

# Contents

<b>Preface</b>	ix
<b>I Introduction</b>	
1 Basic notations	1
1.1 The equations of Navier-Stokes	1
1.2 Further notations	3
1.3 Linearized equations	6
2 Description of the functional analytic approach	7
2.1 The role of the Stokes operator $A$	7
2.2 The stationary linearized case	10
2.3 The stationary nonlinear case	11
2.4 The nonstationary linearized case	13
2.5 The full nonlinear case	16
3 Function spaces	22
3.1 Smooth functions	22
3.2 Smoothness properties of the boundary $\partial\Omega$	25
3.3 $L^q$ -spaces	27
3.4 The boundary spaces $L^q(\partial\Omega)$	30
3.5 Distributions	34
3.6 Sobolev spaces	38
<b>II Preliminary Results</b>	
1 Embedding properties and related facts	43
1.1 Poincaré inequalities	43
1.2 Traces and Green's formula	47
1.3 Embedding properties	52
1.4 Decomposition of domains	55
1.5 Compact embeddings	56
1.6 Representation of functionals	61
1.7 Mollification method	64

2	The operators $\nabla$ and $\operatorname{div}$ . . . . .	67
2.1	Solvability of $\operatorname{div} v = g$ and $\nabla p = f$ . . . . .	67
2.2	A criterion for gradients . . . . .	72
2.3	Regularity results on $\operatorname{div} v = g$ . . . . .	78
2.4	Further results on the equation $\operatorname{div} v = g$ . . . . .	79
2.5	Helmholtz decomposition in $L^2$ -spaces . . . . .	81
3	Elementary functional analytic properties . . . . .	89
3.1	Basic facts on Banach spaces . . . . .	89
3.2	Basic facts on Hilbert spaces . . . . .	93
3.3	The Laplace operator $\Delta$ . . . . .	100
3.4	Resolvent and Yosida approximation . . . . .	104

### III The Stationary Navier-Stokes Equations

1	Weak solutions of the Stokes equations . . . . .	107
1.1	The notion of weak solutions . . . . .	107
1.2	Embedding properties of $\widehat{W}_{0,\sigma}^{1,2}(\Omega)$ . . . . .	110
1.3	Existence of weak solutions . . . . .	112
1.4	The nonhomogeneous case $\operatorname{div} u = g$ . . . . .	114
1.5	Regularity properties of weak solutions . . . . .	116
2	The Stokes operator $A$ . . . . .	127
2.1	Definition and properties . . . . .	127
2.2	The square root $A^{\frac{1}{2}}$ of $A$ . . . . .	132
2.3	The Stokes operator $A$ in $\mathbb{R}^n$ . . . . .	135
2.4	Embedding properties of $D(A^\alpha)$ . . . . .	141
2.5	Completion of the space $D(A^\alpha)$ . . . . .	146
2.6	The operator $A^{-\frac{1}{2}}P \operatorname{div}$ . . . . .	153
3	The stationary Navier-Stokes equations . . . . .	157
3.1	Weak solutions . . . . .	157
3.2	The nonlinear term $u \cdot \nabla u$ . . . . .	159
3.3	The associated pressure $p$ . . . . .	163
3.4	Existence of weak solutions in bounded domains . . . . .	165
3.5	Existence of weak solutions in unbounded domains . . . . .	168
3.6	Regularity properties for the stationary nonlinear system . . . . .	173
3.7	Some uniqueness results . . . . .	178

## IV The Linearized Nonstationary Theory

1 Preliminaries for the time dependent linear theory . . . . .	185
1.1 The nonstationary Stokes system . . . . .	185
1.2 Basic spaces for the time dependent theory . . . . .	186
1.3 The vector valued operator $\frac{d}{dt}$ . . . . .	191
1.4 Time dependent gradients $\nabla p$ . . . . .	198
1.5 A special solution class of the homogeneous system . . . . .	203
1.6 The inhomogeneous evolution equation $u' + Au = f$ . . . . .	212
2 Theory of weak solutions in the linearized case . . . . .	219
2.1 Weak solutions . . . . .	219
2.2 Equivalent formulation and approximation . . . . .	221
2.3 Energy equality and strong continuity . . . . .	225
2.4 Representation formula for weak solutions . . . . .	230
2.5 Basic estimates of weak solutions . . . . .	237
2.6 Associated pressure of weak solutions . . . . .	246
2.7 Regularity properties of weak solutions . . . . .	253

## V The Full Nonlinear Navier-Stokes Equations

1 Weak solutions . . . . .	261
1.1 Definition of weak solutions . . . . .	261
1.2 Properties of the nonlinear term $u \cdot \nabla u$ . . . . .	265
1.3 Integral equation for weak solutions and weak continuity . . . . .	270
1.4 Energy equality and strong continuity . . . . .	272
1.5 Serrin's uniqueness condition . . . . .	276
1.6 Integrability properties of weak solutions in space and time, the scale of Serrin's quantity . . . . .	282
1.7 Associated pressure of weak solutions . . . . .	295
1.8 Regularity properties of weak solutions . . . . .	296
2 Approximation of the Navier-Stokes equations . . . . .	305
2.1 Approximate Navier-Stokes system . . . . .	305
2.2 Properties of approximate weak solutions . . . . .	307
2.3 Regularity properties of approximate weak solutions . . . . .	311
2.4 Smooth solutions of the Navier-Stokes equations with "slightly" modified forces . . . . .	312
2.5 Existence of approximate weak solutions . . . . .	315
2.6 Uniform norm bounds of approximate weak solutions . . . . .	318

3	Existence of weak solutions of the Navier-Stokes system . . . . .	320
3.1	Main result . . . . .	320
3.2	Preliminary compactness results . . . . .	323
3.3	Proof of Theorem 3.1.1 . . . . .	329
3.4	Weighted energy inequalities and time decay . . . . .	334
3.5	Exponential decay for domains for which the Poincaré inequality holds . . . . .	336
3.6	Generalized energy inequality . . . . .	339
4	Strong solutions of the Navier-Stokes system . . . . .	343
4.1	The notion of strong solutions . . . . .	343
4.2	Existence results . . . . .	344
	<b>Bibliography</b> . . . . .	355
	<b>Index</b> . . . . .	365

The Navier-Stokes Equations  
An Elementary Functional Analytic Approach

Sohr, H.

2001, X, 367 p., Softcover

ISBN: 978-3-0348-0550-6

A product of Birkhäuser Basel