2. Partial differential equations: analytic aspects

2.1 Introduction

As seen in the preceding chapter, fluid dynamics is governed by partial differential equations. Therefore knowledge of the numerical analysis of partial differential equations is indispensable in computational fluid dynamics. Introductions to this subject of varying degree of difficulty are Fletcher (1988), Hackbusch (1986), Hall and Porsching (1990), Hirsch (1988), Lapidus and Pinder (1982), Mitchell and Griffiths (1994), Morton and Mayers (1994), Grossmann and Roos (1994), Strikwerda (1989), Quarteroni and Valli (1994), Richtmyer and Morton (1967); the last book in particular is useful for practitioners of computational fluid dynamics.

Of course, in the study of the numerical aspects of partial differential equations, their analytic aspects play an important role. Although the reader is assumed to be familiar with the basics of the analysis of partial differential equations, we will devote this chapter to this subject, in order to highlight a few important topics that receive less attention elsewhere, and that will play a role later. In particular, we will discuss *maximum principles*, in order to put discussions of numerical 'wiggles' (see Chap. 4) on a firm footing. Furthermore, we will treat those aspects of *singular perturbation theory* that are required for a thorough understanding of the convection-diffusion equation. Also, we will pay more attention than usual to the treatment of mixed derivatives. These have often been neglected in the past, presumably because mixed derivatives seldom arise in mathematical physics. However, at present the predominant approach to handle arbitrarily shaped flow domains in the context of finite volume discretization is by means of *boundary fitted coordinates* (see Chap. 11). Because these coordinates are generally non-orthogonal, the governing equations frequently have mixed derivatives in the transformed coordinates.

More advanced information on the analytic aspects of partial differential equations can be found in Courant and Hilbert (1989), Garabedian (1964), Hackbusch (1986), Kreiss and Lorenz (1989), Protter and Weinberger (1967).

We will discuss in this chapter (special cases of) the following linear partial

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differential equation:

$$\begin{aligned} L\varphi &\equiv \varphi_t - (a_{\alpha\beta}\varphi_{,\beta})_{,\alpha} + b_\alpha \varphi_{,\alpha} + c\varphi = s \,, \\ \varphi, s &: \Omega \times (0,T] \to \mathbb{R}, \quad \Omega \subset \mathbb{R}^d, \quad \alpha, \beta = 1, 2, ..., d \,. \end{aligned} \tag{2.1}$$

where t stands for time, φ_t for $\partial\varphi/\partial t$, and d is the number of space dimensions. Boundary conditions will be specified later. The coefficients $a_{\alpha\beta}, b_\alpha$ and c are given functions of t and $\boldsymbol{x} \in \Omega$.

2.2 Classification of partial differential equations

Stationary case

We start with assuming that φ does not depend on time, so that the time-derivative in (2.1) is deleted. Furthermore, for the time being it is convenient to rewrite the equation as

$$L\varphi \equiv -a_{\alpha\beta}\varphi_{,\alpha\beta} + b_\alpha \varphi_{,\alpha} + c\varphi = s \,, \tag{2.2}$$

which implies the following replacement;

$$b_\alpha := b_\alpha - a_{\beta\alpha,\beta} \,.$$

Since $\varphi_{,\alpha\beta} = \varphi_{,\beta\alpha}$ we can assume without loss of generality that

$$a_{\alpha\beta} = a_{\beta\alpha} \tag{2.3}$$

(if (2.3) does not hold, simply redefine $a_{\alpha\beta} = a_{\beta\alpha} := \frac{1}{2}(a_{\alpha\beta} + a_{\beta\alpha})$). Consequently, the matrix A with elements $a_{\alpha\beta}$ is symmetric and has real eigenvalues.

Three special types of partial differential equations are distinguished, according to the following definition.

Definition 2.2.1. *Classification of partial differential equations.*
Equation (2.2) is called

(i) elliptic in $\boldsymbol{x}$ if the eigenvalues of A are nonzero and have the same sign;
(ii) hyperbolic in $\boldsymbol{x}$ if the eigenvalues of A are nonzero and precisely one eigenvalue has sign different from all others;
(iii) parabolic in $\boldsymbol{x}$ if precisely one eigenvalue is zero, while the others have the same sign and $\mathrm{Rank}(A, \boldsymbol{b}) = d$, *with $\boldsymbol{b}$ the vector with elements b_α.*

In the elliptic case the symmetric matrix A is definite. Without loss of generality we assume in this case A to be positive definite, because if this is not the case we may reverse sign in (2.2).

It can be shown that the type of a partial differential equation is invariant under coordinate transformation (see Hackbusch (1986) Sect. 1.2). Note that there are undefined cases.

Nonstationary case

The nonstationary case $\partial\varphi/\partial t + L\varphi = s$ can be subsumed under the classification for the stationary case $L\varphi = s$ by introducing time t as an additional coordinate x_{d+1}, and extending the range of α and β to $1, 2, ..., d+1$. The result is the following change in A and $\boldsymbol{b}$:

$$A = \begin{pmatrix} A_s & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} \boldsymbol{b}_s \\ 1 \end{pmatrix},$$

where the subscript s refers to the stationary case. Note that if the stationary case is elliptic, then the nonstationary case is parabolic.

Two-dimensional stationary case

In this case we find that equation (2.2) is
(i) elliptic in $\boldsymbol{x}$ if

$$a_{11}a_{22} - a_{12}^2 > 0\,, \tag{2.4}$$

(ii) hyperbolic in $\boldsymbol{x}$ if

$$a_{11}a_{22} - a_{12}^2 < 0\,, \tag{2.5}$$

(iii) parabolic in $\boldsymbol{x}$ if

$$a_{11}a_{22} - a_{12}^2 = 0 \quad \text{and} \quad \text{Rank} \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{12} & a_{22} & b_2 \end{pmatrix} = 2\,. \tag{2.6}$$

The archetypes of the three cases are

$$-\varphi_{,\alpha\alpha} = 0\,, \tag{2.7}$$

(*Laplace's equation*, elliptic),

$$-\varphi_{,11} + \varphi_{,22} = 0\,, \tag{2.8}$$

(the *wave equation*, hyperbolic),

$$\varphi_{,1} - \varphi_{,22} = 0\,, \tag{2.9}$$

(the *diffusion equation*, parabolic).
The three types require different boundary conditions (more on this later) and different numerical methods.

The coefficients and hence the type depend on $\boldsymbol{x}$, so that the type may change in Ω. A classic example is the *Tricomi equation*:

$$-\varphi_{,11} - x_2\varphi_{,22} = 0 \ .$$

This equation is elliptic for $x_2 > 0$ and hyperbolic for $x_2 < 0$; for $x_2 = 0$ the type is not defined. This situation arises in transonic flow, where we have ellipticity where the flow is subsonic and hyperbolicity where it is supersonic.

Example 2.2.1. Equation of undefined type
Consider

$$\frac{\partial \varphi}{\partial t} - 2\varphi_{,12} = 0 \ .$$

What is the type of this equation? It is left to the reader to verify, that the matrix to consider is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ .$$

The eigenvalues are $0, \pm 1$, so that the above equation is unclassified. □

Physical significance of classification

The classification corresponds to different types of qualitative behavior of solutions, and hence to differences in the underlying physics. This can be shown as follows. A plane wave solution is given by

$$\varphi = e^{iw(\boldsymbol{x})} \ , \quad w(\boldsymbol{x}) = \boldsymbol{n} \cdot \boldsymbol{x} - b \ ,$$

where the function $w(\boldsymbol{x})$ describes the wave fronts or *characteristic surfaces*, given by $w(\boldsymbol{x})$ = constant. Substitution in (2.2) gives (in order to bring out the physical differences between the three cases it suffices to restrict ourselves to $b_\alpha = c = s = 0$, $a_{\alpha\beta}$ = constant)

$$a_{\alpha\beta} n_\alpha n_\beta = 0 \ , \tag{2.10}$$

with $\boldsymbol{n}$ = grad$w$ the normal to the wave front. It is convenient to rewrite (2.10) as

$$\boldsymbol{n}^T A \boldsymbol{n} = 0 \ . \tag{2.11}$$

Let $\boldsymbol{v}^{(\alpha)}$, $\alpha = 1, 2, ..., d$ be the set of orthonormal eigenvectors of the matrix A, with corresponding eigenvalues λ_α. Because A is symmetric, $\boldsymbol{v}^{(\alpha)}$ and λ_α are real. Let us write

$$\boldsymbol{n} = c_\alpha \boldsymbol{v}^{(\alpha)} \ .$$

Substitution in (2.11) gives

$$\lambda_\alpha c_\alpha^2 = 0 \ . \tag{2.12}$$

If (2.2) is elliptic then $\lambda_\alpha > 0$, which contradicts (2.12), so that there is no solution for $\boldsymbol{n}$; hence wave-like behavior of the solution is not to be expected. If $\lambda_\alpha > 0$ then A is positive definite and vice-versa, so that

$$a_{\alpha\beta}\xi_\alpha\xi_\beta > 0 \ , \quad \forall\, \xi_\alpha \neq 0 \ . \tag{2.13}$$

This is often taken as the definition of ellipticity of (2.2).

In the hyperbolic case one eigenvalue, say λ_1, is negative, and solutions are obtained with

$$c_1^2 = (\sum_{\alpha>1} \lambda_\alpha c_\alpha^2)/\lambda_1 \tag{2.14}$$

so that wave-like solutions exist. Furthermore, $\boldsymbol{n}$ can point in any direction, since there are $d-1$ free parameters in (2.14); hence, wave propagation may take place in any direction. In the parabolic case, let λ_1 be the zero eigenvalue. The only solutions of (2.12) are $c_1 \neq 0$, $c_\alpha = 0$, $\alpha > 1$, so that wave propagation can take place in one specific direction only, corresponding to a *time-like* variable.

Another way to illuminate the physical significance of the classification is to look at the role of the boundary conditions. As we will see, in the elliptic case boundary conditions have to be prescribed all along the boundary of the domain, and a local change in boundary data influences the solution everywhere. One might therefore say that elliptic equations model *equilibrium* phenomena. In the hyperbolic case a local change in boundary data propagates its influence only in part of the domain, showing a wave-like behavior, so that hyperbolic equations model *propagation* phenomena. In the parabolic case, as we just saw, invariably a *time-like* independent variable can be identified. Changes in boundary data at a certain time make themselves felt everywhere, but only at later times. The time-like variable cannot be reversed without jeopardizing the well-posedness of the problem, as will be seen. Note that equations (2.7) and (2.8) are invariant under the transformation $x_\alpha := -x_\alpha$, but that (2.9) is not invariant under the transformation $x_1 := -x_1$. This *irreversibility* is why the variable concerned is called time-like. In short, parabolic equations model *diffusion* processes.

First order systems

Extension of the classification to equations more general than (2.2) can be done on the basis of the criterion whether or not wave-like solutions exist. It is always possible to write a system of partial differential equations as a first order system of m equations with m unknowns:

$$F_0 U_t + F_\alpha U_{,\alpha} = Q \ , \tag{2.15}$$

with $U = (U_1, ..., U_m)$ and where $F_0(t, \boldsymbol{x}, U)$, $F_\alpha(t, \boldsymbol{x}, U)$ are $m \times m$ matrices.

We consider the case where F_0 and F_α are made constant by 'freezing' them to their local value at $t = t_p$, $\boldsymbol{x} = \boldsymbol{x}_p, U = U_p$. We now ask whether plane waves can be solutions of the homogeneous version of (2.15). Plane waves are solutions of the following type:

$$U = \hat{U} e^{iw(t,\boldsymbol{x})}, \quad \hat{U} = \text{constant} \ , \quad w(t, \boldsymbol{x}) = \boldsymbol{n} \cdot \boldsymbol{x} - \lambda t \ .$$

This wave travels in the direction $\boldsymbol{n}$ with velocity $\lambda/|\boldsymbol{n}|$. Substitution shows that plane waves are solutions if

$$(n_\alpha F_\alpha - \lambda F_0)\hat{U} = 0 \ . \tag{2.16}$$

Here $\boldsymbol{n} \in \mathbb{R}^d$ is real. This is called a generalized eigenvalue problem. Eigensolutions $\hat{U}$ may exist if λ and $\boldsymbol{n}$ satisfy some constraint:

$$\lambda = \lambda(\boldsymbol{n}) \ .$$

The classification of (2.15) in $\{t_p, \boldsymbol{x}_p, U_p\}$ depends on the number of plane waves that exist. The classification is as follows.

Definition 2.2.2. *The system (2.15) is called hyperbolic in* $(t_p, \boldsymbol{x}_p, U_p)$ *if all eigenvalues* $\lambda = \lambda(\boldsymbol{n})$ *of (2.16) are real and if there exist* m *linearly independent eigenvectors.*

If, on the other hand, no real eigenvalues exist at all, then wave-like solutions are not possible, and the system is called elliptic:

Definition 2.2.3. *The system (2.15) is called elliptic in* $(t_p, \boldsymbol{x}_p, U_p)$ *if the eigenvalue problem (2.16) has no real eigenvalues* $\lambda = \lambda(\boldsymbol{n})$.

A generalization of parabolicity to the case of first order systems is:

Definition 2.2.4. *The system (2.15) is called parabolic in* $(t_p, \boldsymbol{x}_p, U_p)$ *if all eigenvalues* $\lambda = \lambda(\boldsymbol{n})$ *of (2.16) are real and if there exist only* $m - 1$ *linearly independent eigenvectors.*

These definitions can be applied to the stationary case by deleting λ and F_0, and by checking whether $\boldsymbol{n}$ is real or not, and by counting the eigenvectors.

If the system (2.15) does not fall into one of these three categories it is called *hybrid.* Because higher-order systems can be reformulated as first order systems, this classification scheme can be generally applied.

Example 2.2.2. Second order equation
Consider the following special case of (2.2):

$$-a_{\alpha\beta}\varphi_{,\alpha\beta} = 0 \ . \tag{2.17}$$

This is equivalent to the following first order system:

$$\varphi_{,\alpha} = U_\alpha, \quad -a_{\alpha\beta}U_{\alpha,\beta} = 0 \,. \tag{2.18}$$

In order to obtain a system of type (2.15) we eliminate φ by differentiation and combination:

$$\varphi_{,\alpha d} - \varphi_{,d\alpha} = 0, \quad \alpha = 1, 2, ..., d-1 \,,$$

and we obtain the following system of d equations for d unknowns:

$$\begin{aligned} U_{\alpha,d} - U_{d,\alpha} &= 0, \quad \alpha = 1, 2, ..., d-1 \,, \\ -a_{\alpha\beta}U_{\alpha,\beta} &= 0 \,. \end{aligned} \tag{2.19}$$

The reader may verify that in the case of Laplace's equation and $d = 2$ equations (2.19) are the *Cauchy-Riemann* equations:

$$U_{1,2} - U_{2,1} = 0, \quad U_{\alpha,\alpha} = 0 \,.$$

These equations are equivalent to Laplace's equation, but for $d > 2$ equations (2.19) and (2.17) are not equivalent. Solutions of (2.17) are solutions of (2.19), but not vice-versa. For example, for $d = 3$ a solution of (2.19) is obtained with $U_3 = 0$, $U_\alpha = U_\alpha(x_1, x_2)$, $\alpha = 1, 2$ and $-a_{\alpha\beta}U_{\alpha,\beta} = 0$. However, in general this will not satisfy $U_{1,2} = U_{2,1}$, so that we cannot solve for φ from (2.18). In other words, the transformation (2.18) introduces *spurious solutions.*

By comparing (2.19) and (2.15) and specializing to $d = 3$ we find $F_0 = 0$ and

$$F_1 = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ a_{11} & a_{21} & a_{31} \end{pmatrix}, \quad F_2 = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ a_{12} & a_{22} & a_{32} \end{pmatrix},$$
$$F_3 = -\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

For (2.16) to have a non-trivial solution of the type $e^{iw(\boldsymbol{x})}$, $w(\boldsymbol{x}) = \boldsymbol{n} \cdot \boldsymbol{x}$ we must have $\det(\sum n_\alpha F_\alpha) = 0$, or

$$0 = \begin{vmatrix} n_3 & 0 & -n_1 \\ 0 & n_3 & -n_2 \\ -n_\alpha a_{1\alpha} & -n_\alpha a_{2\alpha} & -n_\alpha a_{3\alpha} \end{vmatrix} = -n_3 n_\alpha n_\beta a_{\alpha\beta} \,. \tag{2.20}$$

One real solution is obtained for $n_3 = 0$, giving $w = w(x_1, x_2)$. From (2.16) it follows that $\hat{U}_3 = 0$, $\sum\limits_{\alpha=1}^{2}\sum\limits_{\beta=1}^{2} w_{,\alpha}a_{\beta\alpha}\hat{U}_\beta = 0$, which has non-zero real solutions for every $w(x_1, x_2)$, giving solutions of (2.17) of the form $U_\alpha = \hat{U}_\alpha e^{iw}$, $\alpha = 1, 2$, $U_3 = 0$. However, in general these will not satisfy $U_{1,2} - U_{2,1} = 0$, or

$w_{,2}\hat{U}_1 - w_{,1}\hat{U}_2 = 0$. Hence, these are spurious solutions. The other possibility following from (2.20) is

$$n_\alpha n_\beta a_{\alpha\beta} = 0 ,$$

which is just (2.11), bringing us back to the classification of second order systems. We see, that apart from spurious solutions, the classification of first order systems agrees with the classification of second order systems. □

Example 2.2.3. Parabolic equation
Consider equation (2.9). It can be written as a first order system by defining

$$U_1 = \varphi , \quad U_2 = \varphi_{,2} .$$

We elemine φ by differentiation, as in Example 2.2.2, and obtain the following first order system:

$$\begin{aligned} U_{1,2} &= U_2 , \\ U_{1,1} - U_{2,2} &= 0 . \end{aligned}$$

This gives us the following first order system:

$$F_1 U_{,1} + F_2 U_{,2} = Q , \quad F_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \quad F_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad Q = \begin{pmatrix} U_2 \\ 0 \end{pmatrix} .$$

The eigenproblem becomes

$$0 = \begin{vmatrix} n_2 & 0 \\ n_1 & -n_2 \end{vmatrix} = -n_2^2 ,$$

giving $\boldsymbol{n} = (n_1, 0)$ with n_1 arbitrary. The eigenvector $\hat{U}$ has to satisfy $\hat{U}_1 = 0$, so that there is only one eigenvector:

$$\hat{U} = \begin{pmatrix} 0 \\ \hat{U}_2 \end{pmatrix}$$

Hence, the system is parabolic according to Definition 2.2.4, and we find this classification to be in agreement with Definition 2.2.1. □

Exercise 2.2.1. Derive the form taken by the wave equation (2.8) after the following change of coordinates:

$$y_1 = x_1 + x_2, \quad y_2 = x_1 - x_2 ,$$

and determine the type of the resulting equation. Show that every solution of the wave equation can be written as

$$\varphi(\boldsymbol{x}) = \psi(x_1 + x_2) + \eta(x_1 - x_2) . \tag{2.21}$$

Exercise 2.2.2. Show that

$$\varphi(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \varphi_0(\xi) \exp(-(x-\xi)^2/4t) d\xi$$

satisfies the diffusion equation (2.9). More difficult (see Hackbusch (1986)): show that

$$\lim_{t \downarrow 0} \varphi(t,x) = \varphi_0(x) \ .$$

Exercise 2.2.3. Prove (2.4)–(2.6).

Exercise 2.2.4. Determine, depending on the parameter c, the type of

$$-\varphi_{,11} + 2c\varphi_{,12} - \varphi_{,22} = 0 \ .$$

Exercise 2.2.5. Let a coordinate transformation $\boldsymbol{x} \to \boldsymbol{y}$ be given by

$$\boldsymbol{x} = B\boldsymbol{y} \ ,$$

with B a constant non-singular $d \times d$ matrix. Show that this transformation leaves the type of equation (2.2) invariant. Hint: use Sylvester's lemma: If $C = B^T AB$ and $A = A^T$, then the number of positive, zero and negative eigenvalues of A and C is the same.

Exercise 2.2.6. Repeat Exercise 2.2.5 for equation (2.1). Next, consider the transformation

$$\begin{pmatrix} t \\ \boldsymbol{x} \end{pmatrix} = B \begin{pmatrix} \tau \\ \boldsymbol{y} \end{pmatrix} ,$$

with B a constant non-singular $(d+1) \times (d+1)$ matrix. Show that the type of equation (2.1) is invariant.

2.3 Boundary conditions

We start with the stationary case of (2.1):

$$L\varphi \equiv -(a_{\alpha\beta}\varphi_{,\beta})_{,\alpha} + b_\alpha \varphi_{,\alpha} + c\varphi = s \ , \tag{2.22}$$

and assume this equation to be elliptic for all $\boldsymbol{x} \in \Omega$.

Well-posed problems

To ensure the existence of a unique solution, suitable boundary conditions have to be imposed at the boundary $\partial\Omega$ of Ω. In order to be amenable to computation in the presence of rounding and truncation errors, and in general also in order to correctly model physical phenomena, the solution φ should depend continuously on the data, i.e. on the right-hand-side s and on the prescribed boundary values. In other words, small perturbations in the data should not cause large changes in the solution. This leads us to the concept of *well-posed problems.* Let the boundary condition be given by

$$B\varphi = f \quad \text{on} \quad \partial\Omega \tag{2.23}$$

with B some operator, typical examples of which are given in (2.26)–(2.28), or B could be the zero operator on part of $\partial\Omega$, in which case no boundary condition is prescribed there. Let $\Phi \ni \varphi$, $S \ni s$ and $F \ni f$ be suitable function spaces. Then the concept of well-posedness can be defined as follows:

Definition 2.3.1. *Well-posedness*
The problem specified by (2.22) and (2.23) is called well-posed if for all $f \in F$, $s \in S$ the following two conditions are satisfied:

(i) There exists a unique solution $\varphi \in \Phi$;
(ii) For every two sets of data f_1, s_1 and f_2, s_2 in F and S the corresponding solutions φ_1 and φ_2 satisfy

$$||\varphi_1 - \varphi_2||_\Phi \leq C\{||f_1 - f_2||_F + ||s_1 - s_2||_S\} \tag{2.24}$$

with C some fixed constant.

For mathematical precision, the function spaces Φ, F and S have to be specified (their choice depends on the problem, notably on the smoothness of $\partial\Omega$), but the general idea is clear.

The following three types of boundary conditions lead to well-posed elliptic boundary value problems, assuming

$$c \geq 0 \tag{2.25}$$

(we will not prove this; there is a huge amount of literature on this subject):

$$\varphi = f \quad \text{on} \quad \partial\Omega \qquad \text{(Dirichlet)}\,, \tag{2.26}$$
$$n_\alpha a_{\alpha\beta}\varphi_{,\beta} = f \quad \text{on} \quad \partial\Omega \qquad \text{(Neumann)}\,, \tag{2.27}$$
$$n_\alpha a_{\alpha\beta}\varphi_{,\beta} + a\varphi = f, \quad a > 0 \quad \text{on} \quad \partial\Omega \quad \text{(Robin)}\,, \tag{2.28}$$

with $\boldsymbol{n}$ the outward unit normal on $\partial\Omega$. Instead of (2.27) often the condition $n_\alpha\varphi_{,\alpha} = f$ is given as Neumann condition, but physics always leads to (2.27);

if $a_{\alpha\beta} = 0$ for $\beta \neq \alpha$ the two versions are equivalent.

When Ω is *simply connected* and $\partial\Omega$ and the data f, s satisfy suitable smoothness conditions, then (2.26) and (2.28) are known to result in a well-posed problem. This is also the case if $\partial\Omega$ is divided in segments, in each of which we have precisely one of the conditions (2.26)–(2.28), with (2.26) or (2.28) in at least one of the seqments.

Compatibility condition

In the case of a pure Neumann boundary value problem the situation is more complicated if $c = 0$ in (2.22). Obviously, if a solution exists an arbitrary constant can be added to it. Usually $\boldsymbol{b} = 0$ in this situation. Assuming this, integration of (2.22) over Ω and application of the divergence theorem gives

$$\int\limits_{\partial\Omega} f dS = -\int\limits_{\Omega} s d\Omega \,. \tag{2.29}$$

If and only if this so-called *compatibility condition* is satisfied, solutions exist. They can be stably computed, and the problem is still considered to be well-posed, although we have no uniqueness in the strict sense. Equation (2.29) expresses a physical conservation principle: transport through the boundary balances production in the interior.

We illustrate these theoretical considerations with some examples.

Example 2.3.1. Hadamard's problem
Let $\Omega = (0,1) \times (0,1)$, and consider the following problem:

$$-\varphi_{,\alpha\alpha} = 0 \quad \text{in} \quad \Omega \,,$$

with boundary conditions

$$\varphi(x_1, 0) = 0, \quad -\varphi_{,2}(x_1, 0) = f(x_1) \,, \tag{2.30}$$

$$\varphi(0, x_2) = \varphi(1, x_2) = 0 \,. \tag{2.31}$$

Notice that these conditions are not of the type discussed before, since we have two conditions at the seqment $x_2 = 0$ of $\partial\Omega$, and none at $x_2 = 1$. With $f(x_1) = -\frac{1}{m}\sinh m\pi x_1$, separation of variables gives the following solution:

$$\varphi(\boldsymbol{x}) = \frac{1}{\pi m^2} f(x_1) \sinh m\pi x_2.$$

With the maximum principle we can establish uniqueness. Nevertheless the problem is *ill-posed*, which can be seen as follows. If $f(x_1) = 0$ the solution is $\varphi(\boldsymbol{x}) = \varphi_1(\boldsymbol{x}) \equiv 0$. With f as above, we have $\varphi(x_1, 1) = \varphi_2(x_1, 1) \equiv$

$\frac{1}{m\pi^2} f(x_1) \sinh m\pi$. Given an ε there is an M such that $|f(x_1)| < \varepsilon$ for all $m > M$. However, due to the exponential growth of sinh we can for arbitrarily large K always find an $m > M$ such that $|\varphi_1(x_1, 1) - \varphi_2(x_1, 1)| > K$, so that (2.24) does not hold. □

Example 2.3.2. Two-dimensional potential flow
Let $\Omega = \{\boldsymbol{x} \in \mathbb{R}^2 : \ x_\alpha x_\alpha \geq 1\}$, i.e. the exterior of the unit circle. Consider the following problem:

$$\begin{aligned} -\varphi_{,\alpha\alpha} &= 0 \quad \text{in} \quad \Omega \ , \\ n_\alpha \varphi_{,\alpha} &= 0 \quad \text{at} \quad x_\alpha x_\alpha = 1 \ , \\ \lim_{|\boldsymbol{x}| \to \infty} \varphi / x_1 &= U \ . \end{aligned}$$

This describes potential flow around a cylinder with free-stream velocity U in the x_1-direction at infinity. At first sight it seems that a difficulty is the prescription of an infinite value for φ at infinity, but this difficulty is easily surmounted by a change of the dependent variable: $\psi = \varphi - Ux_1$. Using a conformal mapping method, well-known in classical fluid dynamics, the following one-parameter family of solutions is found:

$$\varphi = U(r + 1/r) \cos\theta + \gamma\theta/2\pi \ ,$$

with (r, θ) polar coordinates, and γ an arbitrary constant. The physical meaning of γ is that it governs the amount of circulation, and in fluid dynamics it is determined by an additional condition, called the *Kutta condition.* The reason that uniqueness is lacking here is that Ω is not simply connected. □

Example 2.3.3. Helmholtz equation
Let $\Omega = (0, \pi) \times (0, \pi)$, and consider

$$-\varphi_{,\alpha\alpha} - \lambda^2 \varphi = 0 \quad \text{in} \quad \Omega \ , \quad \varphi = 0 \quad \text{on} \quad \partial\Omega \ .$$

Here (2.25) is violated. A solution is obviously $\varphi \equiv 0$. For $\lambda = 1, 2, 3, \ldots$ (*eigenvalues*) we also have non-zero solutions (*eigenfunctions*) given by

$$\varphi = \sin \lambda x_1 \sin \lambda x_2 \ ,$$

so that there is no uniqueness. Although this problem does not satisfy our definition of well-posedness, it is meaningful and can be stably computed as an eigenvalue problem. □

The parabolic case

Let equation (2.1) be parabolic. Then a well-posed problem is obtained if an *initial condition* is given:

$$\varphi(0, \boldsymbol{x}) = g(\boldsymbol{x}), \quad x \in \Omega$$

and, furthermore, boundary conditions on $\partial\Omega$ that would fit an elliptic problem, as given by (2.26)–(2.28), with in this case $f = f(t, \boldsymbol{x})$. If singularities are to be avoided, initial and boundary conditions should agree at $t = 0$ and $\boldsymbol{x} \in \partial\Omega$. Note that at $t = T$ no condition is to be given.

Consider the following pure initial value problem for the diffusion equation (2.9):

$$\varphi_t - \varphi_{,11} = 0 \,, \quad x_1 \in \mathbb{R} \,, \quad \varphi(0, x_1) = f(x_1) \,, \quad 0 < t \leq T \,. \tag{2.32}$$

The solution is given by:

$$\varphi(t, x_1) = \frac{1}{2\sqrt{\pi|t|}} \int_{-\infty}^{\infty} f(y) \exp\{-(x_1 - y)^2/4t\} dy \,,$$

which is easily verified. Clearly, for $t \ll -1$ the solution is very sensitive to perturbations in f, so that the problem is not well-posed. This means that the condition $t > 0$ in (2.32) is essential: time cannot be reversed. This corresponds to the intuitive notion that from a smooth temperature distribution the corresponding, perhaps non-smooth, temperature distribution at a sufficiently removed earlier instant of time cannot be stably determined. Note that for all f,

$$\varphi(\infty, x_1) = \text{constant} = \int_{-\infty}^{\infty} f(y) dy \,.$$

This irreversibility of time is a hallmark of parabolic problems.

Example 2.3.4. Backward solution of the diffusion equation
Consider the diffusion equation

$$\varphi_t - \varphi_{,11} = 0, \quad 0 < t < T, \quad x_1 \in (0, 1) \,,$$

with boundary conditions

$$\varphi(t, 0) = \varphi(t, 1) = 0$$

and initial (or rather 'final') condition

$$\varphi(T, x_1) = \frac{1}{m} \sin m\pi x_1 \,.$$

By separation of variables the following solution is obtained:

$$\varphi = \frac{1}{m} \exp(m^2\pi^2(T - t)) \sin m\pi x_1 \,.$$

so that

$$\varphi(0, x_1) = \frac{1}{m} \exp(m^2 \pi^2 T) \sin m\pi x_1 ,$$

which can be made to differ from zero by an arbitrary amount by choosing m large enough. By an argument similar to that employed in Example 2.3.1, we see that the problem is ill-posed. □

The hyperbolic case will not be discussed here, but in Chap. 8. For the wave equation (2.8) one can determine which boundary conditions lead to a well-posed problem by requiring that the two functions ψ and η in the general representation (2.21) are determined uniquely. Illustrations are given in the following exercises.

Exercise 2.3.1. Using (2.21), show that boundary conditions (2.30), (2.31) lead to a well-posed problem for the wave equation (2.8).

Exercise 2.3.2. Let $\Omega = (0,1) \times (0,1)$, and consider the wave equation (2.8) with a Dirichlet condition prescribed at all of $\partial\Omega$. Using (2.21), show that in general a solution does not exist. Note that according to the theoretical results presented earlier, this boundary condition leads to a well-posed problem for the Laplace equation.

2.4 Maximum principles

Physical interpretation

In this and the following section we discuss qualitative properties of the solution of (2.1), giving a-priori information that can be used advantageously in the development of numerical approximations. It is assumed that (2.13) holds, so that (2.1) is parabolic. An intuitive idea about the behavior of solutions of (2.1) is obtained by associating with this equation a physical interpretation. For example, (2.1) models the temperature distribution in a fluid with temperature φ, heat source distribution q, velocity field $\boldsymbol{u}$ and heat conduction tensor $a_{\alpha\beta}$. If $q \leq 0$, no heating takes place and it is intuitively clear that if φ has a local maximum φ_m at $(t_m, \boldsymbol{x}_m)$, then at $t < t_m$ a value $\varphi \geq \varphi_m$ is to be found somewhere. Furthermore, large values of φ may be imposed on $\partial\Omega$ by a Dirichlet boundary condition. Hence, we arrive at the following hypothesis: local maxima can occur only for $t = 0$ and/or $\boldsymbol{x} \in \partial\Omega$. Such a *maximum principle* can be very useful. We now give it a mathematical basis. More background may be found in Protter and Weinberger (1967) and Sperb (1981).

The one-dimensional stationary case

In this case equation (2.1) may be written as, taking $c = 0$, writing D instead of a_{11} and writing u instead of b,

$$u\frac{d\varphi}{dx} - \frac{d}{dx}(D\frac{d\varphi}{dx}) = s, \quad x \in (a,b) .$$

Suppose φ has a local maximum in $x = x_m \in (a,b)$. Then $d\varphi(x_m)/dx = 0$, and $d^2\varphi(x_m)/dx^2 \leq 0$. Assuming D to be differentiable, we find

$$s = u\frac{d\varphi}{dx} - D\frac{d^2\varphi}{dx^2} - \frac{dD}{dx}\frac{d\varphi}{dx} ,$$

so that $s(x_m) \geq 0$. It follows that if $s < 0$, then φ cannot have a local maximum in (a,b). This result is strengthened in the following theorem.

Theorem 2.4.1. *One-dimensional maximum principle*
Let φ satisfy

$$u\frac{d\varphi}{dx} - \frac{d}{dx}(D\frac{d\varphi}{dx}) \leq 0, \quad x \in (0,1) , \tag{2.33}$$

with $D > 0$ differentiable and bounded and φ not constant. Then φ has a local maximum only in $x = 0$ and/or $x = 1$, and $d\varphi/dn > 0$ in a boundary point with a local maximum, with $d\varphi/dn$ the outward derivative in $x = 0$ or $x = 1$.

Proof. (cf. Sperb (1981)). We have

$$L\varphi \equiv v\frac{d\varphi}{dx} - \frac{d^2\varphi}{dx^2} \leq 0, \quad x \in (0,1) ,$$

with $v = u/D - d\ln D/dx$. Assume $\varphi(c) = M$, $0 < c < 1$, is a local maximum. Let $(a,b) \subset (0,1)$ be a neighbourhood of c in which $\varphi(x) \leq M$. Because φ is not constant it is possible to choose a and b such that $\varphi(a) < M$ and/or $\varphi(b) < M$. Assume $\varphi(b) < M$; if not, the following argument is easily repeated using $\varphi(a) < M$. Then there is a point $x = d$, $c < d < b$ with $\varphi(d) < M$. Let

$$\psi(x) = e^{\alpha(x-c)} - 1 ,$$

with $\alpha > 0$ a constant still to be determined. We have

$$L\psi = (\alpha v - \alpha^2)e^{\alpha(x-c)} .$$

Since v is bounded we can choose α such that $L\psi < 0$ in (a,d). Let

$$\theta(x) = \varphi(x) + \varepsilon\psi(x) ,$$

with ε a constant such that

$$\varepsilon < (M - \varphi(d))/\psi(d).$$

It is easily seen that $\theta(x) < M$ in (a, c), $\theta(c) = M$, $\theta(d) < M$. This means that $\theta(x)$ has a maximum $\bar{M} \geq M$ in (a, d). Furthermore,

$$L\theta = L\varphi + \varepsilon L\psi < 0 \quad \text{in} \ (a, d) .$$

But then θ cannot have a maximum in (a, d), according to the result which preceeds the theorem, and we have a contradiction. Hence, φ cannot have a local maximum in (0, 1). It remains to prove the property of $d\varphi/dn$. Suppose that $\varphi(0) = M$ and $\varphi(x) \leq M$, $x \in (0, 1)$, with $\varphi(d) < M$ for some $d \in (a, b)$. Now define

$$\psi(x) = e^{\alpha x} - 1 ,$$

and choose $\alpha > 0$ such that $L\psi < 0$ in $(0, d)$. Defining $\theta(x)$ and ε as before we have $L\theta < 0$ in $(0, d)$ and the maximum of θ in $[0, d]$ must therefore occur at $x = 0$ or $x = d$. But $\theta(0) = \varphi(0) = M > \theta(d)$ because of our choice of ε. Therefore we have

$$\frac{d\theta(0)}{dx} = \frac{d\varphi(0)}{dx} + \varepsilon\frac{d\psi(0)}{dx} = \frac{d\varphi(0)}{dx} + \varepsilon\alpha \leq 0 ,$$

which implies

$$\frac{d\varphi(0)}{dn} = -\frac{d\varphi(0)}{dx} > 0 .$$

The procedure is similar if $\varphi(1) = M$. □

In a similar way we can prove that if φ satisfies

$$u\frac{d\varphi}{dx} - \frac{d}{dx}(D\frac{d\varphi}{dx}) \geq 0, \quad x \in (0, 1) ,$$

and if the other conditions are satisfied, then φ has a local minimum only in $x = 0$ and/or $x = 1$, and $d\varphi/dn < 0$ in boundary points with a local minimum. Hence, in the frequently occurring case that the equality sign holds in (2.33), there can be no local extrema in (0, 1). This means that interior extrema occurring in numerical solutions, often called numerical wiggles, are numerical artifacts. Furthermore, if we have a homogeneous Neumann boundary condition

$$\frac{d\varphi(0)}{dn} = 0$$

then there can be no local extremum at $x = 0$, and similarly in the case of such a boundary condition at $x = 1$.

The general stationary case

We rewrite (2.2) with $c = 0$ as

$$u_\alpha \varphi_{,\alpha} - a_{\alpha\beta}\varphi_{,\alpha\beta} = s, \quad \boldsymbol{x} \in \Omega \subset \mathbb{R}^d , \tag{2.34}$$

with in every point of $\partial\Omega$ precisely one of the boundary conditions (2.26)–(2.28). By saying that $\partial\Omega$ is *smooth* we mean that for every $\boldsymbol{x} \in \partial\Omega$ there exists an open sphere in Ω that is tangent to $\partial\Omega$ in $\boldsymbol{x}$. The following theorem holds:

Theorem 2.4.2. *(Maximum principle)*
Let (2.34) hold with $s \leq 0$, let any outward derivative $\partial\varphi/\partial\nu \leq 0$ on $\partial\Omega' \subset \partial\Omega$, and let $\partial\Omega$ be smooth. Then local maxima occur only on $\partial\Omega \setminus \partial\Omega'$, or $\varphi = constant$ in Ω.

Proof. See Protter and Weinberger (1967) Sect. 2.3 Theorems 5 and 7. □

By reversing sign we obtain a similar *minimum principle.* Note that $n_\alpha a_{\alpha\beta}\varphi_{,\beta}$ (cf. (2.27)) is an example of an outward derivative. In the frequently occurring case that $s = 0$ and a homogeneous Neumann condition $\partial\varphi/\partial n = 0$ on $\partial\Omega'$, both maxima and minima, i.e. *extrema*, can only occur on $\partial\Omega \setminus \partial\Omega'$. Should extrema occur in Ω in a numerical solution, then we know that such extrema are (undesirable) numerical wiggles. Discretizations that satisfy a maximum principle similar to Theorem 2.4.2, and hence exclude numerical wiggles, are called *monotone schemes.* Monotone schemes will be developed later. The maximum principle can be used in the derivation of *global error estimates*, that we also will present later.

The general nonstationary case

Consider the following version of (2.1):

$$\varphi_t + u_\alpha \varphi_{,\alpha} - a_{\alpha\beta}\varphi_{,\alpha\beta} = a , \quad \boldsymbol{x} \in \Omega , \quad 0 < t \leq T . \tag{2.35}$$

We have

Theorem 2.4.3. *(Maximum principle)*
Let $s \leq 0$ in (2.35), let any outward derivative $\partial\varphi/\partial\nu \leq 0$ on $\partial\Omega' \subset \partial\Omega$, and let $\partial\Omega$ be smooth. Then if a local maximum occurs at $t = t^* > 0$ in the interior of Ω or on $\partial\Omega'$, then $\varphi = constant$, $0 \leq t \leq t^*$.

Proof. See Protter and Weinberger (1967) Sect. 3.3, Theorems 5 and 6. □

Hence if $\varphi \neq$ constant, local maxima can occur only at $t = 0$ and portions of the boundary where $\partial\varphi/\partial\nu > 0$.

We have formulated maximum principles for elliptic and parabolic partial differential equations. For hyperbolic equations there are maximum principles of a different type that will not be discussed; see Protter and Weinberger (1967).

2.5 Boundary layer theory

Consider the following version of (2.1):

$$\varphi_t + u_\alpha\varphi_{,\alpha} - \varepsilon\varphi_{,\alpha\alpha} = s\ , \quad \boldsymbol{x} \in \Omega\ , \quad 0 < t \leq T\ , \quad \varepsilon > 0\ . \tag{2.36}$$

Boundary layer theory or singular perturbation theory studies what happens when $\varepsilon \downarrow 0$. Equation (2.36) is parabolic, and we prescribe initial and boundary conditions in accordance with the rules of Sect. 2.3.

Let the following initial condition be given:

$$\varphi(0, \boldsymbol{x}) = \varphi_0(\boldsymbol{x})\ .$$

At every point of $\partial\Omega$ precisely one of the three types of boundary condition (2.26)–(2.28) is precribed.

The convection equation

When $\varepsilon \ll 1$ it is natural to approximate (2.36) by putting $\varepsilon = 0$. For simplicity we also take $s = 0$, and obtain the *convection equation*:

$$\varphi_t + u_\alpha\varphi_{,\alpha} = 0\ . \tag{2.37}$$

This equation is hyperbolic. Solutions cannot satisfy all of the initial and boundary conditions, as will be seen shortly.

Characteristics

Let us define curves in $(t, \boldsymbol{x})$ space by relations $t = t(s)$, $\boldsymbol{x} = \boldsymbol{x}(s)$, satisfying

$$\frac{dt}{ds} = 1, \quad \frac{dx_\alpha}{ds} = u_\alpha\ . \tag{2.38}$$

Then equation (2.37) reduces to

$$\frac{d\varphi}{ds} = 0 \ . \tag{2.39}$$

Since the curves defined by (2.38) are particle paths, equation (2.39) expresses the fact that φ is constant along particle paths. These curves are called *characteristics* in the theory of hyperbolic systems.

One-dimensional case

Although the one-dimensional case is very simple, we will dwell a bit upon it, in order to elucidate the principles of singular perturbation theory. These principles will be applied subsequently in a more general setting.

Let $\Omega = (0, 1)$. Equation (2.36) reduces to, taking $s = 0$,

$$\varphi_t + u_1\varphi_{,1} - \varepsilon\varphi_{,11} = 0, \quad 0 < x_1 < 1, \quad 0 < t \leq T \ . \tag{2.40}$$

We assume the following initial and boundary conditions:

$$\varphi(0, x_1) = \varphi_0(x_1), \quad 0 < x_1 < 1 \ , \tag{2.41}$$

$$\varphi(t, 0) = f_0(t), \quad \varphi(t, 1) = f_1(t), \quad 0 < t \leq T \ . \tag{2.42}$$

For $\varepsilon \ll 1$ one would expect that equation (2.40) can be approximated by

$$\varphi_t + u_1\varphi_{,1} = 0 \ . \tag{2.43}$$

Solutions of (2.43) are constant along the characteristics. Figure 2.1 gives a sketch of the characteristics of (2.43) (as defined by (2.38)), assuming $u_1 < 0$. Consider the characteristic C_1. On C_1, $\varphi = \varphi(C_1) =$ constant. In P_1 we have

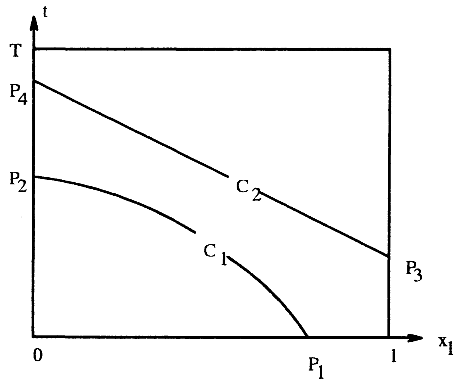

Fig. 2.1. Characteristics for equation (2.43).

the initial condition (2.41), in P_2 the first of the boundary conditions (2.42).

Only one of them can be satisfied in general. What value to take for $\varphi(C_1)$ such that the solution of (2.40)–(2.42) is approximated? The difficulty has to do with the change of type that (2.36) undergoes when ε is replaced by zero. For $\varepsilon = 0$, (2.36) is hyperbolic; for $\varepsilon > 0$, it is parabolic.

Singular perturbation theory

The answer to the foregoing question is provided by *singular perturbation theory.* Introductions to this subject are given in Eckhaus (1973), Kevorkian and Cole (1981) and Van Dyke (1975). In the present case, $\varphi = \varphi(C_1) = \varphi(P_1)$ is a good approximation for $\varepsilon \ll 1$ to the solution of (2.40)–(2.42) in $1 \geq x_1 > \delta = \mathcal{O}(\varepsilon)$, whereas (2.43) has to be replaced by a so-called *boundary layer equation* to obtain an approximation in $\delta > x_1 \geq 0$. This can be seen as follows. First, assume that we indeed have $\varphi(C_1) = \varphi(P_1)$ in $1 \geq x_1 > \delta$ with $\delta \ll 1$. In $\delta > x_1 \geq 0$ we expect a rapid change of φ from $\varphi(P_1)$ to $\varphi(P_2)$. For derivatives of φ we expect

$$\frac{\partial^m \varphi}{\partial x_1^m} = \mathcal{O}(\delta^{-m}) \,, \tag{2.44}$$

so that perhaps the diffusion term in (2.40) cannot be neglected in the boundary layer; this will depend on the size of δ. Assume

$$\delta = \mathcal{O}(\varepsilon^{\alpha}) \,, \tag{2.45}$$

with α to be determined. In order to exhibit the dependence of the magnitude of derivatives on ε we introduce a *stretched coordinate* $\tilde{x}_1$:

$$\tilde{x}_1 = x_1 \varepsilon^{-\alpha} \,, \tag{2.46}$$

which is chosen such that $\tilde{x}_1 = \mathcal{O}(1)$ in the boundary layer. It follows from (2.44)–(2.46) that

$$\frac{\partial^m \varphi}{\partial \tilde{x}_1^m} = \mathcal{O}(1) \tag{2.47}$$

in the boundary layer. In the stretched coordinate equation (2.40) becomes:

$$\varphi_t + \varepsilon^{-\alpha} u_1 \frac{\partial \varphi}{\partial \tilde{x}_1} - \varepsilon^{1-2\alpha} \frac{\partial^2 \varphi}{\partial \tilde{x}_1^2} = 0 \,. \tag{2.48}$$

Letting $\varepsilon \downarrow 0$ and using (2.47), equation (2.48) takes various forms, depending on α. The correct value of α follows from the requirement, that the solution of the $\varepsilon \downarrow 0$ limit of equation (2.48) satisfies the boundary condition at $x_1 = 0$, and the so-called matching principle.

Matching principle

As $\tilde{x}_1$ increases, the solution of (the $\varepsilon \downarrow 0$ limit of) equation (2.48) has to somehow join up with the solution of (2.43), i.e. approach the value $\varphi(C_1)$. In singular perturbation theory this condition is formulated precisely, and is known as the *matching principle*:

$$\lim_{\tilde{x}_1 \to \infty} \varphi_{\text{inner}}(t, \tilde{x}_1) = \lim_{x_1 \downarrow 0} \varphi_{\text{outer}}(t, x) .$$

Here φ_{inner}, also called the *inner solution*, is the solution of the *inner equation* or *boundary layer equation*, which is the limit as $\varepsilon \downarrow 0$ of equation (2.48) for the correct value of α, which we are trying to determine. Furthermore, φ_{outer}, also called the *outer solution*, is the solution of the *outer equation*, which is the limit as $\varepsilon \downarrow 0$ of the original equation, i.e. equation (2.43). In our case the matching principle becomes

$$\lim_{\tilde{x}_1 \to \infty} \varphi_{\text{inner}}(t, \tilde{x}_1) = \varphi(C_1) . \tag{2.49}$$

As already mentioned, the other condition to be satisfied is the boundary condition at $\tilde{x}_1 = 0$:

$$\varphi(t, 0) = f_0(t) . \tag{2.50}$$

For $\alpha < 0$ (corresponding to compression rather than stretching) the limit as $\varepsilon \downarrow 0$ of (2.48) is

$$\varphi_t = 0 . \tag{2.51}$$

Obviously, the solution of (2.51) cannot satisfy (2.50), so that the case $\alpha < 0$ has to be rejected. With $\alpha = 0$ equation (2.43) is obtained, which cannot satisfy both conditions at $t = 0$ and $x_1 = \tilde{x}_1 = 0$, as we saw.

For $0 < \alpha < 1$ the limit of (2.48) is

$$u_1 \frac{\partial \varphi}{\partial \tilde{x}_1} = 0 ,$$

so that the inner solution is given by

$$\varphi(t, \tilde{x}_1) = g(t) ,$$

and in general equations (2.49) and (2.50) cannot be satisfied, so that we cannot have $0 < \alpha < 1$.

For $\alpha = 1$ equation (2.48) becomes as $\varepsilon \downarrow 0$:

$$u_1(t, 0) \frac{\partial \varphi}{\partial \tilde{x}_1} - \frac{\partial^2 \varphi}{\partial \tilde{x}_1^2} = 0 , \tag{2.52}$$

where we have used that $u_1(t,x) = u_1(t, \varepsilon\tilde{x}_1) \to u_1(t,0)$ as $\varepsilon \downarrow 0$. The general solution of (2.52) is

$$\varphi = A(t) + B(t)e^{u\tilde{x}_1} , \tag{2.53}$$

with $u = u_1(t,0)$. We can satisfy both (2.49) and (2.50), remembering that we had assumed $u_1 < 0$, with

$$A(t) = \varphi(C_1), \quad B(t) = f_0(t) - \varphi(C_1) .$$

This gives us the inner solution. In terms of the unstretched variable x_1 the inner solution is given by

$$\varphi = \varphi(C_1) + \{f_1(t) - \varphi(C1)\}e^{ux_1/\varepsilon} .$$

We see a rapid exponential variation from $f_1(t)$ to $\varphi(C_1)$ in a thin layer of thickness $\delta = \mathcal{O}(\varepsilon)$, confirming our earlier statement about the behavior of the solution. Fig. 2.2 gives a sketch of the inner and outer solutions as a function of x_1. An asymptotic approximation for $\varepsilon \downarrow 0$ that is valid everywhere is given by $\varphi_{\text{inner}} + \varphi_{\text{outer}} - \varphi(C_1)$ (not shown in the figure).

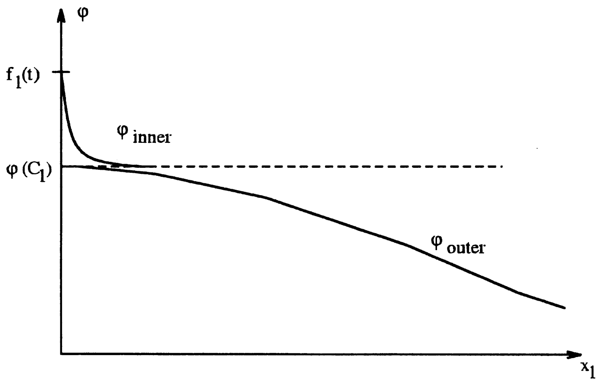

Fig. 2.2. Sketch of inner and outer solutions

The distinguished limit

The limit as $\varepsilon \downarrow 0$ of the stretched equation (2.48) for the special value of $\alpha = 1$ for which the solution of the resulting inner equation can satisfy both the boundary condition and the matching principle is called the *distinguished limit.* In order to show that this limit is unique we will also investigate the remaining values of α that we did not yet consider, namely $\alpha > 1$. Now equation (2.48) gives the following inner equation:

$$\frac{\partial^2 \varphi}{\partial \tilde{x}_1^2} = 0 \, ,$$

with the general solution

$$\varphi = A(t) + B(t)\tilde{x}_1 \, .$$

The limit of φ as $\tilde{x}_1 \to \infty$ does not exist, so that the matching principle cannot be satisfied. Hence, $\alpha = 1$ is the only value that gives a distinguished limit.

The only element of arbitrariness that remains in this analysis is the assumption that we have a boundary layer at $x_1 = 0$. Why no boundary layer at $t = 0$, and $\varphi(C_1) = \varphi(P_2)$ (cf. Fig. 2.1)? This can be investigated by assuming a boundary layer at $t = 0$, and determining whether a distinguished limit exists or not. This is left as an exercise. It turns out there is no boundary layer at $t = 0$. Hence, singular perturbation theory determines uniquely the asymptotic behavior of the solution along the characteristic C_1.

It is left to the reader to verify in a similar way that along characteristic C_2 (cf. Fig. 2.1), which does not originate in $t = 0$ but at the boundary $x_1 = 1$, we have $\varphi(C_2) = \varphi(P_3)$, except in a boundary layer with thickness $\mathcal{O}(\varepsilon)$ at $x_1 = 0$. When $u_1 > 0$ there is a boundary layer at $x_1 = 1$, and no boundary layer at $x_1 = 0$. We see that *boundary layers will not be present at inflow boundaries* (i.e. parts of the domain boundary $\partial\Omega$ where the flow enters the domain, that is where $\boldsymbol{u} \cdot \boldsymbol{n} < 0$ with $\boldsymbol{n}$ the outward normal on $\partial\Omega$). This holds in any number of space dimensions.

The role of boundary conditions

The occurrence of boundary layers is strongly influenced by the type of boundary condition. Let (2.42) be replaced by

$$\frac{\partial \varphi(t,0)}{\partial x_1} = f_0(t), \quad \varphi(t,1) = f_1(t), \quad 0 < t \leq T \, . \tag{2.54}$$

As before, a boundary layer of thickness $\mathcal{O}(\varepsilon)$ is found at $x_1 = 0$, and the boundary layer equation is given by (2.52), with solution (2.53). Taking boundary condition (2.54) into account we find

$$B(t) = \varepsilon f_0(t)/u \, ,$$

so that $B(t) \to 0$ as $\varepsilon \downarrow 0$. Hence, to first order, there is no boundary layer, and the outer solution (solution of (2.43)) is uniformly valid in $0 < x_1 < 1$.

Two-dimensional case

Let $\Omega = (0, 1 \times (0, 1)$. If we take $s = 0$, and consider the time-independent case for brevity, equation (2.36) reduces to

$$u_\alpha \varphi_{,\alpha} - \varepsilon \varphi_{,\alpha\alpha} = 0, \quad \boldsymbol{x} \in (0,1) \times (0,1) \ . \tag{2.55}$$

Assume Dirichlet boundary conditions:

$$\begin{aligned} \varphi(x_1, 0) &= f_1(x_1), \quad \varphi(1, x_2) = f_2(x_2), \quad \varphi(x_1, 1) = f_3(x_1), \\ \varphi(0, x_2) &= f_4(x_2) \ . \end{aligned} \tag{2.56}$$

For $\varepsilon \downarrow 0$ we obtain the following outer equation:

$$u_\alpha \varphi_{,\alpha} = 0 \ . \tag{2.57}$$

This is a hyperbolic equation, with characteristics defined by

$$\frac{dx_\alpha}{ds} = u_\alpha, \quad \alpha = 1, 2 \ . \tag{2.58}$$

For a sketch of possible characteristics we can reuse Fig. 2.1 by replacing t with x_2. Along C_1, $\varphi = \varphi(C_1) =$ constant. Again the question arises, whether $\varphi(C_1)$ will take on the prescribed value $\varphi(P_1)$ or $\varphi(P_2)$. First, let us postulate a boundary layer at $x_1 = 0$. Reasoning as before, we find the following boundary layer equation:

$$u \frac{\partial \varphi}{\partial \tilde{x}_1} - \frac{\partial^2 \varphi}{\partial \tilde{x}_1^2} = 0, \quad \tilde{x}_1 = x_1/\varepsilon, \quad u = u_1(0, x_2) \ . \tag{2.59}$$

The following solution satisfies both the boundary condition at $x_1 = 0$ and the matching principle, *provided* $u < 0$:

$$\varphi(\tilde{x}_1, x_2) = \varphi(P_1) + \{f_4(x_2) - \varphi(P_1)\} e^{u \tilde{x}_1} \ .$$

In case we have a Neumann boundary condition at $x_1 = 0$ there is no boundary layer at $x_1 = 0$. As in the one-dimensional case, when $x_1 = 0$ is an inflow boundary ($u > 0$) there is also no boundary layer at $x_1 = 0$.

Parabolic and ordinary boundary layers

Next, consider the boundary $x_2 = 0$. If it is an outflow boundary, i.e. $v \equiv u_2(x_1, 0) < 0$, then there is again a boundary layer of thickness $\mathcal{O}(\varepsilon)$ in case of a Dirichlet boundary condition. The boundary layer solution is found to be

$$\varphi(x_1, \tilde{x}_2) = \varphi(P_2) + \{f_1(x_1) - \varphi(P_2)\} e^{v \tilde{x}_2}, \quad \tilde{x}_2 = x_2/\varepsilon \ . \tag{2.60}$$

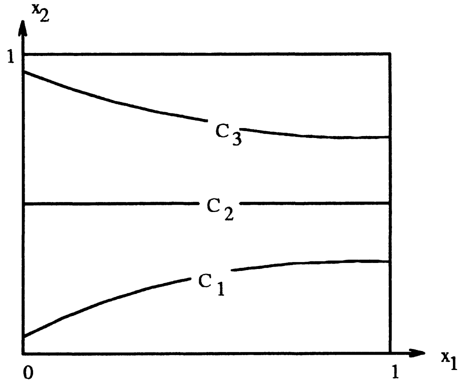

Fig. 2.3. Characteristics of equation (2.57) in a channel flow

Now consider the case that $x_2 = 0$ is a solid wall, so that $v = 0$. The shape of the characteristics of the outer equation (2.57) might be as in Fig. 2.3, where also $x_2 = 1$ is assumed to be a solid wall, so that we have a channel flow. Since $u_2(x_1, 0) = 0$, the curve $x_2 = 0$ is a characteristic of the outer equation (2.57) according to (2.58), so that the solution along this characteristic is given by

$$\varphi(x_1, 0) = f_4(0) \tag{2.61}$$

if $x_1 = 0$ is a inflow boundary, or by

$$\varphi(x_1, 0) = f_2(0) \tag{2.62}$$

if $x_1 = 1$ is an inflow boundary. To verify this using singular perturbation theory is left as an exercise. If both $x_1 = 0$ and $x_1 = 1$ are inflow boundaries the pattern of characteristics has to be qualitatively different from Fig. 2.3, assuming that $u_{\alpha,\alpha} = 0$ (incompressible flow field). This situation will not be considered here. Whether we have (2.61) or (2.62), in both cases the outer solution cannot satisfy boundary condition (2.56) at $x_2 = 0$. Hence, we expect a boundary layer at $x_2 = 0$. Obviously, this boundary layer will be of different type than obtained until now, because the boundary layer solution cannot be given by (2.60), since now we have $v = 0$. In order to derive the boundary layer equation, the same procedure is followed as before. We transform (2.55) to the stretched coordinate $\tilde{x}_2 = x_2 \varepsilon^{-\alpha}$, with α to be determined. Equation (2.55) becomes, with $u_2 = 0$:

$$u_1 \frac{\partial \varphi}{\partial x_1} - \varepsilon \frac{\partial^2 \varphi}{\partial x_1^2} - \varepsilon^{1-2\alpha} \frac{\partial^2 \varphi}{\partial \tilde{x}_2^2} = 0 \, . \tag{2.63}$$

The boundary condition is

$$\varphi(x_1, 0) = f_1(x_1) \, , \tag{2.64}$$

and the matching principle gives

$$\lim_{\tilde{x}_2 \to \infty} \varphi(x_1, \tilde{x}_2) = \lim_{x_2 \downarrow 0} \varphi_0(x_1, x_2) . \tag{2.65}$$

Now we take the limit of (2.63) as $\varepsilon \downarrow 0$. For $\alpha < 1/2$ the outer equation at $x_2 = 0$ is recovered:

$$u_1 \frac{\partial \varphi}{\partial x_1} = 0 ,$$

of which the solution obviously cannot satisfy (2.64) and (2.65). For $\alpha = 1/2$ the limit of (2.63) is

$$u \frac{\partial \varphi}{\partial x_1} - \frac{\partial^2 \varphi}{\partial \tilde{x}_2^2} = 0 , \tag{2.66}$$

with $u = u(x_1) = \lim_{\varepsilon \to 0} u_1(x_1, \tilde{x}_2 \sqrt{\varepsilon}) = u_1(x_1, 0)$. This is a parabolic partial differential equation, which in general cannot be solved explicitly, but for which it is known that boundary conditions at $\tilde{x}_2 = 0$ and $\tilde{x}_2 = \infty$ give a well-posed problem. An analytic expression for the solution of the boundary layer equation (2.66) with $u = 1$ will be given in Sect. 4.7. Hence, $\alpha = 1/2$ gives the distinguished limit, and (2.66) is the boundary layer equation. The thickness of this type of boundary layer is $\mathcal{O}(\sqrt{\varepsilon})$, which is much larger than for the preceding type.

In order to specify a unique solution, in addition an 'initial' condition has to be specified. Assuming $u > 0$, this has to be done at $x_1 = 0$. From (2.56) we obtain the following initial condition for the boundary layer solution:

$$\varphi(0, \tilde{x}_2) = f_4(\tilde{x}_2 \sqrt{\varepsilon}) ,$$

which to the present asymptotic order of approximation (we will not go into higher order boundary layer theory) may be replaced by

$$\varphi(0, \tilde{x}_2) = f_4(0) .$$

It is left to the reader to verify that $\alpha > 1/2$ does not give a distinguished limit.

The cause of the difference between the two boundary layer equations (2.59) (an ordinary differential equation) and (2.66) (a partial differential equation) is the angle which the characteristics of the outer equation (2.57) make with the boundary layer. In the first case this angle is non-zero (cf. Fig. 2.1), in the second case the characteristics do not intersect the boundary layer. The first type is called an *ordinary boundary layer* (the boundary layer equation is an ordinary differential equation), whereas the second type is called a *parabolic boundary layer* (parabolic boundary layer equation).

Summarizing, in the case of the channel flow depicted in Fig. 2.3, for $\varepsilon \ll 1$ there are parabolic boundary layers of thickness $\mathcal{O}(\sqrt{\varepsilon})$ at $x_2 = 0$ and $x_2 = 1$, and an ordinary boundary layer of thickness $\mathcal{O}(\varepsilon)$ at the outflow boundary, unless a Neumann boundary condition is prescribed there.

On outflow boundary conditions

It frequently happens that physically no outflow boundary condition is known, but that this is required mathematically. If $\varepsilon = \mathcal{O}(1)$ such a physical model is incomplete, but for $\varepsilon \ll 1$ an artificial (invented) outflow condition may safely be used to complete the mathematical model, because this does not affect the solution to any significant extent. Furthermore, an artificial condition of Neumann type is to be preferred above one of Dirichlet type. This may be seen as follows.

Consider the following physical situation: an incompressible flow with given velocity field $\boldsymbol{u}$ through a channel, the walls of which are kept at a known temperature. We want to know the temperature of the fluid, especially at the outlet. This leads to the following mathematical model. The governing equation is (2.55), with φ the temperature. Assume $\varepsilon \ll 1$, and $u_1 > 0$. We have φ prescribed at $x_1 = 0$ and at $x_2 = 0, 1$, but at $x_1 = 1$ we know nothing. Hence, we cannot proceed with solving (2.55), either analytically or numerically. Now let us just postulate some temperature profile at $x_1 = 1$:

$$\varphi(1, x_2) = f_2(x_2).$$

An ordinary boundary layer will develop at $x_1 = 1$, with solution, derived in the way discussed earlier, given by

$$\varphi(x_1, x_2) = \varphi_0(1, x_2) + \{f_2(x_2) - \varphi_0(1, x_2)\}e^{u(x_1-1)\varepsilon} \ . \tag{2.67}$$

This shows that the invented temperature profile $f_2(x_2)$ influences the solution only in the thin (artificially generated) boundary layer at $x_1 = 1$. This means that the computed temperature outside this boundary layer will be correct, regardless what we take for $f_2(x_2)$. When $\varepsilon = \mathcal{O}(1)$ this is no longer true, and more information from physics is required. In physical reality there will not be a boundary layer at all at $x_1 = 1$, of course. Therefore a more satisfactory artificial outflow boundary condition is

$$\frac{\partial \varphi(1, x_2)}{\partial x_1} = 0 \ ,$$

since with this Neumann boundary condition there will be no boundary layer at $x_1 = 1$ in the mathematical model.

Exercise 2.5.1. Consider equations (2.40)–(2.42) with $u_1 > 0$. Show that for $\varepsilon \ll 1$ there is a boundary layer of thickness $\mathcal{O}(\varepsilon)$ at and only at $x_1 = 1$.

Exercise 2.5.2. Consider equations (2.55) and (2.56), with the Dirichlet boundary condition at $x_1 = 0$ replaced by a homogeneous Neumann boundary condition. Show that when $u_1 < 0$ and $\varepsilon \ll 1$ there is no boundary layer at $x_1 = 0$.

Exercise 2.5.3. Derive equation (2.67).

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