

KANGER'S IDEAS ON NON-WELL-FOUNDED SETS:
SOME REMARKS

1. INTRODUCTION

1.1 Provability in Logic. Stig Kanger's small book from 1957, *Provability in Logic*, contains eight chapters. The last two chapters are concerned with modal logic. This part has received considerable attention and recognition. Chapters 2–6 treat elementary extensional logic. This part has drawn scantier attention. The present essay contains an exposition and comments on chapters 2–5, with an emphasis on the contributions to set theory and model theory. I take for granted that the reader has access to *Provability in Logic*, either the original edition from 1957 or the reprint in *Collected Papers of Stig Kanger, Vol. I*.

1.2 Content. Kanger develops a calculus LC for one fixed predicate logical language L. His intention seems to be to show that LC is all that is needed for general predicate logic. This forces him to develop new ideas on non-well-founded sets which are of great interest in their own right. These ideas are the main subject of the present essay.

I expose the language L, the calculus LC, the ideas on non-well-founded sets, and the use Kanger makes of them in the model theory for LC.

2. THE CALCULUS LC

2.1 Language. We first indicate the language L.

2.2 Primitive Symbols. The language L is built from the following symbols.

- (1) Parentheses.
- (2) Propositional constants: $p, q, p_1, q_1, p_2, q_2, \dots$
- (3) Set symbols: For $t = 1, 2, \dots$,
variables of type t : x_1^t, x_2^t, \dots
constants of type t : c_1^t, c_2^t, \dots
- (4) Notation for ordered sets: $< >$
- (5) A two-place predicate: \in

- (6) Connectives: \supset (material implication), $\&$ (conjunction), \vee (disjunction), \sim (negation)
- (7) Quantifier symbols: \forall (universal quantification), \exists (existential quantification)
- (8) A symbol for Gentzen entailment: \longrightarrow

2.3 *Formulas.* The atomic formulas of L are all expressions of the form

$$(a \in b) \text{ or } (\langle a_1 \dots a_n \rangle \in b)$$

where a , b , a_1, \dots, a_n are set symbols. The set of formulas are obtained in the usual way by closing the set of atomic formulas under the connectives and quantifiers.

2.4 *Sequents.* Let Γ and Θ be sequences of closed formulas of LC . A *quasi-sequent* is an expression of the form

$$\Gamma \longrightarrow \Theta$$

A quasi-sequent S is a *sequent* if there are infinitely many set constants of each type which do not occur in S . This distinction is relevant for the formulation of the deduction rules *10 and *13 of LC , and we shall not need it in the sequel.

2.5 *Intended Interpretation.* Kanger considers two interpretations of the atomic formula $(\langle a_1 \dots a_n \rangle \in b)$. ' \in ' may be interpreted as denoting an arbitrary 2-place relation, or it may be interpreted as representing the membership relation for sets.

2.6 *Remark.* The language L with the calculus LC is predicate logic without identity. This is important. One of Kanger's completeness theorems for LC cannot be extended to predicate calculus with identity as will be shown in Section 7.

2.7 *Remark.* All variables and constants in L are typed. This complicates somewhat the semantics of L , the deduction rules of LC , and the constructions in the proofs of the completeness theorems. It is not easy to see any justification for having several types rather than just untyped variables and untyped constants. A typing of symbols may be justified when the domain is a type structure of sets. The set universe Kanger eventually chooses for his model

theory contains non-well-founded sets and allows loops like $a \in a$ and $a \in b \in a$. This makes a typing inappropriate.

In the comments later in the essay, I will sometimes make reformulations where types are neglected.

2.8 Remark. The usual way to do general predicate logic is to consider the family $\{L_\alpha\}$ of all predicate logical languages and define a predicate calculus for each L_α . Kanger's intention is clearly to do general predicate logic; but he considers only one fixed language, the language L with one predicate \in and individual constants c_1, c_2, \dots . To motivate this approach, consider, e.g., a language L^* which contains a one-place predicate P and a 2-place predicate R and also constants c and d . Let \mathcal{A} be a model for L^* . Then

$$\begin{aligned} (2-1) \quad \mathcal{A} \models P(c) &\quad \Leftrightarrow \quad c^{\mathcal{A}} \in P^{\mathcal{A}} \\ (2-2) \quad \mathcal{A} \models R(c, d) &\quad \Leftrightarrow \quad (c^{\mathcal{A}}, d^{\mathcal{A}}) \in R^{\mathcal{A}} \end{aligned}$$

We see that atomic formulas in the semantics always are interpreted in terms of the \in -relation. This suggests the possibility of doing general predicate logic by having only one predicate, namely \in . Individual terms like c, d and predicates like P, R should then be represented by constants intended to denote atoms or sets. This is exactly how Kanger's language L is built up.

The difficulties with such an approach are considerable. The \in -relation on the right-hand side of equivalences (2-1) and (2-2) is governed by the axioms of ZF. A logical calculus for L should presumably be incomplete if the ' \in ' of L were interpreted in this way since for completeness, the calculus should have to include at least one of the non-logical ZF axioms. Kanger therefore needs to invent another set universe which contains sets not occurring in Zermelo's cumulative type structure.

It should be pointed out that the considerations stated in the present remark are my own and do not occur in Kanger's work. I nevertheless feel that they must have motivated Kanger in his work. They also make it intelligible that he attaches so great importance to normal models and completeness with respect to normal models (see paragraphs 2.11–2.14 and Section 3 below).

2.9 Semantics. Kanger defines a semantics for L . It consists of a frame together with a valuation. A *frame* for L is an infinite sequence $r = \langle r^1, r^2, \dots \rangle$ of classes where $r^1 \neq \emptyset$ and $r^t \subseteq r^{t+1}$ for $t = 1, 2, \dots$.

2.10 Remark. The frame r is the domain of the model. r^t is the class of entities of type at most t . Since $r^t \subseteq r^{t+1}$, the type structure is cumulative.

2.11 Valuations. A primary valuation is a 2-place function V . The first argument is a frame; the second argument is either a propositional constant, the predicate ' ϵ ', or a set symbol. V satisfies:

- (1) $V(r, P) = 1$ or $V(r, P) = 0$ if P is a propositional constant. 1 and 0 are the truth-values *true* and *false*, respectively;
- (2) $V(r, '\epsilon')$ is a class of finite non-unitary ordered sets of elements of r ;
- (3) $V(r, s)$ is an element of r^t if s is a set symbol of type t .

A primary valuation is *normal* if it holds for each frame r that any ordered set $\langle v_1, \dots, v_n, w \rangle$ of elements of r belongs to $V(r, '\epsilon')$ if and only if $\langle v_1, \dots, v_n \rangle$ is a member of w , for $n \in \mathbb{Z}_+$.

The primary valuation gives rise to a secondary valuation $T(r, V, S)$ which, given a frame r and a primary valuation V , assigns a truth-value, 0 or 1, to each formula or sequent S . The extension of V to T is done in the natural way.

2.12 Remark. If we use modern notation and disregard types, the semantics can be reformulated as follows.

A *structure* (sometimes called an *arbitrary structure*) is a sequence

$$(2-3) \quad \mathcal{A} = (A, \epsilon^{\mathcal{A}}, \dots, c^{\mathcal{A}}, \dots)$$

where $A \neq \emptyset$ is a set, $\epsilon^{\mathcal{A}} \subseteq A^+ \times A$ with $A^+ = \bigcup_{n>0} A^n$, and $c^{\mathcal{A}} \in A$ for each constant c in L . Thus in an arbitrary structure, ' ϵ ' is interpreted as any relation over $A^+ \times A$.

A *normal structure* is a structure \mathcal{A} such that $\epsilon^{\mathcal{A}}$ is a set theoretical membership relation. Thus if A is a class of atoms and sets, then

$$(2-4) \quad \mathcal{A} \models \langle c_1, \dots, c_n \rangle \in d \Leftrightarrow \langle c_1^{\mathcal{A}}, \dots, c_n^{\mathcal{A}} \rangle \text{ is an element of } d^{\mathcal{A}}$$

Note that the definition of a normal structure is vague and ambiguous as is Kanger's concept of a normal primary valuation. This is due to the fact that the exact meaning of "element of" is left open so far. The meaning given to "element of" by the ZF axioms is not adequate for Kanger's purposes. A main task for him is to find a more suitable concept.

2.13 Validity and Logical Truth. Let S be a sequent or sentence of L . S is *valid* if S is true in every structure. S is *logically true* if S is true in every normal structure.

2.14 *Remark.* Thus validity is defined with reference to structures \mathcal{A} where $\in^{\mathcal{A}}$ is any relation. Logical truth is defined with reference to such structures \mathcal{A} where $\in^{\mathcal{A}}$ is a set theoretical membership relation.

2.15 *The Calculus LC.* Kanger defines a *sequent calculus LC* for the language L . The rules of the calculus are essentially the usual ones for a sequent calculus, though with some minor changes. The purpose of one of these is to ensure that the proof procedure will be effective. Another one allows the exclusion of the Structural Rule from LC. The purpose of a third change is to cope with the types in the language L .

2.16 *Proofs.* Let $\Pi = \langle S, T_1, T_2, \dots \rangle$ be a sequence of sequents. Π is a *quasi-deduction* in LC of S from the class Θ of assumption sequents if the following conditions are satisfied:

- (1) Π begins with an occurrence of S ;
- (2) each component U of Π is either
 - an instance of Postulate *1 (the identity axiom), or
 - an element of Θ , or
 - inferable in one step by a rule of inference from one or two succeeding components of Π .

A *proof* in LC of a sequent S is a finite quasi-deduction of S from the empty class of assumptions. A *theorem* of LC is a sequent which has a proof in LC.

2.17 *Remark.* (I) A curiosity about the definition is that Kanger defines a deduction as a linear sequence though what he needs and actually uses is the picture of a deduction as a labeled tree.

(II) If S is a provable sequent, then a proof for S is a finite tree with S at the root and with instances of the *Identity Postulate*

$$(2-5) \quad \Gamma', B, \Gamma \rightarrow \Theta', B, \Theta$$

labeling the leaves. If S is not provable, then the proof tree for S contains a branch Σ which does not begin with an Identity Postulate and is extended as far backward as the rules of LC allow.

(III) Note that Kanger writes a deduction in the opposite order of the usual one, i.e., he writes the conclusion first and not last. When one works in ordinary formulations of the sequent calculus, one soon discovers that the only reasonable way to construct a deduction is to construct it backward starting

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