

Contents

1. Introduction	1
1.1 Motivation	1
1.2 Background	2
1.2.1 Problem Specification and Geometry Preparation	2
1.2.2 Selection of Governing Equations and Boundary Conditions	3
1.2.3 Selection of Gridding Strategy and Numerical Method	3
1.2.4 Assessment and Interpretation of Results	4
1.3 Overview	4
1.4 Notation	4
2. Conservation Laws and the Model Equations	7
2.1 Conservation Laws	7
2.2 The Navier–Stokes and Euler Equations	8
2.3 The Linear Convection Equation	11
2.3.1 Differential Form	11
2.3.2 Solution in Wave Space	12
2.4 The Diffusion Equation	13
2.4.1 Differential Form	13
2.4.2 Solution in Wave Space	14
2.5 Linear Hyperbolic Systems	15
Exercises	17
3. Finite-Difference Approximations	19
3.1 Meshes and Finite-Difference Notation	19
3.2 Space Derivative Approximations	21
3.3 Finite-Difference Operators	22
3.3.1 Point Difference Operators	22
3.3.2 Matrix Difference Operators	23
3.3.3 Periodic Matrices	26
3.3.4 Circulant Matrices	27
3.4 Constructing Differencing Schemes of Any Order	28
3.4.1 Taylor Tables	28
3.4.2 Generalization of Difference Formulas	31

3.4.3	Lagrange and Hermite Interpolation Polynomials	33
3.4.4	Practical Application of Padé Formulas	35
3.4.5	Other Higher-Order Schemes	36
3.5	Fourier Error Analysis	37
3.5.1	Application to a Spatial Operator	37
3.6	Difference Operators at Boundaries	41
3.6.1	The Linear Convection Equation	41
3.6.2	The Diffusion Equation	44
	Exercises	46
4.	The Semi-Discrete Approach	49
4.1	Reduction of PDE's to ODE's	50
4.1.1	The Model ODE's	50
4.1.2	The Generic Matrix Form	51
4.2	Exact Solutions of Linear ODE's	51
4.2.1	Eigensystems of Semi-discrete Linear Forms	52
4.2.2	Single ODE's of First and Second Order	53
4.2.3	Coupled First-Order ODE's	54
4.2.4	General Solution of Coupled ODE's with Complete Eigensystems	56
4.3	Real Space and Eigenspace	58
4.3.1	Definition	58
4.3.2	Eigenvalue Spectrums for Model ODE's	59
4.3.3	Eigenvectors of the Model Equations	60
4.3.4	Solutions of the Model ODE's	62
4.4	The Representative Equation	64
	Exercises	65
5.	Finite-Volume Methods	67
5.1	Basic Concepts	67
5.2	Model Equations in Integral Form	69
5.2.1	The Linear Convection Equation	69
5.2.2	The Diffusion Equation	70
5.3	One-Dimensional Examples	70
5.3.1	A Second-Order Approximation to the Convection Equation	71
5.3.2	A Fourth-Order Approximation to the Convection Equation	72
5.3.3	A Second-Order Approximation to the Diffusion Equation	74
5.4	A Two-Dimensional Example	76
	Exercises	79

6. Time-Marching Methods for ODE'S	81
6.1 Notation	82
6.2 Converting Time-Marching Methods to O Δ E's	83
6.3 Solution of Linear O Δ E's with Constant Coefficients	84
6.3.1 First- and Second-Order Difference Equations	84
6.3.2 Special Cases of Coupled First-Order Equations	86
6.4 Solution of the Representative O Δ E's	87
6.4.1 The Operational Form and its Solution	87
6.4.2 Examples of Solutions to Time-Marching O Δ E's	88
6.5 The λ - σ Relation	89
6.5.1 Establishing the Relation	89
6.5.2 The Principal σ -Root	90
6.5.3 Spurious σ -Roots	91
6.5.4 One-Root Time-Marching Methods	92
6.6 Accuracy Measures of Time-Marching Methods	92
6.6.1 Local and Global Error Measures	92
6.6.2 Local Accuracy of the Transient Solution ($er_\lambda, \sigma , er_\omega$)	93
6.6.3 Local Accuracy of the Particular Solution (er_μ)	94
6.6.4 Time Accuracy for Nonlinear Applications	95
6.6.5 Global Accuracy	96
6.7 Linear Multistep Methods	96
6.7.1 The General Formulation	97
6.7.2 Examples	97
6.7.3 Two-Step Linear Multistep Methods	100
6.8 Predictor-Corrector Methods	101
6.9 Runge-Kutta Methods	103
6.10 Implementation of Implicit Methods	105
6.10.1 Application to Systems of Equations	105
6.10.2 Application to Nonlinear Equations	106
6.10.3 Local Linearization for Scalar Equations	107
6.10.4 Local Linearization for Coupled Sets of Nonlinear Equations	110
Exercises	112
7. Stability of Linear Systems	115
7.1 Dependence on the Eigensystem	115
7.2 Inherent Stability of ODE's	116
7.2.1 The Criterion	116
7.2.2 Complete Eigensystems	117
7.2.3 Defective Eigensystems	117
7.3 Numerical Stability of O Δ E's	118
7.3.1 The Criterion	118
7.3.2 Complete Eigensystems	118
7.3.3 Defective Eigensystems	119
7.4 Time-Space Stability and Convergence of O Δ E's	119

7.5	Numerical Stability Concepts in the Complex σ -Plane	121
7.5.1	σ -Root Traces Relative to the Unit Circle	121
7.5.2	Stability for Small Δt	126
7.6	Numerical Stability Concepts in the Complex λh Plane	127
7.6.1	Stability for Large h	127
7.6.2	Unconditional Stability, A -Stable Methods	128
7.6.3	Stability Contours in the Complex λh Plane	130
7.7	Fourier Stability Analysis	133
7.7.1	The Basic Procedure	133
7.7.2	Some Examples	134
7.7.3	Relation to Circulant Matrices	135
7.8	Consistency	135
	Exercises	138
8.	Choosing a Time-Marching Method	141
8.1	Stiffness Definition for ODE's	141
8.1.1	Relation to λ -Eigenvalues	141
8.1.2	Driving and Parasitic Eigenvalues	142
8.1.3	Stiffness Classifications	143
8.2	Relation of Stiffness to Space Mesh Size	143
8.3	Practical Considerations for Comparing Methods	144
8.4	Comparing the Efficiency of Explicit Methods	145
8.4.1	Imposed Constraints	145
8.4.2	An Example Involving Diffusion	146
8.4.3	An Example Involving Periodic Convection	147
8.5	Coping with Stiffness	149
8.5.1	Explicit Methods	149
8.5.2	Implicit Methods	150
8.5.3	A Perspective	151
8.6	Steady Problems	151
	Exercises	152
9.	Relaxation Methods	153
9.1	Formulation of the Model Problem	154
9.1.1	Preconditioning the Basic Matrix	154
9.1.2	The Model Equations	156
9.2	Classical Relaxation	157
9.2.1	The Delta Form of an Iterative Scheme	157
9.2.2	The Converged Solution, the Residual, and the Error	158
9.2.3	The Classical Methods	158
9.3	The ODE Approach to Classical Relaxation	159
9.3.1	The Ordinary Differential Equation Formulation	159
9.3.2	ODE Form of the Classical Methods	161
9.4	Eigensystems of the Classical Methods	162
9.4.1	The Point-Jacobi System	163

9.4.2	The Gauss–Seidel System	166
9.4.3	The SOR System	169
9.5	Nonstationary Processes	171
	Exercises	176
10.	Multigrid	177
10.1	Motivation	177
10.1.1	Eigenvector and Eigenvalue Identification with Space Frequencies	177
10.1.2	Properties of the Iterative Method	178
10.2	The Basic Process	178
10.3	A Two-Grid Process	185
	Exercises	187
11.	Numerical Dissipation	189
11.1	One-Sided First-Derivative Space Differencing	189
11.2	The Modified Partial Differential Equation	190
11.3	The Lax–Wendroff Method	192
11.4	Upwind Schemes	195
11.4.1	Flux-Vector Splitting	196
11.4.2	Flux-Difference Splitting	198
11.5	Artificial Dissipation	199
	Exercises	200
12.	Split and Factored Forms	203
12.1	The Concept	203
12.2	Factoring Physical Representations – Time Splitting	204
12.3	Factoring Space Matrix Operators in 2D	206
12.3.1	Mesh Indexing Convention	206
12.3.2	Data-Bases and Space Vectors	206
12.3.3	Data-Base Permutations	207
12.3.4	Space Splitting and Factoring	207
12.4	Second-Order Factored Implicit Methods	211
12.5	Importance of Factored Forms in Two and Three Dimensions	212
12.6	The Delta Form	213
	Exercises	214
13.	Analysis of Split and Factored Forms	217
13.1	The Representative Equation for Circulant Operators	217
13.2	Example Analysis of Circulant Systems	218
13.2.1	Stability Comparisons of Time-Split Methods	218
13.2.2	Analysis of a Second-Order Time-Split Method	220
13.3	The Representative Equation for Space-Split Operators	222
13.4	Example Analysis of the 2D Model Equation	225

13.4.1 The Unfactored Implicit Euler Method	225
13.4.2 The Factored Nondelta Form of the Implicit Euler Method	226
13.4.3 The Factored Delta Form of the Implicit Euler Method	227
13.4.4 The Factored Delta Form of the Trapezoidal Method .	227
13.5 Example Analysis of the 3D Model Equation	228
Exercises	230
Appendices	231
A. Useful Relations from Linear Algebra	231
A.1 Notation	231
A.2 Definitions	232
A.3 Algebra	232
A.4 Eigensystems	233
A.5 Vector and Matrix Norms	235
B. Some Properties of Tridiagonal Matrices	237
B.1 Standard Eigensystem for Simple Tridiagonal Matrices	237
B.2 Generalized Eigensystem for Simple Tridiagonal Matrices	238
B.3 The Inverse of a Simple Tridiagonal Matrix	239
B.4 Eigensystems of Circulant Matrices	240
B.4.1 Standard Tridiagonal Matrices	240
B.4.2 General Circulant Systems	241
B.5 Special Cases Found from Symmetries	241
B.6 Special Cases Involving Boundary Conditions	242
C. The Homogeneous Property of the Euler Equations	245
Index	247



<http://www.springer.com/978-3-540-41607-4>

Fundamentals of Computational Fluid Dynamics

Lomax, H.; Pulliam, Th.H.; Zingg, D.W.

2001, XIV, 250 p., Hardcover

ISBN: 978-3-540-41607-4