Chapter 10
Palais and Kobayashi Theorems

§1. Infinite-Dimensional Manifolds and Lie Groups

In general, the concept of differentiability (smoothness) does not require finite dimensionality and can be defined for mappings of open sets of an arbitrary linear topological space $\mathcal{E}$. This allows defining *smooth manifolds with charts in* $\mathcal{E}$ in an obvious way: it suffices to replace open sets of the space $\mathbb{R}^n$ with those of the space $\mathcal{E}$ everywhere in the usual definition of a smooth manifold (see the addendum). We obtain Hilbert, Banach, locally convex, etc., manifolds depending on the type of the space $\mathcal{E}$. All such manifolds are conventionally called *infinite-dimensional manifolds*, although this term seems not very appropriate. The theory of such manifolds (under one or another condition on the space $\mathcal{E}$) almost literally repeats the finite-dimensional smooth manifold theory in its initial part, but, for example, a smooth vector field on an infinite-dimensional manifold might have no integral curves.

Infinite-dimensional Lie groups are defined as usual, as manifolds $\mathcal{G}$ with a smooth multiplication $\mathcal{G}\times\mathcal{G} \to \mathcal{G}$. The *Lie algebra* $\mathfrak{l}\mathcal{G}$ of each such group is also defined as usual (this algebra as a linear space is isomorphic to the base space $\mathcal{E}$). For *Banach Lie groups* (over a Banach space $\mathcal{E}$), the theory can be well developed in parallel to finite-dimensional Lie group theory. For more general groups, the parallelism is violated, and little is known about them. It is even unknown if there always exist one-parameter subgroups with a given initial vector, i.e., if the exponential mapping $\mathfrak{l}\mathcal{G} \to \mathcal{G}$ is always defined (although a counterexample is also apparently unknown). If the exponential mapping exists, it is not injective in general and does not cover any neighborhood of the identity.

Of course, the absence of the general theory does not prevent the study of one or another class of infinite-dimensional Lie groups (moreover, it even stimulates special attention to them). For example, the theory of *current groups*, whose elements are smooth mappings $\mathcal{X} \to \mathcal{G}$ of a given compact manifold $\mathcal{X}$ (the case $\mathcal{X} = \mathbb{S}^1$ is the most interesting and well developed) into a given finite-dimensional group $\mathcal{G}$, has been intensively developed recently. Another important class of infinite-dimensional non-Banach Lie groups, about which essentially less is known, consists of diffeomorphism groups $\operatorname{Diff}\mathcal{X}$ of finite-dimensional smooth manifolds $\mathcal{X}$.

The Lie algebra of the group $\operatorname{Diff}\mathcal{X}$ is the algebra $\mathfrak{a}\mathcal{X}$ of all vector fields on $\mathcal{X}$, and the Lie algebras of subgroups of the group $\operatorname{Diff}\mathcal{X}$ are therefore subalgebras of the algebra $\mathfrak{a}\mathcal{X}$. As in the case of finite-dimensional Lie groups, it is natural to expect the existence (probably, with certain conditions) of a bijective correspondence between subgroups of the group $\operatorname{Diff}\mathcal{X}$ and subalgebras of the algebra $\mathfrak{a}\mathcal{X}$. We consider this question for finite-dimensional subgroups (i.e., for subgroups that are usual Lie groups) and finite-dimensional algebras for which the study can be entirely performed in the framework of finite-dimensional manifolds.

Of course, the restriction to only finite-dimensional manifolds distorts the whole picture and excludes the true perspective from the presentation, but we must follow this line of reasoning. In what follows, we do not explicitly mention infinite-dimensional manifolds, and their theme continues only sotto voce. In particular, we do not introduce a topology and smoothness on the group Diff $\mathcal{X}$.

§2. Vector Fields Induced by a Lie Group Action

If a Lie group $\mathcal{G}$ acts smoothly and effectively on a smooth manifold $\mathcal{X}$, then for any element $a \in \mathcal{G}$, the mapping

$$L_a\colon p \mapsto ap, \quad p \in \mathcal{X},$$

is a diffeomorphism of the manifold $\mathcal{X}$ onto itself, and the correspondence $L\colon a \mapsto L_a$ is a monomorphic mapping of the group $\mathcal{G}$ into the group Diff $\mathcal{X}$ of all diffeomorphisms $\mathcal{X} \to \mathcal{X}$. Therefore, the group $\mathcal{G}$ can be considered as a subgroup of the (abstract) group Diff $\mathcal{X}$. Under this embedding, each one-parameter subgroup $\beta_X\colon t \mapsto \exp tX$, $X \in \mathfrak{l}\mathcal{G}$, of the group $\mathcal{G}$ turns out to be a flow on the manifold $\mathcal{X}$ that is defined for all $t \in \mathbb{R}$. The vector field generating this flow is denoted by $-X^*$. (We emphasize the sign: to avoid introducing it, right-invariant instead of left-invariant fields X should be considered.) As a derivation of the algebra of smooth functions $\mathsf{F}\mathcal{X}$, the field X^* acts according to the formula

$$(X^*f)(p) = \lim_{t\to 0} \frac{f(p) - f((\exp tX)p)}{t}, \quad p \in \mathcal{X}, \quad f \in \mathsf{F}\mathcal{X}.$$

Therefore, for any elements X and Y of the Lie algebra, we have

$$\begin{aligned}(X^*Y^*f)(p) &= \lim_{s\to 0} \frac{(Y^*f)(p) - (Y^*f(\exp sX)p)}{s} \\ &= \lim_{\substack{s\to 0\\ t\to 0}} \frac{f(p) - f((\exp tY)p - f((\exp sX)p) + f((\exp tY)(\exp sX)p)}{st} \\ &= \lim_{t\to 0} \frac{f((\exp tY)(\exp tX)p) - f((\exp tX)p) - f((\exp tY)p) + f(p)}{t^2}\end{aligned}$$

and therefore

$$\begin{aligned}([X^*, Y^*]f)(p) &= (X^*Y^*f)(p) - (Y^*X^*f)(p) \\ &= -\lim_{t\to 0} \frac{f((\exp tX)(\exp tY)p) - f((\exp tY)(\exp tX)p)}{t^2}.\end{aligned}$$

Let $t^1, \dots, t^m$ be normal coordinates in a neighborhood of the identity of the group $\mathcal{G}$ that correspond to a basis $X_1, \dots, X_m$ of the Lie algebra $\mathfrak{g}$, and let $x^1, \dots, x^n$ be local coordinates on $\mathcal{X}$ defined in a neighborhood of the point p and vanishing at p. Further, let $f = f(x)$ be the expression of the function f in the coordinates $x^1, \dots, x^n$, and let $x = x(t)$ be the vector-valued function assigning the mapping $a \mapsto ap$ in the coordinates $t^1, \dots, t^m$ and $x^1, \dots, x^n$. Because $ep = p$, the components $x^i(t)$, $i = 1, \dots, n$, of the latter function have the form

$$x^i(t) = c^i_a t^a + c^i_{ab} t^a t^b + \dots,$$

where the dots denote terms of degree ≥ 3 with respect to $t^1, \dots, t^m$ (and $c^i_{ab} = c^i_{ba}$). Therefore, for any element $X = t^a X_a$ of the Lie algebra $\mathfrak{g}$, we have

$$\begin{aligned} f((\exp X)p) &= f(x(t)) \\ &= f(p) + \left(\frac{\partial f}{\partial x^i}\right)_p x^i(t) + \left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)_p x^i(t) x^j(t) + \cdots \\ &= f(p) + \left(\frac{\partial f}{\partial x^i}\right)_p (c^i_a t^a + c^i_{ab} t^a t^b + \cdots) \\ &\quad + \left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)_p (c^i_a c^j_b t^a t^b + \dots) + \dots, \end{aligned}$$

and therefore

$$(X^* f)(p) = \lim_{t \to 0} \frac{f(p) - f((\exp tX)p)}{t} = -\left(\frac{\partial f}{\partial x^i}\right)_p c^i_a X^a,$$

where we write X^a instead of t^a. Moreover (see (11) and (12) in Chap. 7), we have

$$f((\exp tX)(\exp tY)p) = f(\exp(t(X+Y) + \frac{1}{2}t^2[X,Y] + \cdots)p),$$

which also implies (in a clear notation)

$$\begin{aligned} f((\exp tX)(\exp tY)p) = f(p) &+ t\left(\frac{\partial f}{\partial x^i}\right)_p c^i_a (X^a + Y^a) \\ &+ t^2 \left[\frac{1}{2}\left(\frac{\partial f}{\partial x^i}\right)_p c^i_a [X,Y]^a \right. \\ &+ \left(\frac{\partial f}{\partial x^i}\right)_p c^i_{ab} (X^a + Y^a)(X^b + Y^b) \\ &+ \left. \left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)_p c^i_a c^j_b (X^a + Y^a)(X^b + Y^b)\right] + \dots, \end{aligned}$$

where, as above, the dots stand for terms of degree ≥ 3 with respect to t, and therefore

$$
\begin{aligned}
&f((\exp tX)(\exp tY)p)\\
&\quad - f((\exp tY)(\exp tX)p) = \frac{t^2}{2}\left(\frac{\partial f}{\partial x^i}\right)_p c_a^i([X,Y]^a - [Y,X]^a) + \cdots\\
&\qquad = t^2\left(\frac{\partial f}{\partial x^i}\right)_p c_a^i[X,Y]^a + \cdots\\
&\qquad = -t^2([X,Y]^* f)(p) + \cdots.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
([X^*,Y^*]f)(p) &= -\lim_{t\to 0}\frac{f((\exp tX)(\exp tY)p) - f((\exp tY)(\exp tX)p)}{t^2}\\
&= ([X,Y]^* f)(p).
\end{aligned}
$$

This proves that $[X,Y]^* = [X^*,Y^*]$, i.e., that *the correspondence* $X \mapsto X^*$ *is a homomorphism of the Lie algebra* $\mathfrak{g} = \mathfrak{l}\mathcal{G}$ *of the Lie group* $\mathcal{G}$ *into the Lie algebra* $\mathfrak{a}\mathcal{X}$ *of vector fields on* $\mathcal{X}$.

If $X^* = 0$, then $\exp(-tX)p = p$ for any point $p \in \mathcal{X}$ and any $t \in \mathbb{R}$; because of the effectiveness of the action of the group $\mathcal{G}$ on the manifold $\mathcal{X}$, this is possible only for $X = 0$. Therefore, *the homomorphism* $X \mapsto X^*$ *is a monomorphism.*

This means that identifying X with X^*, we can (and do) consider the Lie algebra $\mathfrak{g}$ a subalgebra of the Lie algebra $\mathfrak{a}\mathcal{X}$. We stress that in contrast to the whole algebra $\mathfrak{a}\mathcal{X}$, this subalgebra is finite dimensional. Moreover, it consists of *complete fields*, i.e., those fields for which the corresponding maximal flow is defined for all $t \in \mathbb{R}$. It turns out that these properties completely characterize subalgebras of the Lie algebra $\mathfrak{a}\mathcal{X}$ that are Lie algebras of Lie groups acting smoothly and effectively on the manifold $\mathcal{X}$. Moreover, this assertion can be slightly strengthened.

§3. Palais Theorem

A subset S of a Lie algebra $\mathfrak{g}$ *generates* $\mathfrak{g}$ if each subalgebra of the algebra $\mathfrak{g}$ containing S coincides with $\mathfrak{g}$. Similarly, a set S *linearly generates* $\mathfrak{g}$ if it contains a basis of the algebra $\mathfrak{g}$. For each complete field $X \in \mathfrak{a}\mathcal{X}$, the flow on $\mathcal{X}$ induced by this field is denoted by $\{\varphi_t^X\}$.

Theorem 10.1. *Let* $\mathfrak{g}$ *be a subalgebra of the Lie algebra* $\mathfrak{a}\mathcal{X}$, *and let* $\mathcal{G}$ *be a subgroup of the group* Diff $\mathcal{X}$ *generated by all diffeomorphisms of the form* φ_t^X, *where* $t \in \mathbb{R}$ *and* X *is an arbitrary complete field from* $\mathfrak{g}$. *If the Lie algebra* $\mathfrak{g}$

1. *is finite dimensional and*
2. *is generated by a set consisting of only complete fields,*

then the group $\mathcal{G}$ admits a smoothness with respect to which it is a connected Lie group that acts smoothly and effectively on the manifold $\mathcal{X}$; moreover, the monomorphism $X \mapsto X^*$ corresponding to this action is an isomorphism of the algebra $\mathfrak{l}\mathcal{G}$ onto the subalgebra $\mathfrak{g}$.

Proof. According to the Cartan theorem, there exists a connected and simply connected Lie group $\widetilde{\mathcal{G}}$ whose Lie algebra $\mathfrak{l}\widetilde{\mathcal{G}}$ is isomorphic to the algebra $\mathfrak{g}$. An element of the Lie algebra $\mathfrak{l}\widetilde{\mathcal{G}}$ (left-invariant vector field on $\widetilde{\mathcal{G}}$) corresponding to the vector field $X \in \mathfrak{g}$ under this isomorphism is denoted by $\widetilde{X}$.

At each point $(a,p) \in \widetilde{\mathcal{G}} \times \mathcal{X}$ of the direct product $\widetilde{\mathcal{G}} \times \mathcal{X}$, the tangent space $\mathbf{T}_{(a,p)}(\widetilde{\mathcal{G}} \times \mathcal{X})$ is naturally decomposed into the direct sum of the tangent spaces $\mathbf{T}_a\widetilde{\mathcal{G}}$ and $\mathbf{T}_p\mathcal{X}$. Therefore, each field $X \in \mathfrak{g}$ defines the vector $(\widetilde{X}_a, X_p)$ in $\mathbf{T}_{(a,p)}(\widetilde{\mathcal{G}} \times \mathcal{X})$ (we recall that $\mathfrak{g} \subset \mathfrak{a}\mathcal{X}$), and all such vectors form a subspace $\mathcal{D}_{(a,p)}$ in $\mathbf{T}_{(a,p)}(\widetilde{\mathcal{G}} \times \mathcal{X})$.

Exercise 10.1. Show that the following statements hold:

1. The subspaces $\mathcal{D}_{(a,p)}$ smoothly depend on the point (a,p), i.e., they compose a distribution $\mathcal{D}$ on $\widetilde{\mathcal{G}} \times \mathcal{X}$.
2. The distribution $\mathcal{D}$ is involutive.

[*Hint*: The vector fields $(a,p) \mapsto (\widetilde{X}_a, X_p)$ generate a $\mathbf{F}(\widetilde{\mathcal{G}} \times \mathcal{X})$-module $\mathfrak{a}\mathcal{D}$.]

Therefore, according to the Frobenius theorem, a unique maximal integral submanifold of the distribution $\mathcal{D}$ passes through any point $(a,p) \in \widetilde{\mathcal{G}} \times \mathcal{X}$. For brevity, these integral manifolds are called *leaves.* We note that by definition *each leaf is a connected submanifold of the manifold* $\widetilde{\mathcal{G}} \times \mathcal{X}$ (in general, it is only immersed). A leaf passing through the point (e,p), where e is the identity of the group $\widetilde{\mathcal{G}}$, is denoted by $\mathcal{L}_p$.

The formula

$$a(b,p) = (ab,p), \quad a,b \in \widetilde{\mathcal{G}}, \quad p \in \mathcal{X},$$

obviously defines a smooth and effective (even free) action of the group $\widetilde{\mathcal{G}}$ on the manifold $\widetilde{\mathcal{G}} \times \mathcal{X}$. For any element $a \in \widetilde{\mathcal{G}}$, the corresponding left translation

$$L_a\colon (b,p) \mapsto (ab,p), \quad (b,p) \in \widetilde{\mathcal{G}} \times \mathcal{X},$$

is such that its differential

$$(dL_a)_{(b,p)}\colon \mathbf{T}_{(b,p)}(\widetilde{\mathcal{G}} \times \mathcal{X}) \to \mathbf{T}_{(ab,p)}(\widetilde{\mathcal{G}} \times \mathcal{X})$$

acts on the vectors $(\widetilde{X}_b, X_p)$ by

$$(dL_a)_{(b,p)}(\widetilde{X}_b, X_p) = ((dL_a)_b\widetilde{X}_b, X_p) = (\widetilde{X}_{ab}, X_p),$$

where L_a is the left translation in the group $\mathcal{G}$. Therefore,

$$(dL_a)_{(b,p)}\mathcal{D}_{(b,p)} = \mathcal{D}_{(ab,p)}$$

for any point $(b,p) \in \widetilde{\mathcal{G}} \times \mathcal{X}$ and any element $a \in \widetilde{\mathcal{G}}$ (by definition, this means that the distribution $\mathcal{D}$ is *invariant* with respect to the action of the group $\widetilde{\mathcal{G}}$

on $\widetilde{\mathcal{G}}\times\mathcal{X}$). This directly implies that *for any leaf* $\mathcal{L}$*, the submanifold* $a\mathcal{L} = L_a\mathcal{L}$ *is also a leaf.*

Therefore, the correspondence $\mathcal{L} \mapsto a\mathcal{L}$ defines the action of the group $\widetilde{\mathcal{G}}$ on the set $\{\mathcal{L}\}$ of all leaves. If $(a,p) \in \mathcal{L}$, then $(e,p) \in a^{-1}\mathcal{L}$ and therefore $a^{-1}\mathcal{L} = \mathcal{L}_p$, i.e., $\mathcal{L} = a\mathcal{L}_p$. This proves that *any leaf* $\mathcal{L}$ *has the form* $a\mathcal{L}_p$*, where* $a \in \widetilde{\mathcal{G}}$ and $p \in \mathcal{X}$. By definition, this means that each orbit of the action of the group $\widetilde{\mathcal{G}}$ on the set $\{\mathcal{L}\}$ contains a leaf of the form $\mathcal{L}_p$.

We note that because each leaf is a conservative submanifold (a submanifold $\mathcal{Y}$ of a manifold $\mathcal{X}$ is said to be *conservative* if for any smooth manifold $\mathcal{Z}$, the mapping $\varphi\colon \mathcal{Z} \to \mathcal{Y}$ iff it is smooth as a mapping into $\mathcal{X}$, i.e., a smooth mapping $\imath \circ \varphi\colon \mathcal{Z} \to \mathcal{X}$, where $\imath\colon \mathcal{Y} \to \mathcal{X}$ is an immersion), *the mapping*

$$\mathcal{L} \to a\mathcal{L}, \quad (b,p) \mapsto (ab,p), \quad (b,p) \in \mathcal{L},$$

of the leaf $\mathcal{L}$ *onto the leaf* $a\mathcal{L}$ *induced by the left translation* L_a *is a diffeomorphism.* Let $\pi_p\colon \mathcal{L}_p \to \widetilde{\mathcal{G}}$ be the restriction of the projection

$$\mathrm{pr}_1\colon \widetilde{\mathcal{G}} \times \mathcal{X} \to \widetilde{\mathcal{G}}, \quad (a,q) \mapsto a,$$

to $\mathcal{L}_p$. Because the submanifold $\mathcal{L}_p$ is conservative, *the mapping* π_p *is smooth.*

Because $(d\,\mathrm{pr}_1)_{(a,p)}(\widetilde{X}_a, X_q) = \widetilde{X}_a$ for any field $X \in \mathfrak{g}$ and any point $(a,q) \in \widetilde{\mathcal{G}} \times \mathcal{X}$, the differential $(d\pi_p)_{(a,q)}$ of the mapping π_p at the point $(a,q) \in \mathcal{L}_p$ is an isomorphism of the space $\mathcal{D}_{(a,q)} = \mathbf{T}_{(a,q)}\mathcal{L}_p$ onto the space $\mathbf{T}_a\widetilde{\mathcal{G}}$. Therefore, *the mapping* π_p *is étale*, and it is therefore invertible on a neighborhood V_p of the point (e,p) in $\mathcal{L}_p$, i.e., there exists a neighborhood U_p of the identity e of the group $\widetilde{\mathcal{G}}$ and a smooth mapping

$$\varphi_p\colon U_p \to \mathcal{L}_p \tag{1}$$

such that $\pi_p \circ \varphi_p = \mathrm{id}$ on U_p.

Lemma 10.1. *The neighborhood* U_p *can be chosen one and the same for all points* $p \in \mathcal{X}$.

The neighborhood U_p chosen in such a way is denoted by U. It is clear that we can assume without loss of generality that *the neighborhood* U *is connected.*

We prove Lemma 10.1 below in order not to interrupt the proof of the theorem.

Let $m \geq 1$, and let U^m be the set of all elements of the group $\widetilde{\mathcal{G}}$ of the form $a_1 \cdot \cdots \cdot a_m$, where $a_1, \ldots, a_m \in U$. We suppose that $U^m \subset \pi_p\mathcal{L}_p$. (Because $\pi_p \circ \varphi_p = \mathrm{id}$ on U, this holds for $m = 1$.) By definition, the inclusion $U^m \subset \pi_p\mathcal{L}_p$ means that for any element $a \in U^m$, there exists a point $q \in \mathcal{X}$ such that $(a,q) \in \mathcal{L}_p$, i.e., $\mathcal{L}_p = a\mathcal{L}_q$. But because $\pi_q \circ \varphi_q = \mathrm{id}$ on U, we have $U \subset \pi_q\mathcal{L}_q$ and therefore

$$aU \subset a(\pi_q\mathcal{L}_q) = \pi_p\mathcal{L}_p.$$

Therefore, because the element $a \in U^m$ is arbitrary, we have $U^{m+1} = U^m U \subset \pi_p \mathcal{L}_p$. By induction, this proves that $U^m \subset \pi_p \mathcal{L}_p$ for any $m \geq 1$ and therefore (because the group $\widetilde{\mathcal{G}}$ is generated by the neighborhood U) $\widetilde{\mathcal{G}} = \pi_p \mathcal{L}_p$. Therefore, *the mapping* $\pi_p \colon \mathcal{L}_p \to \widetilde{\mathcal{G}}$ *is surjective.*

In particular, this implies that for any element $a \in \widetilde{\mathcal{G}}$, we have a point of the form (a^{-1}, q), where $q \in \mathcal{X}$, in the leaf $\mathcal{L}_p$. Moreover, $(e, q) \in a\mathcal{L}_p$ and therefore $\mathcal{L}_q = a\mathcal{L}_p$. Because the leaves $a\mathcal{L}_p$ are (as we know) all leaves of the distribution $\mathcal{D}$, this proves that *any leaf of the distribution* $\mathcal{D}$ *has the form* $\mathcal{L}_p$, $p \in \mathcal{X}$, i.e., the mapping $p \mapsto \mathcal{L}_p$ of the manifold $\mathcal{X}$ onto the set $\{\mathcal{L}_p\}$ is surjective.

We now more carefully consider the open set $V_p = \varphi_p U$ (in $\mathcal{L}_p$ and therefore in $\pi_p^{-1} U = \mathcal{L}_p \cap (U \times \mathcal{X})$) that is diffeomorphically projected on U. Let (a, q) be a point in $\mathcal{L}_p$ such that $\pi_p(a, q) \in U$ but $(a, q) \notin V_p$ (i.e., such that $a \in U$ but $(a, q) \neq \varphi_p(a)$). Because the manifold $\widetilde{\mathcal{G}} \times \mathcal{X}$ is Hausdorff, the manifold $\mathcal{L}_p$ is also Hausdorff (prove this!), and the points (a, q) and $\varphi_p(a)$ therefore admit disjoint neighborhoods $\mathcal{O}_1$ and $\mathcal{O}_2 \subset V_p$ in $\mathcal{L}_p$. Because $\pi_p(a, q) = \pi_p(\varphi_p(a))$, we can assume without loss of generality that $\pi_p \mathcal{O}_1 = \pi_p \mathcal{O}_2$, and therefore $\mathcal{O}_1 \cap V_p = \emptyset$ (because the mapping π_p is injective on V_p). Therefore, each point $(a, q) \in \pi_p^{-1} U$ that does not belong to V_p admits a neighborhood in $\mathcal{L}_p$ that does not intersect V_p. This means that the set V_p is not only open but also closed in $\pi_p^{-1} U$. Because the set V_p (being diffeomorphic to the neighborhood U) is connected, this proves that *the set* V_p *is a connected component of the set* $\pi_p^{-1} U$.

This component is characterized by the fact that $(e, p) \in V_p$. Any other component of the set $\pi_p^{-1} U$ contains a point of the form (e, q) and therefore has the form V_q, where q is a point for which $\mathcal{L}_q = \mathcal{L}_p$. Because V_q is also diffeomorphically projected on U, this proves that *the mapping* π_p *exactly covers the neighborhood* U.

For any point $a \in \widetilde{\mathcal{G}}$, the set aU is its neighborhood, and because any connected component of the set $\pi_p^{-1}(aU)$ has the form aV_q, where $(a, q) \in \mathcal{L}_p$, and, moreover, $\pi_p|_{aV_q} = a \circ \pi_q|_{V_q} \circ a^{-1}$, the mapping π_p regularly covers this neighborhood. Therefore, *the mapping* π_p *is a covering.* (We recall that the leaf $\mathcal{L}_p$, as well as the group $\widetilde{\mathcal{G}}$, is a connected manifold.) Because the group $\widetilde{\mathcal{G}}$ is simply connected by assumption and therefore has only trivial coverings, *the mapping* π_p *is a diffeomorphism.*

In particular, this means that any leaf $\mathcal{L} = \mathcal{L}_p$ contains only one point of the form (e, q), $q \in \mathcal{X}$ (namely, the point (e, p)), i.e., *the mapping* $p \mapsto \mathcal{L}_p$ *of the manifold* $\mathcal{X}$ *onto the set* $\{\mathcal{L}\}$ *of leaves of the distribution* $\mathcal{D}$ *is bijective.*

Therefore, the action of the group $\widetilde{\mathcal{G}}$ on the set $\{\mathcal{L}\}$ is transported to $\mathcal{X}$. (For any element $a \in \widetilde{\mathcal{G}}$ and any point $p \in \mathcal{X}$, the point $ap \in \mathcal{X}$, by definition, is a point q in $\mathcal{X}$ such that $\mathcal{L}_q = a\mathcal{L}_p$.) To study this action, we need to make the choice of the neighborhood U more precise and describe the mappings φ_p explicitly.

Lemma 10.2. *The Lie algebra* $\mathfrak{g}$ *admits a basis*

$$X_1, \quad \dots, \quad X_m, \tag{2}$$

consisting of smooth vector fields.

We also defer the proof of Lemma 10.2.

This lemma implies that the formulas

$$\beta(t_1X_1 + \cdots + t_mX_m) = \exp t_1\widetilde{X}_1 \cdot \cdots \cdot \exp t_m\widetilde{X}_m,$$
$$\varphi(t_1X_1 + \cdots + t_mX_m) = \varphi_{t_m}^{X_m} \circ \cdots \circ \varphi_{t_1}^{X_1}$$

correctly define certain mappings

$$\beta: \mathfrak{g} \to \widetilde{\mathcal{G}} \qquad \text{and} \qquad \varphi: \mathfrak{g} \to \mathcal{G}.$$

It is easy to see that the mapping β is étale at the point 0, i.e., this point admits a neighborhood $U^{(0)}$ in the algebra $\mathfrak{g}$ on which the mapping β is a diffeomorphism onto a certain neighborhood U of the identity of the group $\widetilde{\mathcal{G}}$ (we see below that the latter neighborhood is the neighborhood U_p for any point $p \in \mathcal{X}$; therefore (as the notation hints), we can consider it the neighborhood U in Lemma 10.1).

Lemma 10.3. *The formula*

$$\varphi_p(a) = (a, ((\varphi \circ \beta^{-1})a)p), \quad p \in \mathcal{X}, \quad a \in U, \tag{3}$$

defines a smooth mapping

$$\varphi_p: U \to \mathcal{L}_p.$$

We also defer the proof of this lemma.

Because $\varphi_p(e) = (e, p)$ and $\pi_p \circ \varphi_p = \mathrm{id}$, Lemma 10.1 is a direct consequence of Lemma 10.3 (and the mappings φ_p are mappings (1)).

If $q = ((\varphi \circ \beta^{-1})a)^{-1}p$, then $\varphi_q(a) = (a, p)$ and hence $(a, p) \in \mathcal{L}_q$, i.e., $(e, p) \in a^{-1}\mathcal{L}_q$. Therefore, $\mathcal{L}_q = a\mathcal{L}_p$, i.e., $q = ap$. This means that the action

$$\widetilde{\mathcal{G}} \times \mathcal{X} \to \mathcal{X}, \qquad (a, p) \mapsto ap \tag{4}$$

on $U \times \mathcal{X}$ is given by

$$ap = ((\varphi \circ \beta^{-1})a)^{-1}p,$$

which directly implies that *this action is smooth* (on U and therefore on the whole group $\widetilde{\mathcal{G}}$).

Moreover, we see that for and $a = \beta(t_1X_1 + \cdots + t_mX_m) \in U$, the formula

$$L_a = \varphi_{t_1}^{-X_1} \circ \cdots \circ \varphi_{t_m}^{-X_m}$$

holds for the mapping $L_a: p \mapsto ap$. In particular, this shows that $L_a \in \mathcal{G}$. Because the neighborhood U generates a connected group $\widetilde{\mathcal{G}}$, this inclusion also holds for any element $a \in \widetilde{\mathcal{G}}$. Therefore, *the formula*

$$L(a) = L_a, \quad a \in \widetilde{\mathcal{G}},$$

assigns a mapping

$$L\colon \widetilde{\mathcal{G}} \to \mathcal{G}$$

(which is obviously homomorphic).

For $a = \exp t\widetilde{X}_i$, we have $L_a = \varphi_t^{-X_i}$. By definition, this means that the monomorphism $\mathfrak{l}\widetilde{\mathcal{G}} \to \mathfrak{a}\mathcal{X}$ induced by action (4) is given by $\widetilde{X} \mapsto X$ (on the base vectors and therefore everywhere), i.e., it coincides with the initial isomorphism $\mathfrak{l}\widetilde{\mathcal{G}} \to \mathfrak{g}$. Therefore, first, all vector fields $X \in \mathfrak{g}$ are complete, and, second,

$$L(\exp t\widetilde{X}) = \varphi_t^{-X}$$

for each such field. Because all diffeomorphisms φ_t^X, $t \in \mathbb{R}$, $X \in \mathfrak{g}$, generate the group $\mathcal{G}$ by assumption, this implies that *the homomorphism L is an epimorphism* (and therefore generates the isomorphism $\lambda\colon \widetilde{\mathcal{G}}/K \to \mathcal{G}$, where $K = \operatorname{Ker} L$).

If $\exp t\widetilde{X} \in K$ for all $t \in \mathbb{R}$, then $\varphi_t^X = \mathrm{id}$; this is possible only if $X = 0$. Therefore, $\mathfrak{l}K = 0$, and K is hence a discrete invariant subgroup of the group $\widetilde{\mathcal{G}}$. In particular, the subgroup K is closed, and the quotient group $\widetilde{\mathcal{G}}/K$ is therefore a Lie group acting smoothly on $\mathcal{X}$ (and locally isomorphic to the Lie group $\widetilde{\mathcal{G}}$). Because the isomorphism λ transports the smoothness from $\widetilde{\mathcal{G}}/K$ to $\mathcal{G}$, we obtain a smoothness on the group $\mathcal{G}$ that satisfies the conditions of Theorem 10.1.

Exercise 10.2. Show that the smoothness on $\mathcal{G}$ in Theorem 10.1 is unique.

To complete the proof of Theorem 10.1, it remains to prove Lemmas 10.2 and 10.3 (as already noted, Lemma 10.3 implies Lemma 10.1).

Lemma 10.4. *For any complete vector fields $X_1, \dots, X_k \in \mathfrak{g}$ and for any point $p \in \mathcal{X}$, the formula*

$$\gamma_p(\boldsymbol{t}) = (\exp t_1\widetilde{X}_1 \cdot \dots \cdot \exp t_k\widetilde{X}_k, (\varphi_{t_k}^{X_k} \circ \dots \circ \varphi_{t_1}^{X_1})p), \tag{5}$$

where $\boldsymbol{t} = (t_1, \dots, t_k) \in \mathbb{R}^k$, defines a smooth mapping $\gamma_p\colon \mathbb{R}^k \to \mathcal{L}_p$.

Proof. Because the submanifold $\mathcal{L}_p$ is conservative, it suffices to prove that $\gamma_p(\boldsymbol{t}) \in \mathcal{L}_p$ for all $\boldsymbol{t} \in \mathbb{R}^k$.

We first let $k = 1$. In this case, γ_p is a curve $t \mapsto (\beta(t), \varphi(t))$, where $\beta(t) = \exp t\widetilde{X}$, $\varphi(t) = \varphi_t^X(p)$, and $X = X_1$, passing through the point (e, p) for $t = 0$. Moreover, $\dot{\gamma}_p(t) = (\dot{\beta}(t), \dot{\varphi}(t))$, where $\dot{\beta}(t) = \widetilde{X}_{\beta(t)}$ and $\dot{\varphi}(t) = X_{\varphi(t)}$, and therefore $\dot{\gamma}_p(t) \in \mathcal{D}_{\gamma(t)}$ for any $t \in \mathbb{R}$. This means that γ_p is an integral curve of the distribution $\mathcal{D}$. Therefore, $\gamma_p(t) \in \mathcal{L}_p$ for all $t \in \mathbb{R}$.

We suppose that Lemma 10.4 is already proved for the fields $X_1, \dots, X_{k-1}$, $k > 1$, and let

$$a = \exp t_1\widetilde{X}_1 \cdot \dots \cdot \exp t_{k-1}\widetilde{X}_{k-1},$$

$$q = (\varphi_{t_{k-1}}^{X_{k-1}} \circ \dots \circ \varphi_{t_1}^{X_1})p.$$

Then $(a, q) \in \mathcal{L}_p$ and therefore $a\mathcal{L}_q = \mathcal{L}_p$. On the other hand, applying the just proved case where $k = 1$ to the field X_k and the point q, we obtain

$$(\exp t_k \widetilde{X}_k, \varphi_{t_k}^{X_k}(q)) \in \mathcal{L}_q.$$

Therefore, $\gamma_p(t) \in \mathcal{L}_p$. □

Proof (of Lemma 10.3). Mapping (5) constructed for basis (2) (for $k = m$) is connected with mapping (3) by

$$\gamma_p = \varphi_p \circ \beta \circ \chi^{-1} \quad \text{on } U,$$

where $\chi \colon \mathfrak{g} \mapsto \mathbb{R}^m$ is the coordinate isomorphism corresponding to basis (2). □

We note that Lemma 10.2 is used in this proof. Therefore, everything is reduced to the proof of this lemma, for which we need the following elementary algebraic assertion.

Lemma 10.5. *Let a Lie algebra $\mathfrak{g}$ be generated by a set S. If S is closed with respect to multiplication by numbers (i.e., $tX \in S$ for any $X \in S$ and $t \in \mathbb{R}$) and if*

$$e^{\operatorname{ad} X} Y \in S \quad \text{for } X, Y \in S, \tag{6}$$

then S linearly generates $\mathfrak{g}$.

Proof. Let $\mathcal{S}$ be the linear span of the set S. Because

$$[X, Y] = (\operatorname{ad} X)Y = \lim_{t \to 0} \frac{e^{t \operatorname{ad} X} Y - Y}{t},$$

we have $[X, Y] \in \mathcal{S}$ for any $X, Y \in S$ and therefore for any $X, Y \in \mathcal{S}$ (because of linearity). Therefore, $\mathcal{S}$ is a subalgebra of the Lie algebra $\mathfrak{g}$ that contains S, and then $\mathcal{S} = \mathfrak{g}$. □

Proof (of Lemma 10.2). Lemma 10.2 is equivalent to the assertion that the set S of all complete fields from $\mathfrak{g}$ (which generates the Lie algebra $\mathfrak{g}$ by assumption) linearly generates $\mathfrak{g}$. Because this set is obviously closed with respect to multiplication by numbers, it therefore suffices (by Lemma 10.5) to prove that the set S satisfies condition (6).

Let $X, Y \in S$, $t \in \mathbb{R}$, and let

$$\beta(t) = \exp \widetilde{X} \exp t\widetilde{Y} \exp(-\widetilde{X}),$$
$$\varphi(t) = (\varphi_1^X \circ \varphi_t^Y \circ \varphi_{-1}^X)p.$$

Because $\beta(t) = \operatorname{int}_a(\exp t\widetilde{Y})$, where $a = \exp \widetilde{X}$, and therefore

$$\beta(t) = \exp t(\operatorname{Ad} a)\widetilde{Y} = \exp t\, e^{\operatorname{ad} \widetilde{X}} \widetilde{Y},$$

we have $\dot{\beta}(0) = \widetilde{Z}_e$, where $Z = e^{\operatorname{ad} X} Y$; we recall that $\operatorname{Ad} a = \mathfrak{l}(\operatorname{int}_a))$ by definition. But because the inclusion $(\beta(t), \varphi(t)) \in \mathcal{L}_p$ holds for any $t \in \mathbb{R}$ according to Lemma 10.4, we have $(\dot{\beta}(0), \dot{\varphi}(0)) \in \mathcal{D}_{(e,p)}$ and therefore $(\widetilde{Z}_e, \dot{\varphi}(0)) \in \mathcal{D}_{(e,p)}$, i.e., $\dot{\varphi}(0) = Z_p$. By the arbitrariness of the point p, this means that the flow $\{\varphi_1^X \circ \varphi_t^Y \circ \varphi_{-1}^X\}$ is generated by the vector field Z. Therefore, this field is complete. Moreover, because $(\widetilde{Z}_e, Z_p) \in \mathcal{D}_{(e,p)}$ for any point $p \in \mathcal{X}$, the field Z belongs to $\mathfrak{g}$ by the definition of the subspaces $\mathcal{D}_{(e,p)}$. Being a complete field from $\mathfrak{g}$, the field Z lies in S. □

Therefore, Theorem 10.1 is completely proved. □

Remark 10.1. In proving Theorem 10.1, we showed that *any subalgebra* $\mathfrak{g}$ *of the Lie algebra* $\mathfrak{a}\mathcal{X}$ *satisfying conditions* 1 *and* 2 *of Theorem* 10.1 *consists of complete vector fields.*

Theorem 10.1 is known as the *Palais theorem.*

§4. Kobayashi Theorem

Exercise 10.3. Let $\mathcal{G}$ be a group, and let $\mathcal{G}_0$ be its invariant subgroup. Show that if

1. the group $\mathcal{G}_0$ is a Lie group and
2. all inner automorphisms

$$\mathrm{int}_a\colon x \mapsto axa^{-1}, \quad a, x \in \mathcal{G}$$

are smooth mappings on $\mathcal{G}_0$,

then there exists a unique smoothness on $\mathcal{G}$ with respect to which the group $\mathcal{G}$ is a Lie group whose component of the identity is $\mathcal{G}_0$.

The proof of the following variant of the Palais theorem, which is known as the *Kobayashi theorem,* is based on this assertion.

Theorem 10.2. *Let $\mathcal{G}$ be a subgroup of the diffeomorphism group of a manifold $\mathcal{X}$, and let S be the set of all complete fields $X \in \mathfrak{a}\mathcal{X}$ such that $\varphi_t^X \in \mathcal{G}$ for any $t \in \mathbb{R}$. If S generates a finite-dimensional Lie subalgebra $\mathfrak{g}$ in the Lie algebra $\mathfrak{a}\mathcal{X}$, then $S = \mathfrak{g}$, and the group $\mathcal{G}$ admits a unique smoothness with respect to which it is a Lie group that acts smoothly and effectively on the manifold $\mathcal{X}$; moreover, the monomorphism $X \mapsto X^*$ corresponding to this action is an isomorphism of the Lie algebra $\mathfrak{l}\mathcal{G}$ onto the subalgebra $\mathfrak{g}$.*

Proof. Let $\mathcal{G}_0$ be a subgroup of the group $\mathcal{G}$ generated by all diffeomorphisms of the form φ_t^X, where $t \in \mathbb{R}$ and X is an arbitrary (complete) field from $\mathfrak{g}$. Because the Lie algebra $\mathfrak{g}$ obviously satisfies conditions 1 and 2 of Theorem 10.1, the group $\mathcal{G}_0$ admits the smoothness in this theorem (and according to the assertion in Exercise 10.2, this smoothness is unique). According to Remark 10.1, all fields from $\mathfrak{g}$ are complete; because they generate one-parameter subgroups of the group $\mathcal{G}$, we have $\mathfrak{g} \subset S$ and therefore $S = \mathfrak{g}$.

For any flow $\{\varphi_t^X\}$, $X \in \mathfrak{g}$, and any diffeomorphism $\varphi \in \mathcal{G}$, the diffeomorphism $\varphi^{-1} \circ \varphi_t^X \circ \varphi$ composes a complete flow and is contained in $\mathcal{G}$. Therefore, $\varphi^{-1} \circ \varphi_t^X \circ \varphi \in \mathcal{G}_0$. Because diffeomorphisms of the form φ_t^X generate the subgroup $\mathcal{G}_0$, this implies $\varphi^{-1}\mathcal{G}_0\varphi \subset \mathcal{G}_0$, i.e., the subgroup $\mathcal{G}_0$ is invariant.

Exercise 10.4. Show that condition 2 of Exercise 10.3 holds for the invariant subgroup $\mathcal{G}_0$.

Therefore, there exists a unique smoothness on $\mathcal{G}$ with respect to which the group $\mathcal{G}$ is a Lie group whose component of the identity is $\mathcal{G}_0$. An obvious verification shows that this smoothness satisfies all conditions of Theorem 10.2. □

§5. Affine Automorphism Group

We apply the obtained general theorems to groups that arise in differential geometry.

Proposition 10.1. *The group* Aff $\mathcal{X}$ *of affine automorphisms of an arbirary connected affine connection space $\mathcal{X}$ admits a natural smoothness with respect to which it is a Lie group acting smoothly on $\mathcal{X}$. In the case where the space $\mathcal{X}$ is geodesically complete, the Lie algebra of the group* Aff $\mathcal{X}$ *is the Lie algebra* $\mathfrak{aff}\,\mathcal{X}$ *of affine fields.*

Proof. According to Proposition 8.2, the group Aff $\mathcal{X}$ satisfies the condition of Theorem 10.2. If the space $\mathcal{X}$ is geodesically complete, then according to Proposition 8.3, the corresponding Lie algebra $\mathfrak{g}$ coincides with the algebra $\mathfrak{aff}\,\mathcal{X}$. □

§6. Automorphism Group of a Symmetric Space

Corollary 10.1. *The group* Aut $\mathcal{X}$ *of automorphisms of an arbitrary symmetric space $\mathcal{X}$ is a Lie group with the Lie algebra* $\mathfrak{d}\mathcal{X}$.

Proof. According to Proposition 6.1, the group Aut $\mathcal{X}$ coincides with the group Aff $\mathcal{X}$, and the Lie algebra $\mathfrak{d}\mathcal{X}$ coincides with the Lie algebra $\mathfrak{aff}\,\mathcal{X}$. In addition, according to the assertion in Exercise 5.1, the space $\mathcal{X}$ is geodesically complete. □

Exercise 10.5. Let $\beta: \mathbb{R} \to \mathcal{X}$ be a curve in a punctured symmetric space $\mathcal{X}$ that passes through a point p_0 for $t = 0$. Prove that the following assertions are equivalent:

1. The curve β is a morphism of symmetric spaces (where $\mathbb{R}$ is considered as a symmetric space with the multiplication $s \cdot t = 2s - t$; see Exercise 5.6).
2. The curve β is an integral curve of a certain derivation $X \in \mathfrak{s}\mathcal{X}$.

3. For the flow $\{\varphi_t^X\}$ induced by the field X, we have

$$\varphi_{2t}^X = s_{\beta(t)} \circ s_{p_0}, \quad t \in \mathbb{R}. \tag{7}$$

4. The curve β is a geodesic of the canonical connection.
5. Parallel translations along the curve β are differentials of the mappings φ_t.

§7. Translation Group of a Symmetric Space

Proposition 10.2. *The translation group* $\operatorname{Trans}\mathcal{X}$ *of an arbitrary symmetric space* $\mathcal{X}$ *is a connected subgroup of the Lie group* $\operatorname{Aut}\mathcal{X}$. *Its algebra Lie* $\operatorname{trans}\mathcal{X}$ *is the subalgebra of the Lie algebra* $\mathfrak{d}\mathcal{X}$ *generated by all fields* $X \in \mathfrak{s}\mathcal{X}$.

(Compare this with Proposition 9.3.)

Proof. The subalgebra $\mathfrak{g}$ of the Lie algebra $\mathfrak{d}\mathcal{X}$ generated by all fields $X \in \mathfrak{s}\mathcal{X}$ satisfies conditions 1 and 2 of Theorem 10.1, and the corresponding group $\mathcal{G}$ is therefore a connected Lie group acting smoothly on $\mathcal{X}$. In this case, according to (7), the group $\mathcal{G}$ is generated by all possible transformations of the form $s_{\beta(t)} \circ s_{p_0}$, where p_0 is a distinguished point of the punctured symmetric space $\mathcal{X}$ and $\beta\colon \mathbb{R} \to \mathcal{X}$ is an arbitrary geodesic of the space $\mathcal{X}$ passing through the point p_0 for $t = 0$. Therefore, $\mathcal{G} \subset \operatorname{Trans}\mathcal{X}$.

To prove the converse inclusion, for any point $p \in \mathcal{X}$, we consider the set W_p of all points $q \in \mathcal{X}$ for which $s_q \circ s_p \in \mathcal{G}$. If $p_0 \in W_p$ and U is an arbitrary normal neighborhood of the point p_0, then the transformation $s_q \circ s_{p_0}$ belongs to $\mathcal{G}$ because any point $q \in U$ has the form $\beta(t)$. Moreover, $s_{p_0} \circ s_p \in \mathcal{G}$ by assumption. Therefore, $s_q \circ s_p = s_p \circ s_{p_0} \circ s_{p_0} \circ s_q \in \mathcal{G}$ and hence $U \subset W_p$. Therefore, an arbitrary normal neighborhood of each point from W_p is contained in W_p, which directly implies (compare with the proof of Proposition 8.3) that the set W_p is open and closed in $\mathcal{X}$. Because the set W_p is not empty (because $p \in W_p$) and the space $\mathcal{X}$ is connected by assumption, this proves that $W_p = \mathcal{X}$, i.e., $s_q \circ s_p \in \mathcal{G}$ for any point $q \in \mathcal{X}$, and therefore $\operatorname{Trans}\mathcal{X} \subset \mathcal{G}$.

Therefore, $\operatorname{Trans}\mathcal{X} = \mathcal{G}$, which obviously proves Proposition 10.2. □

We note that the Lie algebra $\operatorname{trans}\mathcal{X}$ is obviously symmetric and is the minimal symmetric Lie algebra with the socle $\mathfrak{s}\mathcal{X}$.

Springer

http://www.springer.com/978-3-540-41108-6

Geometry VI
Riemannian Geometry
Postnikov, M.M.
2001, XVIII, 504 p., Hardcover
ISBN: 978-3-540-41108-6