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THE PROOF THEORY OF STIG KANGER: A PERSONAL RECOLLECTION.*

I. HOW TO TURN GERHARD GENTZEN ON HIS HEAD: THE SEMANTIC COMPLETENESS OF CUT-FREE SYSTEMS

1. Semantics versus Proof Theory

The term *Proof Theory* shows a certain ambiguity. In the fifties when Stig Kanger carried out his logical work it stood for a cluster of topics pertaining to the syntactic turnstile $\vdash$, that is, the syntactic counterpart to the semantical notion of (logical) consequence $\vDash$. On the other hand, and more narrowly, it also stood for investigations of the properties of the syntactic turnstile by means of systematic transformations of derivation trees. Stig Kanger was a proof theorist only in the former sense. For him, model-theoretic semantics, couched in a rich set-theoretic framework, held pride of place, and in this he was very close to the then main European school of logic, namely the Münster School, under the leadership of Heinrich Scholz. There are indeed many questions to be asked with respect to the mere 26 (!!) non-modal pages of *Provability in Logic.*[1] Not the least of these is the question: where did Stig Kanger find his semantics? He admired Alfred Tarski above all other logicians. By the side of *Finnegan's Wake,* Tarski-Mostowski-Robinson, *Undecidable Theories,*[2] and, of course, *Der Wahrheitsbegriff in den formalisierten Sprachen,*[3] would have been with him on the Desert Island. The rare off-print copy of the German (1935) version of Tarski's masterpiece from 1933, formerly in Stockholms Högskolas Humanistiska Bibliotek, now in the University library at Stockholm, bears the mark of careful study, but it does contain the model-theoretic semantics in question only derivatively at pp. 361–62: Tarski's official definition of truth in §3, for the general calculus of classes, is not relativized to a domain of individuals, but quantifies over a universe of everything. The key-concept of model-theoretic semantics, as we now know it, is the three-place relation

$$R(\mathfrak{U}, \varphi, s) =_{\text{def}} s \text{ satisfies } \varphi \text{ in } \mathfrak{U}.\text{[4]}$$

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As far as I know Tarski only gave the explicit definition of R, by a metamathematical recursion over the complexity of the wff φ, for the first time in the classical paper by Tarski and Vaught in 1957. *Provability in Logic* was conceived in 1955, so Kanger did not take the semantics from Tarski and Vaught. If he did not make it himself (which, on the basis of Tarski's remarks in *Der Wahrheitsbegriff,* would certainly have been within his reach), a probable source is the *Mathematische Logik* by Hans Hermes and Heinrich Scholz.[5] Kanger cites the work at the appropriate place in *Provability in Logic* (at p. 16) and it is written in a style that would appeal to him. It is semantically inclined, while syntactically (almost too) precise.[6] Kanger's *Handbok i Logik* shows many similarities with that of Hermes and Scholz.[7]

Anders Wedberg, who held the Chair of Theoretical Philosophy at Stockholm during Kanger's period of study there, was also strongly influenced by Scholz, as is borne out even by a cursory inspection of *Filosofins Historia*, I–III.[8] Indeed, it is no exaggeration to say that, but for Wedberg, only Heinrich Scholz has shown the same appreciation of symbolic logic as a central tool in historical studies, as witnessed by the articles collected in *Mathesis Universalis*.[9] Scholz, however, had no empiricist leanings whatsoever. Wedberg certainly did: in my opinion, he was equally influenced by Moritz Schlick's *Allgemeine Erkenntnislehre*.[10] Scholz, on the other hand, was a classical metaphysician, who began his academic career in the Philosophy of Religion as a favourite pupil of Adolf Harnack. His Platonizing tendencies would not greatly have disturbed Stig Kanger, his fellow Platonist semanticist.[11]

2. *Confluence of Ideas in 1955: Backwards Application of Gentzen Rules*

My Oxford supervisor, the late Robin Gandy, wrote a splendid paper called *The Confluence of Ideas in 1936*, in which he dealt with the origins of recursive function theory and the many different ways of defining the same class of functions by Church, Kleene, Herbrand, Gödel, Hilbert-Bernays, Turing and Post.[12] The early history of contemporary mathematical logic is replete with such confluence of ideas. Post, Lukasiewicz and Wittgenstein provide decision-methods for the propositional calculus. Skolem and Fraenkel emend Zermelo's set theory in similar fashion. Tarski and Herbrand prove the Deduction Theorem. Gödel, Gentzen and Bernays give the intuitionistic Double-negation interpretation in 1932.

Stig Kanger played a central role in two major instances of such confluence. One of these is becoming well-known and concerns the semantics of modal logic.[13] The other concerns the new method for proving completeness of the classical predicate calculus that was independently developed around 1955 by

Ewert Willem Beth,[14] Jaakko Hintikka,[15] Stig Kanger and Kurt Schütte.[16] Beth and Hintikka took the search for a counter-model, where frustration of the search is held to be a proof, as their starting point and worked with, respectively, *semantic tableaux* and *model sets*. [The latter are now also known as 'semi-valuations' (Schütte),[17] '(analytic) consistency properties', and, perhaps even more appropriately, 'Hintikka sets' (Smullyan).] Schütte invented a new book-keeping device in terms of positive and negative parts, but it has never caught on outside his immediate circle. The Beth–Hintikka approach was streamlined by Raymond Smullyan, magician friend of Kanger's, in the 1960's, and codified in his text *First-Order Logic* in terms of 'Consistency properties'.[18] This approach has now become part of standard teaching, above all through the successful Penguin-paperback textbook *Logic* by Wilfrid Hodges.[19]

Of our four inventors only Kanger stayed close to the original Gentzen format.[20] The key observation is the following. Consider the sequent

$$S =_{\text{def}} A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_n,$$

or $\Gamma \Rightarrow \Delta$, for short. It holds (logically) when

$$A_1 \,\&\, \ldots \,\&\, A_n \supset B_1 \vee \ldots \vee B_n$$

is (logically) true. However, and this is the main discovery, the sequent S can also be read as a task to be resolved, namely: make every antecedent formula in Γ true and every succedent formula in Δ false. This is what Kanger does. For instance, in order to resolve the task

$$A \supset B, \Gamma \Rightarrow \Delta$$

(that is, the task: make $A \supset B$ and all of Γ true and make all of Δ false), it is necessary either to make all of Γ true and to make A and all of Δ false or to make B and all of Γ true and all of Δ false. Set out in schematic form this becomes:

$$\frac{\Gamma \Rightarrow \Delta, A \qquad\qquad B, \Gamma \Rightarrow \Delta}{A \supset, \Gamma \Rightarrow \Delta}$$

This, however, is nothing but an instance of $\supset\Rightarrow$, Gentzen's antecedent introduction-rule for $\supset$. Similar backwards applications of the Gentzen rules yield a systematic search-tree, along whose open branches generate certain semi-valuations — Hintikka sets. The search is frustrated when the systematic decomposition of the sequent-tasks issues in an impossible task. These take the form

$$(*) \qquad \Phi, C, \Xi \Rightarrow \Theta, C, \Psi,$$

where one would have to make the wff C both true and false, which is clearly impossible. From the other perspective, though, the sequent (*) is nothing but an axiom of the Gentzen sequent calculus, so if each branch of the search-tree is thus truncated it is converted into a sequent-calculus proof, when read from top to bottom. An open branch in the search-tree, on the other hand, allows one to read off a semi-valuation which — in the case of first-order logic — can be extended immediately to a total valuation: an atomic wff without a value in the semi-valuation is assigned the value true. We have found the desired counter-model to the original sequent. This procedure is perfectly deterministic: at each stage it can be explicitly laid down which wff to attack and, in the case of a backwards application of the quantifier-rules $\forall\Rightarrow$ and $\exists\Rightarrow$, what new free variable to choose as a witness. In this fashion we get a primitive recursive two-place function T(x, u) such that when S is a sequent and $u = \langle k_1, ..., k_n \rangle$ (with $n \geq 0$), that is, a finite sequence of natural numbers considered as a node in the universal spread, $T(\ulcorner S \urcorner, u)$ gives the (Gödel-number $\ulcorner S' \urcorner$ of the) sequent S′ (if any) that has to be placed at the node u in our derivation. (For nodes v such that no sequent is placed there, $T(\ulcorner S \urcorner, v)$ is trivially put $\equiv$ 0.) This information can be squeezed out, in some version or other, from all four approaches to the backwards proof-methods. The treatment becomes particularly smooth when one stays close to the Gentzen format, though, and a beautiful exposition of (essentially) the Kanger treatment can be found in Kleene's *second* text-book *Mathematical Logic* (as Kleene himself came to realise upon completion of his work).[21] Dag Prawitz's contribution to the Schütte-*Festschrift* from 1974 is another useful exposition.[22]

3. *Kanger's Background when Working Backwards*

How did Kanger reach this point? Who, if any, were his precursors? First, and foremost, Gödel and Gentzen. Gödel's original completeness proof can, in retrospect, be seen as working with semi-valuations, that is, minimal counter-models,[23] rather than with the total valuations that are obtained through the Lindenbaum maximalization technique, now very well-known from the Henkin completeness-proof.[24] Gentzen himself, in his dissertation from 1932 (published 1934–35), used the cut-free formalism to prove that intuitionistic propositional logic is decidable, by means of applying the rules backwards in a hypothetical derivation (which, in virtue of the *Hauptsatz*, may be assumed cut-free).[25] Kanger knew Oiva Ketonen's dissertation from 1944,[26] which gives a Gentzen-like refinement of the *Kriterien der Widerlegbarkeit des reinen*

Prädikatkalküls from Hilbert-Bernays[27]. He also knew Erik Stenius' book *Das Interpretationsproblem der formalisierten Zahlenreihe und ihre formale Widerspruchsfreiheit*,[28] which gives a Hilbert-Bernays inspired Herbrand-treatment of the proof-theory of classical predicate logic, and of arithmetic with the omega-rule.

4. How Did the Backwards Method Fare After Kanger?

i) Beth shortly afterwards designed also a version of his semantic tableaux that was complete for an intuitionistic system,[29] but, as was shown by work of Gödel, Kreisel and Dyson, the completeness proof in question was not constructive. After two decades, in the mid-seventies, W. Veldman[31] and H. de Swart[32] at Nijmegen were able to circumvent the Gödel–Kreisel obstacle by considering constructive Beth-models in which $\perp$ was allowed to be true at certain "exploding" nodes. Later refinements were given by Friedman and Dummett.[33]

ii) A decade after the Beth–Hintikka–Kanger–Schütte proof, the method was extended to the then emerging *infinitary logics*. E. G. K. Lopez-Escobar, in particular, gave a Kanger-like completeness-proof for cut-free $L_{\omega_1\omega}$, in his dissertation from 1963, with applications to Craig's Interpolation Theorem and Beth's Theorem.[34] Also Jon Barwise originally developed his version of the theory of admissible sets on a proof-theoretic basis,[35] but Makkai later eliminated the proof theory in favour of an approach in terms of infinitary consistency properties,[36] now conveniently accessible in Keisler's *Model Theory for Infinitary Logic*[37] and Barwise's *Admissible Sets and Structures.*[38]

iii) On a more modest level, it was realised, e. g. by Kent[39] and Lopez-Escobar,[40] that the Shoenfield completeness-theorem for the recursive omega-rule in arithmetic[41] was readily provable using the backwards method of proof, and then it yields even Kalmár-elementary proof-trees.

I have no information as to whether Kanger knew of the work under i)–iii). Lopez-Escobar's thesis was written under the supervision of Dana Scott, a good friend of Kanger's, and it is not improbable that it was known to him.

iv) In the mid-sixties Takeuti's conjecture concerning cut-elimination in second-order predicate calculus[42] was established by Tait,[43] Prawitz(2 ×),[44] and Takahashi.[45] Kanger knew and appreciated Prawitz's extremely elegant *Theoria*-proof: a two-sorted semi-valuation is obtained by running the backwards method. The predicate universes of this semi-valuation need not be closed under definability. The required *total* second-order counter-valuation is obtained by closing the *ramified analytical hierarchy* based on the predicate http://www.springer.com/978-1-4020-0111-6

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