsilon \star u\|_{L^q(\Omega)} &= \left(\int_\Omega \left| \int_{|z|\le 1} \mathcal{F}(z)^{\frac{1}{q'}} \mathcal{F}(z)^{\frac{1}{q}} u(x-\varepsilon z)\,dz \right|^q dx \right)^{\frac{1}{q}} \\
&\le \left(\int_{|z|\le 1} \mathcal{F}(z)\,dz \right)^{\frac{1}{q'}} \left(\int_{|z|\le 1} \mathcal{F}(z) \left(\int_{\mathbb{R}^n} |u(x-\varepsilon z)|^q\,dx \right) dz \right)^{\frac{1}{q}} \\
&\le \left(\int_{|z|\le 1} \mathcal{F}\,dz \right)^{\frac{1}{q'}} \left(\int_{|z|\le 1} \mathcal{F}\,dz \right)^{\frac{1}{q}} \|u\|_{L^q(\Omega)} \\
&= \|u\|_{L^q(\Omega)}.
\end{aligned}$$

This estimate

$$\|\mathcal{F}_\varepsilon \star u\|_{L^q(\Omega)} \;\leq\; \|u\|_{L^q(\Omega)}$$

also holds if $q = 1$.

This shows,

$$\mathcal{F}_\varepsilon\star : u \mapsto \mathcal{F}_\varepsilon \star u \;\;,\quad u \in L^q(\Omega) \tag{1.7.9}$$

is a bounded linear operator from $L^q(\Omega)$ to $L^q(\Omega)$ with operator norm

$$\|\mathcal{F}_\varepsilon \star\| \;\leq\; 1 \;\;,\quad \varepsilon > 0. \tag{1.7.10}$$

Next we use the density

$$\overline{C_0^\infty(\Omega)}^{\|\cdot\|_{L^q(\Omega)}} = L^q(\Omega) \;\;,\quad 1 \leq q < \infty, \tag{1.7.11}$$

the property (1.7.8), which holds for each $u \in C_0^\infty(\Omega)$, and the uniform boundedness (1.7.10). This leads by an elementary calculation to

$$\lim_{\varepsilon \to 0} \|(\mathcal{F}_\varepsilon \star u) - u\|_{L^q(\Omega)} \;=\; 0$$

for all $u \in L^q(\Omega),\ 1 \leq q < \infty$.

Collecting these facts yields the following result:

1.7.1 Lemma *Let $\Omega \subseteq \mathbb{R}^n, n \geq 1$, be any domain, and let $1 \leq q < \infty$, $\varepsilon > \sigma$. Then for all $u \in L^q(\Omega)$ we get*

$$\|(\mathcal{F}_\varepsilon \star u)\|_{L^q(\Omega)} \;\leq\; \|u\|_{L^q(\Omega)} \tag{1.7.12}$$

and

$$\lim_{\varepsilon \to 0}(\mathcal{F}_\varepsilon \star u) = u \tag{1.7.13}$$

with respect to the norm $\|\cdot\|_{L^q(\Omega)}$.

Proof. See [Ada75, II, Lemma 2.18]. □

We mention some further properties of the operator $\mathcal{F}_\varepsilon\star$, see [Ada75, II, 2.17–2.19]. Let Ω and $\Omega_0 \subseteq \Omega$ be as above. Let $x \in \Omega_0$ and

$$0 < \varepsilon < \text{ dist } (\partial\Omega, \Omega_0) \;:=\; \inf_{x \in \partial\Omega, y \in \Omega_0} |x - y| \tag{1.7.14}$$

with $0 < \varepsilon < \infty$ if $\partial\Omega = \emptyset$.

Consider any distribution $u \in C_0^\infty(\Omega)'$ in Ω, for example $u \in L^1_{loc}(\Omega)$. Then for each fixed $x \in \Omega_0$, we let $\mathcal{F}_\varepsilon(x - \cdot)$ be the test function

$$\mathcal{F}_\varepsilon(x - \,\cdot\,) : y \mapsto \mathcal{F}_\varepsilon(x - y) \;\;,\quad y \in \Omega,$$

and we see,

$$u^\varepsilon(x) = (\mathcal{F}_\varepsilon \star u)(x) = \int_\Omega \mathcal{F}_\varepsilon(x-y)u(y)\,dy \ := \ [u, \mathcal{F}_\varepsilon(x - \ \cdot\,)] \tag{1.7.15}$$

is well defined in the sense of distributions. In this case, the "integral" $\int_\Omega \cdots dy$ is only used formally as a notation. An easy calculation yields the properties

$$u^\varepsilon \ = \ \mathcal{F}_\varepsilon \star u \ \ \in C^\infty(\overline{\Omega}_0) \tag{1.7.16}$$

and

$$(D^\alpha u^\varepsilon)(x) \ = \ (\mathcal{F}_\varepsilon \star (D^\alpha u))(x) \ = \ ((D^\alpha \mathcal{F}_\varepsilon) \star u)(x) \tag{1.7.17}$$

for all $x \in \Omega_0$, where $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$. Thus if $x \in \Omega_0$, and ε satisfies (1.7.14), D^α commutes with the operator $\mathcal{F}_\varepsilon\star$.

As an application of this method we prove the following property:

$$\left.\begin{array}{l}\text{If } u \in L^1_{loc}(\Omega) \ \text{ and } \ \nabla u = 0 \ \text{ in the sense of} \\ \text{distributions, then } u \text{ is a constant.}\end{array}\right\} \tag{1.7.18}$$

Indeed, we see that $\nabla u^\varepsilon(x) = (\nabla u)^\varepsilon(x) = 0$ for all $x \in \Omega_0$ and all ε as in (1.7.14). Since u^ε is smooth, see (1.7.16), an elementary argument shows that $u^\varepsilon = C_\varepsilon$ holds in Ω_0 with a constant C_ε depending on ε. Letting $\varepsilon \to 0$ and using (1.7.13) we see that C_ε converges to some constant C. Replacing Ω_0 by the subdomains Ω_j, $j \in \mathbb{N}$, in Lemma 1.4.1, we conclude that u is constant on the whole domain Ω.

The results of this subsection can also be used if u is replaced by a vector field $u = (u_1, \ldots, u_m)$, $m \in \mathbb{N}$. If $n = 1$, $\Omega \subseteq \mathbb{R}$ means any open interval.

2 The operators ∇ and div

2.1 Solvability of div $v = g$ and $\nabla p = f$

The investigation of these operators is the first important step in the theory of the Navier-Stokes system. The construction of the pressure p rests on properties of ∇ and div. Both operators div and ∇ are connected by a duality principle, see the proof of the lemma below. Therefore, it is sufficient to know the basic properties of one of these operators. The approach which we use here is based on the estimates of gradients in Lemma 1.5.4. There are several other approaches to these operators, see [Bog79], [Bog80], [Gal94a, III, Lemma 3.1], [vWa88], and [Pil80].

2.1.1 Lemma *Let $\Omega \subseteq \mathbb{R}^n, n \geq 2$, be a bounded Lipschitz domain, let $\Omega_0 \subseteq \Omega$, $\Omega_0 \neq \emptyset$, be any subdomain, and let $1 < q < \infty$, $q' = \frac{q}{q-1}$. Then we have:*

a) *For each $g \in L^q(\Omega)$ with $\int_\Omega g dx = 0$, there exists at least one $v \in W_0^{1,q}(\Omega)^n$ satisfying*
$$\operatorname{div} v = g \ , \quad \|\nabla v\|_{L^q(\Omega)^n} \leq C \|g\|_{L^q(\Omega)} \, , \tag{2.1.1}$$
where $C = C(q, \Omega) > 0$ is a constant.

b) *For each $f \in W^{-1,q}(\Omega)^n$ so that*
$$[f, v] = 0 \quad \textit{for all} \quad v \in W_0^{1,q'}(\Omega)^n \ \text{with} \ \operatorname{div} v = 0,$$
there exists a unique $p \in L^q(\Omega)$ satisfying
$$\nabla p = f \ , \quad \int_{\Omega_0} p \, dx = 0 \ , \quad \|p\|_{L^q(\Omega)} \ \leq \ C \, \|f\|_{W^{-1,q}(\Omega)^n}, \tag{2.1.2}$$
where $C = C(q, \Omega, \Omega_0) > 0$ is a constant.

Proof. First let $\Omega_0 = \Omega$. We set
$$< p, g > := \int_\Omega pg \, dx \ , \quad p \in L^q(\Omega) \, , \ g \in L^{q'}(\Omega) \, .$$

Since $1 < q < \infty$, $L^q(\Omega)$ and $L^{q'}(\Omega)$ are reflexive Banach spaces. Therefore, $L^q(\Omega)$ is the dual space of $L^{q'}(\Omega)$ if we identify each $p \in L^q(\Omega)$ with the functional $< p, \cdot >$, and $L^{q'}(\Omega)$ is the dual space of $L^q(\Omega)$ if we identify each $g \in L^{q'}(\Omega)$ with the functional $< \cdot, g >$. See [Yos80, IV, 9, (3)] for these notions.

Consider now the closed subspaces
$$\begin{aligned} L_0^q(\Omega) \ &:= \ \{p \in L^q(\Omega); \int_\Omega p \, dx = 0\} \subseteq L^q(\Omega), \\ L_0^{q'}(\Omega) \ &:= \ \{g \in L^{q'}(\Omega); \int_\Omega g \, dx = 0\} \subseteq L^{q'}(\Omega) \, . \end{aligned}$$

As before we set
$$< p, g > = \int_\Omega pg \, dx \ , \quad p \in L_0^q(\Omega) \, , \quad g \in L_0^{q'}(\Omega).$$

Each continuous linear functional defined on $L_0^{q'}(\Omega)$ has a continuous linear extension to $L^{q'}(\Omega)$, see the Hahn-Banach theorem [Yos80, IV, 1], see also Section 3.1. Therefore, each such functional has the form
$$g \mapsto < p, g > \ , \quad g \in L_0^{q'}(\Omega),$$

with some $p \in L^q(\Omega)$. We choose $p_0 \in \mathbb{R}$ in such a way that $\int_\Omega (p - p_0)\, dx = 0$. Then $< p, g > = < p - p_0, g >$ for $g \in L_0^{q'}(\Omega)$ and it holds that $p - p_0 \in L_0^q(\Omega)$.

This shows that $L_0^q(\Omega)$ is the dual space of $L_0^{q'}(\Omega)$ if each $p \in L_0^q(\Omega)$ is identified with the functional $< p, \cdot >$. Correspondingly, $L_0^{q'}(\Omega)$ is the dual space of $L_0^q(\Omega)$. Thus we get

$$L_0^q(\Omega) = L_0^{q'}(\Omega)' \ , \quad L_0^{q'}(\Omega) = L_0^q(\Omega)' \ . \tag{2.1.3}$$

Next we consider the space $W_0^{1,q'}(\Omega)^n$ and its dual space

$$W^{-1,q}(\Omega)^n \ = \ W_0^{1,q'}(\Omega)^{n\prime} \ ,$$

see (3.6.5), I. Let $[f, v]$ denote the value of $f \in W^{-1,q}(\Omega)^n$ at $v \in W_0^{1,q'}(\Omega)^n$. Then $W_0^{1,q'}(\Omega)^n$ is the dual space of $W^{-1,q}(\Omega)^n$ if each $v \in W_0^{1,q'}(\Omega)^n$ is identified with the functional $[\,\cdot\,, v] : f \mapsto [f, v]$.

Let $v \in W_0^{1,q'}(\Omega)^n$. Then from (1.2.5) we see that $v|_{\partial\Omega} = 0$ holds in the sense of traces, and Green's formula (1.2.12), applied with $u = 1$ in Ω, shows that

$$\int_\Omega \text{div}\; v\, dx = 0 \ , \quad \text{div}\; v \in L_0^{q'}(\Omega) \, .$$

The linear operator

$$\text{div} \ : v \mapsto \ \text{div}\; v \ , \quad v \in W_0^{1,q'}(\Omega)^n \tag{2.1.4}$$

from $W_0^{1,q'}(\Omega)^n$ to $L_0^{q'}(\Omega)$ is bounded since

$$\|\text{div}\; v\|_{q'} \ \le \ C_1 \, \|v\|_{W^{1,q'}(\Omega)^n} \tag{2.1.5}$$

with $C_1 = C_1(n) > 0$. Let

$$R(\text{div}) := \{\text{div}\; v \in L_0^{q'}(\Omega) \ ; \ v \in W_0^{1,q'}(\Omega)^n\}$$

denote the range space and let

$$N(\text{div}) := \{v \in W_0^{1,q'}(\Omega)^n \ ; \ \text{div}\; v = 0\}$$

be the null space of div.

Further we consider the operator

$$\nabla : p \mapsto \nabla p \ , \quad p \in L_0^q(\Omega) \tag{2.1.6}$$

from $L^q_0(\Omega)$ to $W^{-1,q}(\Omega)^n$, defined by the relation

$$[\nabla p, v] := - < p, \text{div } v > \ , \quad v \in W_0^{1,q'}(\Omega)^n \ , \ \ p \in L^q_0(\Omega), \tag{2.1.7}$$

with range

$$R(\nabla) := \{\nabla p \in W^{-1,q}(\Omega)^n \ ; \ p \in L^q_0(\Omega)\}. \tag{2.1.8}$$

If $\nabla p = 0$ we see that p is a constant, see (1.7.18), and therefore $p = 0$ since $\int_\Omega p\, dx = 0$. Thus

$$N(\nabla) := \{p \in L^q_0(\Omega) \ ; \ \nabla p = 0\} = \{0\}. \tag{2.1.9}$$

From the estimate

$$\begin{array}{rcl} |[\nabla p, v]| & = & |< p, \text{div } v >| \ \leq \ \|p\|_q \, \|\text{div } v\|_{q'} \\ & \leq & C_1 \, \|p\|_q \, \|v\|_{1,q'} \ , \end{array}$$

with C_1 as in (2.1.5), we see that ∇ is a bounded operator from $L^q_0(\Omega)$ to $W^{-1,q}(\Omega)^n$. It holds that

$$\|\nabla p\|_{-1,q} \ \leq \ C_1 \|p\|_q \ , \quad p \in L^q_0(\Omega). \tag{2.1.10}$$

Next we use a functional analytic argument. The relation (2.1.7) means that $-\nabla$ is the dual operator of div, we write

$$-\nabla = \ \text{div}' \ , \tag{2.1.11}$$

see [Yos80, VII, 1] for this notion.

From Lemma 1.5.4, see (1.5.10), we obtain the estimate

$$\|p\|_q \ \leq \ C_2 \|\nabla p\|_{-1,q} \ , \quad p \in L^q_0(\Omega) \tag{2.1.12}$$

with some constant $C_2 = C_2(q, \Omega) > 0$. This shows that the range $R(-\nabla) = R(\nabla)$ of $-\nabla$ is a closed subspace of $W^{-1,q}(\Omega)^n$. Therefore we conclude that the inverse

$$\nabla^{-1} : \nabla p \mapsto p \ , \quad \nabla p \in R(\nabla)$$

from $R(\nabla)$ onto $L^q_0(\Omega)$ is a bounded operator, see [Yos80, II, 6, Theorem 1].

The closed range theorem, see [Yos80, VII, 5], yields now the following result:
$R(\text{div})$ is a closed subspace of $L_0^{q'}(\Omega)$, we have

$$R(\text{div}) \ = \ \{g \in L_0^{q'}(\Omega); \ < p, g > = 0 \quad \text{for all } p \in N(\nabla)\}, \tag{2.1.13}$$

and

$$R(\nabla) \;=\; \{f \in W^{-1,q}(\Omega)^n;\; [f,v] = 0 \quad \text{for all } v \in N(\text{div})\}. \tag{2.1.14}$$

Since $N(\nabla) = \{0\}$ we conclude that

$$R(\text{div}) = L_0^{q'}(\Omega). \tag{2.1.15}$$

Let

$$W_0^{1,q'}(\Omega)^n / N(\text{div}) := \{[v];\; v \in W_0^{1,q'}(\Omega)^n\} \tag{2.1.16}$$

denote the quotient space (see [Yos80, I, 11]) of all classes $[v] := v + N(\text{div}),\ v \in W_0^{1,q'}(\Omega)^n$, equipped with the norm

$$\|[v]\|_{W_0^{1,q'}(\Omega)^n/N(\text{div})} := \inf_{w \in [v]} \|\nabla(v+w)\|_{q'}. \tag{2.1.17}$$

Recall that $\|\nabla v\|_{q'}$ is an equivalent norm of $W_0^{1,q'}(\Omega)^n$ since Ω is bounded, see (1.1.1).

We see that there exists the well defined inverse operator

$$\text{div}^{-1} :\; \text{div}\ v \mapsto [v] \tag{2.1.18}$$

from $R(\text{div}) = L_0^{q'}(\Omega)$ onto $W_0^{1,q'}(\Omega)^n / N(\text{div})$. The operator div in (2.1.4) is bounded and therefore closed, which means its graph is closed. From the closed graph theorem, see [Yos80, II, 6, Theorem 1], we can now conclude that the operator div^{-1} in (2.1.18) is bounded. This means that

$$\|\,[v]\,\|_{W_0^{1,q'}(\Omega)/N(\text{div})} \;\le\; C_3\, \|\text{div}\ v\|_{q'} \tag{2.1.19}$$

for all $v \in W_0^{1,q'}(\Omega)^n$ with some constant $C_3 = C_3(q,\Omega) > 0$.

Therefore, for each $g \in L_0^{q'}(\Omega)$ we can select a representative $v \in W_0^{1,q'}(\Omega)^n$ such that div $v = g$ and

$$\|\nabla v\|_{q'} \;\le\; C_3\, \|g\|_{q'}.$$

Note that this mapping $g \mapsto v$ need not be linear. This proves assertion a) with q replaced by q'.

To prove b) we use (2.1.14). If $f \in W^{-1,q}(\Omega)^n$ satisfies $[f,v] = 0$ for all $v \in N(\text{div})$, then from (2.1.14) we see that $f \in R(\nabla)$, and therefore there exists some $p \in L_0^q(\Omega)$ with $f = \nabla p$; p is unique since $N(\nabla) = 0$, and the estimate in (2.1.2) follows from (2.1.12) with $C := C_2$.

This proves b) in the case $\Omega_0 = \Omega$. If $\Omega_0 \subseteq \Omega$ is any subdomain, then for given $f \in R(\nabla)$ we first choose $p \in L^q_0(\Omega)$ as above, and then we set $\tilde{p} := p - p_0$ so that

$$p_0 \;:=\; |\Omega_0|^{-1} \int_{\Omega_0} p\, dx, \tag{2.1.20}$$

where $|\Omega_0|$ means the Lebesgue measure of Ω_0. Then $\int_{\Omega_0} \tilde{p}\, dx = 0$, and using Hölder's inequality we get

$$\begin{aligned} \|\tilde{p}\|_q &\leq \|p\|_q + \|p_0\|_q \\ &\leq \|p\|_q + |\Omega_0|^{-1} |\int_{\Omega_0} p\, dx|\, |\Omega|^{\frac{1}{q}} \\ &\leq \|p\|_q (1 + |\Omega_0|^{-\frac{1}{q}}\, |\Omega|^{\frac{1}{q}}) \\ &\leq C\, \|f\|_{W^{-1,q}(\Omega)^n} \end{aligned}$$

with $C = C(q, \Omega, \Omega_0) > 0$. The proof is complete. □

2.2 A criterion for gradients

Lemma 2.1.1 contains in particular a criterion for the property that

$$f \;\in\; W^{-1,q}(\Omega)^n$$

is a gradient of the form $f = \nabla p$ with $p \in L^q(\Omega)$. A sufficient condition is that

$$[f, v] = 0 \quad \text{for all} \quad v \in N(\mathrm{div}) := \{v \in W^{1,q'}_0(\Omega)^n;\;\; \mathrm{div}\; v = 0\}$$

where $[f, v]$ means the value of the functional f at v.

Our aim is to improve this criterion and to show that it is sufficient to require $[f, v] = 0$ only for all

$$v \in C^\infty_{0,\sigma}(\Omega) = \{v \in C^\infty_0(\Omega)^n;\;\; \mathrm{div}\; v = 0\}.$$

This is important since $C^\infty_{0,\sigma}(\Omega)$ is the appropriate space of test functions in the theory of Navier-Stokes equations.

There are several approaches to such criterions. They are based on de Rham's theory [dRh60], see [Tem77, Chap. I, Prop. 1.1], on Bogovski's theory, see [Bog80], or on an elementary argument in [SiSo92]. Here we essentially follow [SiSo92], see also [Gal94a, III, proof of Lemma 1.1].

Further we will admit a general domain $\Omega \subseteq \mathbb{R}^n, n \geq 2$, in the next result. Recall that by definition, see (3.6.13), I, the following holds:

$$f \in W^{-1,q}_{loc}(\Omega)^n \quad \text{iff} \quad f \in W^{-1,q}(\Omega_0)^n$$

for all bounded subdomains $\Omega_0 \subseteq \Omega$ with $\overline{\Omega}_0 \subseteq \Omega$.

2.2.1 Lemma *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be an arbitrary domain, let $\Omega_0 \subseteq \Omega$ be a bounded subdomain with $\overline{\Omega_0} \subseteq \Omega$, $\Omega_0 \neq \emptyset$, and let $1 < q < \infty$. Suppose $f \in W_{loc}^{-1,q}(\Omega)^n$ satisfies*

$$[f, v] = 0 \quad \textit{for all} \quad v \in C_{0,\sigma}^\infty(\Omega). \tag{2.2.1}$$

Then there exists a unique $p \in L_{loc}^q(\Omega)$ satisfying $\nabla p = f$ in the sense of distributions and

$$\int_{\Omega_0} p\,dx = 0. \tag{2.2.2}$$

Proof. The lemma is proved if we show the following property:
For any bounded Lipschitz subdomain $\Omega_1 \subseteq \Omega$ with $\overline{\Omega_0} \subseteq \Omega_1$, $\overline{\Omega_1} \subseteq \Omega$, there exists a unique $p \in L^q(\Omega_1)$ with $\nabla p = f$ in the sense of distributions in Ω_1, and with $\int_{\Omega_0} p\,dx = 0$.

Indeed, using a representation of Ω as a union of bounded Lipschitz domains, see Lemma 1.4.1, and the uniqueness of p in Ω_1, we will see that p can be extended to a well defined function defined on Ω with the desired properties.

Let Ω_1 be such a subdomain. Then we choose, using a similar construction as in the proof of Lemma 1.4.1, another bounded Lipschitz subdomain $\Omega_2 \subseteq \Omega$ satisfying

$$\overline{\Omega_1} \subseteq \Omega_2 \ , \quad \overline{\Omega_2} \subseteq \Omega.$$

From $f \in W_{loc}^{-1,q}(\Omega)^n$ we see that $f \in W^{-1,q}(\Omega_2)^n$, and since Ω_2 is bounded we get by Lemma 1.6.1 a representation of the form

$$f = \text{ div } F \quad \text{with} \quad F = (F_{jl})_{j,l=1}^n \in L^q(\Omega_2)^{n^2} .$$

This was shown in Lemma 1.6.1 only for $q = 2$, however the same proof holds for $1 < q < \infty$.

Next we use the mollification method, see Section 1.7, and set $F^\varepsilon := \mathcal{F}_\varepsilon \star F = (\mathcal{F}_\varepsilon \star F_{jl})_{j,l=1}^n$ with $0 < \varepsilon <$ dist $(\partial\Omega_2, \Omega_1)$. This yields $F^\varepsilon \in C^\infty(\overline{\Omega}_1)^{n^2}$.

Our purpose is to prove the representation

$$\text{div } F^\varepsilon \ = \ \nabla U_\varepsilon \tag{2.2.3}$$

with some function $U_\varepsilon \in C^\infty(\overline{\Omega}_1)$. To prove this we use the following elementary procedure from [SiSo92].

Let $w : \tau \mapsto w(\tau)$, $0 \leq \tau \leq 1$, be a continuous mapping from $[0, 1]$ to $\overline{\Omega}_1$. We assume that the derivative w' exists and is piecewise continuous on $[0, 1]$. Such a function w is called a curve in $\overline{\Omega}_1$; w is called a closed curve if $w(0) = w(1)$.

Further we consider a vector field $g = (g_1, \dots, g_n) \in C^\infty(\overline{\Omega}_1)^n$, and define the curve integral

$$\int_0^1 g(w(\tau)) \cdot w'(\tau)\, d\tau \;:= \int_0^1 \sum_{j=1}^n g_j(w(\tau)) w_j'(\tau)\, d\tau$$

with $w(\tau) = (w_1(\tau), \dots, w_n(\tau)),\; w'(\tau) = (w_1'(\tau), \dots, w_n'(\tau))$.

An elementary classical argument shows that if this integral is zero for each closed curve in $\overline{\Omega}_1$, then g has the form $g = \nabla U$ with $U \in C^\infty(\overline{\Omega}_1)$.

To apply this argument for the proof of (2.2.3), we have to show that

$$\int_0^1 (\text{ div } F^\varepsilon)(w(\tau)) \cdot w'(\tau)\, d\tau = 0 \tag{2.2.4}$$

for each closed curve w in $\overline{\Omega}_1$. To prove this we set

$$V_{w,\varepsilon}(x) := \int_0^1 \mathcal{F}_\varepsilon(x - w(\tau)) w'(\tau)\, d\tau \quad , \quad x \in \Omega_2,$$

and get $V_{w,\varepsilon} \in C_0^\infty(\Omega_2)^n$,

$$\begin{aligned}
\text{div } V_{w,\varepsilon}(x) \quad &= \quad \int_0^1 \sum_{j=1}^n (D_j \mathcal{F}_\varepsilon)(x - w(\tau)) w_j'(\tau)\, d\tau \\
&= \quad -\int_0^1 \frac{d}{d\tau} \mathcal{F}_\varepsilon(x - w(\tau))\, d\tau \\
&= \quad \mathcal{F}_\varepsilon(x - w(0)) \,-\, \mathcal{F}_\varepsilon(x - w(1)) = 0
\end{aligned}$$

if w is a closed curve in $\overline{\Omega}_1$. This leads to $V_{w,\varepsilon} \in C_{0,\sigma}^\infty(\Omega_2)^n$, and using the assumption (2.2.1) and Fubini's theorem we obtain

$$\begin{aligned}
0 \quad &= \quad [f, V_{w,\varepsilon}] \,=\, [\text{ div } F, V_{w,\varepsilon}] \\
&= \quad \sum_{j,l=1}^n \int_{\Omega_2} D_j\, F_{jl}(x) \left(\int_0^1 \mathcal{F}_\varepsilon(x - w(\tau))\, w_l'(\tau)\, d\tau \right) dx \\
&= \quad \int_0^1 \left(\sum_{j,l=1}^n \int_{\Omega_2} \mathcal{F}_\varepsilon(w(\tau) - x) D_j\, F_{jl}(x) dx \right) w_l'(\tau)\, d\tau \\
&= \quad \int_0^1 \left(\sum_{j,l=1}^n \int_{\Omega_2} (D_j\, \mathcal{F}_\varepsilon)(w(\tau) - x)\, F_{jl}(x)) dx \right) w_l'(\tau)\, d\tau \\
&= \quad \int_0^1 (\text{ div } F^\varepsilon(w(\tau)) \cdot w'(\tau)\, d\tau.
\end{aligned}$$

This proves (2.2.4).

Thus we get the representation (2.2.3) with some $U_\varepsilon \in C^\infty(\overline{\Omega}_1)$ which is determined up to a constant. Choosing this constant in an appropriate way we can conclude that $\int_{\Omega_0} U_\varepsilon dx = 0$. Using Lemma 1.5.4, (1.5.10), we obtain

$$\begin{aligned}
\|U_\varepsilon\|_{L^q(\Omega_1)} &\leq C\,\|\nabla U_\varepsilon\|_{W^{-1,q}(\Omega_1)} \\
&= C \sup_{0\neq v\in C_0^\infty(\Omega_1)^n} \left(|[\nabla U_\varepsilon, v]|\,/\,\|\nabla v\|_{q'}\right) \\
&= C \sup_{0\neq v\in C_0^\infty(\Omega_1)^n} \left(|< F^\varepsilon, \nabla v >|\,/\,\|\nabla v\|_{q'}\right) \\
&\leq C\|F^\varepsilon\|_{L^q(\Omega_1)}
\end{aligned}$$

with $C = C(q, \Omega_0, \Omega_1) > 0$ independent of ε.

Since $\|F - F^\varepsilon\|_{L^q(\Omega_1)} \to 0$ as $\varepsilon \to 0$, see Lemma 1.7.1, we obtain, letting $\varepsilon \to 0$, some $U \in L^q(\Omega_1)$ satisfying

$$\int_{\Omega_0} U\,dx = 0 \ \ , \quad \lim_{\varepsilon\to 0} \|U - U_\varepsilon\|_{L^q(\Omega_1)} = 0 \ \ , \quad f = \operatorname{div} F = \nabla U$$

in Ω_1. To prove this, we choose $0 < \eta < \varepsilon$ and replace F^ε by $F^\varepsilon - F^\eta$, U_ε by $U_\varepsilon - U_\eta$ in the last estimate. U is uniquely determined.

Consider now all possible Lipschitz subdomains Ω_1 as defined above with $\overline{\Omega}_0 \subseteq \Omega_1$. Each bounded subdomain $\Omega' \subseteq \Omega$ with $\overline{\Omega'} \subseteq \Omega$ is contained in such a domain Ω_1, see Remark 1.4.2.

Defining p by U constructed above in each such Ω_1, the uniqueness of U because of $\int_{\Omega_0} U\,dx = 0$ yields in this way a uniquely determined function $p \in L^q_{loc}(\Omega)$ with $f = \nabla p$ in the whole domain Ω. This proves the lemma. $\square$

If in particular Ω is a bounded Lipschitz domain, we can improve the above result, see the next lemma, and show that even $p \in L^q(\Omega)$. Moreover p satisfies the important estimate (1.5.10). For the proof we use the **scaling argument**, see, e.g., the proof of [Tem77, Chap. I, Theorem 1.1].

2.2.2 Lemma *Let $\Omega \subseteq \mathbb{R}^n, n \geq 2$, be a bounded Lipschitz domain, let $\Omega_0 \subseteq \Omega$, $\Omega_0 \neq \emptyset$, be any subdomain, and let $1 < q < \infty$. Suppose $f \in W^{-1,q}(\Omega)^n$ satisfies*

$$[f, v] = 0 \quad \textit{for all} \quad v \in C^\infty_{0,\sigma}(\Omega). \tag{2.2.5}$$

Then there exists a unique $p \in L^q(\Omega)$ satisfying

$$\int_{\Omega_0} p\,dx = 0 \ \ , \quad f = \nabla p$$

in the sense of distributions. The estimate

$$\|p\|_{L^q(\Omega)} \ \leq\ C_1\,\|f\|_{W^{-1,q}(\Omega)^n} \ \leq\ C_1 C_2\,\|p\|_{L^q(\Omega)} \tag{2.2.6}$$

holds with constants $C_1 = C_1(q, \Omega_0, \Omega) > 0$ and $C_2 = C_2(n) > 0$.

Proof. First we assume additionally that Ω is **starlike** with respect to some $x_0 \in \Omega$. This means that the line $\{x_0 + te;\ t \in \mathbb{R}\}$ intersects the boundary $\partial\Omega$ in exactly two points for each vector $e \in \mathbb{R}^n$. We may assume, for simplicity, that $x_0 = 0$. This property enables us to apply the following scaling argument.

Let $0 < \varepsilon < 1$,

$$\Omega_\varepsilon := \{x \in \mathbb{R}^n;\ \ \varepsilon x \in \Omega\}$$

and let the functional $f_\varepsilon \in W^{-1,q}(\Omega_\varepsilon)^n$ be defined by $[f_\varepsilon, v] := [f, v_\varepsilon],\ v \in W_0^{1,q'}(\Omega_\varepsilon)^n$, where $v_\varepsilon \in W_0^{1,q'}(\Omega)^n$ is defined by $v_\varepsilon(x) := v(\varepsilon^{-1}x),\ x \in \Omega$.

Let $v \in C_{0,\sigma}^\infty(\Omega_\varepsilon)$. Then $v_\varepsilon \in C_{0,\sigma}^\infty(\Omega)$, and from (2.2.5) we get that $[f_\varepsilon, v] = 0$ for all $v \in C_{0,\sigma}^\infty(\Omega_\varepsilon)$. Applying Lemma 2.2.1 yields a unique $p_\varepsilon \in L_{loc}^q(\Omega_\varepsilon)$ satisfying $\int_{\Omega_0} p_\varepsilon dx = 0$ and $f_\varepsilon = \nabla p_\varepsilon$ in Ω_ε. Note that $\overline{\Omega} \subseteq \Omega_\varepsilon$ and therefore $\overline{\Omega}_0 \subseteq \Omega_\varepsilon$

Since $\overline{\Omega} \subseteq \Omega_\varepsilon$ we get $p_\varepsilon \in L^q(\Omega),\ 0 < \varepsilon < 1$. Therefore we may apply Lemma 1.5.4 and estimate (1.5.10). This yields

$$\|p_\varepsilon\|_{L^q(\Omega)} \ \leq\ C\,\|\nabla p_\varepsilon\|_{W^{-1,q}(\Omega)^n} = C\,\|f_\varepsilon\|_{W^{-1,q}(\Omega)^n}$$

with $C = C(q, \Omega) > 0$ not depending on ε.

Let now $\frac{1}{2} \leq \varepsilon < 1$ and $v \in C_{0,\sigma}^\infty(\Omega)$. Extending v by zero we get $v \in C_{0,\sigma}^\infty(\Omega_\varepsilon)$. Then a calculation shows that

$$\|\nabla v_\varepsilon\|_{L^{q'}(\Omega)^{n^2}} \ \leq\ 2\,\|\nabla v\|_{L^{q'}(\Omega)^{n^2}} \quad , \quad q' = \frac{q}{q-1},$$

and

$$\begin{aligned} |[f_\varepsilon, v]| \ =\ |[f, v_\varepsilon]| \ &\leq\ \|f\|_{W^{-1,q}(\Omega)^n}\|\nabla v_\varepsilon\|_{L^{q'}(\Omega)^{n^2}} \\ &\leq\ 2\,\|f\|_{W^{-1,q}(\Omega)^n}\|\nabla v\|_{L^{q'}(\Omega)^{n^2}}. \end{aligned}$$

This yields

$$\|p_\varepsilon\|_{L^q(\Omega)} \ \leq\ C\,\|f_\varepsilon\|_{W^{-1,q}(\Omega)^n} \ \leq\ 2\,C\,\|f\|_{W^{-1,q}(\Omega)^n} \tag{2.2.7}$$

for $\frac{1}{2} \leq \varepsilon < 1$.

Since C does not depend on ε, we are able to let $\varepsilon \to 1$. Choose $\frac{1}{2} \leq \varepsilon_j < 1,\ j \in \mathbb{N}$, with $\lim_{j\to\infty} \varepsilon_j = 1$, and set $p_j := p_{\varepsilon_j},\ j \in \mathbb{N}$. The uniform boundedness in (2.2.7) shows the existence of a subsequence of $(p_j)_{j=1}^\infty$ which converges weakly in $L^q(\Omega)$ to some $p \in L^q(\Omega)$. We may assume that the sequence itself has this property. With $f_j := f_{\varepsilon_j}$ we get

$$\begin{aligned} < p,\ \operatorname{div}\, v >_\Omega \ &=\ \lim_{j\to\infty} < p_j, \operatorname{div}\, v >_\Omega \ =\ \lim_{j\to\infty}\, [-\nabla p_j, v]_\Omega \\ &=\ \lim_{j\to\infty}\, [-f_j, v]_\Omega = \lim_{j\to\infty}\, [-f, v_j]_\Omega \\ &=\ [-f, v]_\Omega \end{aligned}$$

for all $v \in C_0^\infty(\Omega)^n$, where $v_j := v_{\varepsilon_j}$ is defined as above by $v_{\varepsilon_j}(x) := v(\varepsilon_j^{-1}x)$, $x \in \Omega$. This shows that $f = \nabla p$ in the sense of distributions. The weak convergence of p_j to p yields that $\int_{\Omega_0} p\,dx = 0$. This proves the uniqueness property of p.

The weak convergence property shows, see Section 3.1 or the proof of Lemma 1.5.4, that

$$\|p\|_{L^q(\Omega)} \;\le\; \liminf_{j\to\infty} \|p_j\|_{L^q(\Omega)} \;\le\; 2\,C\,\|f\|_{W^{-1,q}(\Omega)^n}.$$

This proves the lemma for starlike domains.

The case of a general bounded Lipschitz domain Ω can be reduced to the case above by the following localization argument. Using the definition of a Lipschitz domain, we easily find bounded starlike subdomains $\Omega_1, \dots, \Omega_m \subseteq \Omega$ such that

$$\Omega = \Omega_1 \cup \dots \cup \Omega_m.$$

For $j = 1, \dots, m$ let $f_j \in W^{-1,q}(\Omega_j)^n$ be the restriction of f to $W_0^{1,q'}(\Omega_j)^n$.

Consider first the case that $\overline{\Omega}_0 \subseteq \Omega$. Then from Lemma 2.2.1 we obtain a unique $p \in L^q_{loc}(\Omega)$ satisfying $f = \nabla p$, $\int_{\Omega_0} p\,dx = 0$. Since $\Omega_j \subseteq \Omega$ we get in particular that $\nabla p = f_j$, $j = 1, \dots, m$, in the sense of distributions in Ω_j. On the other hand, the result above yields some $p_j \in L^q(\Omega_j)$ with $\nabla p_j = f_j$, $j \in \mathbb{N}$, which is uniquely determined up to a constant. Therefore we get $p + C_j = p_j$, $j = 1, \dots, m$, where C_j is a constant. This proves that $p \in L^q(\Omega)$. If $\Omega_0 \subseteq \Omega$ is any subdomain, we choose a subdomain $\Omega_0' \subseteq \Omega$ with $\overline{\Omega'}_0 \subseteq \Omega$. This yields as above some $\tilde{p} \in L^q(\Omega)$ with $\nabla \tilde{p} = f$ and $\int_{\Omega_0'} \tilde{p}\,dx = 0$. Subtracting a constant from $\tilde{p}$ yields the desired $p \in L^q(\Omega)$ with $\nabla p = f$ and $\int_{\Omega_0} p\,dx = 0$. Since $p \in L^q(\Omega)$, the estimate (2.2.6) now follows from Lemma 1.5.4, (1.5.10). This completes the proof. $\square$

The following density property is an important consequence of Lemma 2.2.2. Note that this property need not hold in unbounded domains, see [Hey76] for counter examples.

2.2.3 Lemma *Let $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, be a bounded Lipschitz domain, and let $1 < q < \infty$. Then $C_{0,\sigma}^\infty(\Omega) = \{v \in C_0^\infty(\Omega)^n; \operatorname{div} v = 0\}$ is dense in the space $N(\operatorname{div}) = \{v \in W_0^{1,q}(\Omega)^n; \operatorname{div} v = 0\}$ with respect to the norm $\|\cdot\|_{W^{1,q}(\Omega)^n} = \|\cdot\|_{1,q}$. Thus*

$$\overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_{1,q}} = N(\operatorname{div}). \tag{2.2.8}$$

Proof. We use a functional analytic argument. To prove (2.2.8), it suffices to show that each functional $f \in W^{-1,q'}(\Omega)^n$, $q' = \frac{q}{q-1}$, from the dual space

$W^{-1,q'}(\Omega)^n$ of $W_0^{1,q}(\Omega)^n$ which vanishes on $C_{0,\sigma}^\infty(\Omega)$ even vanishes on $N(\text{div})$. Then (2.2.8) must be valid, otherwise we would find by the Hahn-Banach theorem some $f \in W^{-1,q'}(\Omega)^n$ with $[f,v]=0$ for all $v \in C_{0,\sigma}^\infty(\Omega)$ and $[f,v_0] \neq 0$ for some $v_0 \in N(\text{div})$.

Thus let $f \in W^{-1,q'}(\Omega)^n$ be given with $[f,v]=0$, $v \in C_{0,\sigma}^\infty(\Omega)$. From Lemma 2.2.2 we see that $f = \nabla p$ with some $p \in L^{q'}(\Omega)$. It follows that

$$[f,v] \;=\; [\nabla p, v] \;=\; -<p,\; \text{div } v> \tag{2.2.9}$$

for all $v \in C_0^\infty(\Omega)^n$. Since f is continuous in $\|\nabla v\|_q$, and since $p \in L^{q'}(\Omega)$, we conclude that (2.2.9) even holds for all $v \in W_0^{1,q}(\Omega)^n$. It follows that

$$[f,v] \;=\; -<p, \text{div } v> = \;0 \;\;, \quad v \in N(\text{div}).$$

This proves the lemma. $\square$

2.3 Regularity results on $\text{div}\, v = g$

Lemma 2.1.1 yields a solution $v \in W_0^{1,q}(\Omega)^n$ of the system

$$\text{div } v = g \;\;, \quad v|_{\partial\Omega} = 0 \tag{2.3.1}$$

for each given $g \in L^q(\Omega)$ with $\int_\Omega g\, dx = 0$. In the regularity theory of the Navier-Stokes equations we need solutions v of (2.3.1) with higher regularity properties if g is sufficiently smooth. The next lemma yields such a result. See [Bog80] or [Gal94a, III, 3] for a different approach to the regularity theory of (2.3.1).

2.3.1 Lemma *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain, and let $1 < q < \infty$, $k \in \mathbb{N}$. Then for each $g \in W_0^{k,q}(\Omega)$ with $\int_\Omega g\, dx = 0$, there exists at least one $v \in W_0^{k+1,q}(\Omega)^n$ satisfying*

$$\text{div } v = g \;\;, \quad \|v\|_{W^{k+1,q}(\Omega)^n} \;\leq\; C\, \|g\|_{W^{k,q}(\Omega)} \tag{2.3.2}$$

with some constant $C = C(q,k,\Omega) > 0$.

Proof. See [Gal94a, III, Theorem 3.2] for another proof. The result also holds for $k=0$ and is contained in this case in Lemma 2.1.1, a). We use the same argument as for $k=0$, now for $k \geq 1$. For $k=0$ the proof rests on inequality (1.5.10) which follows from (1.1.6) by a compactness argument, see the proof of Lemma 1.5.4. The same argument can be used in the case $k \geq 1$. Instead of (1.1.6) we now use the corresponding inequality (1.1.8) for $k \geq 1$. The analogous compactness argument as in the proof of Lemma 1.5.4 yields instead of (1.5.10) the inequality

$$\|u\|_{W^{-k,q}(\Omega)/N(\nabla)} \;\leq\; C_1 \|\nabla u\|_{W^{-k-1,q}(\Omega)^n} \;\leq\; C_1 C_2 \|u\|_{W^{-k,q}(\Omega)} \tag{2.3.3}$$

for all $u \in W^{-k,q}(\Omega)$ with constants $C_1 = C_1(q,k,\Omega) > 0,\ C_2 = C_2(n,k) > 0$. $W^{-k,q}(\Omega)/N(\nabla)$ means the quotient space modulo the null space $N(\nabla)$, which consists of the constants. If $k = 0$, $W^{-k,q}(\Omega)/N(\nabla) = L^q(\Omega)/N(\nabla)$ can be identified with $L_0^q(\Omega) = \{u \in L^q(\Omega);\ \int_\Omega u\,dx = 0\}$.

The proof of Lemma 2.3.1 follows from (2.3.3) with q replaced by $q' = \frac{q}{q-1}$ by the same duality principle as in the proof of Lemma 2.1.1. It follows that the bounded linear operator

$$\operatorname{div}\ : v \mapsto \ \operatorname{div}\, v$$

from $W_0^{k+1,q}(\Omega)^n$ to $W_0^{k,q}(\Omega)$ has the closed range $W_0^{k,q}(\Omega) \cap L_0^q(\Omega)$. Therefore, the inverse operator div^{-1} from $W_0^{k,q}(\Omega) \cap L_0^q(\Omega)$ onto the quotient space $W_0^{k+1,q}(\Omega)^n/N(\operatorname{div})$, $N(\operatorname{div}) := \{v \in W_0^{k+1,q}(\Omega)^n;\ \ \operatorname{div}\, v = 0\}$, is bounded. This proves the existence of some $v \in W_0^{k+1,q}(\Omega)^n$ satisfying (2.3.2). The proof is complete. □

2.4 Further results on the equation $\operatorname{div} v = g$

Modifying the duality argument in the proof of Lemma 2.1.1 we find some other solution classes of this equation. Here we need the traces, see Section 1.2, II, and the exterior normal vector field N at the boundary $\partial\Omega$, see (3.4.7), I.

2.4.1 Lemma *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain with boundary $\partial\Omega$, and let $1 < q < \infty$. Then we have:*

a) *For each $g \in W^{-1,q}(\Omega)$ there exists at least one $v \in L^q(\Omega)^n$ satisfying*

$$\operatorname{div}\, v = g$$

in the sense of distributions, and

$$\|v\|_{L^q(\Omega)^n} \ \leq\ C\,\|g\|_{W^{-1,q}(\Omega)} \tag{2.4.1}$$

with some constant $C = C(q,\Omega) > 0$.

b) *For each $g \in L^q(\Omega)$ with $\int_\Omega g\,dx = 0$, there exists at least one $v \in L^q(\Omega)^n$ satisfying*

$$\operatorname{div}\, v \ =\ g$$

in the sense of distributions, $N \cdot v|_{\partial\Omega} = 0$ in the sense of generalized traces (1.2.24), and

$$\|v\|_{L^q(\Omega)^n} \ \leq\ C\,\|g\|_{L^q(\Omega)} \tag{2.4.2}$$

with some constant $C = C(q,\Omega) > 0$.

Proof. To prove a) we consider the operator

$$\text{div } : v \mapsto \text{ div } v$$

from $L^q(\Omega)^n$ to $W^{-1,q}(\Omega)$, and its dual operator $\text{div}' = -\nabla$,

$$-\nabla : p \mapsto \nabla p,$$

from $W_0^{1,q'}(\Omega)$ to $L^{q'}(\Omega)^n$, $q' = \frac{q}{q-1}$. We get

$$[p, \text{ div } v] = < -\nabla p, v >$$

for all $p \in W_0^{1,q'}(\Omega)$ and $v \in L^q(\Omega)^n$. From Poincaré's inequality (1.1.1) we see that $-\nabla$ has a closed range. Therefore, div has also a closed range which is the whole space $W^{-1,q}(\Omega)$, since $\{0\}$ is the null space of $-\nabla$; see the closed range theorem [Yos80].

The inverse operator div^{-1} from $W^{-1,q}(\Omega)$ to the quotient space $L^q(\Omega)^n/N(\text{div})$, $N(\text{div}) := \{v \in L^q(\Omega)^n;\ \text{div } v = 0\}$, is therefore bounded. This yields a).

To prove b) we define the operator

$$\text{div } : v \mapsto \text{ div } v$$

with domain

$$D(\text{div}) := \{v \in L^q(\Omega)^n;\ \text{div } v \in L^q(\Omega),\ N \cdot v|_{\partial\Omega} = 0\} \ \subseteq\ L^q(\Omega)^n$$

and range $R(\text{div}) \subseteq L^q(\Omega)$. From Green's formula (1.2.25) we conclude that $\int_\Omega \text{ div } v\, dx = 0$ for $v \in D(\text{div})$. To see this we set $u \equiv 1$ in (1.2.25). This yields $R(\text{div}) \subseteq L_0^q(\Omega) = \{g \in L^q(\Omega); \int_\Omega g\, dx = 0\}$. The trace $N \cdot v|_{\partial\Omega}$ is well defined since $D(\text{div}) \subseteq E_q(\Omega)$, see Lemma 1.2.2.

$D(\text{div})$ is dense in $L^q(\Omega)^n$ since $C_0^\infty(\Omega)^n \subseteq D(\text{div})$. We consider div as an operator from $D(\text{div})$ to $R(\text{div}) \subseteq L_0^q(\Omega)$.

$L_0^{q'}(\Omega)$ is the dual space of $L_0^q(\Omega)$, see (2.1.3). Next we define the operator

$$\nabla :\ p \mapsto \nabla p$$

with domain $D(\nabla) := \{p \in L_0^{q'}(\Omega);\ \nabla p \in L^{q'}(\Omega)^n\} \subseteq W^{1,q'}(\Omega)$ and range $R(\nabla) \subseteq L^{q'}(\Omega)^n$. It holds that $N(\nabla) = \{p \in L_0^{q'}(\Omega); \nabla p = 0\} = \{0\}$ since

$$\nabla p = 0 \ \ , \quad \int_\Omega p\, dx = 0$$

implies $p = 0$, see (1.7.18). Green's formula (1.2.25) yields

$$< p, \text{div } v > = \ - < \nabla p, v >$$

for all $p \in D(\nabla)$ and $v \in D(\text{div})$. This means, $-\nabla$ is the dual operator of div. Poincaré's inequality (1.1.2) implies that $R(-\nabla)$ is closed in $L^{q'}(\Omega)^n$. Therefore, $R(\text{div}) \subseteq L_0^q(\Omega)$ is closed too, and since $N(-\nabla) = \{0\}$, we conclude that $R(\text{div}) = L_0^q(\Omega)$ and that

$$\inf_{v_o \in N(div)} \|v + v_0\|_q \ \leq \ C \, \|\text{div } v\|_q$$

with $N(\text{div}) := \{v \in D(\text{div});\ \text{div } v = 0\},\ C = C(q, \Omega) > 0$. Thus we may choose v in such a way that (2.4.2) is satisfied. This proves b). $\square$

2.5 Helmholtz decomposition in L^2-spaces

In this subsection $\Omega \subseteq \mathbb{R}^n$ is an arbitrary domain with $n \geq 2$. We consider the Hilbert space $L^2(\Omega)^n$ with scalar product

$$< f, g >_\Omega = < f, g > = \int_\Omega f \cdot g \, dx,$$

the subspace

$$L^2_\sigma(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_2} \quad , \quad C^\infty_{0,\sigma}(\Omega) := \{f \in C_0^\infty(\Omega)^n;\ \text{div } f = 0\}, \tag{2.5.1}$$

and the space

$$G(\Omega) := \{f \in L^2(\Omega)^n;\ \exists\, p \in L^2_{loc}(\Omega) : f = \nabla p\}. \tag{2.5.2}$$

In other words, $L^2_\sigma(\Omega)$ is the closure of $C^\infty_{0,\sigma}(\Omega)$ in the norm $\|\cdot\|_2 = \|\cdot\|_{L^2(\Omega)^n}$, and $G(\Omega)$ is the space of those $f \in L^2(\Omega)^n$ for which there is some $p \in L^2_{loc}(\Omega)$ satisfying $f = \nabla p$ in the sense of distributions. "$\exists$" means "there exists".

The next lemma shows that $G(\Omega)$ is orthogonal to $L^2_\sigma(\Omega)$, we write

$$G(\Omega) = L^2_\sigma(\Omega)^\perp$$

for this property. This leads to the unique decomposition (2.5.4) of each $f \in L^2(\Omega)^n$ which is called the **Helmholtz decomposition** of f. In particular we see that $G(\Omega)$ is a closed subspace of $L^2(\Omega)^n$. See [Gal94a, III, 1], [FuM77], [SiZ98] concerning the Helmholtz decomposition in L^q-spaces with $1 < q < \infty$.

2.5.1 Lemma *Let $\Omega \subseteq \mathbb{R}^n, n \geq 2$, be any domain. Then*

$$G(\Omega) = \{f \in L^2(\Omega)^n;\ < f, v > = 0 \quad \textit{for all} \quad v \in L^2_\sigma(\Omega)\}, \tag{2.5.3}$$

and each $f \in L^2(\Omega)^n$ has a unique decomposition

$$f \ = \ f_0 + \nabla p \tag{2.5.4}$$

with $f_0 \in L^2_\sigma(\Omega),\ \nabla p \in G(\Omega),\ < f_0, \nabla p > = 0,$

$$\|f\|_2^2 = \|f_0\|_2^2 + \|\nabla p\|_2^2. \tag{2.5.5}$$

Remark As a consequence of this lemma we obtain a bounded linear operator $P : f \mapsto Pf$ from $L^2(\Omega)^n$ onto $L^2_\sigma(\Omega)$ defined by $Pf := f_0$ with f_0 as in (2.5.4). P is called the **Helmholtz projection** of $L^2(\Omega)^n$ onto $L^2_\sigma(\Omega)$.

2.5.2 Lemma *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be any domain, and let $f = f_0 + \nabla p$ be the Helmholtz decomposition of $f \in L^2(\Omega)^n$. Then*

$$P : L^2(\Omega)^n \to L^2_\sigma(\Omega)\ , \tag{2.5.6}$$

defined by $Pf := f_0$ for all $f \in L^2(\Omega)^n$, is a bounded linear operator with operator norm $\|P\| \leq 1$. Thus

$$\|Pf\|_2 \leq \|f\|_2\ \ ,\quad f \in L^2(\Omega)^n. \tag{2.5.7}$$

P has the following properties:

$$P(\nabla p) = 0\ , \qquad (I - P)f = \nabla p\ , \qquad P^2 f = Pf\ ,$$
$$(I-P)^2 f = (I-P)f,\quad < Pf, g > = < f, Pg >,\quad \|f\|_2^2 = \|Pf\|_2^2 + \|(I-P)f\|_2^2$$

for all $f, g \in L^2(\Omega)^n$.

From these properties we easily conclude that P is a selfadjoint operator, and that $P' = P$, where P' means the dual operator of P, see Section 3.2 for this notion.

Proof of Lemma 2.5.1. First we prove the characterization (2.5.3) of the subspace $G(\Omega)$ in (2.5.2). The space on the right side of (2.5.3) is by definition the orthogonal subspace of $L^2_\sigma(\Omega)$. Thus we have to show that

$$G(\Omega)\ =\ L^2_\sigma(\Omega)^\perp\ . \tag{2.5.8}$$

To prove (2.5.8) let $f \in L^2_\sigma(\Omega)^\perp$. Then for any bounded subdomain $\Omega_0 \subseteq \Omega$ with $\overline{\Omega}_0 \subseteq \Omega$ we get, using Poincaré's inequality (1.1.1), that

$$|< f, v >|\ \leq\ \|f\|_2\, \|v\|_{L^2(\Omega_0)^n}\ \leq\ C\, \|f\|_2\, \|\nabla v\|_{L^2(\Omega_0)^{n^2}}$$

for all $v \in C_0^\infty(\Omega_0)^n$ with $C = C(\Omega_0) > 0$. This shows that

$$f\ \in\ W^{-1,2}_{loc}(\Omega)^n.$$

Next we observe that $[f, v] = < f, v > = 0$ for all $v \in C^\infty_{0,\sigma}(\Omega)$. Lemma 2.2.1 yields some $p \in L^2_{loc}(\Omega)$, uniquely determined up to a constant, which satisfies $f = \nabla p$ in the sense of distributions. This shows that $f \in G(\Omega)$.

Conversely, let $f \in G(\Omega)$ with $f = \nabla p,\ p \in L^2_{loc}(\Omega)$. Then $< \nabla p, v > = - < p,\ \mathrm{div}\ v > = 0$ for all $v \in C^\infty_{0,\sigma}(\Omega)$, and since $\nabla p \in L^2(\Omega)^n$, this even holds for all $v \in L^2_\sigma(\Omega)$. This proves (2.5.8).

Using some elementary Hilbert space properties, see Section 3.2, we get the unique orthogonal decomposition $f = f_0 + \nabla p$ for each $f \in L^2(\Omega)^n$ with $f \in L^2_\sigma(\Omega),\ \nabla p \in L^2_\sigma(\Omega)^\perp = G(\Omega)$; (2.5.5) is obvious. This proves the Lemma. $\square$

Proof of Lemma 2.5.2. The Hilbert space theory yields a uniquely determined projection operator P from $L^2(\Omega)^n$ onto the subspace $L^2_\sigma(\Omega)$; the properties of P are obvious. This yields the lemma. $\square$

For special domains we can improve the properties of the Helmholtz decomposition $f = f_0 + \nabla p$. In particular we are interested in bounded Lipschitz domains and in the case $\Omega = \mathbb{R}^n$. In these cases we can give special important characterizations of $L^2_\sigma(\Omega)$ and $G(\Omega)$.

In the following lemma, $N \cdot f|_{\partial\Omega}$ means the generalized trace, see (1.2.24), and N the exterior normal field at $\partial\Omega$, see (3.4.7), I. Note that the trace $N \cdot f|_{\partial\Omega}$ in (2.5.9) is well defined since $f \in E_2(\Omega)$, see (1.2.20).

2.5.3 Lemma *Let $\Omega \subseteq \mathbb{R}^n, n \geq 2$, be a bounded Lipschitz domain with boundary $\partial\Omega$. Then*

$$L^2_\sigma(\Omega) = \{f \in L^2(\Omega)^n;\ \mathrm{div}\ f = 0,\ N \cdot f|_{\partial\Omega} = 0\} \tag{2.5.9}$$

and

$$G(\Omega) := \{f \in L^2(\Omega)^n;\ \exists\, p \in L^2(\Omega) : f = \nabla p\}. \tag{2.5.10}$$

Proof. In other words, $G(\Omega)$ is the space of all $f \in L^2(\Omega)^n$ for which there is some $p \in L^2(\Omega)$ with $f = \nabla p$ in the sense of distributions.

To prove (2.5.10), it suffices to show the following property:

$$p \in L^2_{loc}(\Omega),\ \nabla p \in L^2(\Omega)^n \text{ implies } p \in L^2(\Omega).$$

This is a consequence of Lemma 1.1.5, b). Thus we obtain (2.5.10).

To prove (2.5.9), let L be the space on the right side of (2.5.9). From $G(\Omega) = L^2_\sigma(\Omega)^\perp$ we get by an elementary Hilbert space argument that $G(\Omega)^\perp = L^2_\sigma(\Omega)^{\perp\perp} = L^2_\sigma(\Omega)$. Thus it remains to show that $L = G(\Omega)^\perp$.

To prove this let $f \in G(\Omega)^\perp$. By definition

$$G(\Omega)^\perp := \{f \in L^2(\Omega)^n;\ < f, \nabla p > = 0 \ \text{ for all } \nabla p \in G(\Omega)\},$$

and therefore we obtain in particular $< f, \nabla p > = 0$ for all $p \in C^\infty_0(\Omega)$. This means that $\mathrm{div}\ f = 0$ in the sense of distributions. It follows that $f \in E_2(\Omega)$, see Lemma 1.2.2. Using (2.5.10) we get $< f, \nabla p > = 0$ for all $p \in W^{1,2}(\Omega)$. Green's formula (1.2.25) now yields that

$$0 = < p,\ \text{div}\ f >_\Omega = < p, N \cdot f >_{\partial\Omega} - < \nabla p, f >_\Omega = < p, N \cdot f >_{\partial\Omega}$$

for all $p \in W^{1,2}(\Omega)$. This shows that $N \cdot f|_{\partial\Omega} = 0$ and therefore that $f \in L$. Thus we have $G(\Omega)^\perp \subseteq L$.

Conversely let $f \in L$. Then $f \in E_2(\Omega)$ and Green's formula (1.2.25) yields $< f, \nabla p >_\Omega = < \text{div} f, p >_\Omega = 0$ for all $\nabla p \in G(\Omega)$. This shows that $f \in G(\Omega)^\perp$. Therefore we get $L = G(\Omega)^\perp$ and (2.5.9) holds. The proof is complete. $\square$

In the case $\Omega = \mathbb{R}^n$ we can prove the following characterization of the spaces $L^2_\sigma(\Omega)$ and $G(\Omega)$.

2.5.4 Lemma *Let $n \in \mathbb{N}$, $n \geq 2$. Then*

$$L^2_\sigma(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n)^n;\ \text{div}\ f = 0\}, \tag{2.5.11}$$

and $G(\mathbb{R}^n)$ is the closure of the space

$$\nabla C_0^\infty(\mathbb{R}^n) := \{\nabla p\ ;\ p \in C_0^\infty(\mathbb{R}^n)\} \tag{2.5.12}$$

with respect to the norm $\|\cdot\|_{L^2(\mathbb{R}^n)^n}$. Thus

$$G(\mathbb{R}^n) = \overline{\nabla C_0^\infty(\mathbb{R}^n)}^{\|\cdot\|_2}. \tag{2.5.13}$$

Proof. First we prove (2.5.13). For this purpose we use the scaling method and the mollification method, see Section 1.7.

To prepare the scaling argument we consider a function $\varphi \in C_0^\infty(\mathbb{R}^n)$ with the properties

$$0 \leq \varphi \leq 1\ ,\quad \varphi(x) = 1\ \text{ if }\ |x| \leq 1\ ,\quad \varphi(x) = 0\ \text{ if }\ |x| \geq 2, \tag{2.5.14}$$

and define the functions

$$\varphi_j \in C_0^\infty(\mathbb{R}^n)\ ,\quad \varphi_j(x) := \varphi(j^{-1}x)\ ,\quad x \in \mathbb{R}^n,\ j \in \mathbb{N}. \tag{2.5.15}$$

It follows that $\lim_{j\to\infty} \varphi_j(x) = 1$ for all $x \in \mathbb{R}^n$, and setting

$$B_j := \{x \in \mathbb{R}^n;\ |x| < j\}\ ,\quad G_j := B_{2j} \backslash \overline{B}_j\ , \tag{2.5.16}$$

we get supp $\nabla\varphi_j \subseteq \overline{G}_j$, supp $\varphi_j \subseteq \overline{B}_{2j}$, $j \in \mathbb{N}$. See [SiSo96] for the method concerning φ.

To show (2.5.13) we consider any $\nabla p \in G(\mathbb{R}^n) = \{\nabla p \in L^2(\mathbb{R}^n)^n;\ p \in L^2_{loc}(\mathbb{R}^n)\}$ and choose constants $K_j,\ j \in \mathbb{N}$, such that

$$\int_{G_j} (p - K_j)\,dx = 0 \ \ , \quad j \in \mathbb{N}.$$

Applying Poincaré's inequality (1.1.2) to G_1, we get

$$\|p - K_1\|_{L^2(G_1)} \ \leq \ C\,\|\nabla p\|_{L^2(G_1)^n} \tag{2.5.17}$$

with some constant $C > 0$. Using the transformation formula for integrals with $x = jy,\ dx = j^n dy$, we obtain

$$\begin{aligned}
\|p - K_j\|_{L^2(G_j)} &= \big(\int_{G_j} |p(x) - K_j|^2 dx\big)^{\frac{1}{2}} \ = \ \big(\int_{G_1} |p(jy) - K_j|^2 dy\big)^{\frac{1}{2}}\, j^{\frac{n}{2}} \\
&\leq C j^{\frac{n}{2}}\,\big(\int_{G_1} |\nabla_y p(jy)|^2 dy\big)^{\frac{1}{2}} \\
&= C j^{\frac{n}{2}}\, j^{-\frac{n}{2}}\, j\,\big(\int_{G_j} |\nabla p(x)|^2 dx\big)^{\frac{1}{2}} \\
&= C j\,\|\nabla p\|_{L^2(G_j)^n}
\end{aligned}$$

with C as in (2.5.17) since

$$\int_{G_1} (p(jy) - K_j)\,dy \ = \ j^{-n} \int_{G_j} (p(x) - K_j)\,dx = 0.$$

Thus we get

$$\|p - K_j\|_{L^2(G_j)} \leq jC\,\|\nabla p\|_{L^2(G_j)^n} \ \ , \quad j \in \mathbb{N}. \tag{2.5.18}$$

Setting $p_j := \varphi_j(p - K_j)$ and using $\nabla p_j = (\nabla\varphi_j)(p - K_j) + \varphi_j \nabla(p - K_j) = (\nabla\varphi_j)(p - K_j) + \varphi_j \nabla p$, we obtain

$$\begin{aligned}
\|\nabla p - \nabla p_j\|_{L^2(\mathbb{R}^n)^n} &\leq \|\nabla p - \varphi_j \nabla p\|_{L^2(\mathbb{R}^n)^n} \ + \ \|(\nabla\varphi_j)(p - K_j)\|_{L^2(\mathbb{R}^n)^n} \\
&\leq \|\nabla p - \varphi_j \nabla p\|_{L^2(\mathbb{R}^n)^n} + \frac{C'}{j}\|p - K_j\|_{L^2(G_j)^n}
\end{aligned}$$

with $\nabla\varphi_j(x) = \nabla\varphi(j^{-1}x) = j^{-1}(\nabla\varphi)(j^{-1}x)$ and $C' := \sup_x |\nabla\varphi(x)|$.

Lebesgue's dominated convergence lemma, see [Apo74], yields

$$\begin{aligned}
&\lim_{j\to\infty} \|\nabla p - \varphi_j \nabla p\|_{L^2(\mathbb{R}^n)^n} \\
&= \big(\int_{\mathbb{R}^n} \big(\lim_{j\to\infty} |1 - \varphi_j(x)|^2\big)\,|\nabla p(x)|^2\,dx\big)^{\frac{1}{2}} = 0,
\end{aligned} \tag{2.5.19}$$

since $|1-\varphi_j(x)| = |1-\varphi(j^{-1}x)| \le 2$ and $\lim_{j\to\infty} |1-\varphi(j^{-1}x)| = 0$ for each $x \in \mathbb{R}^n$. Using (2.5.18) we get

$$\|\nabla p - \nabla p_j\|_{L^2(\mathbb{R}^n)^n} \;\le\; \|\nabla p - \varphi_j \nabla p\|_{L^2(\mathbb{R}^n)^n} + C'C\|\nabla p\|_{L^2(G_j)^n}.$$

Together with

$$\lim_{j\to\infty} \|\nabla p\|_{L^2(G_j)^n} \;=\; \lim_{j\to\infty} \Big(\int_{G_j} |\nabla p(x)|_2^2 \, dx\Big)^{\frac{1}{2}} = 0$$

and (2.5.19) we conclude that

$$\lim_{j\to\infty} \|\nabla p - \nabla p_j\|_{L^2(\mathbb{R}^n)^n} = 0. \tag{2.5.20}$$

Next we use the mollification method, see Lemma 1.7.1. Since supp $p_j \subseteq \overline{B}_{2j}$ we can approximate each p_j by C_0^∞-functions in the gradient norm. Using the operator $\mathcal{F}_\varepsilon\star$, $\varepsilon > 0$, see (1.7.5), we find for each $j \in \mathbb{N}$ some $\varepsilon_j > 0$ such that

$$\|\nabla p_j - \mathcal{F}_{\varepsilon_j} \star \nabla p_j\|_{L^2(\mathbb{R}^n)^n} \;\le\; \frac{1}{j}.$$

With $\nabla(\mathcal{F}_{\varepsilon_j} \star p_j) = \mathcal{F}_{\varepsilon_j} \star (\nabla p_j)$, see (1.7.17), we get

$$\|\nabla p_j - \nabla(\mathcal{F}_{\varepsilon_j} \star p_j)\|_{L^2(\mathbb{R}^n)^n} \;\le\; \frac{1}{j} \tag{2.5.21}$$

for all $j \in \mathbb{N}$.

Setting $\tilde{p}_j := \mathcal{F}_{\varepsilon_j} \star p_j$ we see that $\tilde{p}_j \in C_0^\infty(\mathbb{R}^n)$, $j \in \mathbb{N}$, and combining (2.5.20) with (2.5.21) leads to

$$\lim_{j\to\infty} \|\nabla p - \nabla \tilde{p}_j\|_{L^2(\mathbb{R}^n)^n} = 0.$$

This proves (2.5.13).

To prove (2.5.11), let L be the space on the right side of (2.5.11). Recall, div $f = 0$ is understood in the sense of distributions. Since

$$L^2_\sigma(\mathbb{R}^n) \;=\; \overline{C^\infty_{0,\sigma}(\mathbb{R}^n)}^{\|\cdot\|_2} \subseteq L\,,$$

we only have to show that $L \subseteq L^2_\sigma(\mathbb{R}^n)$. For this purpose let $f \in L$. Then

$$< f, \nabla p > = -[\text{ div } f, p] = -< \text{ div } f, p > = 0 \tag{2.5.22}$$

for all $p \in C_0^\infty(\mathbb{R}^n)$. Since $f \in L^2(\mathbb{R}^n)^n$ and since the space of all ∇p with $p \in C_0^\infty(\mathbb{R}^n)$ is dense in $G(\mathbb{R}^n)$ in the norm $\|\cdot\|_2$, see (2.5.13), we see that

$< f, \nabla p >= 0$ holds as well for all $\nabla p \in G(\mathbb{R}^n)$. This means that $f \in G(\mathbb{R}^n)^\perp$, and we see that

$$f \in G(\mathbb{R}^n)^\perp \;=\; L^2_\sigma(\mathbb{R}^n)^{\perp\perp} \;=\; L^2_\sigma(\mathbb{R}^n).$$

Thus we get $f \in L^2_\sigma(\mathbb{R}^n)$ and $L \subseteq L^2_\sigma(\mathbb{R}^n)$ which proves (2.5.11). The proof of the lemma is complete. □

Finally we mention an important density property which follows by the same approximation argument as above.

2.5.5 Lemma *Let $n \in \mathbb{N}$, $n \geq 2$. Then*

$$\overline{C^\infty_{0,\sigma}(\mathbb{R}^n)}^{\|\cdot\|_{W^{1,2}(\mathbb{R}^n)^n}} \;=\; \{v \in W^{1,2}(\mathbb{R}^n)^n;\; \text{div } v = 0\}, \tag{2.5.23}$$

Thus $C^\infty_{0,\sigma}(\mathbb{R}^n) = \{v \in C^\infty_0(\mathbb{R}^n)^n; \text{div } v = 0\}$ is dense in the space on the right side of (2.5.23) *with respect to the norm of $W^{1,2}(\mathbb{R}^n)^n$.*

Proof. Recall that

$$W^{1,2}(\mathbb{R}^n)^n = W^{1,2}_0(\mathbb{R}^n)^n \;=\; \overline{C^\infty_0(\mathbb{R}^n)^n}^{\|\cdot\|_{W^{1,2}(\mathbb{R}^n)^n}}, \tag{2.5.24}$$

see (3.6.17), I.

To prove (2.5.23), let $v \in W^{1,2}_0(\mathbb{R}^n)^n = W^{1,2}(\mathbb{R}^n)^n$ with div $v = 0$. Then we have to construct some $v_j \in C^\infty_{0,\sigma}(\mathbb{R}^n)$, $j \in \mathbb{N}$, such that

$$\lim_{i\to\infty} \|v - v_j\|_{W^{1,2}(\mathbb{R}^n)^n} \;=\; 0. \tag{2.5.25}$$

For this purpose we use the same approximation method as in the last proof, and consider $\varphi_j, B_j, G_j, j \in \mathbb{N}$, as in (2.5.15), (2.5.16), $\mathcal{F}_{\varepsilon_j}$ as in (2.5.21). Then we construct some $w_j \in W^{1,2}_0(G_j)^n$, $j \in \mathbb{N}$, such that

$$\text{div } w_j = \; \text{div } (\varphi_j v) = (\nabla \varphi_j) \cdot v \tag{2.5.26}$$

and

$$\lim_{j\to\infty} \|w_j\|_{W^{1,2}(G_j)} = 0. \tag{2.5.27}$$

Assume for a moment that we already have such a sequence $(w_j)_{j=1}^\infty$. Then a similar argument as in (2.5.19) shows that

$$\lim_{j\to\infty} \|v - \varphi_j v\|_{W^{1,2}(\mathbb{R}^n)} = 0\,,$$

and setting $\tilde{v}_j := \varphi_j v - w_j$, $j \in \mathbb{N}$, we get div $\tilde{v}_j = 0$ and

$$\lim_{j\to\infty} \|v - \tilde{v}_j\|_{W^{1,2}(\mathbb{R}^n)} = 0.$$

A similar argument as in (2.5.21) leads to

$$\lim_{j\to\infty} \|\tilde{v}_j - \mathcal{F}_{\varepsilon_j} \star \tilde{v}_j\|_{W^{1,2}(\mathbb{R}^n)} = 0.$$

Then we set $v_j := \mathcal{F}_{\varepsilon_j} \star \tilde{v}_j$ and obtain

$$v_j \in C_0^\infty(\mathbb{R}^n)^n \quad , \quad \text{div } v_j = \mathcal{F}_{\varepsilon_j} \star \ \text{div } \tilde{v}_j = 0 \quad , \quad j \in \mathbb{N} \, ,$$

see (1.7.17), and (2.5.25) follows.

Thus it remains to construct the above sequence $(w_j)_{j=1}^\infty$. For this purpose we use Lemma 2.1.1, a). First we observe that

$$\int_{B_{2j}} \text{div } (\varphi_j v)\, dx \ = \int_{G_j} (\nabla \varphi_j) \cdot v \, dx = 0. \tag{2.5.28}$$

This follows from Green's formula (1.2.12) with $u \equiv 1$. Then we use the transformation $x = jy,\ x \in G_j,\ y \in G_1$, and setting $\tilde{w}_j(y) = w_j(jy) = w_j(x)$, we get from (2.5.26) the transformed equations

$$\text{div } \tilde{w}_j(y) \ = \ j(\nabla \varphi_j)(jy) \cdot v(jy) \ , \tag{2.5.29}$$

now in G_1 for all $j \in \mathbb{N}$. Using (2.5.28) we see that

$$\begin{aligned} \int_{G_1} (\text{div } \tilde{w}_j)(y)\, dy \quad &= \quad j \int_{G_1} (\nabla \varphi_j)(jy) \cdot v(jy)\, dy \\ &= \quad j j^{-n} \int_{G_j} (\nabla \varphi_j)(x) \cdot v(x)\, dx \ = \ 0 \, , \end{aligned}$$

and Lemma 2.1.1, a), yields a solution $\tilde{w}_j \in W_0^{1,2}(G_1)^n$ satisfying

$$\|\nabla \tilde{w}_j\|_{L^2(G_1)} \ \le \ C \ (\int_{G_1} |j(\nabla \varphi_j)(jy) \cdot v(jy)|^2 \, dy)^{\frac{1}{2}}$$

for all $j \in \mathbb{N}$ with some fixed $C = C(G_1) > 0$. Then $w_j \in W_0^{1,2}(G_j)^n$ defined by $w_j(x) = \tilde{w}_j(y),\ x = jy$, is a solution of (2.5.26), and we get

$$\begin{aligned} \|\nabla w_j\|_{L^2(G_j)} \quad &= \quad \left(\int_{G_j} |\nabla w_j(x)|^2 \, dx \right)^{\frac{1}{2}} = j^{-1} j^{\frac{n}{2}} \left(\int_{G_1} |(\nabla \tilde{w}_j)(y)|^2 \, dy \right)^{\frac{1}{2}} \\ &= \quad j^{-1} j^{\frac{n}{2}} \|\nabla \tilde{w}_j\|_{L^2(G_1)} \le C \, j^{\frac{n}{2}} \left(\int_{G_1} |(\nabla \varphi_j)(jy) \cdot v(jy)|^2 \, dy \right)^{\frac{1}{2}} \\ &= \quad C \left(\int_{G_j} |(\nabla \varphi_j)(x) \cdot v(x)|^2 \, dx \right)^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&= j^{-1}C\left(\int_{G_j} |(\nabla\varphi)(j^{-1}x)\cdot v(x)|^2\,dx\right)^{\frac{1}{2}} \\
&\le j^{-1}C_1\|v\|_{L^2(G_j)}
\end{aligned}
$$

for all $j \in \mathbb{N}$, with some $C_1 = C_1(G_1) > 0$ not depending on j.

Then with Poincaré's inequality for G_1 we obtain

$$
\begin{aligned}
\|w_j\|_{L^2(G_j)} &= \left(\int_{G_j} |w_j(x)|^2\,dx\right)^{\frac{1}{2}} = j^{\frac{n}{2}}\left(\int_{G_1} |\tilde{w}_j(y)|^2\,dy\right)^{\frac{1}{2}} \\
&\le C_2\, j^{\frac{n}{2}}\left(\int_{G_1} |\nabla_y\tilde{w}_j(y)|^2\,dy\right)^{\frac{1}{2}} \\
&= C_2\, j\left(\int_{G_j} |\nabla w_j(x)|^2\,dx\right)^{\frac{1}{2}} \\
&= C_2\, j\,\|\nabla w_j\|_{L^2(G_j)} \\
&\le C_2\, C_1\,\|v\|_{L^2(G_j)}
\end{aligned}
$$

with some $C_2 = C_2(G_1) > 0$ and C_1 as above.

Since obviously

$$\lim_{j\to\infty}\|v\|_{L^2(G_j)^n} = 0\,,$$

we conclude from these estimates that (2.5.27) is satisfied. This completes the proof. □

3 Elementary functional analytic properties

3.1 Basic facts on Banach spaces

For the convenience of the reader, and in order to fix notations, we collect some elementary facts on Banach spaces and in particular on Hilbert spaces. We mainly refer to [Yos80], [HiPh57], [Heu75].

Let X be a (real) **Banach space** with norm $\|v\|_X = \|v\|\,,\ v \in X$. By definition, the **dual space** X' of X is the Banach space of all linear continuous functionals

$$f : v \mapsto [f, v]\ \ ,\quad v \in X$$

with norm

$$\|f\|_{X'} := \sup_{0\ne v\in X}(|[f,v]|/\|v\|_X).$$

Sometimes we write $f = [f, \cdot\,]$; $[f, v]$ always means the value of the functional f at v.

A linear functional $f : v \mapsto [f, v],\ \ v \in X$, is continuous iff there is a constant $C = C(f) > 0$ such that

$$|[f, v]| \ \leq \ C\|v\|_X \quad \text{for all } v \in X. \tag{3.1.1}$$

It holds that $\|f\|_{X'} = \inf C(f)$, which is the infimum over all such constants $C(f)$ for fixed f. Therefore, if (3.1.1) holds with any $C = C(f)$, then

$$\|f\|_{X'} \ \leq \ C. \tag{3.1.2}$$

A sequence $(v_j)_{j=1}^{\infty}$ in X converges **strongly** to some $v \in X$ iff

$$\lim_{j\to\infty} \|v - v_j\| = 0;$$

we write $v = s - \lim_{j\to\infty} v_j$ in this case. The sequence $(v_j)_{j=1}^{\infty}$ in X converges **weakly** to $v \in X$ iff

$$\lim_{j\to\infty} [f, v_j] = [f, v]$$

for all $f \in X'$; we write $v = w - \lim_{j\to\infty} v_j$ in this case.

X is **reflexive** iff each linear continuous functional on X' has the form $f \mapsto [f, v],\ f \in X'$, with some fixed $v \in X$. We write $[\,\cdot, v]$ for this functional. Usually we identify each $v \in X$ with the functional $[\,\cdot, v]$. Then X can be identified with $(X')' = X''$ and we write $X'' = X$ if X is reflexive.

If X is reflexive, each bounded sequence $(v_j)_{j=1}^{\infty}$ in X contains a subsequence which converges weakly to some $v \in X$. For simplicity we will always assume that the sequence itself has this property. In this case

$$\|v\| \ \leq \ \lim_{j\to\infty} \inf \|v_j\| \ \leq \ \sup_j \|v_j\|. \tag{3.1.3}$$

Let $D \subseteq X$ be any subspace of X and let $\overline{D}$ denote the closure of D in the norm $\|\cdot\|$. D is called **dense** in X iff $\overline{D} = X$. We also write $\overline{D}^{\|\cdot\|} = X$ in this case.

Consider two Banach spaces X and $\mathcal{Y}$ with norms $\|\cdot\|_X,\ \|\cdot\|_{\mathcal{Y}}$, respectively. Let

$$B : v \mapsto Bv \ \ , \quad v \in D(B)$$

be any linear operator with **domain** $D(B) \subseteq X$ and **range** $R(B) := \{Bv;\ v \in D(B)\} \subseteq \mathcal{Y}$. $N(B) := \{v \in D(B);\ Bv = 0\}$ means the **null space** of B, and

$G(B) := \{(v, Bv);\ v \in D(B)\} \subseteq X \times \mathcal{Y}$ means the **graph** of B. If $\overline{D(B)} = X,\ B$ is called densely defined. The norm

$$\|v\|_{D(B)} := \|v\|_X + \|Bv\|_{\mathcal{Y}} \ , \quad v \in D(B) \tag{3.1.4}$$

is called the **graph norm** of $D(B)$. B is called **closed** if the graph $G(B)$ is closed in $X \times \mathcal{Y}$ with respect to the norm $\|v\|_X + \|w\|_{\mathcal{Y}},\ (v, w) \in X \times \mathcal{Y}$. If B is closed, $D(B)$ is a Banach space in the graph norm $\|\cdot\|_{D(B)}$.

Let $N(B) = \{0\}$. Then B is injective and

$$\|v\|_{\widehat{D}(B)} := \|Bv\|_{\mathcal{Y}} \ , \quad v \in D(B) \tag{3.1.5}$$

is called the **homogeneous graph norm** of $D(B)$. Even if B is closed, $D(B)$ need not be a Banach space in this norm. The **completion** $\widehat{D}(B)$ of $D(B)$ consists of all (classes of) Cauchy sequences $(v_j)_{j=1}^{\infty}$ in $D(B)$ with respect to this norm.

Let $v = (v_j)_{j=1}^{\infty}$ be any element of $\widehat{D}(B)$. Then, by definition, $(Bv)_{j=1}^{\infty}$ is a Cauchy sequence in $R(B) \subseteq \mathcal{Y}$. Setting

$$Bv := s - \lim_{j \to \infty} Bv_j \ , \quad v \in \widehat{D}(B) \tag{3.1.6}$$

we get a (well defined) linear operator from $\widehat{D}(B)$ to $\mathcal{Y}$ which is an extension of the given operator $v \mapsto Bv,\ v \in D(B)$. This extension is called the **closure extension** of B from $D(B)$ to $\widehat{D}(B)$, we simply use the same notation B for this extension. Note that $\widehat{D}(B) \supseteq D(B) \subseteq X$, but $\widehat{D}(B)$ need not be a subspace of X.

Let $B : D(B) \to \mathcal{Y},\ D(B) \subseteq X$, be a densely defined closed operator. Then the **dual operator** $B' : f \mapsto B'f$ with domain $D(B') \subseteq \mathcal{Y}'$ and range $R(B') \subseteq X'$ is well defined by the following property:

It holds that $[f, Bv] = [B'f, v]$ for all $f \in D(B'),\ v \in D(B)$, and B' is maximal with this property (that is, $D(B')$ is the totality of all $f \in \mathcal{Y}'$ such that $v \mapsto [f, Bv],\ v \in D(B)$, is continuous in $\|v\|_X$).

If one of the spaces $R(B),\ R(B')$ is closed, then both are closed and $R(B) = \{w \in \mathcal{Y};\ [f, w] = 0$ for all $f \in N(B')\},\ R(B') = \{g \in X';\ [g, v] = 0$ for all $v \in N(B)\}$; see the closed range theorem [Yos80, VII, 5]. If $R(B)$ is closed, then there is a constant $C > 0$ with

$$\|Bv\|_{\mathcal{Y}} \geq C \,\|\, [v] \,\|_{X/N(B)} \tag{3.1.7}$$

for all $v \in D(B)$, where

$$\|\, [v] \,\|_{X/N(B)} := \inf_{v_0 \in N(B)} \|v + v_0\|_X$$

means the quotient norm of $[v] = v + N(B)$; see [Yos80, I, 11] and the closed graph theorem [Yos80, II, 6, Theorem 1].

Let X and $\mathcal{Y}$ be reflexive Banach spaces and let $B : v \mapsto Bv,\ v \in D(B)$, be a closed linear operator with dense domain $D(B) \subseteq X$ and range $R(B) \subseteq \mathcal{Y}$. Suppose $(v_j)_{j=1}^\infty$ is a sequence in $D(B)$ with the following property:

$$\begin{aligned} &(v_j)_{j=1}^\infty \text{ converges weakly in } X \text{ to some } v \in X, \\ &\text{and } \sup_j \|Bv_j\|_{\mathcal{Y}} < \infty. \end{aligned} \tag{3.1.8}$$

Then $v \in D(B)$ and we get the estimate

$$\|Bv\|_{\mathcal{Y}} \ \le\ \liminf_{j\to\infty} \|Bv_j\|_{\mathcal{Y}} \ \le\ \sup_j \|Bv_j\|_{\mathcal{Y}}. \tag{3.1.9}$$

The proof of (3.1.9) rests on the following facts, see [Yos80, V, 1]. The pairs $(v_j, Bv_j),\ j \in \mathbb{N}$, yield a bounded sequence with respect to the graph norm (3.1.4), and the graph $G(B)$ is a reflexive Banach space with this norm. Therefore we get a subsequence which converges weakly in $G(B)$ to some element $(\tilde v, B\tilde v) \in G(B)$, and we may assume that the sequence itself has this property. Since $(v_j)_{j=1}^\infty$ converges to $v \in X$ weakly, we get $\tilde v = v,\ B\tilde v = Bv$ and $v \in D(B)$; (3.1.9) now follows from (3.1.3).

Let $B : v \mapsto Bv$ be any closed linear operator with dense domain $D(B) \subseteq X$ and range $R(B) \subseteq \mathcal{Y}$, and suppose that $N(B) = \{0\}$. This means that B is injective. Then the inverse operator $B^{-1} : D(B^{-1}) \to X$ with domain $D(B^{-1}) = R(B) \subseteq \mathcal{Y}$ and range $R(B^{-1}) = D(B) \subseteq X$, is well defined by $B^{-1}Bv = v$ for all $v \in D(B)$. B^{-1} is a closed operator.

Suppose $B : v \mapsto Bv$ is a bounded linear operator from X to $\mathcal{Y}$. Thus $D(B) = X$, and

$$\|B\| := \sup_{0\ne v\in X} (\|Bv\|_{\mathcal{Y}} / \|v\|_X) < \infty.$$

Then $\|B\|$ is called the norm of B. B is called **compact** iff for each bounded sequence $(v_j)_{j=1}^\infty$ in X, the sequence $(Bv_j)_{j=1}^\infty$ contains a subsequence which converges strongly in $\mathcal{Y}$ to some element of $\mathcal{Y}$.

Finally we consider an operator $B : X \to X$ which is only a mapping and need not be linear. B is called **completely continuous** iff

$$\left.\begin{array}{l} B \text{ is continuous and for each bounded sequence } (v_j)_{j=1}^\infty \text{ in } X, \\ \text{the sequence } (Bv_j)_{j=1}^\infty \text{ contains a subsequence which} \\ \text{converges strongly to some element of } X. \end{array}\right\} \tag{3.1.10}$$

We need the following result.

3.1.1 Lemma (Leray-Schauder principle) *Let X be a Banach space and let $B : X \to X$ be a completely continuous operator. Assume there exists some $r > 0$ with the following property:*

$$\text{If} \quad v \in X \ , \ \ 0 \le \lambda \le 1 \ , \ \ v = \lambda B v, \ \ \text{then} \ \ \|v\|_X \le r. \tag{3.1.11}$$

Then there exists at least one $v \in X$ with $v = Bv$, $\|v\|_X \le r$.

Proof. See [LeSch34], [Lad69, Chap. 1, Sec. 3], [Zei76, 6.5, Theorem 6.1]. □

3.2 Basic facts on Hilbert spaces

Here we mainly refer to [Yos80], [Kat66], [ReSi75], [Heu75] and [Wei76]. Let H be a (real) **Hilbert space** with scalar product $< u, v >_H = < u, v >$ and norm $\|u\|_H = \|u\| = < u, u >^{\frac{1}{2}}$, $u, v \in H$. Then H' denotes the dual space of all continuous linear functionals defined on H.

The Riesz representation theorem, see [Yos80, III, 6], shows that each element of H' has the form

$$v \mapsto < u, v > \ , \quad v \in H$$

with some fixed $u \in H$. As usual, this functional $< u, \cdot >$ will be identified with u, and we therefore obtain that $H' = H$.

Let $B : v \mapsto Bv$ be a closed linear operator with dense domain $D(B) \subseteq H$ and range $R(B) \subseteq H$. Then the **dual (adjoint)** operator B' with (dense) domain $D(B') \subseteq H$ and range $R(B') \subseteq H$ is determined by the property

$$< u, Bv > = < B'u, v > \quad \text{for all } v \in D(B) \ , \ \ u \in D(B'), \tag{3.2.1}$$

and $D(B')$ is the totality of all $u \in H$ such that the functional $v \mapsto < u, Bv >$, $v \in D(B)$, is continuous in $\|v\|_H$.

If $B = B'$, that is if $D(B) = D(B')$ and $Bv = B'v$ for all $v \in D(B)$, B is called a **selfadjoint** operator. A selfadjoint operator B is called **positive** if $< v, Bv > \ \ge \ 0$ for all $v \in D(B)$.

If $N(B) = \{v \in D(B); \ Bv = 0\} = \{0\}$, B is injective and we define the inverse operator $B^{-1} : D(B^{-1}) \to H$ by $D(B^{-1}) = R(B)$, $R(B^{-1}) = D(B)$, $B^{-1}Bv = v$ for all $v \in D(B)$. If B is positive selfadjoint, B^{-1} is also positive selfadjoint. See [Yos80, VII, 3] concerning these facts.

B is bounded iff $D(B) = H$ and there exists some $C = C(B) > 0$ such that

$$\|Bv\| \le C \, \|v\| \quad \text{for all } \ v \in H. \tag{3.2.2}$$

The operator norm $\|B\|$ is the infimum of all $C(B)$ with (3.2.2). Thus

$$\|B\| \ \le \ C \tag{3.2.3}$$

for all $C = C(B) > 0$ with (3.2.2).

Let $D \subseteq H$ be any closed subspace of H. Then

$$D^\perp := \{u \in H;\ < u, v >= 0 \ \text{ for all } v \in D\} \tag{3.2.4}$$

is called the **orthogonal subspace** of D. Each $u \in H$ has a unique decomposition $u = u_1 + u_2$ with $u_1 \in D,\ u_2 \in D^\perp$.

The operator $P : u \mapsto Pu$, defined by $Pu := u_1$ for all $u \in H$, is called the **projection** of H onto D. P is a positive selfadjoint operator with $P^2 = P$ and operator norm $\|P\| \le 1$.

Let I denote the identity. If P is the projection of H onto D, then $I - P$ is the projection onto $D^\perp$, and

$$\|u\|^2 \ = \ \|Pu\|^2 + \|(I-P)u\|^2 \quad \text{for all } \ u \in H. \tag{3.2.5}$$

Let $D \subseteq H$ be a dense subspace, and let $S(u,v) \in \mathbb{R}$ be defined for all $u, v \in D$ with the following properties:

$$\begin{aligned} &v \mapsto S(u,v),\ v \in D, \ \text{is a linear functional for each } u \in D \\ &S(u,v) = S(v,u) \ \text{ and } \ S(u,u) \ge 0 \text{ for all } u,v \in D. \end{aligned}$$

Then $S : (u,v) \mapsto S(u,v)$ is called a **positive symmetric bilinear form** with dense domain $D = D(S) \subseteq H$.

By

$$< u, v > + S(u,v) \ \ , \quad u,v \in D, \tag{3.2.6}$$

we obtain a scalar product and by

$$(\|u\|^2 + S(u,u))^{\frac{1}{2}} \ \ , \quad u \in D, \tag{3.2.7}$$

we get the corresponding norm in D. S is called **closed** if D is complete with respect to this norm. This means that D is a Hilbert space with the scalar product (3.2.6). We need the following result:

3.2.1 Lemma *Let H be a Hilbert space with scalar product $< \cdot, \cdot >$ and norm $\|\cdot\|$, and let $S : (u,v) \mapsto S(u,v)$ be a closed positive symmetric bilinear form with dense domain $D = D(S) \subseteq H$.*

Then there exists a uniquely determined positive selfadjoint operator $B : D(B) \to H$ with dense domain $D(B) \subseteq D$, satisfying:

$$\left.\begin{aligned} &D(B) \textit{ is the totality of all } \ u \in D \textit{ such that the} \\ &\textit{functional } v \mapsto S(u,v),\ v \in D, \textit{ is continuous in } \|v\|, \\ &\textit{and } S(u,v) =< Bu, v > \ \textit{ for all } u \in D(B),\ v \in D. \end{aligned}\right\} \tag{3.2.8}$$

Proof. See [Kat84, VI, Theorem 2.6] or [Wei76, Satz 5.37]. The proof rests on the Riesz representation theorem, applied to the scalar product (3.2.6). □

We need this lemma in order to define the Stokes operator A for arbitrary domains $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$.

Next we mention some facts on the spectral representation of selfadjoint operators, see [Yos80, XI, 5-7 and 12], [Kat84, Chap. V], [Wei76, 7.2]. Here we only need the special case of positive selfadjoint operators.

For each $\lambda \in [0, \infty)$, let E_λ be a projection operator which projects H onto a subspace $D_\lambda \subseteq H$. We call $\{E_\lambda;\, \lambda \geq 0\}$ a family of projections. Let $0 \leq \lambda_0 \leq \infty$. Then we write

$$E_{\lambda_0} \;=\; s - \lim_{\lambda \to \lambda_0} E_\lambda \tag{3.2.9}$$

iff $E_{\lambda_0} v = s - \lim_{\lambda \to \lambda_0} E_\lambda v$ holds for all $v \in H$ (strong convergence of operators).

Suppose $\{E_\lambda;\, \lambda \geq 0\}$ has the following properties:

a) $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda \;\;,\quad 0 \leq \lambda \leq \mu < \infty$

b) $E_\lambda = s - \lim_{\mu \to \lambda} E_\mu \;\;,\quad 0 < \mu < \lambda < \infty$

c) $E_0 = 0 \;\;,\quad s - \lim_{\lambda \to \infty} E_\lambda = I$.

Then $\{E_\lambda; \lambda \geq 0\}$ is called a **resolution of the identity** I on $[0, \infty)$. Condition a) means that E_λ and E_μ commute and that $D_\lambda \subseteq D_\mu$ for $\lambda \leq \mu$. It follows that $E_\mu - E_\lambda$, $\lambda \leq \mu$, is again a projection operator, and that $\lambda \mapsto \|E_\lambda v\|^2$ is monotonously increasing for each $v \in H$. Condition b) means that $\lambda \mapsto E_\lambda$ is left continuous in the interval $(0, \infty)$ with respect to the strong convergence of operators. $E_0 = 0$ means zero as an operator, and the last condition means that $\lim_{\lambda \to \infty} \|v - E_\lambda v\| = 0$ for all $v \in H$.

For each continuous function $g : \lambda \mapsto g(\lambda)$, $\lambda \geq 0$, we can define the usual Stieltjes integral

$$\int_0^b g(\lambda)\, d\|E_\lambda v\|^2 \;\;,\quad v \in H \;,\;\; 0 < b < \infty$$

as a limit of Riemann-Stieltjes sums of the form

$$\sum_{j=1}^m g(\lambda_j)\, (\|E_{\lambda_j} v\|^2 - \|E_{\lambda_{j-1}} v\|^2) = \sum_{j=1}^m g(\lambda_j)\, \|(E_{\lambda_j} - E_{\lambda_{j-1}}) v\|^2$$

where $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_m = b$, $\max |\lambda_j - \lambda_{j-1}| \to 0$, see [Apo74, 7.3].

If $g(\lambda) \geq 0$ for all $\lambda \geq 0$, and if

$$\int_0^\infty g(\lambda)\, d\|E_\lambda v\|^2 = \lim_{b \to \infty} \int_0^b g(\lambda)\, d\|E_\lambda v\|^2$$

exists for some $v \in H$, we simply write $\int_0^\infty g(\lambda)\, d\|E_\lambda v\|^2 \, < \, \infty$.

Let $g : \lambda \mapsto g(\lambda),\ \lambda \geq 0$, be a continuous real function. Then the integral

$$\int_0^b g(\lambda)\, dE_\lambda v \ \in H \quad , \quad 0 < b < \infty \ , \ \ v \in H$$

is well defined as the strong limit of the usual Riemann sums of the form $\sum_{j=1}^m g(\lambda_j)\,(E_{\lambda_j} - E_{\lambda_{j-1}})v,\ 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_m = b$, and

$$\| \int_0^b g(\lambda)\, dE_\lambda v\|^2 = \int_0^b g^2(\lambda)\, d\|E_\lambda v\|^2.$$

If $\int_0^\infty g^2(\lambda)\, d\|E_\lambda v\|^2 < \infty$ for some $v \in H$, then the integral

$$\int_0^\infty g(\lambda)\, dE_\lambda v \ := \ s - \lim_{b\to\infty} \int_0^b g(\lambda)\, dE_\lambda v$$

exists. We thus obtain a well defined operator

$$\int_0^\infty g(\lambda)\, dE_\lambda \ : \ v \mapsto \int_0^\infty g(\lambda)\, dE_\lambda v \tag{3.2.10}$$

which is selfadjoint and has the dense domain

$$D\left(\int_0^\infty g(\lambda)\, dE_\lambda\right) \ := \ \{v \in H;\ \int_0^\infty g^2(\lambda) d\|E_\lambda v\|^2 < \infty\}. \tag{3.2.11}$$

We see that

$$\| \int_0^\infty g(\lambda)\, dE_\lambda v\|^2 \ = \ \int_0^\infty g^2(\lambda) d\|E_\lambda v\|^2 \tag{3.2.12}$$

and that

$$< \left(\int_0^\infty g(\lambda)\, dE_\lambda\right)\ v, v > \ = \ \int_0^\infty g(\lambda) d\|E_\lambda v\|^2 \tag{3.2.13}$$

for all $v \in D(\int_0^\infty g(\lambda)\, dE_\lambda)$. In particular for all $v \in H$ we get

$$v = \int_0^\infty dE_\lambda v \ \ , \quad \|v\|^2 = \int_0^\infty d\|E_\lambda v\|^2. \tag{3.2.14}$$

If $g(\lambda) \geq 0$ for all $\lambda \geq 0$, then with (3.2.13) we see that $\int_0^\infty g(\lambda)\, dE_\lambda$ is positive selfadjoint, and if

$$\sup_{\lambda\geq 0} |g(\lambda)| < \infty\,,$$

we conclude from (3.2.11) and (3.2.12), that $\int_0^\infty g(\lambda)dE_\lambda$ is a bounded operator with $D(\int_0^\infty g(\lambda)\,dE_\lambda) = H$ and operator norm

$$\|\int_0^\infty g(\lambda)\,dE_\lambda\| \;\le\; \sup_{\lambda\ge 0} |g(\lambda)|. \tag{3.2.15}$$

In particular,

$$\int_0^\infty \lambda\,dE_\lambda \quad \text{with} \;\; D\left(\int_0^\infty \lambda\,dE_\lambda\right) = \{v \in H;\; \int_0^\infty \lambda^2\,d\|E_\lambda v\|^2 < \infty\} \tag{3.2.16}$$

is a positive selfadjoint operator.

Let now $B : D(B) \to H$ be any positive selfadjoint operator with (dense) domain $D(B) \subseteq H$. Then there exists a uniquely determined resolution

$$\{E_\lambda;\; \lambda \ge 0\}$$

of identity such that

$$B = \int_0^\infty \lambda\,dE_\lambda \;\;,\quad D(B) = \{v \in H;\; \int_0^\infty \lambda^2\,d\|E_\lambda v\|^2 < \infty\}. \tag{3.2.17}$$

This is called the **spectral representation** of B; see [Yos80, XI, 5], [Kat66, VI, 5.1].

For each continuous real function $g : [0,\infty) \to \mathbb{R}$, we define as above the selfadjoint operator

$$g(B) := \int_0^\infty g(\lambda)\,dE_\lambda \tag{3.2.18}$$

with domain

$$D(g(B)) = \{v \in H;\; \int_0^\infty g^2(\lambda)\,d\|E_\lambda v\|^2 < \infty\}.$$

If $\sup_{\lambda\ge 0}|g(\lambda)| < \infty$, $g(B)$ is bounded with $D(g(B)) = H$, and we see that

$$v \in D(B) \quad \text{implies} \quad g(B)v \in D(B) \quad \text{and} \quad Bg(B)v = g(B)Bv. \tag{3.2.19}$$

This property means that $g(B)$ **commutes** with B; see [Yos80, XI, 12]. Then

$$Bg(B)v = \int_0^\infty \lambda g(\lambda)\,dE_\lambda v \quad \text{for all } v \in D(B). \tag{3.2.20}$$

In particular we define the **fractional powers**

$$B^\alpha := \int_0^\infty \lambda^\alpha\,dE_\lambda \;\;,\quad D(B^\alpha) := \{v \in H;\; \int_0^\infty \lambda^{2\alpha} d\|E_\lambda v\|^2 < \infty\} \tag{3.2.21}$$

for all $\alpha \ge 0$. It holds that $B^\alpha = I$ for $\alpha = 0$.

For all $\mu > 0$, we consider the **resolvent**

$$(\mu I + B)^{-1} = \int_0^\infty (\mu + \lambda)^{-1}\, dE_\lambda, \tag{3.2.22}$$

which is the inverse of $\mu I + B$. This operator is bounded with norm

$$\|(\mu I + B)^{-1}\| \;\le\; \sup_{\lambda \ge 0}\, (\mu + \lambda)^{-1} \;\le\; \mu^{-1}. \tag{3.2.23}$$

If there is a $\delta > 0$ with $E_\lambda = 0$ for $0 \le \lambda \le \delta$, then B is obviously invertible and has the bounded inverse operator

$$B^{-1} = \int_\delta^\infty \lambda^{-1}\, dE_\lambda \tag{3.2.24}$$

with $\|B^{-1}\| \;\le\; \sup_{\lambda \ge \delta} \lambda^{-1}$.

Let $N(B) = \{v \in D(B); Bv = 0\}$ be the null space of B and let P_0 be the projection operator from H onto $N(B)$. Then we conclude that

$$P_0 \;=\; s - \lim_{\lambda \to 0} E_\lambda \quad , \quad \lambda > 0, \tag{3.2.25}$$

holds in the strong sense. This means that $N(B) = \bigcap_{\lambda > 0} D_\lambda$.
Therefore, the jump of $\lambda \mapsto E_\lambda$ at $\lambda = 0$ determines the null space $N(B)$ of B. B is injective, i.e., $N(B) = \{0\}$, iff $\lambda \mapsto E_\lambda$ is right continuous at $\lambda = 0$ with respect to the strong convergence.

Let now $N(B) = \{0\}$. Then for each $v \in H$ the function $\lambda \mapsto \|E_\lambda v\|^2, \lambda \ge 0$, is right continuous at $\lambda = 0$. This enables us to obtain an integral representation of the inverse operator

$$B^{-1} : D(B^{-1}) \to H \quad , \quad D(B^{-1}) = R(B),$$

although $\lambda \mapsto \lambda^{-1}$ is not a continuous function defined on the whole interval $[0, \infty)$ as in (3.2.18). We obtain (with $\delta > 0$) the representation

$$B^{-1} v = \int_0^\infty \lambda^{-1}\, dE_\lambda v = s - \lim_{\delta \to 0} \int_\delta^\infty \lambda^{-1}\, dE_\lambda v \quad , \quad v \in D(B^{-1}), \tag{3.2.26}$$

B^{-1} is positive selfadjoint, and

$$D(B^{-1}) = \{v \in H;\ \|B^{-1} v\|^2 = \int_0^\infty \lambda^{-2}\, d\|E_\lambda v\|^2 < \infty\}. \tag{3.2.27}$$

More generally, in the case $N(B) = \{0\}$ we can define the operator $B^{-\alpha} : D(B^{-\alpha}) \to H$ for $\alpha \ge 0$ by

$$B^{-\alpha} v \;=\; \int_0^\infty \lambda^{-\alpha}\, dE_\lambda v \;:=\; s - \lim_{\delta \to 0} \int_\delta^\infty \lambda^{-\alpha}\, dE_\lambda v \quad , \quad v \in D(B^{-\alpha}) \tag{3.2.28}$$

with domain

$$D(B^{-\alpha}) = \{v \in H;\ \|B^{-\alpha}v\|^2 = \int_0^\infty \lambda^{-2\alpha} d\|E_\lambda v\|^2 < \infty\}. \tag{3.2.29}$$

Then $N(B) = \{0\}$ implies $N(B^\alpha) = \{0\}$, $D(B^{-\alpha}) \subseteq H$ is dense, $B^{-\alpha}$ is positive selfadjoint, and

$$B^{-\alpha} = (B^{-1})^\alpha = (B^\alpha)^{-1}.$$

Thus $B^{-\alpha}$ is the inverse operator of B^α, and therefore we get $D(B^\alpha) = R(B^{-\alpha})$ and $D(B^{-\alpha}) = R(B^\alpha)$. If $0 \le \alpha \le 1$ we obtain

$$D(B) \subseteq D(B^\alpha) \ , \quad D(B^{-1}) \subseteq D(B^{-\alpha}). \tag{3.2.30}$$

These properties follow from the integral representations above.

Next we assume that the given positive selfadjoint operator B is defined by the form S with domain $D(S)$ as in Lemma 3.2.1. In this case we get

$$\begin{aligned} S(u,u) &= \ < Bu, u > \ = \ < B^{\frac12}u, B^{\frac12}u > \ = \ \|B^{\frac12}u\|^2 \\ &= \int_0^\infty \lambda\, d\|E_\lambda u\|^2 \end{aligned}$$

for all $u \in D(B)$. Then a closure argument shows that

$$D(B^{\frac12}) = D(S) \ , \quad S(u,u) = \|B^{\frac12}u\|^2 \quad \text{for all} \ \ u \in D(S). \tag{3.2.31}$$

We conclude from the spectral representation $B = \int_0^\infty \lambda\, dE_\lambda$ that $Bu = 0$ holds for $u \in D(B)$ iff $S(u,u) = 0$. Therefore,

$$N(B) = \{0\} \quad \text{iff} \quad \{u \in D(S);\ S(u,u) = 0\} = \{0\}\,. \tag{3.2.32}$$

This means that B is injective iff $S(u,u) = 0$ implies that $u = 0$.

The next lemma yields the **interpolation inequality** for fractional powers.

3.2.2 Lemma *Let $B : D(B) \to H$, $D(B) \subseteq H$, be a positive selfadjoint operator in the Hilbert space H, and let $0 \le \alpha \le 1$. Then*

$$\|B^\alpha v\| \ \le \ \|Bv\|^\alpha \|v\|^{1-\alpha} \ \le \ \alpha\|Bv\| + (1-\alpha)\|v\| \tag{3.2.33}$$

for all $v \in D(B)$.

Proof. Using the spectral representation and Hölder's inequality, see [Yos80, I, 3, (5)], we obtain

$$\begin{aligned} \|B^\alpha v\|^2 &= \int_0^\infty \lambda^{2\alpha}\, d\|E_\lambda v\|^2 \\ &\leq \Big(\int_0^\infty \lambda^2\, d\|E_\lambda v\|^2\Big)^\alpha \Big(\int_0^\infty d\|E_\lambda v\|^2\Big)^{1-\alpha} \\ &= \|Bv\|^{2\alpha}\, \|v\|^{2(1-\alpha)}, \end{aligned}$$

and apply Young's inequality (3.3.8), I. This proves the lemma. □

Finally we need a special result on fractional powers which is due to Heinz [Hei51].

3.2.3 Lemma (Heinz) *Let H_1, H_2 be two Hilbert spaces with norms $\|\cdot\|_1, \|\cdot\|_2$, respectively. Let $B : H_1 \to H_2$ be a bounded linear operator from H_1 into H_2 with operator norm $\|B\|$, and let*

$$A_1 : D(A_1) \to H_1 \quad , \quad A_2 : D(A_2) \to H_2$$

be positive selfadjoint injective operators with domains $D(A_1) \subseteq H_1$, $D(A_2) \subseteq H_2$. Suppose B maps $D(A_1)$ into $D(A_2)$ and

$$\|A_2 Bv\|_2 \;\leq\; C\,\|A_1 v\|_1 \quad \textit{for all} \;\; v \in D(A_1) \tag{3.2.34}$$

with some constant $C > 0$.

Then for $0 \leq \alpha \leq 1$, B maps $D(A_1^\alpha)$ into $D(A_2^\alpha)$, and the inequality

$$\|A_2^\alpha Bv\|_2 \;\leq\; C^\alpha\,\|B\|^{1-\alpha}\|A_1^\alpha v\|_1 \tag{3.2.35}$$

holds for all $v \in D(A_1^\alpha)$.

Proof. See [Hei51] or [Tan79, Theorem 2.3.3], [Kre71, Chap. I, Theorem 7.1]. Inequality (3.2.35) is called the **Heinz inequality**. □

3.3 The Laplace operator Δ

After discussing the operators div and ∇, see Section 2, the Laplacian

$$\Delta = \operatorname{div} \nabla = D_1^2 + \cdots + D_n^2$$

is the next important operator which occurs in the Navier-Stokes equations (1.1.1), I. The purpose of this subsection is to consider some basic facts on Δ mainly for the whole space $\mathbb{R}^n$, $n \geq 1$. These are potential theoretic properties.

We need the Riesz potential and the Bessel potential. For the proofs we refer to [Ste70], [Tri78], [Ada75], [SiSo96].

First let $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$, be an arbitrary domain. We consider the Hilbert space $L^2(\Omega)$ with scalar product

$$< u, v > = < u, v >_\Omega := \int_\Omega uv\, dx\,,$$

norm $\|u\|_{L^2(\Omega)} = \|u\|_2 = \|u\|_{2,\Omega} = < u, u >^{\frac{1}{2}}$, and define the bilinear form S with domain $D(S) \subseteq L^2(\Omega)$ by setting

$$D(S) := W_0^{1,2}(\Omega) \quad , \quad S(u,v) := < \nabla u, \nabla v > := \int_\Omega (\nabla u) \cdot (\nabla v)\, dx \qquad (3.3.1)$$

for $u, v \in D(S)$. Recall that $< \nabla u, \nabla v > = \sum_{j=1}^n \int_\Omega (D_j u)(D_j v)\, dx$. Since $W_0^{1,2}(\Omega)$ is complete with respect to the norm

$$(\|u\|_2^2 + S(u,u))^{\frac{1}{2}} = (\|u\|_2^2 + \|\nabla u\|_2^2)^{\frac{1}{2}}\ , \qquad (3.3.2)$$

the form S is closed. S is obviously symmetric and positive. Therefore, by Lemma 3.2.1 we obtain a positive selfadjoint operator $B : D(B) \to L^2(\Omega)$ with dense domain $D(B) \subseteq W_0^{1,2}(\Omega)$ satisfying the relation

$$< \nabla u, \nabla v > = < Bu, v > \quad \text{for all } u \in D(B)\,,\ v \in W_0^{1,2}(\Omega).$$

Setting $v \in C_0^\infty(\Omega)$, we see that

$$Bu = -\Delta u = -\ \mathrm{div}\ \nabla u$$

holds in the sense of distributions. Therefore we set $B = -\Delta$. Thus the operator

$$-\Delta : D(-\Delta) \to L^2(\Omega)$$

is defined by

$$D(-\Delta) = \{u \in W_0^{1,2}(\Omega);\ v \mapsto < \nabla u, \nabla v > \quad \text{is continuous in } \|v\|_2\} \qquad (3.3.3)$$

and by

$$< (-\Delta)u, v > = < \nabla u, \nabla v > \quad \text{for } u \in D(-\Delta)\ ,\ \ v \in W_0^{1,2}(\Omega). \qquad (3.3.4)$$

Obviously $\nabla u = 0$ implies $u = 0$ for all $u \in W_0^{1,2}(\Omega)$. Therefore, see (3.2.21) and (3.2.28), the fractional powers

$$(-\Delta)^{\frac{\alpha}{2}} = \int_0^\infty \lambda^{\frac{\alpha}{2}}\, dE_\lambda\,, \qquad (3.3.5)$$

with domain

$$D((-\Delta)^{\frac{\alpha}{2}}) = \{v \in L^2(\Omega);\ \int_0^\infty \lambda^\alpha \, d\|E_\lambda v\|_2^2 < \infty\},$$

are well defined for all $\alpha \in \mathbb{R}$. Here $\{E_\lambda; \lambda \geq 0\}$ denotes the resolution of identity for $-\Delta$, see Section 3.2.

An equivalent characterization is

$$D(-\Delta) = D(\Delta) = \{u \in W_0^{1,2}(\Omega);\ \Delta u \in L^2(\Omega)\} \tag{3.3.6}$$

with $\Delta u \in L^2(\Omega)$ in the sense of distributions.

Consider now the case $\Omega = \mathbb{R}^n$, $n \geq 1$. Then we have $W_0^{1,2}(\mathbb{R}^n) = W^{1,2}(\mathbb{R}^n)$, see (3.6.17), I. In this case there exists an explicit characterization of the spectral representation (3.3.5) which is obtained by using the **Fourier transform** $\mathcal{F}$. $\mathcal{F}$ is defined by

$$(\mathcal{F}u)(y) := \int_{\mathbb{R}^n} e^{-2\pi i x \cdot y} \, u(x) \, dx \quad , \quad y \in \mathbb{R}^n,$$

in the sense of distributions, see [Yos80, VI, 1], [Ste70, III, 1.2], [Tri78, 2.2.1]. For this purpose we have to work for the moment in the corresponding complex function spaces. This requires us to use complexifications of the real function spaces.

Then a calculation shows, see [Ste70, Chap. V, 1.1, (4)], that u and $(-\Delta)^{\frac{\alpha}{2}} u$ satisfy the integral equation

$$u(x) = \frac{1}{\gamma(\alpha, n)} \int_{\mathbb{R}^n} |x-y|^{-n+\alpha} (-\Delta)^{\frac{\alpha}{2}} u(y) \, dy \quad , \quad x \in \mathbb{R}^n \tag{3.3.7}$$

for $0 < \alpha < n$, where $\gamma(\alpha, n) := \pi^{\frac{n}{2}} \, 2^\alpha \, \Gamma(\frac{\alpha}{2}) / \Gamma(\frac{n}{2} - \frac{\alpha}{2})$. Γ means the Gamma function. The expression (3.3.7) is called the **Riesz potential**; it can be directly estimated by the Hardy-Littlewood theorem, see [Tri78, 1.18.8, Theorem 3]. The result is the following lemma.

3.3.1 Lemma *Let* $n \in \mathbb{N}$, $0 < \alpha < n$, $2 \leq q < \infty$,

$$\alpha + \frac{n}{q} = \frac{n}{2} \ , \tag{3.3.8}$$

and suppose that $u \in D((-\Delta)^{\frac{\alpha}{2}})$. *Then* $u \in L^q(\mathbb{R}^n)$ *and*

$$\|u\|_{L^q(\mathbb{R}^n)} \ \leq \ C \, \|(-\Delta)^{\frac{\alpha}{2}} u\|_{L^2(\mathbb{R}^n)} \tag{3.3.9}$$

with some constant $C = C(\alpha, n) > 0$.

Proof. See [Ste70, Chap. V, 1.2, Theorem 1]. It is shown that in this case the integral (3.3.7) converges absolutely for almost all $x \in \mathbb{R}^n$, the Hardy-Littlewood theorem, see also [Tri78, 1.18.8], yields the result. □

The following lemma concerns the special case $n = 1$. In this case, we write $(-\Delta)^{\frac{\alpha}{2}} u = f$, $u = (-\Delta)^{-\frac{\alpha}{2}} f$, and we are mainly interested in the estimate (3.3.9). Now we admit that $f \in L^r(\mathbb{R})$ with $1 < r < \infty$. The following result rests again on the Hardy-Littlewood theorem.

3.3.2 Lemma *Let* $0 < \alpha < 1$, $1 < r < q < \infty$ *with*

$$\alpha + \frac{1}{q} = \frac{1}{r} \; , \tag{3.3.10}$$

and suppose $f \in L^r(\mathbb{R})$. *Then the integral*

$$u(t) := \int_{\mathbb{R}} |t - \tau|^{\alpha - 1} f(\tau)\, d\tau$$

converges absolutely for almost all $t \in \mathbb{R}$, *and*

$$\|u\|_{L^q(\mathbb{R})} \;\le\; C\, \|f\|_{L^r(\mathbb{R})} \tag{3.3.11}$$

with some constant $C = C(\alpha, q) > 0$.

Proof. See [Ste70, Chap. V, 1.2] or [Tri78, 1.18.9, Theorem 3]. $\square$

Next we consider the positive selfadjoint operator $I - \Delta$ with domain $D(I - \Delta) = D(\Delta)$. We can define $I - \Delta$ also directly by using the form

$$< u, v > \; + \; < \nabla u, \nabla v > \tag{3.3.12}$$

instead of (3.3.1), see Lemma 3.2.1.

In this case u and $(I - \Delta)^{\frac{\alpha}{2}} u$ satisfy for $\alpha \ge 0$ the integral equation

$$u(x) = \int_{\mathbb{R}^n} G_\alpha(x - y)((I - \Delta)^{\frac{\alpha}{2}} u)(y)\, dy \;\; , \quad x \in \mathbb{R}^n, \tag{3.3.13}$$

where G_α is defined by

$$G_\alpha(z) := (4\pi)^{-\frac{\alpha}{2}}\, \Gamma(\alpha/2)^{-1} \int_0^\infty e^{-\pi |z|^2/t}\; e^{-t/4\pi}\; t^{-1 + (-n+\alpha)/2}\, dt, \tag{3.3.14}$$

$z \in \mathbb{R}^n$, see [Ste70, Chap. V, 3, (26)]. The expression (3.3.13) is called the **Bessel potential**. There are similar estimates as for the Riesz potential (3.3.7). We only need the following special case.

3.3.3 Lemma *Let* $n \in \mathbb{N}$, $1 \le \alpha \le 2$, $2 \le q < \infty$, *with*

$$\alpha + \frac{n}{q} = 1 + \frac{n}{2}, \tag{3.3.15}$$

and suppose that $u \in D((I - \Delta)^{\frac{\alpha}{2}})$. *Then* $u \in W^{1,q}(\mathbb{R}^n)$ *and*

$$\|u\|_{W^{1,q}(\mathbb{R}^n)} \;\le\; C\, \|(I - \Delta)^{\frac{\alpha}{2}} u\|_{L^2(\mathbb{R}^n)} \tag{3.3.16}$$

with some constant $C = C(\alpha, n) > 0$.

Proof. A direct proof follows using [Ste70, Chap. V, (29), (30)] and the Hardy-Littlewood estimate [Tri78, 1.18.8, Theorem 3] in the same way as before. It is based on the estimate of the potential (3.3.13). Another proof rests on the following argument. First we use [Ste70, V, 3, Theorem 3] or [Tri78, 2.3.3, (2)], [Ada75, Theorem 7.63, (f)] in order to show that the norms

$$\|u\|_{W^{1,q}(\mathbb{R}^n)} \quad \text{and} \quad \|(I-\Delta)^{\frac{\alpha}{2}} u\|_{L^q(\mathbb{R}^n)} \tag{3.3.17}$$

are equivalent. Then we use the embedding inequality

$$\|(I-\Delta)^{\frac{1}{2}} u\|_{L^q(\mathbb{R}^n)} \;\le\; C\, \|(I-\Delta)^{\frac{\alpha}{2}} u\|_{L^2(\mathbb{R}^n)} \tag{3.3.18}$$

with q, α as in (3.3.15); this follows from [Ada75, Theorem 7.63, (d)] or [Tri78, 2.8.1, Remark 2]. See also [Tri78, 2.8.1, (15)]. This yields the result. □

3.4 Resolvent and Yosida approximation

In the theory of the Navier-Stokes equations the Yosida approximation is used for technical reasons as a "smoothing" procedure which approximates L^2- functions by more regular functions. See [Ama95, II.6.1] concerning general properties, and see [Soh83], [Soh84], [MiSo88] concerning applications to the Navier-Stokes equations.

Let H be a Hilbert space and let $B : D(B) \to H$ be a positive selfadjoint operator as in (3.2.17). Then we consider the resolvent

$$(\mu I + B)^{-1} = \int_0^\infty (\mu + \lambda)^{-1}\, dE_\lambda \;\; , \quad \mu > 0 \tag{3.4.1}$$

as defined in (3.2.22). The relation

$$\begin{aligned} (\mu I + B)^{-1}(\mu I + B)v &= (\mu I + B)(\mu I + B)^{-1} v \\ &= \int_0^\infty (\mu+\lambda)(\mu+\lambda)^{-1}\, dE_\lambda v \\ &= \int_0^\infty dE_\lambda v = v \end{aligned}$$

holds for all $v \in D(B)$. For each $k \in \mathbb{N}$ we define the operator

$$J_k = J_{k,B} := (I + k^{-1}B)^{-1} = k(kI + B)^{-1} = \int_0^\infty (1 + k^{-1}\lambda)^{-1} dE_\lambda \; . \tag{3.4.2}$$

This representation shows that

$$J_k v \in D(B) \quad \text{for all} \;\; v \in H \; , \quad k \in \mathbb{N}, \tag{3.4.3}$$

and that

$$BJ_k = \int_0^\infty \lambda(1 + k^{-1}\lambda)^{-1}\, dE_\lambda \tag{3.4.4}$$

is a bounded operator with operator norm

$$\|BJ_k\| \leq \sup_{\lambda\geq 0} |\lambda(1 + k^{-1}\lambda)^{-1}| \leq k, \tag{3.4.5}$$

see (3.2.15). In the same way we get

$$\|J_k\| \leq \sup_{\lambda\geq 0} |(1 + k^{-1}\lambda)^{-1}| \leq 1. \tag{3.4.6}$$

The operators $J_k,\ k \in \mathbb{N}$, are called the **Yosida approximation** of the identity I. We have the following result; see [Yos80, IX, 9 and 12] or (in a slightly modified formulation) the proof of [Fri69, Part 2, Theorem 1.2] for more details.

3.4.1 Lemma *Let H be a Hilbert space and let $B : D(B) \to H$ be a positive selfadjoint operator with (dense) domain $D(B) \subseteq H$. Let $J_k,\ k \in \mathbb{N}$, be defined by* (3.4.2)*.*

Then we have:

$$\left.\begin{array}{l} J_k v \in D(B) \ \text{ for all } v \in H,\ BJ_k \text{ is bounded with (3.4.5)}, \\ BJ_k v = J_k B v \text{ for all } v \in D(B),\ J_k \text{ is bounded with (3.4.6)}, \end{array}\right\} \tag{3.4.7}$$

and

$$v = s - \lim_{k\to\infty} J_k v \quad \text{for all } v \in H, \tag{3.4.8}$$

$$Bv = s - \lim_{k\to\infty} BJ_k v \quad \text{for all } v \in D(B). \tag{3.4.9}$$

Proof. The properties (3.4.7) immediately follow from the spectral representation (3.4.1), see Section 3.2.

The property (3.4.8) means that $\lim_{k\to\infty} \|v - J_k v\| = 0$. To prove this we use (3.2.12), get

$$\begin{aligned} \|v - J_k v\|^2 &= \|(I - J_k)v\|^2 = \|\int_0^\infty (1 - (1 + k^{-1}\lambda)^{-1})\, dE_\lambda v\|^2 \\ &= \int_0^\infty (1 - (1 + k^{-1}\lambda)^{-1})^2\, d\|E_\lambda v\|^2, \end{aligned}$$

$(1 - (1 + k^{-1}\lambda)^{-1})^2 \leq 1$, and obtain

$$\lim_{k\to\infty} (1 - (1 + k^{-1}\lambda)^{-1})^2 = \lim_{k\to\infty} (\frac{\lambda}{k + \lambda})^2 = 0$$

for all $\lambda \geq 0$. Then we use Lebesgue's dominated convergence theorem [Apo74], and see that

$$\lim_{k\to\infty} \|v - J_k v\|^2 \;=\; \int_0^\infty \lim_{k\to\infty} \left(\frac{\lambda}{k+\lambda}\right)^2 d\|E_\lambda v\|^2 = \;0.$$

Let $v \in D(B)$. Then $BJ_k v = J_k Bv$, and from above we get

$$\lim_{k\to\infty} \|Bv - BJ_k v\|^2 \;=\; \lim_{k\to\infty} \|(I - J_k)Bv\|^2 = \;0.$$

This proves the lemma. $\square$

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