

NONSTANDARD CONSTRUCTION OF STABLE TYPE EUCLIDEAN RANDOM FIELD MEASURES

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Abstract: A nonstandard construction of stable type Euclidean random fields via hyperfinite flat integrals and stable white noise is given. Moreover, a brief account on an extension of Cutland's flat integral formula for (centered) Gaussian measures on the Hilbert space l_2 to the case of Banach spaces l_p , $1 \leq p < \infty$, is presented.

Introduction

The aim of this paper is to derive a nonstandard flat integral representation for certain stable type Euclidean random field measures. In the case of Gaussian Euclidean random field measures, this was done in [3] where a flat integral formula for Nelson's free field measure has been given. In his seminal paper [6], Cutland studied the nonstandard flat integral representation of Wiener measure on the classical Wiener space $C_0[0, 1]$, which gives a nonstandard justification of Donsker's (heuristic) "flat integral". He then used such representation to give a fairly simple

and intuitive nonstandard proof of a Schilder's large deviation principle for the Wiener measure. Furthermore, in [7, 8, 9, 10], he extended such investigations to various (centered) Gaussian measures, which provides a shorter (and nonstandard) version of the large deviation principle discussed in general, for centered Gaussian measures on separable Banach spaces, in section III.3.4 of [11]. Let us also mention the interesting work [19] by Osswald, where he presents a further nonstandard construction of Brownian motion in abstract Wiener spaces based on [6]. In the last section of this paper, we will present shortly an extension of Cutland's work [9] on flat integral representation for measures on l_2 to the case of the Banach spaces l_p , $1 \leq p < \infty$.

The study of large deviations for (non Gaussian) Euclidean random field measures seems delicate but possible. Also, one may expect to be able to discuss the scaling limits for such random field measures. To this purpose, a flat integral formula is apparently useful. Our strategy is to find an appropriate (hyperfinite) lattice setting and to construct certain lattice measures via inverse Fourier transform, then utilizing the fact that the stable Euclidean random field measures are induced by Euclidean random fields independent at each point to make a product measure of all such lattice measures, and finally to use Loeb measure structure to get the flat integral formula. This idea has been further utilized in [4] to investigate a functional integral realization for the class of Euclidean random field models for constructive quantum field theory developed in recent years in all space-time dimensions by Albeverio, Gottschalk and Wu.

In this paper, we take for granted the familiarity with the preliminaries on nonstandard analysis and the Loeb measure construction presented e.g. in [1], [5] and [18].

1. EUCLIDEAN RANDOM FIELD MEASURES

Let $\mathcal{D} := C_0^\infty(\mathbb{R}^d)$ be the vector space consisting of all C^∞ -smooth functions on \mathbb{R}^d ($d \in \mathbb{N}$) with compact support endowed with Schwartz topology and \mathcal{D}' its (topological) dual space. Let \mathcal{B} denote the Kolmogorov σ -algebra on \mathcal{D}' generated by cylindrical sets of \mathcal{D}' (which coincides with the topological σ -algebras generated by the strong or weak topologies of \mathcal{D}'). Let $p \in (0, 2]$ be arbitrarily fixed. From [13],

$$f \in \mathcal{D} \mapsto e^{-\int_{\mathbb{R}^d} |f(x)|^p dx} \in \mathbb{R}(\subset \mathbb{C})$$

is a characteristic functional on the nuclear space \mathcal{D} . By the well-known Bochner-Minlos' theorem (see e.g. [13]), there exists a unique probability measure μ on $(\mathcal{D}', \mathcal{B})$ such that

$$C(f) := \int_{\mathcal{D}'} e^{i\langle f, \omega \rangle} d\mu(\omega) = e^{-\int_{\mathbb{R}^d} |f(x)|^p dx}, \quad f \in \mathcal{D}. \quad (1.1)$$

Moreover, there is a Euclidean random field¹ $F : \mathcal{D} \times (\mathcal{D}', \mathcal{B}, \mu) \rightarrow \mathbb{R}$ determined by $F(f, \omega) = \langle f, \omega \rangle$, $f \in \mathcal{D}$, $\omega \in \mathcal{D}'$. We call F a stable random field and its probability law μ a stable random field measure.

Now let $m > 0$ if $d = 1, 2$ and $m \geq 0$ if $d \geq 3$ and let $\alpha \in (0, 1)$. Then the following stochastic pseudo-differential equation

$$(-\Delta + m^2)^\alpha X = F$$

induces a stable type Euclidean random field $X : \mathcal{D} \times (\mathcal{D}', \mathcal{B}, \mu_X) \rightarrow \mathbb{R}$ via $X(f, \omega) = \langle f, \omega \rangle$, $f \in \mathcal{D}$, $\omega \in \mathcal{D}'$, where $\mu_X(D) := \mu((\Delta + m^2)^{-\alpha} D)$, $D \in \mathcal{B}$, whose characteristic functional

$$C_X(f) := \int_{\mathcal{D}'} e^{i\langle f, \omega \rangle} d\mu_X(\omega) = e^{-\int_{\mathbb{R}^d} |(\Delta + m^2)^{-\alpha} f(x)|^p dx}, \quad f \in \mathcal{D}. \quad (1.2)$$

Let us point out that if $p = 2$, μ and μ_X are Gaussian measures on \mathcal{D}' supported by certain Sobolev spaces with negative indices (while μ_X is just Nelson's free field measure if $\alpha = \frac{1}{2}$, which was already studied using methods of nonstandard analysis in [1] and [3]). Also F introduced here is an interesting special case of infinitely divisible (Euclidean) random fields discussed e.g. in [2] (in the terminology of [13], an infinitely divisible random field is called "a generalized random process with independent value at every point"). Our main aim here is to give a representation formula for (the non Gaussian measures) μ and μ_X . Since there is no inverse Fourier transform for probability measures on ∞ -dimensional spaces, we will realize our aim by using nonstandard analysis.

Similar methods will also be used to discuss Gaussian measures on l_p for $1 \leq p < \infty$, see Section 3.

2. NONSTANDARD CONSTRUCTION OF μ AND μ_X

Let us first give a hyperfinite representation of \mathcal{D}' by following [14, 15]. Fix a polysaturated nonstandard model. Let $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ be arbitrarily

¹By "Euclidean", we mean that the probability law is invariant under the (proper) Euclidean transformation group.

fixed and $\delta := \frac{1}{N}$, an infinitesimal. We set

$$T := \{-N, -N + \frac{1}{N}, \dots, -\frac{1}{N}, 0, \frac{1}{N}, \dots, N - \frac{1}{N}, N\} \subset {}^*\mathbb{R}$$

and $\mathcal{L} := T^d \equiv \underbrace{T \times \dots \times T}_{d \text{ times}} \subset {}^*\mathbb{R}^d, d \in \mathbb{N}$. Let ${}^*\mathbb{R}^{\mathcal{L}}$ stands for the internal space of all internal functions from \mathcal{L} into ${}^*\mathbb{R}$. We set

$$\langle f, g \rangle := \sum_{t \in \mathcal{L}} \delta^d f(t) g(t), \quad f, g \in {}^*\mathbb{R}^{\mathcal{L}}.$$

Definition 2.1 (Keßler [15]) *$f \in {}^*\mathbb{R}^{\mathcal{L}}$ is called S -continuous whenever g is infinitesimal in ${}^*\mathcal{D}(K)$ for some compact set $K \subset \mathbb{R}^d$ implies that $\langle f, g \rangle$ is infinitesimal in ${}^*\mathbb{R}$. Moreover, $f \in {}^*\mathbb{R}^{\mathcal{L}}$ is said to be \mathcal{D}' -nearstandard if $\langle f, \cdot \rangle|_{\mathcal{D}(K)}$ is S -continuous for any compact set $K \subset \mathbb{R}^d$.*

$\langle f, \cdot \rangle$ being linear on ${}^*\mathbb{R}^{\mathcal{L}}$ for $f \in {}^*\mathbb{R}^{\mathcal{L}}$, the necessary and sufficient condition for $\langle f, \cdot \rangle|_{\mathcal{D}(K)}$ to be S -continuous for any compact set $K \subset \mathbb{R}^d$ is that $\langle f, g \rangle$ is finite whenever g is finite in ${}^*\mathcal{D}(K)$. Thus, $f \in {}^*\mathbb{R}^{\mathcal{L}}$ is \mathcal{D}' -nearstandard if $\langle f, g \rangle$ is finite for any compact set $K \subset \mathbb{R}^d$ and for any $\mathcal{D}(K)$ -finite $g \in {}^*\mathcal{D}(K)$ (where $g \in {}^*\mathcal{D}(K)$ is said to be $\mathcal{D}(K)$ -finite if the internal suprema $\sup_{x \in {}^*K} |g^{(n)}(x)|, n \in \mathbb{N}$, are finite).

We denote by $Ns({}^*\mathbb{R}^{\mathcal{L}})$ the totality of \mathcal{D}' -nearstandard functions. We define the (weak) standard part mapping $st : Ns({}^*\mathbb{R}^{\mathcal{L}}) \rightarrow \mathcal{D}'$ via duality:

$$\langle st(f), g \rangle = {}^\circ(\langle f, {}^*g \rangle), \quad \forall g \in \mathcal{D}.$$

$\langle st(f), \cdot \rangle$ defines a distribution essentially because of the definition of the linear induction limit topology. The standard part mapping is continuous on each $\mathcal{D}(K)$ and hence on \mathcal{D} . On the other hand, from [14], every standard distribution $g \in \mathcal{D}'$ has a hyperfinite representation $f \in Ns({}^*\mathbb{R}^{\mathcal{L}}) : st(f) = g$. Therefore $st[Ns({}^*\mathbb{R}^{\mathcal{L}})] = \mathcal{D}'$.

Let us now turn to the construction of μ and μ_X . We begin to argue formally. In the hyperfinite lattice setting, we have $f = (f_t)_{t \in \mathcal{L}} \in {}^*\mathbb{R}^{\mathcal{L}}$ and $g = (g_t)_{t \in \mathcal{L}} \in {}^*\mathbb{R}^{\mathcal{L}}$ as hyperfinite sequences (or vectors). Since μ is the probability distribution of “a generalized random process with independent value at every point”, we have $\mu = \prod_{t \in \mathcal{L}} \mu_t$, where $\mu_t := Proj_t \mu, t \in \mathcal{L}$, the marginal probability distribution of μ . Taking a hint

from (1.1), we have for any $(f_t)_{t \in \mathcal{L}} \in {}^*\mathbb{R}^{\mathcal{L}} \cap {}^*\mathcal{D}$ (i.e., the hyperfinite segment of *f for $f \in \mathcal{D}$),

$$\int_{{}^*\mathbb{R}^{\mathcal{L}}} e^{i \sum_{t \in \mathcal{L}} \delta^d f_t q_t} \prod_{t \in \mathcal{L}} d\mu_t(q_t) = e^{-\sum_{t \in \mathcal{L}} \delta^d |f_t|^p}$$

namely

$$\prod_{t \in \mathcal{L}} \int_{{}^*\mathbb{R}} e^{i \delta^d f_t q_t} d\mu_t(q_t) = \prod_{t \in \mathcal{L}} e^{-\delta^d |f_t|^p}$$

which further implies that

$$\int_{{}^*\mathbb{R}} e^{i \delta^d f_t q_t} d\mu_t(q_t) = e^{-\delta^d |f_t|^p}.$$

Setting $\mu_t^\delta(\cdot) := \mu_t(\delta^d \cdot)$, then

$$\int_{{}^*\mathbb{R}} e^{i f_t q_t} d\mu_t^\delta(q_t) = e^{-\delta^d |f_t|^p}.$$

Remarking that the above equality is a one-dimensional Fourier transform, one can take inverse Fourier transform to get the following expression for the density of μ_t (i.e. the Radon-Nikodym derivative with respect to one-dimensional Lebesgue measure)²

$$\frac{d\mu_t^\delta(q_t)}{dq_t} = \frac{1}{2\pi} \int_{{}^*\mathbb{R}} e^{-i f_t q_t} e^{-\delta^d |f_t|^p} df_t, \quad (f_t)_{t \in \mathcal{L}} \in {}^*\mathbb{R}^{\mathcal{L}} \cap {}^*\mathcal{D}$$

where dq_t and df_t stand for one-dimensional Lebesgue measure. Clearly this paves a way for us to construct μ .

Let $(\Omega, \mathcal{A}(\Omega), P)$ be a given internal probability space. The associated Loeb space is denoted by $(\Omega, \mathcal{A}_L(\Omega), P_L)$. Let $\{\eta_t(\omega) : \omega \in \Omega\}_{t \in \mathcal{L}}$ be an internal family of independent, identically distributed ${}^*\mathbb{R}$ -valued random variables on $(\Omega, \mathcal{A}(\Omega), P)$, each $\eta_t : \Omega \rightarrow {}^*\mathbb{R}$ has (internal) density h given by

$$h(q_t) = \frac{1}{2\pi} \int_{{}^*\mathbb{R}} e^{-(i f_t q_t + \delta^d |f_t|^p)} df_t, \quad (f_t)_{t \in \mathcal{L}} \in {}^*\mathbb{R}^{\mathcal{L}} \cap {}^*\mathcal{D}, (q_t)_{t \in \mathcal{L}} \in {}^*\mathbb{R}^{\mathcal{L}}$$

² Actually this inverse Fourier transform can be computed explicitly, for instance, by using the formulae of items 82 and 83 on page 25 of F. Oberhettinger: *Fourier Transforms of Distributions and Their Inverses. A Collection of Tables.* Academic Press, New York, London, 1973. But we do not need it here.

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