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LOGIC, TRUTH AND NUMBER: THE ELEMENTARY GENESIS OF ARITHMETIC

Abstract. Following Paul Gilmore's *LK*-style natural deduction based set theories, we present a first order logic of type-free abstraction which employs "truth value gaps" to maintain consistency. By adjoining *LK* bivalence axioms governing the definitions of numeric concepts, we deduce arithmetic from a combination of first order logic + type-free abstraction.

0. FIRST-ORDER NUMBER-THEORETIC LOGICISM

0.1. *Truth and Number*

Crispin Wright (1983) articulates a position he calls "number-theoretic logicism [1]", the claim that arithmetic is derivable from logic in the strong sense that:

it is possible to so define arithmetic concepts in terms of logical ones so that every statement of number-theory has a content preserving transcription in terms of a purely logical vocabulary, and every axiom or theorem of number theory can be so transcribed into a theorem of logic. (page 137)

Several authors (Wright 1983, Boolos 1993, Demopoulos and Bell 1993) have reconstructed Frege's *Grundlagen* program so that the required derivations of Peano's postulates are obtained from a consistent fragment of the *Grundlagen* system based upon Frege's principle of equality for the natural numbers, referred to as "Hume's principle" in Boolos 1993. However, it is commonly agreed that the reconstruction fails to logicize number-theory in the required sense due to the apparently extra-logical nature of Hume's principle.

This paper demonstrates the truth of number-theoretic logicism by deriving first-order Peano Arithmetic from a purely logical first-order theory. Although the present program is not a reconstruction of Frege's, it deploys some of Frege's central insights concerning the role of definition in the development of arithmetic and the role of the bound variable of "set", or more neutrally, "relational", abstraction in the formalization of mathematical definition. By bringing together techniques from several branches of modern logic, notably analytic proof theory, recursion theory, the lambda calculus, programming language semantics and truth theory, we are able to deploy

relational abstraction in a first-order setting in such a way that the arithmetic of the natural numbers is derived as a consequence of instances of the principle of bivalence applied to the definition of arithmetical concepts.

The idea that the mathematical practice of defining concepts can be analyzed in terms of the operation of *relational abstraction* and the cognate logical device of the *bound variable of abstraction* is a theme which unifies the logicist programs of Frege and Russell. The analysis of mathematical definition as relational abstraction is central to the logicist derivations of mathematics because definition was one of the twin pillars upon which the derivation was to be based; mathematics was to be formally derived from the definition of mathematical concepts and logical laws by logical inference alone. Relational abstraction and conversion were to effect the introduction and elimination, respectively, of definitions in the process of deduction, so that classical truth-preserving inferences could be applied to the conceptual contents of those definitions. Although logicism is now commonly regarded of at most historical interest, the theme of assimilating mathematical definition to the device of the bound variable of abstraction has continued, through the foundational programs of Church (1941), Curry (1930), Fitch (1948) and Gilmore (1980, 1986) on into the contemporary analysis of mathematical reasoning in computer languages.

Schütte (1960) showed that second-order Peano Arithmetic may be interpreted in second-order logic based upon typed relational abstraction and a single descriptive function symbol. However, it is commonly believed that there is no corresponding derivation of first-order Peano Arithmetic from a purely logical first-order theory based on abstraction. This is a point that could be of moment within the debate over number theoretic logicism, as some philosophers view the existential commitment incurred by second-order quantification on par with the posit of extra-logical set existence principles. The insufficiency of first-order logic to yield a “purely logical” derivation of arithmetic would be taken by most philosophers to be characteristic of the family of first-order languages for which Tarski defined his notion of truth on a relational structure: it is a triviality that these languages admit of finite models in the absence of descriptive vocabulary and extra-logical axioms. However, my notion of a “first-order language” is more liberal than the now proprietary sense in which that term applies to the Tarskian family of languages. By “a first-order theory” I simply mean a logical theory which does not quantify over any objects other than individuals. It is, after all, the resources of second-order *quantification* that are commonly credited with the successful derivation of the induction scheme from the classical logicist definition of the set of natural numbers as the intersection of all zero-successor sets. And, as mentioned, it is the existential commitment of such quantification that purports to undercut that success.

The first-order language L in which this derivation of arithmetic takes place fails to belong to the Tarskian language family by virtue of the type-

free conception of predication that underlies it. Because within the long shadow cast by the Tarskian tradition, the concept of a first-order language requires more than the abstinence from quantification over non-individuals. It also requires allegiance to a type-theoretic restriction on the grammar of predication that mirrors a semantic regimentation of the types of objects that can stand in the satisfaction relation. A first-order Tarskian language types syntactic combination so that the semantic function performed by the category of formulas, viz., to be predicated of, or satisfied by, an individual, cannot be performed by the category of singular terms. Singular terms, or rather the individuals they denote, bear identity conditions and existential witness, but they cannot be predicated of, or satisfied by, other individuals.

From the perspective of categorial grammar, a Tarskian first-order language thus assigns two quite distinct semantic tasks or functions to its two syntactic categories *singular term* and *formula*. The semantic value of a *singular term* is a first level individual, where the logical characteristic of a first level individual is its ability to bear identity conditions and serve existential witness, but not to be predicated of other individuals. The distinctive logical role of singular terms is realized primarily through the logic of identity and quantification. In contrast, the semantic value of a *formula* is a set, or characteristic function, of first level individuals (or tuples thereof). The semantic role of a formula is to bear satisfaction conditions, to be satisfied by first level individuals. By stratifying predication, Tarskian languages ensure that the entities that the logic of identity and quantification treat of are distinct from the entities that can bear recursively defined satisfaction conditions.

In the present essay, we interpret first-order Peano Arithmetic in a first-order language, L , designed to express type-free relational abstraction. Although L 's surface syntax indeed stratifies syntactic combination, distinguishing between the syntactic categories of *term* and *formula* in the manner of Tarskian languages, this superficial stratification masks the intended interpretation of L , whereupon abstraction terms denote *relations* and " ε " expresses the relation of *satisfaction* or *membership* holding between relations and ordered pairs of first level individuals. Our semantic intentions might be more faithfully represented by using the syntactic combination of singular terms to indicate predication, in the fashion of applicative grammars, viz., $\lambda xy. R(x, y)(a, b)$. Hence, the abstraction terms of L play simultaneously the distinctive semantic roles of Tarskian singular term and formulas: they bear identity conditions and existential witness, but also embed the meta-theoretic satisfaction relation, that is, they are general predicates of individuals.

It is easy to see that the simultaneous deployment of these two logical roles induces numerical existence principles unavailable on the Tarskian conception. For example, by applying the indiscernibility of identicals, we see that $0 =_{\text{df}} \{x : x \neq x\}$ is provably distinct from $1 =_{\text{df}} \{x : x = 0\}$, since the latter has a property not shared by the former, namely being satisfied by some object, i.e., having a member. $2 =_{\text{df}} \{x : x = 1\}$ is provably distinct

from 0 for the same reason, but also from 1, since it has a property not shared by the latter, namely being satisfied by 1. In this way, the logic of identity combines with type-free abstraction to generate the domain of natural numbers under the encoding of 0 by the empty set and the successor function by the operation of passing from an object to its singleton set. It is crucial to the generation of the sequence

$$0, \{0\}, \{\{0\}\}, \{\{\{0\}\}\} \dots$$

that the satisfaction relation we embed in L is type-free in the sense that complex singular terms can satisfy other complex singular terms, since every successor term in the sequence is satisfied by its predecessor.

0.2. The grounding of semantics and the semantics of grounding

Church's early hope was to marry type-free *functional* abstraction with full quantificational logic. In the sixties Paul Gilmore showed that type-free *relational* abstraction may be consistently combined with first-order quantification theory if one is willing to give up the principle of excluded middle and work with the notion of a "partial relation". In effect, Gilmore was taking a leaf from the recursion theorists' book; a decade or so later, this leaf would bloom in the philosophical community in the form of theories of a partial truth predicate.

Gilmore (1980, 1986) presented his theory as a first-order Gentzen-style sequent calculus, NaDSet I. The deductive rules of NaDSet I are *precisely* those of an LK identity calculus with the addition of a pair of LK -style introduction and elimination rules which formalize a very general notion of relational abstraction. However, NaDSet I is based upon a proper subset of the LK axiom sequents. While LK is based upon a complete set of "bivalence axioms" (axiom sequents of the form $A \vdash A$ for any atomic sentence A), NaDSet I's bivalence axioms are restricted to those treating identity statements and those of the form $t \in c \vdash t \in c$ for non-complex individual constants c only. In effect, consistency is maintained in NaDSet I by the strategy of evaluating only *grounded sentences*, sentences whose truth conditions reduce, under logical decomposition, to the truth conditions of simple sentences which are evaluated on the basis on "non-semantic" or "non-definition-theoretic" facts alone.

Bivalence axioms for identity statements provide a sufficient basis for a NaDSet I derivation of the theory of zero and successor (using the familiar set-theoretic encoding of those operations mentioned in the previous section). However, the absence of bivalence axioms of the form $t \in p \vdash t \in p$ governing free-object variables p undercuts a NaDSet I derivation of the scheme of mathematical induction. In the first part of this paper, we show that, although NaDSet I does not yield an induction scheme, it contains the resources for inductive definition. In section 1, we set out the calculus G , a

sub-logic of NaDSet I based on binary abstraction and bivalence axioms for identity statements. The significance of binary abstraction is shown in section 2, where it underwrites the definition of a variable binding “fixed-point” operator fix_y , a relational version of the “paradoxical” Y combinator of the lambda calculus, forming terms which compute the fixed-points of recursive functionals. We are then able to derive *LK*-style deduction rules which implement a relational version of the second recursion theorem in terms of introduction and elimination conditions for the operator fix_y .

In section 4 we use fix_y to construct a set term $\underline{\omega}$ which represents the set \mathbb{N} of natural numbers in the weak sense that $\underline{\omega}$ enumerates the set of all numerals. In section 6, $\underline{\omega}$ figures in the construction of set terms t_f that represent arbitrary primitive recursive functions $f : \mathbb{N} \rightarrow \mathbb{N}$ in the strong sense of enumerating the graph of f . Here, we use the second recursion theorem to simulate primitive recursion. Existential quantification is then used to simulate unbounded search over \mathbb{N} , allowing G to enumerate the graphs of all general recursive functions. G is thus “computationally complete”, but lacks the resources to *reason* about the relations it computes. Specifically, although $\underline{\omega}$ is infinite, G has no theorem to this effect. Similarly, G does not yield the Peano axioms which recursively specify t_+ and t_\times , the set terms that represent the addition and multiplication relations over the natural numbers.

This limitation is entirely due to the failure of bivalence in G for atomic sentences which feature those terms as predicates, and contrasts with the semantic setting of section 3, where term models validate bivalence principles for all primitive recursive terms, hence realizing the full theory of 0, S , $+$ and \times . Thus, we pass successively in sections 5 and 7 to extensions G^ω and $G^{+, \times}$ of G by adjoining bivalence axioms governing $\underline{\omega}$ and then t_+ and t_\times . For example, G^ω is the extension of G obtained by adjoining as new axioms all sequents of the form $t \varepsilon \underline{\omega} \vdash t \varepsilon \underline{\omega}$ for arbitrary closed terms t . As a result, G^ω yields “the axiom of infinity”—the statement that $\underline{\omega}$ is a 0-successor set—as a theorem, and $G^{+, \times}$ sustains relational versions of the Peano axioms specifying the inductive definition of $+$ and \times .

The following foundational paradigm emerges: in the setting of a partial classical logic which maintains unrestricted type-free abstraction by rejecting excluded middle, the posit of bivalence axioms for a primitive recursive set term can *function logically as an existence postulate*. This is seen graphically in the simplicity of the derivation of the axiom of infinity for $\underline{\omega}$ in section 7. Note that adjoining excluded middle axioms to G maintains logical purity: G^ω and $G^{+, \times}$ are obtained from sub-logics of first-order logic by closure under abstraction. And the language of these calculi is just the language L of G . For these reasons, G^ω and $G^{+, \times}$ are referred to as logical extensions of G ; the terminology is chosen purposely to contrast with the standard notion of an extra-logical extension of a logic.

Within this paradigm, progressively stronger logics are countenanced by progressively extending the “classical fragment” of G through the addition of

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